July 6, 2015

SISSA
Entrance
Examination

PhD in Theoretical Particle Physics
Academic Year 2015/2016

SOLVE ONE PROBLEM AMONG I AND II AND ONE PROBLEM AMONG III AND IV.

PROBLEM I

Consider the Lagrangian

$$\mathcal{L} = y_U \ \bar{u}_{R\alpha} Q^{\alpha}_{La} H^a + y_D \ \bar{d}_{R\alpha} Q^{\alpha}_{La} \varepsilon^{ab} \, H^*_b + y_L \ \bar{e}_R L_a \varepsilon^{ab} \, H^*_b + h.c.$$

with $y_{U,D,L}$ real couplings ($\alpha = 1, 2, 3$ and $a = 1, 2$ are $SU(3)_C$ and $SU(2)_L$ indices respectively, while ε^{ab} is the two-dimensional antysimmetric tensor). Four component spinor notation is used ($\bar{\psi} = \psi^\dagger \gamma_0$). The lagrangian $\mathcal{L}$ describes the dimension-four $SU(3)_C \times SU(2)_L \times U(1)_Y$ invariant Yukawa interactions of the first generation quark and leptons with a complex scalar doublet H_a (notation in footnote 1).

1. Does $\mathcal{L}$ exhibit any additional global symmetry? *[Hint: do leptons and quarks mix?].* What is the possible physical interpretation of the conserved charges?

Consider adding a complex scalar transforming as a $SU(3)_C$ triplet, specifically $T^\alpha(3, 1)^{-1/3}$.

2. What additional Yukawa interaction terms can you add to $\mathcal{L}$? Is there any global symmetry remaining? *[Hint: take into account that the antisymmetric product of N fundamental $SU(N)$ representations is a group singlet and that, given four-component spinors ψ_a, $\psi^T_a \mathcal{C} \psi_b$ is a Lorentz scalar, $\mathcal{C}$ being the charge conjugation matrix.]*

3. Given the results above, draw a tree level Feynman diagram that leads to proton decay.

4. Purely based on dimensional arguments estimate the proton decay width Γ_P (use natural units, $\hbar = c = 1$ and assume T much heavier than the proton, $m_P \approx 1$ GeV). Assuming a proton lifetime $\tau_P \equiv \Gamma_P^{-1} > 10^{33}$ years and $\mathcal{O}(1)$ Yukawa couplings compute the lower bound on the relevant T_a mass ($\hbar \approx 10^{-24}$ GeV sec).

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[1] The quark fields $Q^{\alpha}_{La} = (u, d)^{\alpha}_L$ transform as a triplet of $SU(3)_C$ and a doublet of $SU(2)_L$ carrying $U(1)$ charge $Y = 1/6$: in short-hand notation, $Q^{\alpha}_{La}(3, 2)^{1/6}$. The fields $u^{\alpha}_R(3, 1)^{2/3}$ and $d^{\alpha}_R(3, 1)^{-1/3}$ are triplets of $SU(3)_C$ and singlets of $SU(2)_L$. The lepton fields are given by $L_a(1, 2)^{-1/2} = (\nu, e)$ and $e_R(1, 1)^{-1}$. Finally, $H^a(1, \bar{2})^{1/2} = (h^+, h^0)$ is a doublet of complex scalars (the electric charge Q within the $SU(2)_L$ doublets is given by $Y \pm 1/2$, for the up and down components respectively, while $Q = Y$ for the $SU(2)_L$ singlets).

Problem II

Let $\phi(x)$ be a complex scalar field with Lagrangian density

$$\mathcal{L}(x) = |\partial_\mu \phi(x)|^2 + m^2 |\phi(x)|^2 - \lambda |\phi(x)|^4 \,, \qquad (1)$$

where $\lambda > 0$, $m^2 > 0$, so that ϕ acquires a vacuum expectation value $|\langle \phi \rangle|^2 = v^2/2$ and we can write without loss of generality

$$\phi(x) = \frac{v + \sigma(x) + i\,\pi(x)}{\sqrt{2}},$$

where $v > 0$ and σ, π are real scalar fields.

1. Determine the squared masses of σ and π and the interactions of order three in either fields σ, π. Is there an underlying reason why the mass of π vanishes?

2. Let

   $$\delta\mathcal{L} = \frac{z}{2}\,(\partial_\mu \sigma)^2 + \xi \sigma$$

   be a correction to the lagrangian density of σ, with $0 < z \ll 1$, $0 \leq \xi \ll \lambda v^3$ Determine the correction to the squared mass of σ at leading order in z, ξ.

3. The parameters z and ξ are generated at quantum level in the theory defined by the Lagrangian (1). Draw the diagrams contributing to z and ξ at the one-loop level.

4. What is the degree of divergence of the one-loop graphs found above?

PROBLEM III

Consider a particle of mass m propagating in one spatial dimension subject to the well potential

$$V(x) = \begin{cases} 0\,, & |x| \geq a \\ -V_0\,, & |x| \leq a \end{cases}$$

1. Compute the eigenfunctions and derive the eigenvalue equation for $-V_0 \leq E \leq 0$.
2. Determine the total number of bound states associated to this system.

Consider now a particle of mass m propagating in three spatial dimensions subject to the radial potential

$$\widetilde{V}(r) = -\frac{\lambda}{r^s}\,, \qquad \lambda > 0\,. \tag{1}$$

3. Evaluate the L-dependence (sign included) of the average kinetic, potential and total energy on a trial normalizable wave function of the form $\psi(r) = N\phi(r/L)$, where L is a positive reference scale, with

   $$\lim_{r\to\infty} r\frac{d\psi(r)}{dr} = \lim_{r\to 0} r\frac{d\psi(r)}{dr} = 0.$$

   Verify that the result is in agreement with an estimate obtained by using dimensional analysis and the uncertainty principle.

4. Using the substitution $\psi(r) = \chi(r)/r$, the Schrodinger equation in 3 dimensions associated to the potential (1) can be mapped to a Schrodinger equation in 1 dimension with $x \to -x$ symmetry. Using this identification and the answer to point 2. above with $a = L$, prove that for $0 < s < 2$ the potential (1) admits an infinite number of bound states.

   [Hints: i) Focus on the asymptotic value of the potential $\widetilde{V}(r)$ *at large distances. ii) Notice that if a wave function* ψ *has energies* E *and* $\widetilde{E}$ *for an Hamiltonian with potentials* V *and* $\widetilde{V}$ *and* $\widetilde{V}(x) < V(x)\ \forall x$*, then* $\widetilde{E} < E$*.]*

Problem IV

1. Consider the Hamiltonian of the quantum one-dimensional harmonic oscillator
$$H = \frac{p^2}{2m} + \frac{1}{2} m\,\omega^2 x^2 \tag{1}$$
where x is the position operator and p the momentum operator, $p = -i\hbar\partial_x$. This Hamiltonian can be written in terms of a set of creation and annihilation operators a and $a^\dagger$, satisfying the commutation relation $[a, a^\dagger] = 1$ and defined as
$$a = \sqrt{\frac{m\omega}{2\hbar}}\Big(x + \frac{i}{m\omega}p\Big) \quad , \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\Big(x - \frac{i}{m\omega}p\Big)$$
Find the energy eigenvalues of the system and, starting from the lowest-energy normalised eigenstate $|0\rangle$, derive the expression of all higher energy (correctly normalised) eigenstates $|n\rangle$.

2. Consider now the following Hamiltonian
$$H_d = \alpha\, a^\dagger a + \beta\,(a + a^\dagger)$$
where α and β are (dimensionful) real constants and, as before, $[a, a^\dagger] = 1$. Find all energy eigenvalues of H_d.

3. Add a perturbation to the Hamiltonian (1) as
$$H_p = H + \frac{1}{4}\lambda\, x^4\,,$$
where $\lambda \ll m^2\omega^3/\hbar$. Find the *approximate* ground state energy of H_p to second order in the perturbation λ.

4. Consider a particle of mass m subject to the potential
$$V = \frac{1}{2} m\,\omega^2 x^2 \left(\frac{|x|}{L}\right)^{\epsilon} \quad , \quad |\epsilon| < 1$$
where L is a reference length scale. Estimate the energy of the ground state to first order in ϵ.

   [Hint: you might find useful the formula
$$\int_0^\infty dx\; e^{-\alpha x} x^2 \ln x = \frac{\sqrt{\pi}}{4}\alpha^{-3/2}\left[1 - \frac{1}{2}(c + \ln(4\alpha))\right]$$
   where c is Euler's constant.]