July 10, 2014

SISSA
Entrance
Examination

PhD in Theoretical
Particle Physics

SOLVE two out of the four exercises given below.

Problem 1

In a D–dimensional flat Minkowskian space–time, you are given a field theory having an energy–momentum tensor $T_{\mu\nu}$ which is: *i)* conserved $\partial^\mu T_{\mu\nu} = 0$, *ii)* symmetric $T_{\mu\nu} = T_{\nu\mu}$, and *iii)* traceless $T^\mu{}_\mu = 0$. We also assume that the operator $E = \int d^{D-1}x \, T_{00}$ (energy) is well defined with a spectrum $E \geq 0$. Given a state $|\Psi\rangle$, let

$$\mathcal{E}(t, \vec{x}) \equiv \langle\Psi| \, T_{00}(t, \vec{x}) \, |\Psi\rangle$$

be its spatial energy distribution at the time t. Show that for all positive energy state $|\Psi\rangle$ the average square radius of the region in which $\mathcal{E}(t, \vec{x})$ is not zero grows with time at a speed which rapidly approaches the speed of light. Comments on the implications of the result for the particle spectrum of the theory are welcomed.

Problem 2

Consider a quantum field theory model involving two real scalars ϕ_i, $i = 1, 2$, with canonical kinetic term and potential

$$V(\phi_1, \phi_2) = -\frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) + \frac{\lambda}{4}(\phi_1^2 + \phi_2^2)^2 + \frac{\lambda'}{2}\phi_1^2\phi_2^2, \tag{1}$$

where $\mu^2 >0$ and $\lambda > 0$.

1. Under which conditions the potential is bounded from below?
2. Identify the internal global symmetries, continuous and discrete, in the two cases $\lambda' = 0$ and $\lambda' \neq 0$.
3. Find the minimum of the potential and the corresponding mass spectrum of the physical particles.
4. Identify the symmetries unbroken in the vacuum in the two cases $\lambda' = 0$ and $\lambda' \neq 0$.

Problem 3

The one-dimensional Schroedinger equation for a particle of mass m subject to a potential $V(x)$ is

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] \psi(x,t). \tag{1}$$

Suppose that $V(x)$ has the following form:

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 \le x < a \\ v_0 & a \le x < b \\ 0 & x \ge b \end{cases} \tag{2}$$

where v_0 and a are positive.

1. Write down and describe the generic form of the solution with energy $E < v_0$.
2. Compute the integration constants in the previous solution.
3. Consider the limit $b \to a$. Find the phase of the reflected wave in this case.
4. Find the phase of the reflected wave to the lowest order in $b - a$.
5. Without doing any explicit calculation: what is the probability of finding the particle in any finite interval of the positive real axis?

Problem 4

The 3-neutrino mixing matrix in the charged current weak interaction Lagrangian – the Pontecorvo, Maki, Nakagawa, Sakata (PMNS) matrix, $U = (U_{lj})$, $l = e, \mu, \tau$, $j = 1, 2, 3$, can be written as:

$$U = (U_{lj}) = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}. \tag{1}$$

Here $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$, θ_{12}, θ_{13} and θ_{23} are neutrino mixing angles which are measured in neutrino experiments, $0 \leq \theta_{ij} \leq \pi/2$, δ is a Dirac CP violation (CPV) phase, and we have neglected the possible presence of Majorana CPV phases in U. The angles θ_{12}, θ_{13} and θ_{23} have been found experimentally to have the values corresponding to

$$\sin^2\theta_{12} \cong 0.31\,, \quad \sin^2\theta_{23} \cong 0.48\,, \quad \sin^2\theta_{13} \cong 0.024\,. \tag{2}$$

In an approach aiming to describe the neutrino mixing using symmetries, the PMNS mixing matrix in eq. (1) is assumed to originate from:

$$U = R_{12}(\theta^e_{12})\, P(\phi)\, U_{TBM}\,, \tag{3}$$

where θ^e_{12} is a real angle, ϕ is a real phase, $P(\phi)$ is a digonal phase matrix, $P(\phi) = \mathrm{diag}(1, e^{i\phi}, 1)$,

$$R_{12}(\theta^e_{12}) = \begin{pmatrix} \cos\theta^e_{12} & \sin\theta^e_{12} & 0 \\ -\sin\theta^e_{12} & \cos\theta^e_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}\,, \quad U_{TBM} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{4}$$

1. Express $\sin^2\theta_{12}$, $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$ in terms of θ^e_{12} and ϕ. What correlation between $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$ do you find? Comment the result taking into account the values quoted in eq. (2).

2. Find an expression for $\cos\phi$ in terms of θ_{12} and θ_{13}. Obtain an estimate for the value of $\cos\phi$ using the values of $\sin\theta_{12} \cong 0.557$ and $\sin\theta_{13} \cong 0.155$ from eq. (2).

3. Find a relation between ϕ and δ involving also only the angles θ_{12} and θ_{13}. Use the fact that the imaginary and the real parts of the product $U^*_{e1} U^*_{\mu 3} U_{e3} U_{\mu 1}$ of elements of U, obtained using the expressions for U given in eq. (1) and in eq. (3), should be equal.

4. Using the results obtained by solving problems 2 and 3, derive an expression for $\cos\delta$ in terms of the angles θ_{12} and θ_{13}. Using the values of $\sin\theta_{12} \cong 0.557$ and $\sin\theta_{13} \cong 0.155$ from eq. (2), obtain predictions for the values $\cos\delta$ and δ in the approach considered. Comment the result found for δ.

5. In a similar way, derive an expression for $\cos\delta$ in terms of the angles θ_{12} and θ_{13} and obtain predictions for the values $\cos\delta$ and δ in the case when the matrix U_{TBM} in eq. (3) is replaced by the bi-maximal mixing matrix U_{BM},

$$U_{BM} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} . \tag{5}$$

   Comment the result obtained for δ.