Entrance Examination

Theoretical Particle Physics

Trieste, 18 July 2012

TPP entrance examination (2012)

Three problems and a set of questions are given. You are required to solve either two problems or one problem and the set of questions. Please, write clearly. Good luck!

Problem 1

A 4D FIELD THEORY of four real scalar fields A_1, A_2, B_1, B_2 is defined by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{2}\partial_\mu A_i \partial^\mu A_i + \frac{1}{2}\sum_{i=1}^{2}\partial_\mu B_i \partial^\mu B_i - m^2(A_1 B_2 - B_1 A_2)$$

1. Derive the equations of motion.
2. What is the (continuous) symmetry group G of the kinetic term? Identify its subgroup, say H, that preserves the full Lagrangian.
3. Diagonalize the mass term and find its eigenvalues and eigenstates. Write the Lagrangian in terms of such mass eigenstates.
4. Consider now adding a new term to $\mathcal{L}$ so that the total potential becomes

$$V = m^2(A_1 B_2 - B_1 A_2) + \lambda(A_1 B_2 - B_1 A_2)^2\,, \quad \lambda > 0$$

   Find the extremes of this potential and discuss their nature.

   [Hint: in order to answer this question, it is convenient to use a polar representation for the fields]

Problem 2

A uniform beam of particles of energy E and mass m is incident from the left on a potential wall at $x = 0$ given by

$$V(x) = \frac{\hbar^2}{m} \Omega\, \delta(x)\,,$$

where $\delta(x)$ is Dirac's delta-function, and Ω is real and positive.

1. Find the amplitudes R and T of the reflected and transmitted beams.
2. Consider the cases of the opaque ($\Omega \to \infty$) and the almost transparent wall ($\Omega \to 0$). What happens in these two cases? Give your answer in terms of the reflected and transmitted amplitudes R and T you found in the previous step.
3. Assume now that $\Omega < 0$ and study the possible bound states of this potential. Are there any? If yes, give their number and energy.

Problem 3

Part A. Consider the leptonic part of the charged current (CC) weak interaction Lagrangian:

$$\mathcal{L}^{\rm CC}_{lep}(x) = -\frac{g}{2\sqrt{2}} \sum_{l=e,\mu,\tau}\sum_{j=1,2,3} \bar{l}(x)\,\gamma_\alpha(1-\gamma_5)\,U_{lj}\,\chi_j(x)\,(W^\alpha(x))^\dagger$$
$$-\frac{g}{2\sqrt{2}} \sum_{l=e,\mu,\tau}\sum_{j=1,2,3} \bar{\chi}_j(x)\,\gamma_\alpha(1-\gamma_5)\,U^*_{lj}\,l(x)\,W^\alpha(x)\,.$$

Here g is a real constant, $W^\alpha(x)$ is the the $W^\pm-$boson field, $l(x) = e(x), \mu(x), \tau(x)$ are the charged lepton fields, U is the 3×3 unitary lepton mixing matrix and $\chi_j(x)$, $j = 1, 2, 3$, are 3 Majorana neutrino fields having definite masses $m_j > 0$ and satisfying the Majorana condition:

$$C(\overline{\chi}_j(x))^T = \chi_j(x),\ j = 1, 2, 3\,.$$

Derive the constraints on the lepton mixing matix U following from the requirement of CP-invariance of $\mathcal{L}^{\rm CC}(x)$: $U_{\rm CP}\mathcal{L}^{\rm CC}(x)U^\dagger_{\rm CP} = \mathcal{L}^{\rm CC}(x_p)$, where $U_{\rm CP}$ is the unitary CP-transformation operator and $x = (x_0, \mathbf{x})$, $x_p = (x_0, -\mathbf{x})$. Assume that $l(x)$, $W^\alpha(x)$ and $\chi_j(x)$ transform as follows under the CP-symmetry operation:

$$\begin{aligned}
U_{\rm CP}\,l(x)\,U^\dagger_{\rm CP} &= \eta_l\,\gamma_0\,C(\bar{l}(x_p))^T\,,\quad |\eta_l|^2 = 1\,,\quad l = e,\mu,\tau\,,\\
U_{\rm CP}\,W_\alpha(x)\,U^\dagger_{\rm CP} &= \eta_W\,\kappa_\alpha(W_\alpha(x_p))^\dagger\,,\ |\eta_W|^2 = 1\,,\ \kappa_0 = -1\,,\kappa_{1,2,3} = +1\,,\\
U_{\rm CP}\,\chi_j(x)\,U^\dagger_{\rm CP} &= \eta^{CP}_j\,\gamma_0\,\chi_j(x_p)\,,\quad \eta^{CP}_j = i\rho_j = \pm i\,,\quad j = 1,2,3\,.
\end{aligned}$$

Here η_l, η_W are unphysical phases, $\eta^{CP}_j = i\rho_j = \pm i$ is the CP-parity of the Majorana neutrino χ_j and C is the charge conjugation matrix: $C^{-1}\gamma_\alpha C = -\gamma^{\rm T}_\alpha$, γ_α being the Dirac gamma matrices ($\alpha = 0,1,2,3$), $C^{\rm T} = -C$, $C^\dagger = C^{-1}$.

Set $\eta_l = i$ and $\eta_W = 1$ and comment the constraints thus obtain for U_{lj}. In this case what are the CP conserving values of the phases $0 \le \delta, \beta_{1,2} \le 2\pi$, present in the elements $U_{e2} = e^{i\beta_1/2}\sin\theta_{12}\cos\theta_{13}$ $U_{e3} = e^{i(\beta_2/2-\delta)}\sin\theta_{13}$ and $U_{\mu 3} = e^{i\beta_2/2}\sin\theta_{23}\cos\theta_{13}$ if $\theta_{23} = \pi/4$, $\theta_{12} = \pi/6$, and $\theta_{13} = \pi/20$?

PART B. Consider the effective interaction Lagrangian of two Majorana neutrinos χ_1 and χ_2 with the photon:

$$\mathcal{L}_{\text{eff}}(x) = \bar{\chi}_1(x)\, \sigma_{\alpha\beta}(a - b\gamma_5)\, \chi_2(x)\, F^{\alpha\beta}(x) + \text{h.c.}\,,$$

where a and b are, in general, complex constants, $F^{\alpha\beta}(x) = \partial^\alpha A^\beta(x) - \partial^\beta A^\alpha(x)$, $A^\mu(x)$ being the 4-vector potential of the photon field. The latter transforms as follows under CP: $U_{\text{CP}}\, A^\alpha(x)\, U^\dagger_{\text{CP}} = \kappa_\alpha\, A^\alpha(x_p)$, $\kappa_0 = -1\,, \kappa_{1,2,3} = +1$. Derive the constraints on the constants a and b:

i) Following from the fact that χ_1 and χ_2 are Majorana particles.

ii) Following from the requirement of CP invariance of $\mathcal{L}_{\text{eff}}(x)$; comment the result.

iii) In the case of one Majorana neutrino (particle) coupled to the photon: $\chi_1(x) \equiv \chi_2(x) = \chi(x)$. Give a physical interpretation of the result.

[Use the Bjorken-Drell representation for the γ-matrices: $(\gamma_{1,2,3})^\dagger = -\gamma_{1,2,3}$, $(\gamma_0)^\dagger = \gamma_0$, $(\gamma_0)^2 = \mathbf{1}$ - the unit 4×4 matrix. Note also that, e.g., $U_{CP}\,\gamma_\alpha\, U^\dagger_{CP} = \gamma_\alpha$.]

Questions

- An electron has total energy equal to four times its rest energy. What is the momentum of the electron?

- The observer A, who is moving with velocity v with respect to observer B, says that B's clock is running slower than his. What does observer B says about A's clock?

- A spin-1/2 particle is in a state described by the spinor

$$\chi = A\begin{pmatrix} 1+i \\ 2 \end{pmatrix}$$

where A is a normalization constant. What is the probability of finding the particle with spin projection $S_z = -1/2\hbar$?

- Can a spin 1 particle decay into two photons?

- Which of the following is the principal decay mode of the positive muon μ^+?

  (a) $\mu^+ \to e^+ + \nu_e$

  (b) $\mu^+ \to p + \nu_\mu$

  (c) $\mu^+ \to n + e^+ + \nu_e$

  (d) $\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$

  (e) $\mu^+ \to \pi^+ + \nu_e + \bar{\nu}_\mu$

- How many independent states of polarization has the vector gauge boson Z^0 of the electroweak interactions?

- A particle of mass M decays from rest into two particles. One particle has mass m and the other particle is massless. What is the momentum of the massless particle?

- Which of the following is NOT a correct assignment for the quark composition of the corresponding hadrons?

  (a) $p = uud$

  (b) $n = udd$

  (c) $\pi^+ = u\bar{d}$

  (d) $K^+ = \bar{u}s$

  (e) $J = \bar{c}c$

- What is the value of the commutator $[H,\, x]$ for the quantum mechanical Hamiltonian $H = p^2/2m + V(x)$?

- The standard model Higgs boson is electrically neutral. How can it then decay at the LHC into two photons? Draw at least one Feynman diagram for its decay into two photons.