July 19, 2010

SISSA
Entrance
Examination

PhD in Elementary Particle Theory

SOLVE TWO OUT OF THE FIVE EXERCISES

Problem 1

1) Consider the Quantum Field Theory of a real scalar φ in d space–time dimensions specified by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\,\partial^\mu\varphi\,\partial_\mu\varphi - \frac{1}{2}\,m^2\,\varphi^2 - \frac{\lambda}{4!}\,\varphi^4 \tag{1}$$

from the point of view of perturbation theory (as an asymptotic power series in the small quartic coupling λ).

Let $N_k(d)$ be the number of *all* (namely connected and *disconnected)* vacuum Feymann diagrams at order λ^k counted (*i.e.* weighted) with the weight corresponding to the statistical factor of each diagram.

1. Discuss the dependence of $N_k(d)$ on the number of space–time dimensions d.
2. Exploiting the discussion in 1, give an estimate of $N_k(d)$ which is valid (at least) for large order k.
3. Argue that the number $N_k(d)^{\text{conn.}}$ of the *connected* diagrams of order k is of the same order (for large k) as $N_k(d)$.

2) Rescale the field as $\phi = \lambda^{1/2}\,\varphi$, so that the action reads

$$S[\phi] = \frac{1}{\lambda}\int d^dx\,\left(\frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\phi^4\right) \tag{2}$$

and λ may be identified with $\hbar$.

How many loops have the $N_k(d)^{\text{conn.}}$ vacuum diagrams appearing at order λ^k?

3) Consider the particular case $d = 1$ (Quantum Mechanics) at a finite temperature T, and assume $m^2 > 0$. Let $\Gamma_k = \Gamma_k(m,T)$ be the coefficient of λ^k in the perturbative expansion of the thermal expectation value of the operator φ^2

$$\langle\varphi^2\rangle_T \equiv \frac{\mathrm{Tr}\big[\varphi^2\,e^{-H/kT}\big]}{\mathrm{Tr}\,e^{-H/kT}} \approx \Gamma_0 + \Gamma_1\,\lambda + \Gamma_2\,\lambda^2 + \cdots\,, \tag{3}$$

and consider the modified power series

$$\sum_{k=0}^{\infty} \frac{\Gamma_k}{k!} \lambda^k \equiv G(\lambda; T). \tag{4}$$

Using the ideas underlying the computations in **1)**, determine the radius of convergence $\varrho(T)$ of the power series (4) and discuss its dependence on the temperature T. In particular, compute its behaviour as $T \to \infty$, and for low temperature $T \sim 0$.

4) In the set–up of item **3)**, extend the definition of the function $G(\lambda; T)$ beyond the disk $|\lambda| < \varrho(T)$ by analytic continuation in $\lambda \in \mathbb{C}$. Assuming that $G(\lambda; T)$ is regular on a neighborhood of the *positive* real axis, prove the equality

$$\langle \varphi^2 \rangle_T = \frac{1}{\lambda} \int_0^{\infty} e^{-z/\lambda} \, G(z; T) \, dz. \tag{5}$$

Problem 2.

Consider a particle on the real plane with Lagrangian function

$$\mathcal{L} = \frac{1}{2}\left[\left(\frac{dq_1}{dt}\right)^2 + \frac{1}{1+\frac{q_1^2}{q^2}}\left(\frac{dq_2}{dt}\right)^2\right] \tag{1}$$

where (q_1, q_2) are the coordinates of the particle in the plane and q is a real non vanishing parameter.

- Compute the Hamiltonian of the system.
- Discuss the symmetries of the system.
- Compute the transition amplitude between two states at different fixed positions in the q_1 direction and fixed momenta in the q_2 direction, that is $< (q_1)^f, (p_2)^f; t^f | (q_1)^i, (p_2)^i; t^i >$.

Problem 3.

The one-dimensional Schroedinger equation for a particle of mass m subject to a potential $V(x)$ is

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x) \right] \psi(x,t) \tag{1}$$

Suppose that $V(x)$ has the following form:

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 \leq x < a \\ v_0 & x \geq a \end{cases} \tag{2}$$

where v_0 and a are positive.

1. Write down the generic solution for a bound state and the conditions its parameters must satisfy in terms of m, v_0 and a.

2. Argue that there exists a condition on m, v_0 and a so that no bound state solutions are possible, and try to determine it as accurately as possible.

3. Evaluate the limit $v_0 \to \infty$ and determine the eigenfunctions and their energies.

4. In the limit $v_0 \to \infty$ find the average position of the particle trapped in a generic bound state.

Problem 4. Properties of Majorana Fermion Couplings

Consider the effective neutral current (NC) interaction Lagrangian:

$$\mathcal{L}_{\rm eff}(x) = -\,\lambda\,\bar{N}_1(x)\,\gamma_\mu(V - A\gamma_5)\,N_2(x)\,Z^\mu(x) + {\rm h.c.}\,, \qquad (1)$$

where λ is a real couping constant, $N_{1,2}(x)$ are the fields of two spin 1/2 Majorana fermions having masses $M_{1,2} > 0$, $Z^\mu(x)$ is the field of the Standard Model (SM) weak Z^0-boson, and V and A are complex, in general, (vector and axial current) constants. The fields $N_{1,2}(x)$ satisfy the Majorana condition:

$$C(\overline{N}_j(x))^T = \rho_j\,N_j(x),\;\; j = 1,2;\;\;\; \rho_1 = \pm 1,\;\; \rho_2 = \pm 1\,, \qquad (2)$$

where C is the charged conjugation matrix: $C^{-1}\gamma_\mu C = -\gamma_\mu^{\rm T}$, γ_μ being the Dirac gamma matrices ($\mu = 0,1,2,3$), $C^{\rm T} = -C$, $C^\dagger = C^{-1}$.

1. Derive the constraints on the constants V and A following from the fact that $N_{1,2}$ are Majorana fermions (i.e. $N_{1,2}(x)$ satisfy the condition (2)).

2. Derive the constraints which the requirement of CP invariance of $\mathcal{L}_{\rm eff}(x)$ imposes on the constants V and A, knowing that $N_j(x)$ and $Z_\mu(x)$ transform as follows under the CP-symmetry operation:

$$U_{\rm CP}\,N_j(x)\,U_{\rm CP}^\dagger \;=\; \eta_j^{CP}\,\gamma_0\,N_j(x')\,,\;\;\; \eta_j^{CP} = i\rho_j = \pm i\,, \qquad (3)$$

$$U_{\rm CP}\,Z_\mu(x)\,U_{\rm CP}^\dagger \;=\; \kappa_\mu\,Z_\mu(x')\,,\;\; \kappa_0 = -1, \kappa_{1,2,3} = +1\,, \qquad (4)$$

where $U_{\rm CP}$ is the unitary CP-transformation operator, $x = (x_0, \mathbf{x})$, $x' = (x_0, -\mathbf{x})$, and $\eta_j^{CP} = i\rho_j = \pm i$ is the CP-parity of the heavy Majorana neutrino N_j. Comment the combined result obtained from the constraints derived in points 1. and 2.

3. Consider the case of $\mathcal{L}_{\rm eff}(x) = \mathcal{L}_{\rm eff}^N(x)$ when N_2 coincides with N_1: $N_2(x) \equiv N_1(x) \equiv N(x)$, $M_2 = M_1 \equiv M$, $\rho_2 = \rho_1 \equiv \rho$. What is the CP-invarinat form of $\mathcal{L}_{\rm eff}^N(x)$?

a) Consider the process $N + N \rightarrow e^- + e^+$, generated by the CP-invariant $\mathcal{L}_{\rm eff}^N(x)$ and the Standard model coupling of $e^\pm$ to Z^0:

$$\mathcal{L}_{\rm NC}^e(x) = -\,\frac{\sqrt{g^2 + g'^2}}{2}\,\bar{e}(x)\,\gamma_\mu(v_e - a_e\gamma_5)\,e(x)\,Z^\mu(x) + {\rm h.c.}\,, \qquad (5)$$

where $v_e = -0.5 + 2\sin^2\theta_W$, $a_e = -0.5$. Assume that the initial $N+N$ are in a state with total spin $S(N+N) = 1$ and orbital momentum $L(N+N) = 0$.

Taking into account the conservation of CP-parity and of the total angular momentum, determine the orbital momentum of the $e^- + e^+$ pair, knowing that the CP-parity of the latter is given by $\eta^{CP}(e^-e^+) = (-1)^{S+1}$, S being the total spin of the $e^- + e^+$ pair.
b) Derive the additional constraints one obtains on the CP-invariant $\mathcal{L}^N_{\rm eff}(x)$ from the requirement that $\mathcal{L}^N_{\rm eff}(x)$ is i) P-invariant, ii) C-invariant (P = reflection of the 3 spatial coordinates; C = charge conjugation), knowing that under the P- and C- symmetry transformations,

$$U_{\rm P}\, N(x)\, U^\dagger_{\rm P} = \eta^P \gamma_0\, N(x'),\ \eta^P = \pm i\,;\quad U_{\rm P} Z^\mu(x)\, U^\dagger_{\rm CP} = -\,\kappa_\mu\, Z^\mu(x')\,; \tag{6}$$
$$U_{\rm C}\, N(x)\, U^\dagger_{\rm C} = \eta^C\, N(x)\,,\ \ \eta^C = \pm 1\,;\quad U_{\rm C}\, Z^\mu(x)\, U^\dagger_{\rm C} = -\, Z^\mu(x)\,. \tag{7}$$

[N.B. Use the Bjorken-Drell representation for the γ-matrices: $(\gamma_{1,2,3})^\dagger = -\gamma_{1,2,3}$, $(\gamma_0)^\dagger = \gamma_0$, $(\gamma_0)^2 = \mathbf{1}$ - the unit 4×4 matrix. Note also that, e.g., $U_{CP}\ \gamma_\mu\ U^\dagger_{CP} = \gamma_\mu$.]

Problem 5.

The three dominant decay channels of the charged pion are $\pi^+ \to \mu^+\nu_\mu$, $\pi^+ \to e^+\nu_e$, $\pi^+ \to e^+\nu_e\pi^0$.

1. Based on kinematical considerations, identify the dominant decay channel and motivate the suppression of the two subdominant ones. Discuss also the relative importance of the two subdominant channels and estimate the partial decay width ratio $\Gamma(\pi^+ \to \mu^+\nu_\mu)/\,\Gamma(\pi^+ \to e^+\nu_e)$.

2. Consider the matrix element

$$\mathcal{M}^\mu(p) \equiv \langle\Omega|\overline{u}\gamma^\mu P_L d|\pi^+(p)\rangle,$$

   where $\pi^+(p)$ is the one particle state corresponding to a pion with 4-momentum p^μ and Ω is the vacuum state. Write $\mathcal{M}^\mu(p)$ in terms of p and $f_\pi \equiv 2i(\partial\mathcal{M}^0/\partial p^0)$. Which of the two terms in $P_L = 1/2 - \gamma_5/2$ contributes to the matrix element and why?

3. Calculate $\Gamma(\pi^+ \to \mu^+\nu_\mu)$ in terms of G_V, $m_{\pi^\pm}$, m_μ, f_π.

Reminder. The charged pion $\pi^\pm$ is a pseudoscalar meson with mass $m_{\pi^\pm} \approx 140\,\text{MeV}$. Together with the neutral pion π^0, with mass $m_{\pi^0} \approx 135\,\text{MeV}$, it forms an isospin triplet. The muon and electron masses are $m_\mu \approx 105\,\text{MeV}$ and $m_e \approx 0.5\,\text{MeV}$. Below the electroweak scale, the charged pion weak interactions can be described by the effective hamiltonian density

$$\mathcal{H} = 4\frac{G_V}{\sqrt{2}}\,\overline{u}\gamma^\mu P_L d\,(\overline{e}\gamma_\mu P_L\nu_e + \overline{\mu}\gamma_\mu P_L\nu_\mu),$$

where P_L is the chirality projector on left-handed chirality and $G_V \approx 10^{-5}/\text{GeV}^2$ (which includes the effect of quark mixing). For the purpose of this exercise, the neutrinos can be considered to be massless.