SISSA
Entrance
Examination

Theory of
Elementary Particles

Trieste, 16 July 2009

Five problems are given. You are expected to solve two of them. Please, do not try to solve more than two problems; only two are going to be marked for each candidate. You have four hours of time. Please, write clearly.

Problem 1.

Consider a toy model version of the Standard Model Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu H|^2 + \bar{q}_L\gamma^\mu D_\mu q_L + \bar{q}_R\gamma^\mu\partial_\mu q_R - \lambda(H\bar{q}_L q_R + h.c.) - V(|H|)\,, \tag{1}$$

where $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative, A_μ is a $U(1)$ gauge field and $q_{L,R}$ is a quark-like fermion multiplet. $V(H)$ induces a non-vanishing vacuum expectation value $\langle |H| \rangle = v$ for the Higgs field H.

1. Show that the interactions of the physical Higgs field H^0 ($|H| = v + H^0$) with the gauge field and with the fermion are proportional to the gauge field mass and to the fermion mass, respectively.

2. Add to the Lagrangian (1) a non-abelian gauge field under which H is neutral and $q_{L,R}$ are charged (QCD–like interactions). The interaction between H and the "gluons" is not explicitly present in the Lagrangian. Where does it come from? Draw the Feynman diagram and write the mathematical expression corresponding to it (no explicit evaluation is required).

3. Using the result of point **1.**, show that the interaction between two gluons and the Higgs boson discussed at point **2.** can be deduced from the gluon polarization tensor, which can be written as

$$\Pi_{\mu\nu}(p^2) = -i\left(g_{\mu\nu}p^2 - p_\mu p_\nu\right)\left[1 + \frac{\alpha_s}{6\pi}\sum_f \ln\frac{\Lambda^2}{m_q^2} + \mathcal{O}(p^2)\right] \tag{2}$$

where Λ is the cutoff and m_q is the quark mass.
(Hint: $A(m+\delta m) = A(m) + \delta m \frac{\partial A(m)}{\partial m}$ for any mass m).

Show that this interaction can be written as an effective term in the Lagrangian density

$$\mathcal{L}' = \frac{\alpha_s}{12\pi v} H^0 G^a_{\mu\nu} G_a^{\mu\nu}\,,$$

where α_s is the strong coupling and $G^a_{\mu\nu}$ is the gluon field strength.

Problem 2.

Consider the following Lagrangian density for a single scalar field in 3+1 space-time dimensions:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{g}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4 - \frac{h}{6!}\phi^6\,, \tag{1}$$

where $m^2 > 0$.

1. For which values of the parameters m^2, g, λ and h is the theory defined by (1) renormalizable? Motivate the answer.

2. Take $\lambda = 0$. Identify the one-loop graphs responsible for the generation of the ϕ^5 and ϕ^8 operators and compute their divergent part.

3. Which ϕ^n ($n > 6$) operators can be generated when $g = 0$?

PROBLEM 3.

CONSIDER a one-dimensional quantum mechanical system with a potential $V(x)$ which can be written in terms of a function $A(x)$ as

$$V(x) = A'(x)^2 - \frac{\hbar}{\sqrt{2m}} A(x)'' \tag{1}$$

where the symbol $'$ denotes derivative with respect to x.

1. Show that the Hamiltonian of the system, $H = \frac{p^2}{2m} + V$, is semi-positive definite.

Hint: Recall that for a function $f(x)$ we have $[p, f] = -i\hbar f'$.

2. Compute the eigenfunction of the lowest energy state, ψ_0, and discuss the condition $A(x)$ should satisfy to let ψ_0 being normalizable.

3. Consider the following two cases for $A(x)$: $A \sim \alpha \log|x|$ and $A \sim \frac{1}{2}|x|^n$ (with $n \in \mathbf{Z}$). Discuss in the two cases the values of α and n for which ψ_0 is normalizable. Put $\hbar/\sqrt{2m} = 1$, for simplicity. What is the explicit form of $V(x)$ for $n = 1$?

Problem 4.

Consider the scattering of two electrons in QED at the tree level, that is at the lowest order in perturbation theory. Let the incoming electrons have momentum p_1 and p_2 and the outgoing electrons have momentum p_a and p_b. (Disregard overall numerical factors like 2, π).

1. Draw the Feynman diagrams describing the scattering amplitude and write its expression in momentum space in terms of the relativistic spinors $u_1(p_1)$, $u_2(p_2)$ describing the incoming states and $u_a(p_a)$, $u_b(p_b)$ describing the final states.

Say which normalization you have assumed for the incoming and outgoing states and for the spinors.

2. Taking the center of mass frame, derive the non-relativistic limit of the scattering amplitude, that is the limit in which the ratio of the space component of the momentum over the mass goes to zero: $\vec{p}/m \to 0$. Write it as a function of the momentum in the center of mass frame and of non-relativistic spinors.

3. Derive a non-relativistic potential which would give the same amplitude in the non-relativistic Born approximation.

Problem 5.

Consider the classical Action in 1 + 1 dimensions for a classical field $\phi(t, x)$:

$$A = -\int dt \int_0^{2\pi} dx \, \frac{\partial \phi}{\partial x} \sqrt{1 - \left(\frac{\partial \phi}{\partial t}\right)^2} \tag{1}$$

Assume periodic boundary conditions in x : $\phi(t, 0) = \phi(t, 2\pi)$.

1. Derive the equation of motion.

2. Derive the Hamiltonian.

3. Find a zero-energy classical solution for $\phi(t, x)$ which is not constant in t nor in x.