SISSA
Entrance
Examination

Theory of
Elementary Particles

Trieste, 18 July 2007

FOUR PROBLEMS are given. You are expected to solve completely two of them. Please, do not try to solve more than two problems; only two for each candidate are going to be marked. You have four hours. Please, write clearly and in block letters.

Problem 1.

This problem is about the standard model of elementary particles. It includes three questions.

1. Before knowing about the experimental existence of neutral currents, a perhaps more economical choice for the gauge structure of the standard model would have been to try and build the group $SU(2)$ with charges identified (in the lepton sector of the first generation) by the currents

$$J_\mu^- = \bar{\nu}_e \gamma_\mu (1-\gamma_5) e\,, \quad J_\mu^+ = \left(J_\mu^-\right)^\dagger$$

and

$$J_\mu^{em} = -\bar{e}\gamma_\mu e\,.$$

In this alternate standard model, neutrinos only interact with charged currents. Is this choice possible? Explain by considering the algebra of the charges.

2. Consider again only the lepton sector of the first generation of the standard model fermions, that is the electron e and the neutrino ν_e. What is the decay width for the W^- gauge boson?

Hint: Unless you happen to know it by heart, answering requires a computation. The leading order result is sufficient and you can neglect subleading terms proportional to m_e/m_W. Useful formulas:

$$\mathcal{L}_{int} = -\frac{g}{2\sqrt{2}}\bar{\psi}_{\nu_e}\gamma^\mu(1-\gamma_5)\psi_e W_\mu$$

$$\Gamma(W^- \to e\,\bar{\nu}_e)\Big|_{\text{CM frame}} = \frac{1}{8\pi^2}|T|^2\frac{|\mathbf{p}_e|}{m_W^2} \quad \text{with} \quad \langle p|p'\rangle = 2E(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')$$

$$\text{Tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})\,, \qquad g^2 = 8m_W^2 G_F/\sqrt{2}$$

3. In the gaugeless limit of the standard model in which we take both the $SU(2)$ coupling g and the $U(1)$ coupling g' to zero, what particles are stable among the gauge boson W^-, the electron e and the neutrino ν_e? Please, motivate your answer.

Problem 2.

Consider a massive fermion ψ with mass m interacting with a massive vector field Z_μ of mass M_Z and a massive scalar field H of mass M_H. The vector is like the Z^0 vector meson, the scalar is like the physical Higgs and the fermion is like a heavy fermion of the standard model after symmetry breaking, except that there are no the other fields like gluons, photon or W's and for symplicity we take the interaction to be parity conserving:

$$L_{int} = g_Z \bar{\psi} \gamma^\mu \psi Z_\mu + g_H \bar{\psi} \psi H \, .$$

Take m to be much larger than M_Z and M_H.

1. Write the elastic fermion-fermion scattering amplitude from the lowest order tree-level Feynman diagrams in terms of the spinor wave-functions in momentum space $u_i(p_i)$.

2. Take the spin of the two fermions in the initial and those in the final state to be the same. Compute the scattering amplitude for the process in 1. as a function of the scattering angle (in the center-of-mass frame) at the leading order in the non-relativistic limit—that is, take the momentum of the fermions to be negligible with respect to their mass m, but not negligible with respect to M_Z and M_H.

[Hint: Use the non-relativistic approximation for $\bar{u}\gamma_\mu u$ and $\bar{u}u$.]

3. What is the dependence on the distance r of the potential that—in the lowest order approximation for a non-relativistic scattering—reproduces the momentum dependence of the amplitude of question 2.? Is the potential for the exchange of the vector particle attractive or repulsive? And that for the exchange of the scalar particle?

Problem 3.

Consider the usual Hamiltonian of a one-dimensional harmonic oscillator:

$$H_0 = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}\,.$$

1. Add to H_0 the term ϵx^3 and compute in perturbation theory the energy spectrum at first order in ϵ.

2. Add to H_0 the term ηx^4 and compute in perturbation theory the energy spectrum at first order in η.

3. Take $\epsilon = \eta = 0$ and assume that the harmonic oscillator is in its ground state. Suddenly, at $t = 0$, a perturbation of the form $-Fx$ is added to H_0. Compute the probability to find the oscillator in the old ground state for $t > 0$ in perturbation theory at leading order in F and then exactly, to all orders in F.

PROBLEM 4.

CONSIDER the lagrangian for a scalar field $\phi(t,x)$ in two space-time dimensions

$$\mathcal{L} = \frac{1}{2}\left(\partial_t\phi\right)^2 - \frac{1}{2}\left(\partial_x\phi\right)^2 - V(\phi)\,,$$

where

$$V(\phi) = \frac{\mu^4}{2\lambda} - \mu^2\phi^2 + \frac{\lambda}{2}\phi^4\,,$$

and μ^2, λ are positive parameters. The potential is a mexican-hat like potential, with a maximum at $\phi = 0$ and two minima at $\phi = \pm a$ with $a = \sqrt{\mu^2/\lambda}$.

1. Derive the equation of motion for ϕ and show that for a static (i.e. time-independent) configuration $\phi = \phi(x)$ it can be easily integrated to

$$\frac{1}{2}\left(\partial_x\phi\right)^2 - V(\phi) = c\,,$$

with c an integration constant. Fix the value of c by requiring the time-independent field configuration to have finite total energy E.

[Hint: Express the energy density ρ as a function of ϕ and $V(\phi)$ and impose the integral $E = \int_{-\infty}^{+\infty} \rho(x)dx$ to be finite.]

2. Using the previous results, find an analytic expression for such a finite energy, time-independent but non-trivial (that is x-dependent) solution.

[Hint: Recall that $\int \frac{dy}{1-y^2} = \text{arctanh}y$, for $y^2 < 1$.]

3. Compute the total energy E of the static solution $\phi(x)$ and draw the qualitative shape of the energy density $\rho(x)$.