SISSA
Entrance
Examination

Elementary
Particle Theory
Sector

Trieste, 21 July 2004

FOUR PROBLEMS are given. You are expected to solve completely at least two problems. You have three hours to solve them.

Problem 1.

The quantum Hamiltonian of a one-dimensional harmonic oscillator with mass m and frequency ω can be written as

$$H = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right) , \tag{1}$$

where $a, a^\dagger$ are related to the position $\hat{x}$ and momentum $\hat{p}$ operators by

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)$$
$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a).$$

1. Show that the set of states $|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$ (where $a|0\rangle = 0$, $\langle 0|0\rangle = 1$) provides the eigenstates of H.

2. Let us define the state $|\alpha\rangle = e^{\alpha a^\dagger - \alpha^* a}|0\rangle$, where α is a complex number. Show that $|\alpha\rangle$ is an eigenstate of a and determine its eigenvalue.

3. Find the value(s) of α for which the probability that an energy measurement gives the value $\hbar\omega/2$ is 1/2.

4. For any operator A and any state ψ define $\langle\psi|\Delta A^2|\psi\rangle = \langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2$. Show that

$$\langle\alpha|\Delta\hat{x}^2|\alpha\rangle\langle\alpha|\Delta\hat{p}^2|\alpha\rangle = \left(\frac{\hbar}{2}\right)^2 . \tag{2}$$

5. Add a constant force term to the Hamiltonian

$$H' = H - f\hat{x} \tag{3}$$

Draw the new potential. Find the new energy spectrum. Find the fundamental eigenstate in terms of the old vacuum $|0\rangle$.

[Hint: Use the relation $e^{A+B} = e^A e^B e^{\frac{1}{2}[A,B]}$, which is valid for any two operators A and B such that $[A, B]$ is a c-number.]

Problem 2.

Some of the parameters of the standard model seem to be adjusted at values just right to make it possible for us to exist. This has lead to the so-called anthropic principle, according to which these parameters have their values in the universe we inhabit because life would not be possible for any other values and there would be nobody to ask the question.

As an example, consider the values of the masses of the up and down quark

$$m_u \simeq 4 \text{ MeV} \quad \text{and} \quad m_d \simeq 7 \text{ MeV} .$$

Changing their values—while leaving everything else the same—leads to dire consequences, as illustrated by the following exercises you are asked to discuss.

Electromagnetic energy of the proton

1 The electromagnetic energy of a proton is about $E_{em} = 1.7$ MeV. To convince yourself that this value is reasonable (that is, of the right order of magnitude), try and estimate by means of a back-of-the-envelope computation the electromagnetic self-energy of the proton and compare it with E_{em}.

[Hint: since this is a divergent quantity use a cut-off]

Why is the neutron heavier than the proton?

2.1 Given the value of E_{em} and that the neutron is expected to have smaller if not negligible electromagnetic energy (it is neutral after all), how do you explain the experimental fact that it is heavier than the proton?

2.2 Having explained to your satisfaction that $m_N > m_P$, estimate by what amount we can increase the mass of the up quark and decrease that of the down quark before the neutron becomes unstable.

[Hint: Take for the baryon masses the approximated formula

$$m_{baryon} = \sum_q m_q + m_{glue} + E_{em} ,$$

with m_q the quark masses amd m_{glue} the gluon contribution to the baryon energy]

Hydrogen burning in the Sun

The pp reaction

$$p + p \rightarrow D + e^+ + \nu_e$$

begins the conversion of Hydrogen to Helium in the Sun.

3.1 Find the condition that the binding energy of the Deuterium B_d must satisfy for the reaction to be exothermic (that is, energetically allowed).

3.2 Try to estimate B_d by assuming a Yukawa potential

$$V(r) = -V_0 e^{-r/a}$$

for the system proton-neutron, with $V_0 = 32$ MeV and $a = 2.2$ fm. Verify that the value you found satisfies the exothermic condition for the pp reaction in the Sun.

[Hint: use a variational principle with trial wave function $\psi(\alpha) = Ae^{-\alpha r/a}$. Useful integral: $\int_0^\infty x^{\nu-1} e^{-\mu x} dx = \mu^{-\nu}\Gamma(\nu)$ (Re μ, Re $\nu > 0$). Take $\hbar c \simeq 200$ MeV fm]

The universe just so

Using the results of the previous exercises:

4.1 Discuss (briefly) what would be the most devastating effect on the world as we know it of having $m_N < m_P$.

4.2 On the other hand, what would happen if $m_P - m_N$ were to be bigger by only 0.5 MeV?

Problem 3.

Consider the following Euclidean Lagrangian for a system of three scalar fields ϕ_1, ϕ_2 and ϕ_3:

$$\begin{aligned} \mathcal{L} &= \frac{5}{2}(\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2 + 2\,(\partial_\mu\phi_1)(\partial_\mu\phi_2) + 3\,(\partial_\mu\phi_3)^2 \\ &+ m^2\phi_1^2 - 4\,m^2\phi_1\phi_2 + 4\,m^2\phi_2^2 + 8\,m^2\phi_3^2\,, \end{aligned} \tag{1}$$

where m is a constant parameter.

1. Find the combination of fields—expressed in terms of ϕ_1, ϕ_2 and ϕ_3—with definite mass and explicitly compute the corresponding masses.

2. Add now a potential term to the Lagrangian (1), of the form

$$V(\phi_1,\phi_2) = \frac{1}{20}\left(\phi_1^2 + 4\,\phi_2^2 - 4\,\phi_1\phi_2 - v^2\right)^2, \tag{2}$$

where v is a constant parameter.

As before, find the combinations of fields with definite mass and compute the corresponding masses.

3. What are the global symmetries of the system described by the Lagrangian above, with and without the potential $V(\phi_1,\phi_2)$? What happens if $m \to 0$?

PROBLEM 4.

SUPPOSE that the main contribution to the energy density of the Universe is provided by a fluid defined in terms of a classical real scalar field $\phi(x^\mu)$, with lagrangian

$$L = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi) \tag{1}$$

(focus on the case of flat metric $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, and assume that $V(\phi)$ is positive definite).

1. Compute the energy momentum tensor

$$T^\mu_\nu \equiv \frac{\partial L}{\partial(\partial_\mu\phi)}\,\partial_\nu\phi - L\delta^\mu_\nu\,, \tag{2}$$

and verify that, in the limit of spatial homogeneity of the field, it takes the form

$$T^\mu_\nu = \mathrm{diag}(\rho, -p, -p, -p). \tag{3}$$

Here, ρ and p are respectively the energy density and the pressure of the fluid. It follows also that the equation of state $p = p(\rho)$ can be written in the form:

$$p = w\rho \tag{4}$$

What is the allowed range of variation for w?

2. Consider the case of ϕ close to the minimum V_0 of the potential $V(\phi)$, so that it is a fair approximation to replace $V(\phi)$ with an expansion to second order in ϕ. Introduce a plane wave expansion and focus on one single mode with momentum $\vec{k}$; substitute this mode in the expression of the energy momentum tensor found in the first step of question 1. Assume isotropy and homogeneity (i.e. average over the angular variables and over volume) and find the corresponding expression for w. In what limits the fluid defined by ϕ behaves respectively as radiation, matter or vacuum energy (i.e., it has an equation of state with w equal to, respectively, 1/3, 0 and -1)?

3. Suppose now that the lagrangian for the scalar field is given, in place of Eq. (1), by:

$$L = -V\left(\phi\right) \sqrt{1 - \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi} \, , \tag{5}$$

assuming again that $V(\phi)$ is positive definite. Derive the expression of the energy density and pressure, assuming spatial homogeneity of the field. What is the equation of state in the limit of constant potential, i.e. $V(\phi) \equiv V_0$? Under such condition, can you guess how the fluid would behave in the early Universe and at late times?