Problem A. A Field Theory with a Nonlocal Lagrangian

Consider the following field theory for a scalar field $\phi(t,\vec{x})$ with the Action:

$$S_0 = \int dt \int d^3x \frac{1}{2}(\partial_t \phi(t,\vec{x}))^2 - \frac{1}{2}\int dt \int d^3x \int d^3x' \phi(t,\vec{x}) K(\vec{x}-\vec{x}')\phi(t,\vec{x}') \quad (1)$$

where $K(\vec{z}) = \mu^2 \delta^{(3)}(z) - K_0 e^{-\alpha \vec{z}^2}$.

1. Find the spectrum and determine the region of the parameters where the theory is stable. In which limiting case does the spectrum resamble the one of a free relativistic particle?

2. Add a source term to the Action, such that the total Action is now:

$$S = S_0 - \int dt \int d^3x J(t,\vec{x})\phi(t,\vec{x}) \quad (2)$$

   Write the classical equation for the field ϕ. Determine a solution $\phi(t,\vec{x})$ in the case $J(t,\vec{x}) = J_0 e^{i\vec{p}\cdot\vec{x}}$, i.e. J is time-independent. Discuss how general the solutions you found are.

(It may be useful to remember that $\int_{-\infty}^{+\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$)

Problem B. Classical solutions in Field Theory

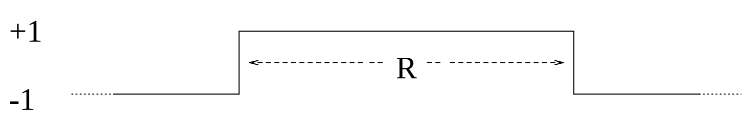
Field Theories may have degenerate vacua and kinks, i.e. interpolating field configurations among them. Let

$$H = \int dx \left[\frac{1}{2} \left(\frac{\partial \varphi}{\partial x} \right)^2 + U(\varphi) \right] \tag{1}$$

be the classical hamiltonian for the static (time–independent) configurations of φ, a scalar field in one space dimension.

1. Take $U(\varphi) = \frac{g}{4}(\varphi^2 - 1)^2$ $(g > 0)$. Derive the differential equation satisfied by the static configurations and prove that $\phi(x) = \pm \tanh[\sqrt{\frac{g}{2}}(x - x_0)]$ is a solution ($\phi_+(x)$ being a kink and $\phi_-(x)$ being an anti–kink). Compute its energy density, interpret physically the profile of this curve and determine how the total energy of the kink depends on g.

2. For the above model, consider the configuration made of a kink and an anti–kink separated by a large distance R (the precise mathematical expression of this configuration is inessential and can be taken as a step function of jumps ± 1, see figure below). Let us switch an external source h on, so that $U \to U + \delta U$, with $\delta U = h\varphi$. Evaluate the effect of the new interaction on the (unperturbed) configuration drawn in the Figure by estimating the behaviour of its energy for large R and determine whether an attractive or repulsive potential between the kink and the antikink will be generated.



Problem C. Reflection and Transmission Amplitudes

Consider a $(1+1)$ free massive boson theory but with a point of defect of coupling g at the origin. Its lagrangian density is given by

$$\mathcal{L} = \frac{1}{2}\left[\,(\partial_\mu \varphi)^2 - m^2\varphi^2 - g\delta(x)\varphi^2\,\right] \quad . \tag{1}$$

The two–dimensional relativistic momenta may be expressed as $p_0 = m\cosh\beta$; $p_1 = m\sinh\beta$. As shown in the space–time diagram of Figure 1, the particle of the field φ propagates freely until it reaches the origin: here, the particle will be reflected and transmitted with amplitudes $R(\beta)$ and $T(\beta)$ respectively. Let $\varphi_\pm(x,t)$ be the field values for $x > 0$ and $x < 0$, respectively, i.e. $\varphi(x,t) = \theta(x)\varphi_+(x,t) + \theta(-x)\varphi_-(x,t)$, where $\theta(x)$ is the usual step–function.

1. Derive the boundary conditions for $\varphi_\pm(x,t)$ at the origin by imposing the equation of motion.

2. Using the free field expansion of $\varphi_\pm(x,t)$,

$$\varphi_\pm(x,t) = \int d\beta\left[A_\pm(\beta)e^{-im(t\cosh\beta - x\sinh\beta)} + A^\dagger_\pm(\beta)e^{im(t\cosh\beta - x\sinh\beta)}\right]$$

and considering the set of oscillators $A_\pm(\beta)$, impose the boundary conditions of point (1) to derive the explicit expressions for the reflection and transmission amplitudes $R(\beta)$ and $T(\beta)$.

(*Hint.* It may be convenient to assume $A_-(-\beta) = 0$, see Figure 1)

Will the answer be the same by considering the other oscillators $A^\dagger_\pm(\beta)$?

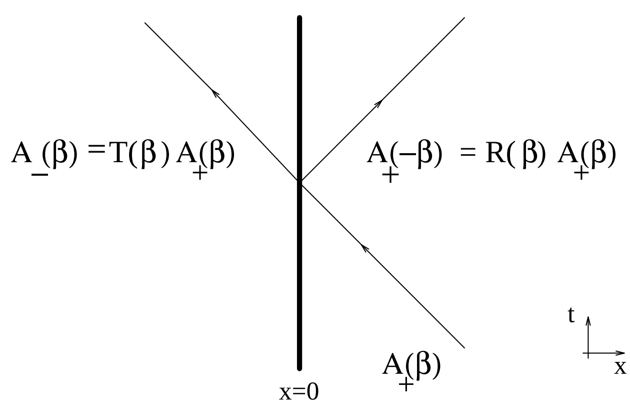


Figure 1.

Problem D. Amplitudes for baryon number violation

Consider the four-fermion operator

$$O = \frac{1}{M^n} uude$$

where u, d and e denote the up-quark, the down-quark and the electron, respectively. M is a mass.

1. Taking into account that the above operator has to be considered part of a lagrangian density, find the value of n and justify your result. The above operator may lead to proton decay. Given the experimental lower bound on the proton lifetime of $\mathcal{O}(10^{32}$ yrs), find the order of magnitude of the corresponding lower bound on M (recall that $\hbar = 6.6 \times 10^{-22}$ MeV $\times$ sec.).

2. If baryon number is not conserved, is it conceivable that oscillations between a proton and an antiproton or a neutron and an antineutron may occur? If yes, which operators may give rise to them?

Problem E. Dirac Equation with an Electromagnetic Field

Consider a relativistic electron in presence of a constant and uniform magnetic field B in the z-direction and a constant and uniform electric field E in the x-direction (in the usual spacetime with 3 space coordinates x, y, z).

1. Find the gauge field A_μ (in this case one can take $A_x = 0$ and $A_z = 0$, A_μ time independent), and write the covariant form of the Dirac equation (i.e. with the γ matrices) for the energy eigenfunctions with energy $\mathcal{E}$. Determine which momentum components are constant of motion and write the equation for the corresponding eigenfunctions.

   (Bear in mind that the free Dirac equation is given by

   $$(i\gamma^\mu \partial/\partial x^\mu - m)\psi = 0)$$

2. Using the chiral basis for the γ matrices given by

   $$\gamma_0 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} \ ; \ \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

   (where I is the 2×2 identity matrix and $\vec{\sigma}$ the usual Pauli matrices), write the coupled equations for the two 2-spinors χ and ϕ which form the 4-spinor $\psi = \begin{pmatrix} \chi \\ \phi \end{pmatrix}$ and obtain the equation involving only one of the 2-spinors. For which range of B and E is the equation meaningful?

Problem F. Particle flavour oscillations

There exist several cases in particle physics where particles produced in physical interactions are not energy eigenstates, but rather result as a coherent superposition of mass eigenstates (neutrino oscillations, kaon-antikaon mixing, etc.). Consider two stable particles A and B which are not mass eigenstates, but have the mass matrix

$$(\overline{A}, \overline{B}) \begin{pmatrix} m & \delta m \\ \delta m & m \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \tag{1}$$

1. Find the mass eigenstates and eigenvalues. Suppose that at $t = 0$ a beam containing only particles A is produced in a physical process. Considering the relativistic case and a beam of fixed spatial momentum $\mathbf{p}$, compute the probability to find a particle B at the time t.

2. Consider a beam of electron antineutrinos emitted from a nuclear reactor with an energy of 5 MeV. Take the neutrino masses to be much smaller than 5 MeV. What is the order of magnitude of the difference of the neutrino (squared) masses yielding oscillation probability of $\mathcal{O}(1)$ if the detector is placed at 20 meters or 2 Km, respectively? (recall that $\hbar \times c = 197 \times 10^{-15}$ MeV $\times$ m.).