Anomalies in Quantum Field Theory

Marco Serone

SISSA, via Bonomea 265, I-34136 Trieste, Italy

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1 Introduction

There are different ways of regularizing a Quantum Field Theory (QFT). The best choice of regulator is the one which keeps the maximum number of symmetries of the classical action unbroken. Cut-off regularization, for instance, breaks gauge invariance while this is unbroken in the somewhat more exotic dimensional regularization. It might happen, however, that there exists *no* regulator that preserves a given classical symmetry. When this happens, we say that the symmetry is anomalous, namely the quantum theory necessarily breaks it, independently of the choice of regulator.

Roughly speaking, anomalies can affect global or local symmetries. The latter case is particularly important, because local symmetries are needed to decouple unphysical states. Indeed, anomalies in linearly realized local gauge symmetries lead to inconsistent theories. Theories with anomalous global symmetries are instead consistent, yet the effect of the anomaly can have important effects. The first anomaly, discovered by Adler, Bell and Jackiw [1], was associated to the non-conservation of the axial current in QCD. Among other things, the axial anomaly resolved a puzzle related to the $\pi^0 \to 2\gamma$ decay rate, predicted by effective Lagrangian considerations to be about three orders of magnitude smaller than the observed one.

In this course we will study in some detail a particularly relevant class of anomalies, those associated to chiral currents (so called chiral anomalies) and their related anomalies in local symmetries. Emphasis will be given to certain mathematical aspects, in particular the close connection between anomalies and index theorems, while most physical aspects will not be discussed. The computation of anomalies will be mapped to the evaluation of the partition function of a certain (supersymmetric) quantum mechanical model, following the seminal paper [2]. Although this might sound unusual, if compared to the more standard treatments using one-loop Feynman diagrams, and requires a bit of background material to be attached, the ending result will be very rewarding. This computation will allow us to get the anomalies associated to gauge currents (so called gauge anomalies) and stress-energy tensor (so called gravitational anomalies) in any number of space-time dimensions! Moreover, it is the best way to reveal the above mentioned connection with index theorems. Since the notion of chirality is restricted to spaces with an even number of space-time dimensions, and the connection with index theorems is best seen for euclidean spaces, we will consider in the following QFT on even dimensional *euclidean* spaces.

Some exercises are given during the lectures (among rows and in bold face in the text). They are simple, yet useful, and the reader should be able to solve them without any problem.

These notes are a slightly revised version of my lecture notes [3], mostly based on the review paper [4]. Basic, far from being exhaustive, references to some of the original papers are given in the text.

2 Basics of Differential Geometry

We recall here some basic aspects of differential geometry that will be useful in understanding anomalies in QFT. This is not a comprehensive review. There are several excellent introductory books in differential geometry, see e.g. ref.[5]. A basic well written overview with a clear physical perspective can be found e.g. in refs.[6] and [7].

2.1 Vielbeins and Spinors

Given a manifold M_d of dimension d, a tensor of type (p,q) is given by

$$T = T^{\mu_1\ldots\mu_p}{}_{\nu_1\ldots\nu_q}\frac{\partial}{\partial x^{\mu_1}}\ldots\frac{\partial}{\partial x^{\mu_p}}dx^{\nu_1}\ldots dx^{\nu_q}\,, \tag{2.1}$$

where $\mu_i, \nu_i = 1,\ldots,d$. A manifold is called Riemannian if it admits a positive definite symmetric (0,2) tensor. We call such tensor the metric $g = g_{\mu\nu}dx^\mu dx^\nu$. On a general M_d a vector acting on a (p,q) tensor does not give rise to a well-defined tensor, e.g. if V_ν are the components of a $(0,1)$ tensor, $\partial_\mu V_\nu$ are not the components of a $(0,2)$ tensor. This problem is solved by introducing the notion of connection and covariant derivative. If M_d has no torsion (as will be assumed from now on), then $\partial_\mu V_\nu \to \nabla_\mu V_\nu$, where

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\rho_{\mu\nu}V_\rho\,, \qquad \Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})\,. \tag{2.2}$$

The (not tensor) field $\Gamma^\rho_{\mu\nu}$ is called the Levi-Civita connection. In terms of it, we can construct the fundamental curvature (or Riemann) tensor

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho\Gamma^\mu_{\nu\sigma} - \partial_\sigma\Gamma^\mu_{\nu\rho} + \Gamma^\alpha_{\sigma\nu}\Gamma^\mu_{\alpha\rho} - \Gamma^\alpha_{\rho\nu}\Gamma^\mu_{\alpha\sigma}\,. \tag{2.3}$$

Other relevant tensors constructed from the Riemann tensor are the Ricci tensor $R_{\nu\sigma} = R^\mu{}_{\nu\mu\sigma}$ and the scalar curvature $R = g^{\nu\sigma}R_{\nu\sigma}$.

The transformation properties under change of coordinates of (p,q) tensors is easily obtained by looking at eq.(2.1). For instance, under a diffeomorphism $x^\mu \to x'^\mu(x)$, a vector transforms as

$$V^\mu(x) \to V'^\mu(x') = Z^\mu_\nu(x)V^\nu(x)\,. \tag{2.4}$$

where

$$Z^\mu_\nu(x) = \frac{\partial x'^\mu}{\partial x^\nu}\,. \tag{2.5}$$

At each space-time point x, the matrix Z^μ_ν is an element of $GL(d,R)$ and we can say that a (1,0) vector transforms in the fundamental representation of $GL(d,R)$. Similarly a (0,1) vector transforms in the dual fundamental representation given by Z^{-1}. A generic (p,q) tensor transforms in the appropriate product of these basic representations. In flat space tensors are defined in terms of their transformation properties under the Lorentz $SO(d)$ group. Since $SO(d) \supset GL(d,R)$, given a $GL(d,R)$ representation, we can always decompose it in terms of $SO(d)$ ones. Contrary to tensors, spinors are representations of $SO(d)$ that do not arise from representations of $GL(d,R)$. For this reason, spinors in presence of gravity require to replace diffeomorphisms with local Lorentz transformations. At each point of the manifold, we can write the metric as

$$g_{\mu\nu}(x) = e^a_\mu(x)e^b_\nu(x)\eta_{ab}\,, \tag{2.6}$$

where $a, b = 1, \ldots, d$.[1] The fields $e^a_\mu(x)$ are called vielbeins. They are not uniquely defined by eq.(2.6). If we perform the local Lorentz transformation

$$e^a_\mu(x) \to (\Lambda^{-1})^a{}_b(x) e^b_\mu(x) \,, \tag{2.7}$$

the metric is left invariant. It is important to distinguish the index μ form the index a. The first transforms under diffeomorphisms and is invariant under local Lorentz transformations, viceversa for the second: invariant under diffeomorphisms and transforms under local Lorentz transformations. They are sometimes denoted curved and flat indices, respectively. We will denote a curved index with a greek letter, and a flat index with a roman letter. From eq.(2.6) we have $g \equiv \det g_{\mu\nu} = (\det e^a_\mu)^2$. Since the metric is positive definite, this implies that e^a_μ is a non-singular $d \times d$ matrix at any point. We denote its inverse by e^μ_a: $e^\mu_a e^b_\mu = \eta^{ab}$, $e^\mu_a e^a_\nu = \delta^\mu_\nu$. It is easy to see that e^μ_a is obtained from e^a_μ by using the inverse metric $g^{\mu\nu}$: $e^\mu_a = g^{\mu\nu} e_{\nu a}$. Curved indices are lowered and raised by the metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$. The vielbeins e^μ_a can be seen as a set of orthonormal (1,0) vectors in the tangent space. As such, we can decompose any vector V^μ in their basis: $V^\mu = e^\mu_a V^a$. Similarly for the (0,1) vectors in cotangent space: $V_\mu = e^a_\mu V_a$. So any tensor with curved indices can be converted to a tensor with flat indices. In other words, we can project the (p, q) tensor (2.1) in the basis

$$T = T^{a_1 \ldots a_p}{}_{b_1 \ldots b_q} E_{a_1} \ldots E_{a_p} e^{b_1} \ldots e^{b_q} \,, \tag{2.8}$$

where

$$E_a = e^\mu_a \frac{\partial}{\partial x^\mu} \,, \qquad e^a = e^a_\mu dx^\mu \,. \tag{2.9}$$

We can then trade, as already mentioned, diffeomorphisms for local Lorentz transformations. In order to ensure invariance under local Lorentz transformations, we should introduce a gauge field connection ω_μ transforming in the adjoint representation of $SO(d)$:

$$\omega_\mu(x) \to \Lambda^{-1}(x) \omega_\mu(x) \Lambda(x) + \Lambda^{-1}(x) \partial_\mu \Lambda(x) \,. \tag{2.10}$$

The components of an $SO(d)$ connection can be represented with a pair of antisymmetric indices in the fundamental representation: $\omega^{ab}_\mu = -\omega^{ba}_\mu$, $a, b = 1, \ldots d$. This field is called the spin connection and is the analogue of the Levi-Civita connection $\Gamma^\rho_{\mu\nu}$ for diffeomorphisms. Derivative of tensors with flat indices are promoted to covariant derivatives:

$$D_\mu T^a = \partial_\mu T^a + \omega^a_{\mu\ b} T^b \,. \tag{2.11}$$

[1]For euclidean spaces $\eta_{ab} = \delta_{ab}$, but we write it here more generally η_{ab} so our considerations will also apply for Lorentzian spaces.

As the Levi-Civita connection is a function of the metric, the spin connection is a function of the vielbeins and cannot be an independent field (otherwise we would have too many gravitational degrees of freedom).

Exercise 1: Prove that demanding the compatibility of the covariant derivative (2.11) with the one defined in eq.(2.2),

$$\nabla_\mu T_\nu = e_\nu^a D_\mu T_a \,, \tag{2.12}$$

leads to the following equation:

$$\nabla_\mu e_\nu^a + \omega_{\mu\;b}^{\;a} e_\nu^b = 0 \,. \tag{2.13}$$

Using eq.(2.13) one could determine the explicit expression of $\omega_{\mu\;\;b}^{\;a} = \omega_{\mu\;b}^{\;a}(e)$ (as well the expression of $\Gamma_{\mu\nu}^\rho$ in eq.(2.2)). We will not report this expression, not needed in what follows. The curvature tensor is given by

$$R_{\mu\nu\;b}^{\;\;a} = D_\mu \omega_{\nu\;b}^{\;a} = \partial_\mu \omega_{\nu\;b}^{\;a} - \partial_\nu \omega_{\mu\;b}^{\;a} + \omega_{\mu\;c}^{\;a} \omega_{\nu\;b}^{\;c} - \omega_{\nu\;c}^{\;a} \omega_{\mu\;b}^{\;c} \,. \tag{2.14}$$

By using eq.(2.12) and the properties $[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\mu\nu\sigma} V^\sigma$, $[D_\mu, D_\nu] V^a = R_{\mu\nu\;b}^{\;\;a} V^b$, we have

$$R_{\mu\nu\;b}^{\;\;a} = e_\sigma^a e_b^\rho R^\sigma{}_{\mu\nu\rho} \,. \tag{2.15}$$

Let us come back to spinors. First of all, recall that the definition of gamma matrices in d euclidean dimensions is a straightforward generalization of the usual one known in four dimensions:

$$\{\gamma^a, \gamma^b\} = 2\delta^{ab} \,. \tag{2.16}$$

We have written γ^a and not γ^μ because in a curved space $\gamma^\mu(x) = e_a^\mu(x)\gamma^a \neq \gamma^a$. The algebra defined by eq.(2.16) is called a Clifford algebra. For even d, the case relevant in these lectures, the lowest dimensional representation in d dimensions require $2^{d/2} \times 2^{d/2}$ matrices. Hence a (Dirac) spinor in d dimensions has $2^{d/2}$ components. A linearly independent basis of matrices is given by antisymmetric products of gamma matrices:

$$I \,, \gamma^a \,, \gamma^{a_1 a_2} \,, \ldots \gamma^{1\ldots d} \,, \tag{2.17}$$

where

$$\gamma^{a_1\ldots a_p} = \frac{1}{p!}\Big(\gamma^{a_1} \ldots \gamma^{a_p} \pm (p-1)! \text{ perms}\Big) \tag{2.18}$$

is completely antisymmetrized in the indices $a_1 \dots a_p$. The total number of elements of the base is

$$\sum_{p=0}^{d} \begin{pmatrix} d \\ p \end{pmatrix} = 2^d = 2^{d/2} \times 2^{d/2}\,, \tag{2.19}$$

which is then complete. Notice the close analogy between gamma matrices and differential forms, analogy that we will use in a crucial way in the next sections. In d euclidean dimensions, the gamma matrices can be taken all hermitian: $(\gamma^a)^\dagger = \gamma^a$. We can also define a matrix γ_{d+1}, commuting with all other γ^a's:

$$\gamma_{d+1} = i^n \prod_{a=1}^{d} \gamma^a\,. \tag{2.20}$$

The factor i^n ensures that $\gamma_{d+1}^2 = 1$ for any d. Needless to say, γ_{d+1} is the generalization of the usual chiral gamma matrix γ_5 in 4d QFT. A spinor ψ can be decomposed in two irreducible components: $\psi = \psi_+ + \psi_-$, with $\gamma_{d+1}\psi_\pm = \pm\psi_\pm$. Weyl spinors in d dimensions have then $2^{d/2-1}$ components. Using eq.(2.16), the matrices

$$J^{ab} = \frac{1}{4}[\gamma^a, \gamma^b] = \frac{1}{2}\gamma^{ab} \tag{2.21}$$

are shown to be (anti-hermitian) generators of $SO(d)$. Under a global Lorentz transformation, a spinor ψ transforms as follows:

$$\psi \to e^{\frac{1}{2}\lambda_{ab}J^{ab}}\psi\,, \tag{2.22}$$

where λ_{ab} are the $d(d-1)/2$ parameters of the transformation. The covariant derivative of a fermion in curved space is obtained by replacing

$$\partial_\mu \to D_\mu = \partial_\mu + \frac{1}{2}\omega_\mu^{ab} J_{ab}\,. \tag{2.23}$$

The Lagrangian density of a massive Dirac fermion coupled to gravity is finally given by

$$\mathcal{L}_\psi = e\,\bar\psi\Big(i e_a^\mu \gamma^a D_\mu - m\Big)\psi\,, \tag{2.24}$$

where $e \equiv \det e_\mu^a$.

A manifold is generally described by different open sets (charts) $U_{(\alpha)}$ and transition functions between the different charts. In general it might not be possible to define e_μ^a globally on the manifold, but locally on $U_{(\alpha)}$ we can always found one $e^a_{\mu\,(\alpha)}$. At intersections $U_{(\alpha\beta)} = U_{(\alpha)} \bigcap U_{(\beta)}$ we can use either (omitting indices) the vielbein $e_{(\alpha)}$ or $e_{(\beta)}$. The two are equivalent provided they are related by a local Lorentz transformation $\Lambda_{(\alpha\beta)}$:

$$e_{(\alpha)}(x) = \Lambda_{(\alpha\beta)}(x) e_{(\beta)}(x)\,. \tag{2.25}$$

Similarly we have

$$\omega_{(\alpha)}(x) = \Lambda^{-1}_{(\alpha\beta)}(x)\omega_{(\beta)}(x)\Lambda_{(\alpha\beta)}(x) + \Lambda^{-1}_{(\alpha\beta)}(x)\partial\Lambda_{(\alpha\beta)}(x)\,. \tag{2.26}$$

The local Lorentz transformation $\Lambda_{(\alpha\beta)}$ is called a transition function and characterizes the non-triviality of the manifold. A manifold is topologically trivial if it admits a unique globally defined vielbein e^a_μ, namely one can choose $\Lambda_{(\alpha\beta)} = 1$ at any intersection $U_{(\alpha\beta)}$. A similar analysis can be made for gauge fields. A topologically non-trivial gauge field configuration is one in which the connection A_μ cannot be specified uniquely over the manifold. We define a connection $A_{(\alpha)}$ on $U_{(\alpha)}$, and on intersections

$$A_{(\alpha)}(x) = g^{-1}_{(\alpha\beta)}(x)A_{(\beta)}(x)g_{(\alpha\beta)}(x) + g^{-1}_{(\alpha\beta)}(x)\partial g_{(\alpha\beta)}(x)\,, \tag{2.27}$$

where $g_{(\alpha\beta)}(x)$ is a gauge transformation. Non-trivial gauge configurations, as well as nontrivial manifolds, are classified by topological invariant quantities. A subset of these are given by characteristic classes, namely integrals of antisymmetric tensors of type $(0, p)$, also denoted differential forms. In the next two subsections we review basics of differential forms and characteristic classes, respectively.

2.2 Differential Forms

A differential form ω_p is commonly denoted a p-form and is represented as

$$\omega_p = \frac{1}{p!}\omega_{\mu_1\ldots\mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}\,. \tag{2.28}$$

The wedge product $\wedge$ in eq.(2.28) represents the completely antisymmetric product of the basic 1-forms dx^{μ_i}:

$$dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} = \sum_{P\in S_p} \mathrm{sign}\,(P) dx^{\mu_{P(1)}} \ldots dx^{\mu_{P(p)}}\,, \tag{2.29}$$

where P denotes a permutation of the indices and S_p is the group of permutations of p objects. For each point in a manifold of dimension d, the dimension of the vector space spanned by ω_p equals

$$\left(\begin{array}{c} d \\ p \end{array}\right) = \frac{d!}{(d-p)!p!}\,. \tag{2.30}$$

The total vector space spanned by all p-forms, $p = 0, \ldots, d$, equals

$$\sum_{p=0}^{d} \left(\begin{array}{c} d \\ p \end{array}\right) = 2^d\,. \tag{2.31}$$

Given a p-form ω_p and a q-form χ_q, their wedge product $\omega_p \wedge \chi_q$ is a $(p+q)$-form. From eq.(2.29) one has

$$\begin{aligned} \omega_p \wedge \chi_q &= (-)^{pq} \chi_q \wedge \omega_p \\ (\omega_p \wedge \chi_q) \wedge \eta_r &= \omega_p \wedge (\chi_q \wedge \eta_r)\,. \end{aligned} \tag{2.32}$$

One has clearly $\omega_p \wedge \chi_q = 0$ if $p+q > d$ and $\omega_p \wedge \omega_p = 0$ if p is odd. Given a p-form ω_p, we define the exterior derivative d_p as the $(p+1)$-form given by

$$d_p \omega_p = \frac{1}{p!} \partial_{\mu_{p+1}} \omega_{\mu_1 \ldots \mu_p} dx^{\mu_{p+1}} \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}\,. \tag{2.33}$$

Notice that in eq.(2.33) we have the ordinary, rather than the covariant derivative, because upon antisymmetrization their action is the same:

$$\nabla_{[\mu_{p+1}} \omega_{\mu_1 \ldots \mu_p]} = \partial_{[\mu_{p+1}} \omega_{\mu_1 \ldots \mu_p]} - \sum_{i=1}^{p} \Gamma^{\alpha}_{[\mu_{p+1}\mu_i} \omega_{\alpha|\mu_1 \ldots \hat{\mu}_i \ldots \mu_p]} = \partial_{[\mu_{p+1}} \omega_{\mu_1 \ldots \mu_p]}\,. \tag{2.34}$$

The hat in the index μ_i means that the index should be removed, while the square brackets indicate complete antisymmetrization in the $p+1$ μ_i indices (the | indicates that the index α is excluded by the antisymmetrization). Since the Levi-Civita connection is symmetric in its two lower indices, the second equality in eq.(2.34) immediately follows. The exterior derivative d is covariant independently of the metric. A differential operator like d can only be defined for differential forms and this explains their importance in the classification of the topological properties of manifolds.

Given eq.(2.32), we have

$$d_{p+q}(\omega_p \wedge \chi_q) = d_p \omega_p \wedge \chi_q + (-)^p \omega_p \wedge d_q \chi_q\,. \tag{2.35}$$

From the definition (2.33) it is clear that $d_{p+1} d_p = 0$ on any p-form.

A p-form is called closed if $d_p \omega_p = 0$ and exact if $\omega_p = d_{p-1}\chi_{p-1}$ for some $(p-1)$-form χ_{p-1}. Clearly, if $\omega_p = d_{p-1}\chi_{p-1}$, then $d_p \omega_p = 0$ (exact $\rightarrow$ closed) but not viceversa. Correspondingly, the image of d_{p-1}, the space of p-forms such that $\omega_p = d_{p-1}\chi_{p-1}$, is included in the kernel of d_p, the space of p-forms such that $d_p\omega_p = 0$: Im $d_{p-1} \subset$ Ker d_p. The coset space Ker d_p/Im d_{p-1} is called the de Rham cohomology group on the manifold M_d. By Stokes theorem, the integral of an exact p-form on a p-dimensional cycle $\mathcal{C}_p$ of M_d with no boundaries necessarily vanish:

$$\int_{\mathcal{C}_p} \omega_p = \int_{\mathcal{C}_p} d_p \chi_{p-1} = \int_{\partial \mathcal{C}_p} \chi_{p-1} = 0\,. \tag{2.36}$$

Notice that on $U_{(\alpha)}$ a closed form can always be written as $\omega_{p\,\alpha} = d_{p-1}\chi_{p-1\,\alpha}$ for some $(p-1)$-form. Exactness means that this decomposition applies globally over the manifold.

Since the action of d_p on a p-form is clear from the context, we will drop from now on the subscript p in the exterior derivative and denote it simply with d.

Gauge fields are conveniently written in terms of differential forms. For a $U(1)$ gauge theory the connection is a 1-form $A = A_\mu dx^\mu$, with

$$F = dA = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \wedge dx^\nu = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \tag{2.37}$$

being the associated field strength 2-form. In order to generalize this geometric picture of gauge theories to the non-abelian case, we have to introduce the notion of Lie-valued differential forms. We then define

$$A = A^\alpha T^\alpha, \qquad A^\alpha = A^\alpha_\mu dx^\mu \,, \tag{2.38}$$

where T^α are the generators of the corresponding Lie algebra.[2] We take them to be *anti-hermitian*: $(T^\alpha)^\dagger = -T^\alpha$. This unconventional (in physics) choice allows us to get rid of some factors of i's in the formulas that will follow. Given two Lie-valued forms ω_p and χ_q, we define their commutator as

$$[\omega_p, \chi_q] \equiv \omega_p^\alpha \wedge \chi_q^\beta [T^\alpha, T^\beta] = \omega_p^\alpha \wedge \chi_q^\beta T^\alpha T^\beta - (-)^{pq} \chi_q^\beta \wedge \omega_p^\alpha \, T^\beta T^\alpha \,. \tag{2.39}$$

The square of a Lie valued p-form does not necessarily vanish when p is odd, like ordinary p-forms. One has

$$\omega_p \wedge \omega_p = \frac{1}{2}[T^\alpha, T^\beta] \omega^\alpha_{\mu_1 \ldots \mu_p} \omega^\beta_{\mu_{p+1} \ldots \mu_{2p}} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{2p}} = \frac{1}{2}[\omega_p, \omega_p] \,. \tag{2.40}$$

The action of the exterior derivative d on a Lie-valued p-form does not give rise to a well-defined covariant $p+1$-form. Under a gauge transformation g, we have

$$A \to g^{-1} A g + g^{-1} dg \,, \tag{2.41}$$

and $F = dA$ does not transform properly. A covariant transformation is provided by the covariant derivative

$$F = DA = dA + A \wedge A \,. \tag{2.42}$$

Exercise 2: i) Show that $F \to g^{-1} F g$ under a gauge transformation. ii) Show that $dF = -[A, F]$, so that $DF = dF + [A, F]$=0 (Bianchi identity).

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[2] Needless to say, the gauge index α here should not be confused with the index (α) in eqs.(2.25)-(2.27), that was used to label the charts covering a manifold.

In complete analogy to eq.(2.42), we can view the components of the Riemann tensor (2.14) as defining a curvature two-form

$$R^a{}_b = D\omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \,. \tag{2.43}$$

A differential form notation can also be introduced to describe the Levi-Civita connection (2.2) and the Riemann tensor (2.3). One defines the matrix of one-forms $\Gamma^\mu_\nu = \Gamma^\mu_{\rho\nu} dx^\rho$ and the curvature two-form

$$R^\mu{}_\nu = d\Gamma^\mu{}_\nu + \Gamma^\mu{}_\sigma \wedge \Gamma^\sigma{}_\nu = \frac{1}{2} R^\mu{}_{\nu\rho\sigma} dx^\rho \wedge dx^\sigma \,. \tag{2.44}$$

Mathematically speaking, a gauge field is a connection on a fibre bundle. A fiber F is the vector space that can be associated to each point of the manifold M_d (called also the base space in this context). The total space given by the base space and the fibre is called the fibre bundle. It can always be locally written as a direct product $M_d \times F$. Globally only trivial gauge bundles (those where $g_{(\alpha\beta)}$ in eq.(2.27) can be chosen to be 1 at any intersection $U_{(\alpha\beta)}$) admit such decomposition, very much like the choice of local coordinates in a non-trivial manifold. We will not discuss the formulation of non-abelian gauge theories in terms of fibre bundles.

2.3 Characteristic Classes

Characteristic classes is the name for certain cohomology classes that measure the non-triviality of a manifold or of its gauge bundle. They are essentially given by integrals of gauge invariant combinations of curvature two-forms, like (omitting flat $SO(d)$ indices) $\mathrm{tr}\, R \wedge \ldots \wedge R$ or $\mathrm{tr}\, F \wedge \ldots \wedge F$. For simplicity of notation we will omit from now the wedge product among forms. Given the covariant transformation of F and R under gauge or local Lorentz transformations, $\mathrm{tr}\, F^n$ and $\mathrm{tr}\, R^n$ are well defined over the manifold, namely $\mathrm{tr}\, F^n_{(\alpha)} = \mathrm{tr}\, F^n_{(\beta)}$ on $U_{(\alpha\beta)}$ so we can write $\mathrm{tr}\, F^n$ globally over the manifold. In a topologically non-trivial situation, gauge fields are classified in terms of families belonging to the same topological class. More precisely, two gauge fields A and A' are of the same topological class if on $U_{(\alpha\beta)}$ they transform as in eq.(2.27), with the *same* transition function $g_{(\alpha\beta)}(x)$. The same considerations apply to $\mathrm{tr}\, R^n$. In fact, since the spin connection can be viewed as an $SO(d)$ gauge connection, there is no need of a separate discussion for the gravitational case and all the analysis that follow will automatically apply.

The simplest example of characteristic class is provided by a $U(1)$ gauge field, where we can consider the integral of F over a compact two-dimensional subspace of M_d. The field strength F is well-defined on M_d and by the Bianchi identity it is closed: $dF = 0$.

If it is exact, namely there is a globally defined connection A for which $F = dA$, then by Stokes theorem $\int F = 0$. In general F might not be exact, in which case $\int F \neq 0$ measures the topological non-triviality of the space (the $U(1)$ gauge bundle).

In the $U(1)$ case, the difference of two gauge fields A and A' in the same topological class is an exact, globally defined, one-form. We have $F' - F = d(A' - A)$ and hence for any two-dimensional compact subspace of M_d

$$\int F = \int F' \tag{2.45}$$

is a topological invariant quantity.

A non-trivial $U(1)$ gauge configuration is provided by a magnetic monopole. Given (say) a two-sphere S^2 surrounding the monopole, we should have

$$\int_{S^2} F \neq 0 \tag{2.46}$$

and proportional to the monopole charge. A monopole induces a magnetic field of the form $\vec{B} = g\vec{r}/r^3$ and correspondingly there is no globally defined connection on S^2.

Exercise 3: Show that in radial coordinates the connection 1-forms in the northern and southern hemispheres can be taken as

$$A_{(N)} = -ig(1 - \cos\theta)d\phi\,, \quad A_{(S)} = ig(1 + \cos\theta)d\phi\,. \tag{2.47}$$

More precisely, we split $S^2 = \Sigma_{(N)} + \Sigma_{(S)}$, $\Sigma_{(N),(S)}$ being the northern and southern hemispheres, and compute

$$\int_{S^2} F = \int_{\Sigma_{(N)}} dA_{(N)} + \int_{\Sigma_{(S)}} dA_{(S)} = \oint (A_{(N)} - A_{(S)})\,, \tag{2.48}$$

where the closed integral is around the equator at $\theta = \pi/2$. From the exercise 3 we have $A_{(N)} - A_{(S)} = -2igd\phi$ and hence

$$\frac{i}{2\pi}\int_{S^2} F = \frac{1}{2\pi}\int_0^{2\pi} d\phi(2g) = 2g\,. \tag{2.49}$$

We now show that $2g$ must be an integer in order to have a well-defined quantum theory. If a charged particle moves along the equator S^1, its action will contain the minimal coupling of the particle with the gauge field. In a path-integral formulation, it corresponds to a term

$$e^{\oint_{S^1} A} = e^{\int_D F}\,, \tag{2.50}$$

where we used Stokes' theorem and D is any two-dimensional space with S^1 as boundary: $\partial D = S^1$. Both $\Sigma_{(N)}$ and $\Sigma_{(S)}$ are possible choices for D. We get

$$e^{\oint_{S^1} A} = e^{\int_{\Sigma_{(N)}} F} = e^{-\int_{\Sigma_{(S)}} F} \Longrightarrow e^{(\int_{\Sigma_{(N)}} + \int_{\Sigma_{(S)}})F} = e^{\int_{S^2} F} = 1\,. \tag{2.51}$$

Combining with eq.(2.49), we get the condition

$$g = \frac{n}{2}, \quad n \in \mathbf{Z}\,. \tag{2.52}$$

We have assumed here the electric charge to be unit normalized. For a generic charge q, eq.(2.52) gives the known Dirac quantization condition between electric and magnetic charges:

$$qg = \frac{n}{2}, \quad n \in \mathbf{Z}\,. \tag{2.53}$$

We can now show the topological nature of characteristic classes more in general. Consider a G-valued two form field-strength F and define the gauge-invariant $2n$-form $Q_{2n}(F) \equiv \mathrm{tr}\, F^n$. Using the results of Exercise 1, it is easy to show that $Q_{2n}(F)$ is closed:

$$dQ_{2n}(F) = -\sum_{i=0}^{n-1} \mathrm{tr}\, F^i [A, F] F^{n-i-1} = -\mathrm{tr}\,[A, F^n] = 0\,. \tag{2.54}$$

In general, $Q_{2n}(F)$ is not exact, but so it will be the difference $Q_{2n}(F) - Q_{2n}(F')$, where F an F' are two arbitrary field strengths in the same topological class. Hence the integral of $Q_{2n}(F)$ over a $2n$-dimensional compact sub-manifold $\mathcal{C}_{2n}$ of M_d (or over the entire M_d) does not depend on F:

$$\int_{\mathcal{C}_{2n}} Q_{2n}(F) = \int_{\mathcal{C}_{2n}} Q_{2n}(F')\,, \tag{2.55}$$

and defines the cohomology class called the characteristic class of the polynomial $Q_{2n}(F)$. Let us prove that $Q_{2n}(F) - Q_{2n}(F')$ is an exact form. Although the proof is a bit involved, it will turn out to be very useful when we will discuss the so called Wess-Zumino consistency conditions for anomalies. Let A and A' be one-form connections associated to F and F' and define an interpolating connection A_t as

$$A_t = A + t(A' - A)\,, \qquad t \in [0,1]\,. \tag{2.56}$$

We clearly have $A_0 = A$, $A_1 = A'$. Let us also denote by $\theta = A' - A$ the difference of the two connections. We have (recall eq.(2.39))

$$F_t = dA_t + A_t^2 = dA + td\theta + (A + t\theta)^2 = F + t(d\theta + [A,\theta]) + t^2\theta^2\,. \tag{2.57}$$

We can also write

$$Q_{2n}(F') - Q_2(F) = \int_0^1 dt\, \frac{d}{dt} Q_{2n}(F_t) = n \int_0^1 dt\ \mathrm{tr}\left(\frac{dF_t}{dt} F_t^{n-1}\right) \\ = n \int_0^1 dt \Big(\mathrm{tr}\ D\theta F_t^{n-1} + 2t\,\mathrm{tr}\ \theta^2 F_t^{n-1}\Big)\,. \tag{2.58}$$

On the other hand, we have

$$d\mathrm{tr}\ \theta F_t^{n-1} = \mathrm{tr}\ d\theta F_t^{n-1} - \sum_{i=0}^{n-2} \mathrm{tr}\ \theta F_t^i dF_t F_t^{n-i-2} \tag{2.59}$$
$$= \mathrm{tr}\ D\theta F_t^{n-1} - \mathrm{tr}\ [A,\theta] F_t^{n-1} - \sum_{i=0}^{n-2}\Big(\mathrm{tr}\ \theta F_t^i D F_t F_t^{n-i-2} - \mathrm{tr}\ \theta F_t^i [A,F_t] F_t^{n-i-2}\Big)\,.$$

The second and fourth terms in eq.(2.59) combine into a total commutator that vanishes inside a trace: $\mathrm{tr}\ [A, \theta F_t^{n-1}] = 0$. The derivative DF_t equals

$$DF_t = dF_t + [A, F_t] = dF_t + [A_t - A_t + A, F_t] = D_t F_t - t[\theta, F_t] = -t[\theta, F_t]\,, \tag{2.60}$$

given that $D_t F_t = 0$. We also have

$$0 = \mathrm{tr}\ [\theta, \theta F_t^{n-1}] = \mathrm{tr}\ 2\theta^2 F_t^{n-1} - \sum_{i=0}^{n-2} \mathrm{tr}\ \theta F_t^i [\theta, F_t] F_t^{n-i-2}\,. \tag{2.61}$$

Plugging eqs.(2.60) and (2.61) in eq.(2.59) gives

$$d\mathrm{tr}\ \theta F_t^{n-1} = \mathrm{tr}\ D\theta F_t^{n-1} + 2t\,\mathrm{tr}\ \theta^2 F_t^{n-1}\,. \tag{2.62}$$

We have then proved that

$$Q_{2n}(F') - Q_2(F) = d\chi_{2n-1}\,, \tag{2.63}$$

with

$$\chi_{2n-1} = n \int_0^1 dt\, \mathrm{tr}\ \theta F_t^{n-1}\,. \tag{2.64}$$

Notice that χ_{2n-1} is globally defined, because both θ and F_t transform covariantly.

The invariant polynomials we will be interested in comes from generating functions of polynomials defined in any number of dimensions. We will briefly mention here the ones featuring in anomalies. The first is given by the Chern character

$$\mathrm{ch}\,(F) \equiv \mathrm{tr}\ e^{\frac{iF}{2\pi}}\,. \tag{2.65}$$

The Taylor expansion of the exponential gives rise to the series of 2n-forms denoted by n^{th} Chern characters

$$\mathrm{ch}_n(F) = \frac{1}{n!}\mathrm{tr}\left(\frac{iF}{2\pi}\right)^n\,. \tag{2.66}$$

It is clear that for any given d, $\mathrm{ch}_n(F) = 0$ for $n > d/2$.

The second involves the curvature two-form $R^a{}_b$ defined in eq.(2.43). For simplicity of notation we will denote it just by R, omitting the flat indices a and b. It should not be confused with the scalar curvature that will never enter in our considerations! An interesting characteristic class is the Pontrjagin class defined by

$$p(R) = \det\left(1 + \frac{R}{2\pi}\right). \tag{2.67}$$

The expansion of $p(R)$ in invariant polynomials is obtained by bringing the curvature 2-form R_{ab} into a block-diagonal form (this can always be done by an appropriate local Lorentz rotation) of the type

$$R_{ab} = \begin{pmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & & \\ & & \ldots & & \\ & & & 0 & \lambda_{d/2} \\ & & & -\lambda_{d/2} & 0 \end{pmatrix}, \tag{2.68}$$

where λ_i are 2-forms. In this way it is not difficult to find the terms in the expansion of $p(R)$. Due to the antisymmetry of R, there are only even terms in R and hence the invariant polynomials are $4n$-forms. The first two terms with $n = 1, 2$ are

$$\begin{aligned} p_1(R) &= -\frac{1}{2}\left(\frac{1}{2\pi}\right)^2 \mathrm{tr}\, R^2\,, \\ p_2(R) &= \frac{1}{8}\left(\frac{1}{2\pi}\right)^4 \left((\mathrm{tr}\, R^2)^2 - 2\mathrm{tr}\, R^4\right). \end{aligned} \tag{2.69}$$

Like for the Chern class, on a manifold with dimension d all $4n$-forms with $4n > d$ are trivially vanishing. There is however a way in which we can define an other class from the formally vanishing $p_{2d}(R)$ term. We see from eq.(2.67) that the $2d$-form equals just $\det R/2\pi$ (if the factor 1 is taken in any diagonal entry in evaluating the determinant, one necessarily gets a form with lower degree). The determinant of an antisymmetric matrix is always a square of a polynomial called the Pfaffian. Combining these two facts, we can define a d-form $e(R)$ called the Euler class as

$$p_{2d}(R) = e(R)^2\,. \tag{2.70}$$

The integral over the manifold of $e(R)$ is an important invariant called the Euler characteristic of the manifold.

Exercise 4: Compute $e(R)$ for a two-sphere S^2 and show that

$$\chi(S^2) = \int_{S^2} e(R) = +2\,. \tag{2.71}$$

The characteristic class entering directly in the evaluation of anomalies is the so-called roof-genus, defined as

$$\hat{A}(R) = \prod_{k=1}^{d/2} \frac{x_k/2}{\sinh(x_k/2)} = 1 - \frac{1}{24}p_1(R) + \frac{1}{5760}(7p_1(R)^2 - 4p_2(R)) + \ldots \tag{2.72}$$

where $x_k = \lambda_k/2\pi$.

An important theorem by Atiyah and Singer, called index theorem, relates the spectral properties of differential operators on a manifold M_d with its topology (fibers included). This is a remarkable property, because at first glance the two concepts might seem unrelated. We will not discuss index theorems in general, but just states the result for a specific operator that will feature in the following: the dirac operator

$$i\not{D}_{\mathcal{R}} = i e^\mu_a \gamma^a \Big(\partial_\mu + \frac{1}{2}\omega_\mu^{bc} J_{bc} + A^\alpha T^\alpha_{\mathcal{R}}\Big)\,. \tag{2.73}$$

In eq.(2.73) the subscript $\mathcal{R}$ refers to the representation of the fermion under the (unspecified) gauge group G, and $T^\alpha_{\mathcal{R}}$ are the generators in the corresponding representation. The index of the Dirac operator is nothing else that the difference between the number of zero energy eigenfunctions of positive and negative chirality. One has

$$n_+ - n_- = \text{index}\, i\not{D}_{\mathcal{R}} = \int_{M_d} \text{ch}_{\mathcal{R}}(F)\hat{A}(R)\,, \tag{2.74}$$

where

$$\text{ch}_{\mathcal{R}}(F) \equiv \text{tr}_{\mathcal{R}} e^{\frac{iF}{2\pi}}\,. \tag{2.75}$$

As we will see in the next sections, eq.(2.74) gives the contribution of a Dirac fermion to the so called chiral anomaly in d dimensions. It is a remarkable compact formula that should be properly understood. For any given dimension d, one should expand the integrand in eq.(2.74) and selects the form of degree d, which is the only one that can be meaningfully integrated over the manifold M_d.

Let us conclude by spending a few words on the character $\text{ch}_{\mathcal{R}}(F)$. If a representation $\mathcal{R} = \mathcal{R}_1 \otimes \mathcal{R}_2$, one has $\text{ch}_{\mathcal{R}}(F) = \text{ch}_{\mathcal{R}_1}(F)\text{ch}_{\mathcal{R}_2}(F)$. For example, for $G = SU(N)$, we have fund. $\otimes$ $\overline{\text{fund.}}$ = adj. $\oplus$ 1. Correspondingly,

$$\text{ch}_{\text{Adj.}}(F) = \text{ch}(F)\text{ch}(-F) - 1\,, \tag{2.76}$$

where the Chern character without subscript refers to the fundamental representation and we used the fact that $\mathrm{ch}_{\overline{\mathrm{fund.}}}(F) = \mathrm{ch}(-F)$. Expanding eq.(2.76) we have

$$N^2 - 1 - \frac{1}{2}\mathrm{tr}_{\mathrm{Adj.}} F^2 + \ldots = (N - \frac{1}{2}\mathrm{tr}\, F^2 + \ldots)^2 - 1\,, \tag{2.77}$$

from which we deduce the known result

$$\mathrm{tr}_{\mathrm{Adj.}} t^\alpha t^\beta = 2N\, \mathrm{tr}\, t^\alpha t^\beta\,. \tag{2.78}$$

3 Supersymmetric Quantum Mechanics

For reasons that will be clear in the next section, the most elegant computation of anomalies, that makes clear its connection with index theorems and eq.(2.74), is essentially mapped to a computation in a quantum mechanical model whose Hamiltonian equals the square of the Dirac operator (2.73):

$$H = (i\not{D}_{\mathcal{R}})^2\,. \tag{3.1}$$

Let us start by considering the simplest situation of flat space R^d with vanishing gauge potential. The hamiltonian (3.1) reduces to $H = (i\not\partial)^2 = -\partial_\mu^2 = -\nabla = p^2$, namely twice the hamiltonian of a free particle moving in R^d with unit mass. Despite the hamiltonian is the correct one, there is no obvious way to see p^2 as coming from $\not{p}^2$. The "spinor" structure is simply absent. The problem is fixed by adding fermionic variables ψ^μ and write the lagrangian[3]

$$L = \frac{1}{2}\dot{x}^\mu \dot{x}^\mu + \frac{i}{2}\psi^\mu \dot{\psi}^\mu\,. \tag{3.2}$$

The first term is the usual free kinetic term for a particle, the second is a somewhat more unusual "free kinetic term" for fermion coordinates.[4] Notice that ψ^μ are Grassmann variables in one dimension but transform as vectors on R^d. The system described by the lagrangian (3.2) is invariant under the following supersymmetric (SUSY) transformation:

$$\begin{aligned} \delta x^\mu &= i\epsilon\psi^\mu\,, \\ \delta\psi^\mu &= -\epsilon\dot{x}^\mu\,, \end{aligned} \tag{3.3}$$

where ϵ is a constant Grassmann variable, $\epsilon^2 = 0$, anticommuting with ψ^μ. It is easy to check that. We have

$$\delta L = \dot{x}^\mu i\epsilon\dot{\psi}^\mu + \frac{i}{2}(-\epsilon\dot{x}^\mu)\dot{\psi}^\mu + \frac{i}{2}\psi^\mu(-\epsilon\ddot{x}^\mu) = \frac{i}{2}\epsilon\frac{d}{dt}(\dot{x}^\mu\psi^\mu)\,. \tag{3.4}$$

[3] We are in flat euclidean space, so upper and lower vector indices are equivalent.

[4] Notice that we are not discussing a QFT here, so ψ^μ do not represent fermion particles. The interpretation of ψ^μ will be given shortly.

Since the Lagrangian transforms as a total derivative the action $S = \int dt L$ is invariant and eq.(3.3) is a good symmetry. Let us define canonical momenta and impose quantization:

$$\begin{aligned} p^\mu &= \frac{dL}{d\dot{x}^\mu} = \dot{x}^\mu \,, \\ \pi^\mu &= \frac{dL}{d\dot{\psi}^\mu} = \frac{i}{2}\psi^\mu \,. \end{aligned} \tag{3.5}$$

In the bosonic sector we can straightforwardly proceed with the usual replacement of Poisson brackets with quantum commutators, i.e. $[x^\mu, x^\nu] = [p^\mu, p^\nu] = 0$, $[x^\mu, p^\nu] = i\delta^{\mu\nu}$. In the fermion sector one has to be more careful because the naive anti-commutators

$$\{\psi^\mu, \psi^\nu\}_P = \{\pi^\mu, \pi^\nu\}_P = 0 \,, \qquad \{\psi^\mu, \pi^\nu\}_P = i\delta^{\mu\nu} \tag{3.6}$$

are clearly incompatible with the actual form of π^μ in eq.(3.5). This problem is solved by looking at the definition of π^μ in eq.(3.5) as a constraint

$$\chi^\mu \equiv \pi^\mu - \frac{i}{2}\psi^\mu = 0 \,. \tag{3.7}$$

The quantization of theories with constrains is known (see e.g. section 7.6 of ref.[8] for an excellent introduction). If the matrix of the Poisson brackets among the constraints χ^μ is non-singular (computed using the brackets (3.6)), we say that the constraints are of second class. This is our case, since $\{\chi^\mu, \chi^\nu\}_P \equiv c^{\mu\nu} = \delta^{\mu\nu}$. A consistent quantization in presence of second class constraints is obtained by replacing the Poisson bracket with the so called Dirac bracket. More in general, for two anti-commuting operators A and B subject to a series of second class constraints of the form $\chi^M = 0$, we have

$$\{A, B\}_D \equiv \{A, B\}_P - \{A, \chi^M\}_P \, c^{-1}_{MN} \{\chi^N, B\}_P \,, \tag{3.8}$$

where c^{-1} is the inverse of the matrix of brackets $c^{MN} = \{\chi^M, \chi^N\}_P$. A similar formula applies for bosonic fields, with commutators replacing anti-commutators. Coming back to our case, if we take $A = \chi^\rho$ or $B = \chi^\rho$ in eq.(3.8) we get a vanishing result, consistently with the constraints $\chi^\mu = 0$. The consistent anti-commutators for fermions are then

$$\{\psi^\mu, \psi^\mu\}_D = \delta^{\mu\nu} \,, \qquad \{\pi^\mu, \pi^\mu\}_D = -\frac{1}{4}\delta^{\mu\nu} \,, \qquad \{\psi^\mu, \pi^\nu\}_D = \frac{i}{2}\delta^{\mu\nu} \,. \tag{3.9}$$

The Hamiltonian of the system is given by (pay attention to the order of the fermion fields)

$$H = p_\mu \dot{x}^\mu + \pi_\mu \dot{\psi}^\mu - L = \frac{1}{2}p_\mu^2 = -\frac{1}{2}\nabla \,, \tag{3.10}$$

namely it is just the standard free hamiltonian in absence of fermions! The fermion generator of SUSY is given by

$$Q = -\psi^\mu \dot{x}^\mu = -\psi^\mu p^\mu \,. \tag{3.11}$$

We can easily check that it leads to the correct transformation properties:

$$\begin{aligned}\delta x^\mu &= [\epsilon Q, x^\mu] = -\epsilon\psi^\nu[p^\nu, x^\mu] = i\epsilon\psi^\mu\,, \\ \delta\psi^\mu &= [\epsilon Q, \psi^\mu] = -[\epsilon\psi^\nu p^\nu, \psi^\mu] = -\epsilon p^\nu\{\psi^\nu, \psi^\mu\} = -\epsilon p^\mu = -\epsilon\dot{x}^\mu\,.\end{aligned} \tag{3.12}$$

The fermion Dirac brackets in eq.(3.9) generate the same Clifford algebra of gamma matrices in d dimensions. The action in the Hilbert space of ψ_μ is then that of a gamma matrix:

$$\psi^\mu \to \frac{1}{\sqrt{2}}\gamma^\mu\,. \tag{3.13}$$

The anti-commutator of two supercharges equal

$$\{Q, Q\} = \{\psi^\mu p^\mu, \psi^\nu p^\nu\} = p_\mu^2 = 2H \Longrightarrow Q^2 = H\,. \tag{3.14}$$

Plugging eq.(3.13) in the supercharge Q in eq.(3.11) gives

$$Q = \frac{i}{\sqrt{2}}\gamma^\mu\partial_\mu = \frac{1}{\sqrt{2}}i\partial\!\!\!/\,, \tag{3.15}$$

namely the supercharge acts in the Hilbert space as the Dirac operator. Thanks to the fermion operators, we can now reinterpret the hamiltonian (3.10) as the square of the Dirac operator. In quantum mechanics, the existence of SUSY is the statement about the possibility of classifying the spectrum of the system using a fermion number operator. If $|B\rangle$ is a bosonic state, then the state $\psi^\mu|B\rangle$ is fermionic. Correspondingly, given a bosonic state $|B\rangle$, $Q|B\rangle$ is a fermionic one. Viceversa, if $|F\rangle$ is a fermionic state, then $Q|F\rangle$ is a bosonic one. States are grouped in multiplets that rotate as spinors do in d dimensions upon the action of the operator ψ^μ. For d even, we might then split the spectrum into "boson" and "fermion" states, according to the action of the matrix γ_{d+1} defined in eq.(2.20). States with $\gamma_{d+1} = 1$ and $\gamma_{d+1} = -1$ will be denoted bosons and fermions, respectively. The chiral matrix γ_{d+1} is then equivalent to a fermion number operator $(-)^F$.

Let us finally compute the SUSY transformation of the generator Q itself:

$$\delta Q = -\delta\psi^\mu\dot{x}^\mu - \psi^\mu\delta\dot{x}^\mu = \epsilon\dot{x}^\mu\dot{x}^\mu - \psi^\mu(i\epsilon\dot{\psi}^\mu) = 2\epsilon L\,. \tag{3.16}$$

We notice two important properties of Q: its square is proportional to the Hamiltonian and its SUSY variation is proportional to the Lagrangian.[5] In turn, the variation of the Lagrangian is proportional to the time derivative of Q, see eq.(3.4).

[5]This is true in the Lagrangian formulation. In the Hamiltonian formulation we have $\delta Q = [\epsilon Q, Q] = \epsilon\{Q, Q\} = 2\epsilon H$.

Consider now a particle moving on a general manifold M_d. The SUSY transformations (3.3) do not depend on the metric of M_d and are unchanged. Correspondingly Q should be given by the straightforward generalization of eq.(3.11):

$$Q = -g_{\mu\nu}(x)\psi^\mu \dot{x}^\nu \,. \tag{3.17}$$

The simplest way to get the curved space generalization of the Lagrangian (3.2) is to use eq.(3.16).

Exercise 5: Compute the curved space lagrangian L_g by using eqs.(3.16) and (3.17) and show that

$$L_g = \frac{1}{2}g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu + \frac{i}{2}g_{\mu\nu}(x)\psi^\mu\Big(\dot{\psi}^\nu + \Gamma^\nu_{\rho\sigma}(x)\psi^\rho\dot{x}^\sigma\Big)\,, \tag{3.18}$$

where $\Gamma^\nu_{\rho\sigma}$ is the Levi-Civita connection (2.2).

It is important for our purposes to rewrite eq.(3.18) in terms of vielbeins. We have $\psi^\mu = e^\mu_a\psi^a$, so that

$$\frac{d}{dt}\psi^\nu = \frac{d}{dt}(e^\nu_a\psi^a) = (\partial_\mu e^\nu_a)\dot{x}^\mu\psi^a + e^\nu_a\dot{\psi}^a\,, \tag{3.19}$$

where from now on we omit the dependence of vielbeins, metric, etc. on the coordinates x^μ. The terms in the Lagrangian (3.18) involving the fermions can be rewritten as

$$\begin{aligned} g_{\mu\nu}\psi^\mu\Big(\dot{\psi}^\nu + \Gamma^\nu_{\rho\sigma}(x)\psi^\rho\dot{x}^\sigma\Big) &= \psi_a e^a_\nu\Big((\partial_\mu e^\nu_b)\dot{x}^\mu\psi^b + e^\nu_b\dot{\psi}^b + \Gamma^\nu_{\rho\mu}e^\rho_b\psi^b\dot{x}^\mu\Big) \\ &= \psi_a\dot{\psi}^a + \psi_a\psi^b\dot{x}^\mu(e^a_\nu\partial_\mu e^\nu_b + e^a_\rho\Gamma^\rho_{\nu\mu}e^\nu_b) \\ &= \psi_a\dot{\psi}^a + \psi_a\psi^b\dot{x}^\mu e^\nu_b(-\partial_\mu e^a_\nu + \Gamma^\rho_{\nu\mu}e^a_\rho) \\ &= \psi_a\dot{\psi}^a + \frac{1}{2}[\psi_a,\psi_b]\dot{x}^\mu\omega^{ab}_\mu\,, \end{aligned} \tag{3.20}$$

where in the last step we used eq.(2.13). The Lagrangian (3.18) becomes

$$L_g = \frac{1}{2}g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu + \frac{i}{2}\psi_a\dot{\psi}^a + \frac{i}{4}[\psi_a,\psi_b]\omega^{ab}_\mu\dot{x}^\mu\,. \tag{3.21}$$

The momentum conjugate to x^μ is modified:

$$p_\mu = g_{\mu\nu}\dot{x}^\nu + \frac{i}{4}[\psi_a,\psi_b]\omega^{ab}_\mu\,. \tag{3.22}$$

Using eq.(3.22) and replacing $p_\mu \to -i\partial_\mu$, $\psi^a \to \gamma^a/\sqrt{2}$ in the expression for Q in eq.(3.17), we get the natural curved space generalization of eq.(3.15):

$$Q = \frac{1}{\sqrt{2}}ie^\mu_a\gamma^a\Big(\partial_\mu + \frac{1}{8}\omega^{bc}_\mu[\gamma_b,\gamma_c]\Big) = \frac{1}{\sqrt{2}}i\not{D}\,. \tag{3.23}$$

3.1 Adding Gauge Fields

The gauge field terms in the curved space covariant derivative (2.73) are reproduced by adding new fermionic degrees of freedom c^*_A and c^A, where A runs over the dimension of the representation $\mathcal{R}$ associated to eq.(2.73). The fields c^A and c^*_A transform under the representation $\mathcal{R}$ and its complex conjugate $\overline{\mathcal{R}}$ of the gauge group G, respectively. The Lagrangian associated to these fields is

$$L_A = \frac{i}{2}(c^*_A \dot{c}^A - \dot{c}^*_A c^A) + i c^*_A A^A_{\mu\, B} \dot{x}^\mu c^B + \frac{1}{2} c^*_A c^B \psi^\mu \psi^\nu F^A_{\mu\nu B}\,, \tag{3.24}$$

where $A^A_{\mu\;B} = A^\alpha_\mu (T^\alpha)^A{}_B$, $F^A_{\mu\nu\;B} = F^\alpha_{\mu\nu}(T^\alpha)^A{}_B$ are the connection and field strength associated to the group G, with generators T^α. In order to simplify the notation, it is quite convenient to define the one-form $A \equiv A_\mu \psi^\mu$ and two-form $F = F_{\mu\nu}\psi^\mu\psi^\nu$ and suppress the gauge indices. In this more compact notation the Lagrangian (3.24) reads

$$L_A = \frac{i}{2}(c^* \dot{c} - \dot{c}^* c) + i c^* A_\mu c\, \dot{x}^\mu + c^* F c\,. \tag{3.25}$$

The Lagrangian (3.25) is invariant under SUSY transformations of the form

$$\delta c = -i\epsilon A c\,, \qquad \delta c^* = -i\epsilon c^* A\,, \tag{3.26}$$

together with the transformations (3.3) that are left unchanged.

Exercise 6: Show that L_A is invariant under the transformations (3.3) and (3.26). Hint: use a compact notation and recall that $DF = dF + [A, F] = 0$.

The canonical momenta of the fields c^* and c read

$$\pi_c = \frac{i}{2} c^*\,, \qquad \pi_{c^*} = \frac{i}{2} c\,. \tag{3.27}$$

Like the fermions ψ^μ before, eq.(3.27) should be interpreted as a set of constraints among the fields c, c^* and their momenta π_c and π_{c^*}: $\chi_c = \pi_c - ic^*/2 = 0$, $\chi_{c^*} = \pi_{c^*} - ic/2 = 0$. Using eq.(3.8), it is straightforward to find the consistent Dirac (anti)commutation relations between the fields c^B and c^*_A:

$$\left\{c^\star_A, c^B\right\} = \delta^B_A\,. \tag{3.28}$$

Considering c^* and c as creation and annihilation operators, respectively, the Hilbert space of the system includes (in addition to excited states) the vacuum $|0\rangle$, "1-particle" states

$c^\star_A|0\rangle$, "2-particle" states $c^\star_A c^\star_B|0\rangle$, etc. Among all these states, only the 1-particle states correspond to the representation $\mathcal{R}$ of the gauge group G, the vacuum being a singlet of G and "multiparticle" states leading to tensor products of the representation $\mathcal{R}$. On such particle states, the operator $c^*_A(T^\alpha)^A{}_B c^B$ acts effectively as $(T^\alpha)^A{}_B$. The supercharge is given by eq.(3.17), but the momentum conjugate to x^μ gets another term, coming from L_A:

$$p_\mu = g_{\mu\nu}\dot{x}^\nu + \frac{i}{4}[\psi_a,\psi_b]\omega_\mu^{ab} + ic^* A_\mu c\,. \tag{3.29}$$

It is straightforward to check that the transformations (3.26) are reproduced by acting with Q on the fields c^* and c.

Summarizing, the total Lagrangian is given by $L = L_g + L_A$, with

$$\begin{aligned} L =& \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \frac{i}{2}\psi_a\dot{\psi}^a + \frac{i}{4}\big[\psi_a,\psi_b\big]\omega_\mu^{ab}\dot{x}^\mu \\ &+ \frac{i}{2}(c^\star_A\dot{c}^A - \dot{c}^\star_A c^A) + ic^*_A A^A_{\mu\,B}\,\dot{x}^\mu c^B + \frac{1}{2}c^\star_A c^B\psi^a\psi^b F^A_{ab\;B}\,, \end{aligned} \tag{3.30}$$

and is invariant under a SUSY transformation Q that acts on the Hilbert space as

$$Q = \frac{i}{\sqrt{2}}e^\mu_a\gamma^a\Big[\Big(\partial_\mu + \frac{1}{8}\omega_\mu^{ab}\big[\gamma_a,\gamma_b\big]\Big)\delta^A_B + A^\alpha_\mu\,T^A_{\alpha\;B}\Big] = \frac{1}{\sqrt{2}}\,i\not{D}_{\mathcal{R}}\,. \tag{3.31}$$

We have finally determined a SUSY quantum mechanical model with Hamiltonian given by eq.(3.1).

4 The Chiral Anomaly

In the path-integral formulation of QFT, anomalies arise from the transformation of the measure used to define the fermion path integral [9].

Let $\psi_A(x)$ be a massless Dirac fermion on a $d = 2n$-dimensional manifold M_{2n} in an arbitrary representation $\mathcal{R}$ of a gauge group G ($A = 1,\ldots,\dim\mathcal{R}$). The minimal coupling of the fermion to the gauge and gravitational fields is described by the Lagrangian

$$\mathcal{L} = e\,\bar{\psi}(x)_A i(\not{D})^A{}_B\,\psi^B\,, \tag{4.1}$$

where e is the determinant of the vielbein, $i\not{D}$ is the Dirac operator (2.73) and for simplicity we drop from now on the subscript $\mathcal{R}$ in the covariant derivative.

The classical Lagrangian (4.1) is invariant under the global chiral transformation

$$\psi \to e^{i\alpha\gamma_{2n+1}}\psi\,, \qquad \bar{\psi} \to \bar{\psi}e^{i\alpha\gamma_{2n+1}}\,, \tag{4.2}$$

where γ_{2n+1} is given in eq.(2.20) and α is a constant parameter. The associated classically conserved chiral current reads $J^\mu_{2n+1} = \bar{\psi}_A\gamma_{2n+1}\gamma^\mu\psi^A$. At the quantum level, however, this conservation law can be violated. Consider the quantum effective action Γ defined by

$$e^{-\Gamma(e,\omega,A)} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{-\int d^{2n}x\,\mathcal{L}}\,, \tag{4.3}$$

and study its behavior under an infinitesimal version of the chiral transformation (4.2) with a space-time-dependent parameter $\epsilon(x)$, given by[6]

$$\delta_\epsilon\psi = i\epsilon(x)\gamma_{2n+1}\psi(x)\,, \quad \delta_\epsilon\bar{\psi} = i\epsilon(x)\bar{\psi}(x)\gamma_{2n+1}\,. \tag{4.4}$$

Since the external fields e, ω and A are inert, the transformation (4.4) represents a redefinition of dummy integration variables, and should not affect the effective action: $\delta_\epsilon\Gamma = 0$. This statement carries however a non-trivial piece of information, since neither the action nor the integration measure is invariant under eq.(4.4). The variation of the classical action under eq.(4.4) is non-vanishing only for non-constant ϵ, and has the form $\delta_\epsilon \int \mathcal{L} = \int J^\mu_{2n+1}\partial_\mu\epsilon$. The variation of the measure is instead always non-vanishing, because the transformation (4.4) leads to a non-trivial Jacobian factor, which has the form $\delta_\epsilon[\mathcal{D}\psi\mathcal{D}\bar{\psi}] = [\mathcal{D}\psi\mathcal{D}\bar{\psi}](-2i\int\epsilon\mathcal{A})$, as we will see below. In total, the effective action therefore transforms as

$$\delta_\epsilon\Gamma = \int d^{2n}x e\,\epsilon(x)\Big[2i\mathcal{A}(x) - \langle\partial_\mu J^\mu_{2n+1}(x)\rangle\Big]\,. \tag{4.5}$$

The condition $\delta_\epsilon\Gamma = 0$ implies the anomalous Ward identity:

$$\langle\partial_\mu J^\mu_{2n+1}\rangle = 2i\mathcal{A}\,. \tag{4.6}$$

In order to compute the anomaly $\mathcal{A}$, we need to define the integration measure more precisely. This is best done by considering the eigenfunctions of the Dirac operator $i\not{D}$. Since the latter is Hermitian, the set of its eigenfunctions $\psi_k(x)$ with eigenvalues λ_k, defined by $i\not{D}\psi_n = \lambda_n\psi_n$, form an orthonormal and complete basis of spinor modes:

$$\int d^{2n}x\, e\,\psi_k^\dagger(x)\psi_l(x) = \delta_{kl}\,, \quad \sum_k e\,\psi_k^\dagger(x)\psi_k(y) = \delta^{(2n)}(x-y)\,. \tag{4.7}$$

The fermion fields ψ and $\bar{\psi}$, which are independent from each other in Euclidean space, can be decomposed as

$$\psi = \sum_k a_k\psi_k\,, \quad \bar{\psi} = \sum_k \bar{b}_k\psi_k^\dagger\,, \tag{4.8}$$

so that the measure becomes

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} = \prod_{k,l} da_k d\bar{b}_l\,. \tag{4.9}$$

[6]For simplicity of the notation, we omit the gauge index A in the following equations. It will be reintroduced later on in this section.

Under the chiral transformation (4.4), we have

$$a'_k = a_k + i\int d^{2n}x\, e\,\sum_l \psi_k^\dagger \epsilon \gamma_{2n+1}\psi_l a_l\,,\quad \bar{b}'_k = \bar{b}_k + i\int d^{2n}x\, e\,\sum_l \bar{b}_l \psi_l^\dagger \epsilon\gamma_{2n+1}\psi_k\,, \tag{4.10}$$

and the measure (4.9) transforms as

$$\begin{aligned} \mathcal{D}\psi'\mathcal{D}\bar{\psi}' =&\mathcal{D}\psi\mathcal{D}\bar{\psi}\det(\delta_{kl} + i\int d^{2n}x\, e\,\psi_k^\dagger\epsilon\gamma_{2n+1}\psi_l)^{-2} \\ \Longrightarrow \delta_\epsilon[\mathcal{D}\psi\mathcal{D}\bar{\psi}] =&[\mathcal{D}\psi\mathcal{D}\bar{\psi}]\Big(-2i\sum_k\int d^{2n}x\, e\,\psi_k^\dagger\epsilon\gamma_{2n+1}\psi_k + \mathcal{O}(\epsilon^2)\Big)\,. \end{aligned} \tag{4.11}$$

We can now take ϵ to be constant. The expression (4.11) is ill-defined as it stands since it decomposes into a vanishing trace over spinor indices ($\mathrm{tr}\,\gamma_{2n+1} = 0$) times an infinite sum over the modes ($\sum_k 1 = \infty$). A convenient way of regularizing this expression is to introduce a gauge-invariant Gaussian cut-off. The integrated anomaly $\mathcal{Z} = \int \mathcal{A}$ can then be defined as

$$\begin{aligned} \mathcal{Z} &= \lim_{\beta\to 0}\sum_k \psi_k^\dagger\gamma_{2n+1}\psi_k e^{-\beta\lambda_k^2/2} \\ &= \lim_{\beta\to 0}\mathrm{Tr}\left[\gamma_{2n+1}e^{-\beta(i\not{D})^2/2}\right], \end{aligned} \tag{4.12}$$

where the trace has to be taken over the mode and the spinor indices, as well as over the gauge indices. Equation (4.12) finally provides the connection between anomalies and quantum mechanics. Indeed, it represents the high-temperature limit ($T = 1/\beta$) of the partition function of the quantum mechanical model (3.30) that has as Hamiltonian $H = (i\not{D})^2/2$ and as density matrix $\rho = \gamma_{2n+1}e^{-\beta H}$: $\mathcal{Z} = \mathrm{Tr}\,\rho$. $\mathcal{Z}$ is not clearly the ordinary thermal partition function of the system, because of the presence of the chirality matrix γ_{2n+1}. Its effect is rather interesting. As we mentioned in the previous section, γ_{2n+1} acts as the fermion counting operator $(-)^F$. In supersymmetric quantum mechanics, with $H = Q^2$, any state $|E\rangle$ with strictly positive energy $E > 0$ is necessarily paired with its supersymmetric partner $|\tilde{E}\rangle$: $Q|E\rangle = \sqrt{E}|\tilde{E}\rangle$. Of course, $|E\rangle$ and $|\tilde{E}\rangle$ have the same energy but a fermion number F differing by one unit. Independently of the bosonic or fermionic nature of $|E\rangle$, the contribution to $\mathcal{Z}$ of $|E\rangle$ and $|\tilde{E}\rangle$ is equal and opposite and always cancels. The only states that might escape this pairing are the ones with zero energy. In this case $Q|E = 0\rangle = 0$ and hence these states are not necessarily paired. We conclude that

$$\mathcal{Z} = n_+ - n_- \tag{4.13}$$

independently of β, where $n_\pm$ are the number of zero energy bosonic and fermionic states. Interestingly enough, $\mathcal{Z}$ does not depend on smooth deformations of the system. If a state

$|E\rangle$ with $E > 0$ approaches $E = 0$, so has to do its partner $|\tilde{E}\rangle$. Viceversa, zero energy states can be deformed to $E > 0$ only in pairs. So, while n_+ and n_- individually might depend on the details of the theory, their difference is an invariant object, called Witten index [10]. Since $n_\pm$ correspond to the number of zero energy eigenfunctions of positive and negative chirality of the Dirac operator, we see that $\mathcal{Z}$ coincides with the index of the Dirac operator defined in eq.(2.74). Using the Atiyah-Singer theorem we could read the result for the chiral anomaly from the right hand side of eq.(2.74). However, it is actually not so difficult to compute $\mathcal{Z}$. This computation can be seen as a physical way to prove the Atiyah-Singer theorem applied to the Dirac operator [11].

The integrated anomaly $\mathcal{Z}$ can be derived by computing directly the partition function of the above supersymmetric model in a Hamiltonian formulation. In order to obtain the correct result, however, one has to pay attention when tracing over the fermionic operators $c^\star_A$ and c^A. As discussed below eq.(3.28), only the 1-particle states $c^\star_A|0\rangle$ correspond to the representation $\mathcal{R}$ of the gauge group G. In order to compute the anomaly for a single spinor in the representation $\mathcal{R}$, it is therefore necessary to restrict the partition function to such 1-particle states. On the other hand, the partition function of a system at finite temperature $T = 1/\beta$ can be conveniently computed in the canonical formalism by considering an euclidean path integral where the (imaginary) time direction is compactified on a circle of radius $1/(2\pi T)$. The insertion of the fermion operator $(-)^F$ amounts to change the fermion boundary conditions from anti-periodic (the usual one in canonical formalism) to periodic. So the best way to proceed is to use a hybrid formulation, which is Hamiltonian with respect to the fields $c^\star_A$ and c^A, and Lagrangian with respect to the remaining fields x^μ and ψ^a. Starting from the hamiltonian H, we then define a modified Lagrange transform (technically called Routhian R) with respect to the fields x^μ and ψ^a only. After Wick-rotating to Euclidean time $\tau \to -i\tau$, the index $\mathcal{Z}$ can be written as

$$\mathcal{Z} = \mathrm{Tr}_{c,c^\star} \int_P \mathcal{D}x^\mu \int_P \mathcal{D}\psi^a \exp\left\{ -\int_0^\beta d\tau\, \mathcal{R}\Big(x^\mu(\tau), \psi^a(\tau), c^\star_A, c^A\Big)\right\}. \tag{4.14}$$

The subscript P on the functional integrals stands for periodic boundary conditions along the closed time direction τ, $\mathrm{Tr}_{c,c^\star}$ represents the trace over the 1-particle states $c^\star_A|0\rangle$, and the Euclidean Routhian $\mathcal{R}$ is given by

$$\begin{aligned}\mathcal{R} = \frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + \frac{1}{2}\psi_a \dot{\psi}^a + \frac{1}{4}[\psi_a, \psi_b]\omega_\mu^{ab}\dot{x}^\mu \\ + c^\star_A A_{\mu\ B}^{A}\dot{x}^\mu c^B - \frac{1}{2} c^\star_A c^B \psi^a\psi^b F_{ab\ B}^{A}\,.\end{aligned} \tag{4.15}$$

Equation (4.14) should be understood as follows: after integrating over the fields x^μ and ψ^a, one gets an effective Hamiltonian $\hat{H}(c, c^\star)$ for the operators $c^\star_A$ and c^A, from which one computes $\mathrm{Tr}_{c,c^\star} e^{-\beta \hat{H}(c,c^\star)}$. Although this procedure looks quite complicated, we will see that it drastically simplifies in the high-temperature limit we are interested in.

The computation of the path-integral is greatly simplified by using the background-field method and expanding around constant bosonic and fermionic configurations in normal coordinates [12]. These are defined as the coordinates for which around any point x_0 the spin-connection (or equivalently, the Christoffel symbol) and all its symmetric derivatives vanish. It is convenient to rescale $\tau \to \beta\tau$ and define

$$x^\mu(\tau) = x_0^\mu + \sqrt{\beta}\, e_a^\mu(x_0)\xi^a(\tau)\,, \quad \psi^a(\tau) = \frac{1}{\sqrt{\beta}}\psi_0^a + \lambda^a(\tau)\,. \tag{4.16}$$

In this way, it becomes clear that it is sufficient to keep only quadratic terms in the fluctuations, which have a β-independent integrated Routhian, since higher-order terms in the fluctuations come with growing powers of β. In normal coordinates one has

$$\begin{aligned}
\omega_\mu^{ab}(x)\dot{x}^\mu(\tau) &= \sqrt{\beta}\omega_\mu^{ab}(x_0)e_d^\mu(x_0)\dot{\xi}^d(\tau) + \beta\partial_\nu\omega_\mu^{ab}(x_0)e_d^\mu(x_0)\dot{\xi}^d(\tau)e_c^\nu(x_0)\,\xi^c(\tau) + \mathcal{O}(\beta^{3/2}) \\
&= 0 + \beta\partial_{[\nu}\omega_{\mu]}^{ab}(x_0)e_d^\mu(x_0)e_c^\nu(x_0)\dot{\xi}^d(\tau)\xi^c(\tau) + \mathcal{O}(\beta^{3/2}) \\
&= \beta R_{cd}^{ab}(x_0)\dot{\xi}^d(x_0)\xi^c(x_0) + \mathcal{O}(\beta^{3/2})\,.
\end{aligned} \tag{4.17}$$

The term proportional to A_μ in eq.(4.15) vanishes in this limit because it is of order β. Using these results, eq. (4.15) reduces to the following effective quadratic Routhian, in the limit $\beta \to 0$:

$$\mathcal{R}^{\rm eff} = \frac{1}{2}\Big[\dot{\xi}_a\dot{\xi}^a + \lambda_a\dot{\lambda}^a + R_{ab}(x_0,\psi_0)\xi^a\dot{\xi}^b\Big] - c_A^\star F^A_{\ B}(x_0,\psi_0)c^B\,, \tag{4.18}$$

where

$$\begin{aligned}
R_{ab}(x_0,\psi_0) &= \frac{1}{2}R_{abcd}(x_0)\psi_0^c\psi_0^d\,, \\
F^A_{\ B}(x_0,\psi_0) &= \frac{1}{2}F_{ab\ \ B}^{\ \ A}(x_0)\psi_0^a\psi_0^b\,.
\end{aligned} \tag{4.19}$$

Since the fermionic zero modes ψ_0^a anticommute with each other,[7] they define a basis of differential forms on M_{2n}, and the above quantities behave as curvature 2-forms.

From eq.(4.18) we see that the gauge and gravitational contributions to the chiral anomaly are completely decoupled. The former is determined by the trace over the 1-particle states $c_A^\star|0\rangle$, and the latter by the determinants arising from the Gaussian path integral over the bosonic and fermionic fluctuation fields:

$$\mathcal{Z} = \int d^{2n}x_0 \int d^{2n}\psi_0\, \mathrm{Tr}_{c,c^\star}\Big[e^{c_A^\star F^A_{\ B}c^B}\Big]\, \mathrm{det}_P^{-1/2}\Big[-\partial_\tau^2\delta_{ab} + R_{ab}\partial_\tau\Big]\, \mathrm{det}_P^{1/2}\Big[\partial_\tau\delta_{ab}\Big]\,. \tag{4.20}$$

[7]The anticommuting ψ_0^a's are simply Grassmann variables in a path integral and should not be confused with the operator-valued fields ψ^a entering eq.(3.30), which satisfy the anticommutation relations (3.9).

The trace yields simply

$$\mathrm{Tr}_{c,c^\star} e^{c^\star_A F^A_{\ B} c^B} = \mathrm{tr}_{\mathcal{R}} e^F \,. \tag{4.21}$$

The determinants can be computed by decomposing the fields on a complete basis of periodic functions of τ on the circle with unit radius, and using the standard ζ-function regularization. Let us quickly recall how ζ-function regularization works by computing the partition function of a free periodic real boson x on a line:

$$Z_B^{\rm free} = \int_P \mathcal{D}x e^{-\frac{m}{2}\int_0^t \dot{x}^2 d\tau} \,. \tag{4.22}$$

We expand x in Fourier modes x_n, $x_{-n} = x_n^*$ and rewrite

$$Z_B^{\rm free} = \int dx_0 \prod_{n=1}^\infty dx_n dx_n^* e^{-mt\sum_{n=1}^\infty \left(\frac{2\pi n}{t}\right)^2 |x_n|^2} = \int dx_0 \prod_{n=1}^\infty \Big(\frac{t}{2\pi m n^2}\Big) \,. \tag{4.23}$$

Using the identity

$$\prod_{n=1}^\infty f_n = \exp\Big(-\lim_{s\to 0}\frac{d}{ds}\sum_{n=1}^\infty f_n^{-s}\Big) \,, \tag{4.24}$$

we compute

$$\prod_{n=1}^\infty \Big(\frac{t}{2\pi m n^2}\Big) = \exp\Big(-\frac{d}{ds}\Big(\frac{t}{2\pi m}\Big)^s \sum_{n=1}^\infty n^{2s}\Big)_{s=0} = \exp\Big(\zeta(0)\log\frac{t}{2\pi m} + 2\zeta'(0)\Big) \,, \tag{4.25}$$

where $\zeta(0) = -1/2$, $\zeta' = -\log 2\pi/2$ are the value in zero of the ζ function and its first derivative. We then find

$$Z_B^{\rm free} = L\Big(\frac{m}{2\pi t}\Big)^{1/2} \,, \tag{4.26}$$

where L is the length of the line. The partition function of a free periodic fermion ψ on a line vanishes because of the integration over the Grassmannian zero mode ψ_0. Inserting one ψ_0 in the path integral and proceeding as in the bosonic case, one gets

$$Z_F^{\rm free} = \int_P \mathcal{D}\psi\, \psi\, e^{\frac{i}{2} m \int_0^t \psi\dot{\psi} d\tau} = m^{-1/2} \,. \tag{4.27}$$

Coming back to the evaluation of the determinants in eq.(4.20), it is useful to bring the curvature 2-form R_{ab} into the block-diagonal form (2.68), so that the bosonic determinant decomposes into n distinct determinants with trivial matrix structure.

Exercise 7: Compute the determinants appearing in eq.(4.20) and show that

$$\det\nolimits_P^{-1/2}\Big[-\partial_\tau^2\delta_{ab} + R_{ab}\partial_\tau\Big] = (2\pi)^{-n}\prod_{i=1}^n \frac{\lambda_i/2}{\sin(\lambda_i/2)} \,, \tag{4.28}$$

$$\det\nolimits_P^{1/2}\Big[\partial_\tau\delta_{ab}\Big] = (-i)^n \,. \tag{4.29}$$

The final result for the anomaly is obtained by putting together eqs.(4.21), (4.28) and (4.29), and integrating over the zero modes. The Berezin integral over the fermionic zero modes vanishes unless all of them appear in the integrand, in which case it yields

$$\int d^{2n}\psi_0 \, \psi_0^{a_1} \dots \psi_0^{a_{2n}} = (-1)^n \, \epsilon^{a_1 \dots a_{2n}} \,, \tag{4.30}$$

and automatically selects the $2n$-form component from the expansion of the integrand in powers of the 2-forms F and R. Since this is a homogeneous polynomial of degree n in F and R, the factor $(i/(2\pi))^n$ arising from the normalization of eqs.(4.28), (4.29) and (4.30) amounts to multiplying F and R by $i/(2\pi)$. The final result for the integrated anomaly can therefore be rewritten more concisely as

$$\mathcal{Z} = \int_{M_{2n}} \mathrm{ch}_{\mathcal{R}}(F) \, \hat{A}(R) \,, \tag{4.31}$$

where $\mathrm{ch}(F)$ and $\hat{A}(R)$ have been defined in eqs.(2.72) and (2.75). As explained below eq.(2.75), only the $2n$-form component of the integrand has to be considered.

The anomalous Ward identity (4.6) for the chiral symmetry can be formally written, in any even dimensional space, as

$$\langle \partial_\mu J^\mu_{2n+1} \rangle = 2i \, \mathrm{ch}_{\mathcal{R}}(F) \hat{A}(R)|_{2n-\mathrm{form}} \,, \tag{4.32}$$

where the volume form is understood to be omitted on the right hand side of eq.(4.32). In 4 space-time dimensions, for instance, eq.(4.32) gives

$$\langle \partial_\mu J^\mu_5 \rangle = -i\epsilon^{\mu\nu\rho\sigma} \left[\frac{1}{16\pi^2} \mathrm{tr} \, F_{\mu\nu} F_{\rho\sigma} + \frac{\dim \mathcal{R}}{384\pi^2} R_{\mu\nu}{}^{\alpha\beta} R_{\rho\sigma\alpha\beta} \right] . \tag{4.33}$$

5 Consistency Conditions for Gauge and Gravitational Anomalies

Anomalies can also affect currents related to space-time dependent symmetries. For simplicity, we will consider in the following anomalies related to spin gauge fields only, i.e. gauge anomalies. Gravitational anomalies can be analyzed in almost the same way, modulo some subtleties we will mention later on. In presence of a gauge anomaly, the theory is no longer gauge-invariant. This is best seen by considering the effective action Γ in eq.(4.3). Under an infinitesimal gauge transformation $A \to A - D\epsilon_1$

$$\delta_{\epsilon_1} \Gamma(A) = -\int d^{2n}x \, \mathrm{tr} \, \frac{\delta\Gamma(A)}{\delta A_\mu} D_\mu \epsilon_1 = \int d^{2n}x \, \mathrm{tr} \, D_\mu J^\mu \epsilon_1 = \int d^{2n}x \, \mathrm{tr} \, a\epsilon_1 \equiv \mathcal{I}(\epsilon_1) \,, \tag{5.1}$$

where a_α is the gauge anomaly and we have defined its integrated form (including the gauge parameter ϵ) by $\mathcal{I}$.[8] The WT identities associated to the conservation of currents in a gauge theory allows us to define a consistent theory, where unphysical states decouple. The anomaly term a that would appear in the WT identities associated to the (would-be) conservation of the current would have the devastating effect of spoiling this decoupling. Gauge anomalies lead to inconsistent theories, where unitarity is broken at any scale.

The structure of the gauge anomalies that can occur in local symmetries is strongly constrained by the group structure of these symmetry transformations. In particular, two successive transformations δ_{ϵ_1} and δ_{ϵ_2} with parameters ϵ_1 and ϵ_2 must satisfy the basic property: $\big[\delta_{\epsilon_1}, \delta_{\epsilon_2}\big] = \delta_{[\epsilon_1,\epsilon_2]}$. We should then have

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\Gamma(A) = \delta_{[\epsilon_1,\epsilon_2]}\Gamma(A)\,, \tag{5.2}$$

or in terms of the anomaly $\mathcal{I}$ defined in eq.(5.1):

$$\delta_{\epsilon_1}\mathcal{I}(\epsilon_2) - \delta_{\epsilon_2}\mathcal{I}(\epsilon_1) = \mathcal{I}([\epsilon_1, \epsilon_2])\,. \tag{5.3}$$

The above relations are called Wess–Zumino consistency conditions [13].

The general solution of this consistency condition can be characterized in an elegant way in terms of a $(2n+2)$-form with the help of the so-called Stora–Zumino descent relations [14]. For any local symmetry with transformation parameter ϵ (a 0-form), connection A (a 1-form) and curvature F (a 2-form), these are defined as follows. Starting from a generic closed and invariant $(2n+2)$-form $\Omega_{2n+2}(F)$, one can define an equivalence class of Chern–Simons $(2n+1)$-forms $\Omega^{(0)}_{2n+1}(A,F)$ through the local decomposition $\Omega_{2n+2} = d\Omega^{(0)}_{2n+1}$, like in section 2.3. In this way, we specify $\Omega^{(0)}_{2n+1}$ only modulo exact $2n$-forms, implementing the redundancy associated to the local symmetry under consideration. One can then define yet another equivalence class of $2n$-forms $\Omega^{(1)}_{2n}(\epsilon, A, F)$, modulo exact $(2n-1)$-forms, through the transformation properties of the Chern–Simons form under a local symmetry transformation: $\delta_\epsilon\Omega^{(0)}_{2n+1} = d\Omega^{(1)}_{2n}$. It is the unique integral of this class of $2n$-forms $\Omega^{(1)}_{2n}$ that gives the relevant general solution of eq.(5.3):

$$\mathcal{I}(\epsilon) = 2\pi i\int_{M_{2n}}\Omega^{(1)}_{2n}(\epsilon)\ . \tag{5.4}$$

To understand this, notice that by Stokes theorem

$$\int_{M_{2n}}\Omega^{(1)}_{2n}(\epsilon) = \int_{M'_{2n+1}} d\Omega^{(1)}_{2n}(\epsilon) = \delta_\epsilon\int_{M'_{2n+1}}\Omega^{(0)}_{2n+1}\,, \tag{5.5}$$

[8]We denote the anomaly by a and not $\mathcal{A}$ as in the last section, because we will later on denote by $\mathcal{A}$ a certain connection entering the Stora-Zumino descent relations.

where M'_{2n+1} is an arbitrary $2n+1$-dimensional manifold whose boundary is M_{2n}: $M_{2n} = \partial M'_{2n+1}$. For instance, if $M_{2n} = S^{2n}$, M_{2n+1} can be taken to be the $2n+1$ dimensional ball B^{2n+1}. Notice that eq.(5.5) is valid only if $\Omega^{(1)}_{2n}(\epsilon)$ is globally well-defined, so we have to assume that either the gauge transformation ϵ is non-vanishing on a single chart only, or that the gauge bundle on M_{2n} is trivial. This does not necessarily imply that the formal $2n+2$ dimensional gauge field is trivial [6]. Eq. (5.4) can be rewritten as

$$\mathcal{I}(\epsilon) = 2\pi i\, \delta_\epsilon \int_{M'_{2n+1}} \Omega^{(0)}_{2n+1} \,, \tag{5.6}$$

and clearly provides a solution of eq.(5.3), since it is manifestly in the form of the variation of some functional under the symmetry transformation. The descent construction also insures that eq.(5.6) characterizes the most general non-trivial solution of eq.(5.3), modulo possible local counterterms. This can be understood more precisely within the BRST formulation of the Stora–Zumino descent relations, which we described in the next section.

The above reasoning applies also to gravitational anomalies. Both gauge and gravitational anomalies in a $2n$-dimensional theory are then characterized by a gauge-invariant $(2n{+}2)$-form. We shall see in the next section that the chiral anomaly form Ω_{2n+2} provides precisely the $(2n+2)$-form defined above through which we can solve the Wess-Zumino consistency relations. The gauge and gravitational anomalies in $2n$ dimensions are then obtained through the descent procedure, defined respectively with respect to gauge transformations and diffeomorphisms (or local Lorentz transformations), as $\mathcal{I} = 2\pi i \int_{M_{2n}} \Omega^{(1)}_{2n}$.

6 The Stora–Zumino Descent Relations

The Stora–Zumino descent relations are best analyzed by using a differential-form notation.[9] As we have seen in the previous section, starting from a gauge-invariant $(2n{+}2)$-form $\Omega_{2n+2}(F)$ we can define a Chern–Simons $(2n+1)$-form through the local decomposition $\Omega_{2n+2}(F) = d\Omega_{2n+1}(A,F)$. The gauge variation of the latter defines a $2n$-form $\Omega_{2n}(A,F)$ through the transformation law $\delta_v \Omega_{2n+1}(A,F) = d\Omega_{2n}(v,A,F)$.

Under gauge transformations $g(x,\theta)$ depending on the coordinates x^μ and on some (ordinary, not Grassmann) parameters θ^α, A and F transform as

$$\bar{A}(x,\theta) = g^{-1}(x,\theta)\big(A(x)+d\big)g(x,\theta) \,, \tag{6.1}$$

$$\bar{F}(x,\theta) = g^{-1}(x,\theta)F(x)g(x,\theta) \,. \tag{6.2}$$

We can define, besides the usual exterior derivative $d = dx^\mu \partial_\mu$ with respect to the coordinates x^μ, an additional exterior derivative $\hat{d} = d\theta^\alpha \partial_\alpha$ with respect to the parameters

[9] This section closely follows section 3.C of ref.[6], adopting the same notation.

θ^α. The range of values of the index α is for the moment undetermined. The operators d and $\hat{d}$ anticommute and are both nilpotent, $d^2 = \hat{d}^2 = 0$. This implies that their sum $\Delta = d + \hat{d}$ is also nilpotent: $\Delta^2 = 0$. The operator $\hat{d}$ naturally defines a transformation parameter $\hat{v}$ through the expression:

$$\hat{v}(x,\theta) = g^{-1}(x,\theta)\hat{d}g(x,\theta)\,. \tag{6.3}$$

Exercise 8: Verify that

$$\hat{d}\hat{v} = -\hat{v}^2\,, \;\; \hat{d}\bar{A} = -\bar{D}\,\hat{v}\,, \;\; \hat{d}\bar{F} = -[\hat{v},\bar{F}]\,. \tag{6.4}$$

Equations (6.4) show that $\hat{d}$ generates an infinitesimal gauge transformation with parameter $\hat{v}$ on the gauge field $\bar{A}$ and its field-strength $\bar{F}$. Interestingly enough, these can also be interpreted as BRST transformations, the ghost fields being identified with $\hat{v}$. At this point, it is possible to define yet another connection $\mathcal{A}$ and field-strength $\mathcal{F}$ as

$$\mathcal{A} = \; g^{-1}\big(A + \Delta\big)g = \bar{A} + \hat{v}\,, \tag{6.5}$$

$$\mathcal{F} = \; \Delta\mathcal{A} + \mathcal{A}^2 = g^{-1}Fg = \bar{F}\,. \tag{6.6}$$

The last relation is easily proved. Using eqs.(6.4) one has

$$\begin{aligned}
\mathcal{F} &= (d+\hat{d})(\bar{A}+\hat{v}) + (\bar{A}+\hat{v})^2 \\
&= d\bar{A} + d\hat{v} - \bar{D}\hat{v} - \hat{v}^2 + \hat{v}^2 + \bar{A}\hat{v} + \hat{v}\bar{A} + \bar{A}^2 \qquad (6.7)\\
&= d\bar{A} + \bar{A}^2 \\
&= \bar{F}\,. \qquad (6.8)
\end{aligned}$$

The crucial point is that $\mathcal{A}$ and $\mathcal{F}$ are defined with respect to Δ exactly in the same way as $\bar{A}$ and $\bar{F}$ are defined with respect to d. Therefore, the corresponding Chern–Simons decompositions must have the same form:

$$Q_{2n+2}(\mathcal{F}) = \Delta Q_{2n+1}(\mathcal{A},\mathcal{F})\,, \tag{6.9}$$

$$Q_{2n+2}(\bar{F}) = dQ_{2n+1}(\bar{A},\bar{F})\,. \tag{6.10}$$

On the other hand, eq.(6.6) implies that the left-hand sides of these two equations are identical. Equating the right-hand sides and using eq.(6.5) yields:

$$(d+\hat{d})Q_{2n+1}(\bar{A}+\hat{v},\bar{F}) = dQ_{2n+1}(\bar{A},\bar{F})\,. \tag{6.11}$$

In order to extract the information carried by this equation, it is convenient to expand $Q_{2n+1}(\bar{A}+\hat{v},\bar{F})$ in powers of $\hat{v}$ as

$$Q_{2n+1}(\bar{A}+\hat{v},\bar{F}) = Q^{(0)}_{2n+1}(\bar{A},\bar{F}) + Q^{(1)}_{2n}(\hat{v},\bar{A},\bar{F}) + \ldots + Q^{(2n+1)}_{0}(\hat{v},\bar{A},\bar{F})\,, \tag{6.12}$$

where the superscripts denote the powers of $\hat{v}$ and the subscript the dimension of the form in real space. Substituting this expansion in eq.(6.11) and equating terms with the same power of $\hat{v}$, we finally find the Stora–Zumino descent relations [14]:

$$\begin{aligned}
&\hat{d}Q^{(0)}_{2n+1} + dQ^{(1)}_{2n} = 0\,,\\
&\hat{d}Q^{(1)}_{2n} + dQ^{(2)}_{2n-1} = 0\,,\\
&\ldots\\
&\hat{d}Q^{(2n)}_{1} + dQ^{(2n+1)}_{0} = 0\,,\\
&\hat{d}Q^{(2n+1)}_{0} = 0\,.
\end{aligned} \tag{6.13}$$

For gauge anomalies $\mathcal{I}(v) = \int \mathrm{tr}(v\,a)$, eq.(5.3) reads

$$\delta_{v_1}\int \mathrm{tr}(v_2\,a) - \delta_{v_2}\int \mathrm{tr}(v_1\,a) - \int \mathrm{tr}([v_1,v_2]\,a) = 0\,. \tag{6.14}$$

The two transformations with parameters v_1 and v_2 can be incorporated into a family of transformations parametrized by θ^1 and θ^2, with parameter

$$\hat{v} = v_1 d\theta^1 + v_2 d\theta^2 = v_\alpha d\theta^\alpha. \tag{6.15}$$

In this way, $v_\alpha = g^{-1}\partial_\alpha g$. At $\theta^\alpha = 0$, $g(x,0) = 1$ and therefore $\bar{A}(x,0) = A(x)$ and $\bar{F}(x,0) = F(x)$. At that point, $\hat{d}$ generates ordinary gauge transformations on A and F, with $\hat{d} = d\theta^\alpha \delta_{v_\alpha}$. For instance, eq.(5.1) can be rewritten as

$$\hat{d}\,\Gamma = \int \mathrm{tr}\,\hat{v}a \tag{6.16}$$

The condition (6.14) can be multiplied by $d\theta^1 d\theta^2$ and rewritten as

$$\begin{aligned}
0 &= d\theta^1 d\theta^2\Big(\int \mathrm{tr}(v_2\delta_{v_1}\,a) - \int \mathrm{tr} v_1\delta_{v_2}\,a) - \int \mathrm{tr}[v_1,v_2]a\Big)\\
&= d\theta^\alpha d\theta^\beta\Big(\int \mathrm{tr}(v_\beta\delta_{v_\alpha}\,a) - \int \mathrm{tr}[v_\alpha,v_\beta]a\Big)\\
&= -\int \mathrm{tr}(\hat{v}\hat{d}\,a) - \int \mathrm{tr}\hat{v}^2 a\,.
\end{aligned} \tag{6.17}$$

Since $\hat{d}\hat{v} = -\hat{v}^2$, this can be rewritten simply as

$$\hat{d}\int \mathrm{tr}(\hat{v}\,a) = 0\,. \tag{6.18}$$

The Wess–Zumino consistency condition is therefore the statement that the anomaly is $\hat{d}$-closed,

$$\hat{d}\,\mathcal{I}(\hat{v}) = 0\,. \tag{6.19}$$

It is clear that the trivial $\hat{d}$-exact solutions $\mathcal{I}(\hat{v}) = \hat{d}\int f(A)$ in terms of a local functional $f(A)$ of the gauge field correspond to the gauge variation of local counterterms that can be added to the theory. Anomalies emerge from the *non-local* part of the effective action and are therefore encoded in the cohomology of $\hat{d}$. Locality is crucial. Any anomaly could otherwise be cancelled by the use of a non-local counterterm. For instance, we will see in the next section that for $U(1)$ gauge fields the anomaly is proportional to

$$\delta_\epsilon \Gamma(A) \propto \int \epsilon F^n\,. \tag{6.20}$$

If we would add to $\Gamma(A)$ the non-local functional[10]

$$f(A) \propto \int F^n \frac{1}{\Box}\partial^\mu A_\mu \tag{6.21}$$

with the appropriate coefficient, the new action $\Gamma(A)+f(A)$ would be gauge invariant. But counterterms must be local, because non-local terms are finite and free of UV-ambiguities. In other words, adding a non-local functional to the action amounts to a change in the physics.

From the second relation appearing in the Stora–Zumino descent relations (6.13), we see that the general non-trivial element of the $\hat{d}$-cohomology is of the form $\mathcal{I}(\hat{v}) = \int Q^{(1)}_{2n}$. With the above definitions, we have $Q_{2n+2} = dQ^{(0)}_{2n+1}$ and $\delta_v Q^{(0)}_{2n+1} = -dQ^{(1)}_{2n}$. We can therefore identify

$$\Omega_{2n+2} \leftrightarrow Q_{2n+2}, \quad \Omega^{(0)}_{2n+1} \leftrightarrow Q^{(0)}_{2n+1}\,, \quad \Omega^{(1)}_{2n} \leftrightarrow -Q^{(1)}_{2n}\,, \tag{6.22}$$

where Ω are the forms defined at the end of the last section.

7 Path Integral for Gauge and Gravitational Anomalies

Gauge and gravitational anomalies in $2n$ dimensions can be computed starting from the chiral anomaly in $2n+2$ dimensions using the Stora-Zumino descent relations. They arise from the Jacobian of the transformation in the integration measure, since the classical action is invariant. Differently from the chiral anomaly, gauge and gravitational anomalies can arise only from massless chiral fermions, with given chiralities $\pm$. For a Dirac fermion

[10] We are using in eq.(6.21) a mixed notation in terms of differential forms and explicit components, but hopefully the point should be clear.

in euclidean space, if ψ transforms under a representation $\mathcal{R}$ of a gauge group G, $\bar{\psi}$ should transform in the complex conjugate representation $\mathcal{R}^\star$, otherwise the mass term $\bar{\psi}\psi$ would not be invariant. In this case, the Jacobian factors associated to ψ and $\bar{\psi}$ exactly cancel, resulting in no anomaly. Similarly for local Lorentz transformations. The Lagrangian to start with is then that of a chiral fermion

$$\mathcal{L} = e\,\bar{\psi}(x) i\not{D}\mathcal{P}_\eta\,\psi\,, \tag{7.1}$$

where

$$\mathcal{P}_\eta = \frac{1}{2}\big(1+\eta\gamma_{2n+1}\big)\,, \qquad \eta = \pm 1\,, \tag{7.2}$$

is the chiral projector. The computation is technically analogous to the one we performed for the chiral anomaly, except that the transformation law is now different and acts with opposite signs on ψ and $\bar{\psi}$. Moreover, the full Dirac operator is now $i\not{D}_\eta = i\not{D}\mathcal{P}_\eta$, and is not Hermitian. For this reason we have to use the eigenfunctions ϕ^η_n of the Hermitian operator $(i\not{D}_\eta)^\dagger i\not{D}_\eta$ to expand ψ and the eigenfunctions φ^η_n of $i\not{D}_\eta(i\not{D}_\eta)^\dagger$ to expand $\bar{\psi}$:

$$\psi = \sum_k a_k\phi^\eta_k\,, \qquad \bar{\psi} = \sum_k b_k\varphi^{\eta\dagger}_k\,. \tag{7.3}$$

Let us first consider the case of gauge anomalies. Under an infinitesimal gauge transformation with parameter $v = v^\alpha T_\alpha$, the fermion fields transform as

$$\delta_v\psi = -v\psi\,,\;\; \delta_v\bar{\psi} = \bar{\psi}v\,. \tag{7.4}$$

Given the commutation properties of γ_{2n+1}, we clearly have $\phi^\pm_k = \varphi^\mp_k$, This induces a variation of the integration measure given by

$$\delta_v\Big[\mathcal{D}\psi\mathcal{D}\bar{\psi}\Big] = \mathcal{D}\psi\mathcal{D}\bar{\psi}\Big(\sum_k\int d^{2n}x\,e\,\Big(\phi^{\eta\dagger}_k v_\alpha T^\alpha\phi^\eta_k - \varphi^{\eta\dagger}_k v_\alpha T^\alpha\varphi^\eta_k\Big)\Big)\,. \tag{7.5}$$

As in the case of the chiral anomaly, this formal expression needs to be regularized, and we can define the integrated gauge anomaly to be

$$\mathcal{I}^{\rm gauge}(v) = -\lim_{\beta\to0}\sum_k\Big(\phi^{\eta\dagger}_k v_\alpha T^\alpha e^{-\beta(i\not{D}_\eta)^\dagger i\not{D}_\eta/2}\phi^\eta_k - \varphi^{\eta\dagger}_k v_\alpha T^\alpha\; e^{-\beta i\not{D}_\eta(i\not{D}_\eta)^\dagger/2}\varphi^\eta_k\Big)\,. \tag{7.6}$$

The trace over the two chiral eigenspinor basis can be combined in one single trace by inserting the chirality matrix γ_{2n+1} to give

$$\mathcal{I}^{\rm gauge}(v) = -\eta\lim_{\beta\to0}{\rm Tr}\Big[\gamma_{2n+1}Q^{\rm gauge}e^{-\beta(i\not{D})^2/2}\Big]\,. \tag{7.7}$$

The operator $Q^{\rm gauge}$ is defined in such a way to act as $v_\alpha T^\alpha$ on the Hilbert space. A concrete realization of it within the supersymmetric quantum mechanics introduced in the previous subsection is

$$Q^{\rm gauge} \to c^* v c\,. \tag{7.8}$$

The computation of eq.(7.7) is similar to that of eq.(4.12), the only difference being the insertion of the operator (7.8) into the trace (4.21). This insertion is equivalent to substitute $F \to F + v$ in eq.(4.21) and take the linear part in v of the result. Hence the anomaly would read (neglecting for the moment the gravitational term $\hat{A}(R)$):

$$\mathcal{I}^{\rm gauge}_{\rm cov}(v) = -\eta \frac{1}{n!}\Big(\frac{i}{2\pi}\Big)^n \int_{M_{2n}} {\rm tr}\, v F^n\,. \tag{7.9}$$

The anomaly computed in this form is called "covariant", because it transforms covariantly under the local symmetry, with no gauge connections appearing explicitly. The covariant anomaly however *does not* satisfy the Wess-Zumino consistency conditions. The general form of the anomaly $Q^{(1)}_{2n}$ satisfying the Wess–Zumino consistency conditions (called "consistent") depends explicitly on A, as we will see later on. Notice that if one could compute directly the anomaly from a given non-local functional Γ by making an infinitesimal gauge transformation $\delta_\epsilon \Gamma$, the result would satisfy the Wess-Zumino consistency conditions, because eq.(5.2) would be automatically satisfied. This implies that the anomaly (7.9) cannot arise from the variation of an action. So, what went wrong in our computation? The "mistake" can be traced to the violation of the Bose symmetry among the external states. In the path integral derivation one computes the gauge anomaly associated to a single external gauge field, while a Bose-symmetric result would require to distribute democratically the anomaly over all the external legs. Bose symmetry can be restored by focusing on the part of the anomaly containing $n+1$ fields, which corresponds effectively to take the dA from any F in eq.(7.9) and divide the result by $1/(n+1)$. In this way the integrand of eq.(7.9) gives

$$-\eta \frac{1}{(n+1)!}\Big(\frac{i}{2\pi}\Big)^n {\rm tr}\, v (dA)^n\,. \tag{7.10}$$

It is straightforward to find the $2n{+}2$ gauge invariant form $\Omega_{2n+2}(F)$ whose part containing $n{+}1$ fields gives eq.(7.10) as descent $2n$-form $\Omega^{(1)}_{2n}$, because the descent procedure is trivial at this order (all fields are effectively abelian). One has

$$\Omega_{2n+2}(F) = \frac{2i\pi}{(n+1)!}\Big(\frac{i}{2\pi}\Big)^{n+1} {\rm tr}\, F^{n+1}\,. \tag{7.11}$$

There is no need to Bose-symmetrize the terms containing more than $n+1$ fields. Given $\Omega_{2n+2}(F)$ in eq.(7.11), the Stora-Zumino descent relations automatically give the correct consistent form of the entire anomaly. We clearly see the close connection between chiral anomalies in $2n+2$ dimensions and gauge anomalies in $2n$ dimensions. The consistent form of the integrated gauge anomaly is therefore

$$\mathcal{I}^{\rm gauge}(v) = 2\pi i\eta \int_{M_{2n}} \Big[{\rm ch}_{\mathcal{R}}(F)\Big]^{(1)} \hat{A}(R)\,. \tag{7.12}$$

Notice that covariant and consistent anomalies contain the same information, and there is a well-defined procedure to switch from one to the other [6].

The case of gravitational anomalies is similar. Under an infinitesimal diffeomorphism with parameter ϵ^μ, the fermion fields transform as scalars, $\delta_\epsilon\psi = -\epsilon^\mu\partial_\mu\psi$, $\delta_\epsilon\bar\psi = -\epsilon^\mu\partial_\mu\bar\psi$. We can turn the ordinary derivative into a covariant one, by performing a local Lorentz transformation with parameter $\alpha^{ab}\propto\epsilon^\mu\omega_\mu^{ab}$. In this way we get

$$\delta_\epsilon\psi = -\epsilon^\mu D_\mu\psi\,, \quad \delta_\epsilon\bar\psi = -\epsilon^\mu D_\mu\bar\psi\,, \tag{7.13}$$

which induces the following variation of the integration measure (recall eq.(7.3)):

$$\begin{aligned}\delta_\epsilon\Big[\mathcal{D}\psi\mathcal{D}\bar\psi\Big] &= \mathcal{D}\psi\mathcal{D}\bar\psi\Big(\sum_k\int d^{2n}x\, e\,\Big(\phi_k^{\eta\dagger}\epsilon^\mu D_\mu\phi_k^\eta + (D_\mu\varphi_k^{\eta\dagger})\epsilon^\mu\varphi_k^\eta\Big)\Big) \\ &= \mathcal{D}\psi\mathcal{D}\bar\psi\Big(\sum_k\int d^{2n}x\, e\,\Big(\phi_k^{\eta\dagger}\epsilon^\mu D_\mu\phi_k^\eta - \varphi_k^{\eta\dagger}\epsilon^\mu D_\mu\varphi_k^\eta\Big)\Big)\,. \end{aligned} \tag{7.14}$$

The regularized expression for the integrated gravitational anomaly is then

$$\begin{aligned}\mathcal{I}^{\rm grav}(\epsilon) &= -\lim_{\beta\to0}\sum_k\Big(\phi_k^\dagger e^{-\beta(i\not{D}_\eta)^\dagger i\not{D}_\eta/2}\epsilon^\mu D_\mu\phi_k - \varphi_k^\dagger e^{-\beta i\not{D}_\eta(i\not{D}_\eta)^\dagger/2}\epsilon^\mu D_\mu\varphi_k\Big) \\ &= -\eta\lim_{\beta\to0}{\rm Tr}\Big[\gamma_{2n+1}Q^{\rm grav}e^{-\beta(i\not{D})^2/2}\Big]\,. \end{aligned} \tag{7.15}$$

The operator $Q^{\rm grav}$ must act as $\epsilon^\mu D_\mu$ on the Hilbert space. Since $i\dot{x}^\mu\to D^\mu$ upon canonical quantization, we can identify

$$Q^{\rm grav}\to i\epsilon_\mu\dot{x}^\mu\,. \tag{7.16}$$

We have (recall the rotation to euclidean time $\tau\to -i\tau$):

$$\mathcal{I}^{\rm grav}(\epsilon) = \eta{\rm Tr}_{c,c^\star}\int_P\mathcal{D}x^\mu\int_P\mathcal{D}\psi^a\epsilon_\mu(x)\frac{dx^\mu(\tau)}{d\tau}\exp\Big\{-\int_0^\beta d\tau\,\mathcal{R}\Big(x^\mu(\tau),\psi^a(\tau),c_A^\star,c^A\Big)\Big\}\,. \tag{7.17}$$

Thanks to the periodic boundary conditions, the path integral does not depend on the point where we insert $Q_{\rm grav}$, so we can replace in eq.(7.17)

$$\epsilon_\mu(x)\frac{dx^\mu(\tau)}{d\tau}\to\frac{1}{\beta}\int_0^\beta d\tau\,\epsilon_\mu(x)\frac{dx^\mu(\tau)}{d\tau}\,. \tag{7.18}$$

Rescaling $\tau \to \beta\tau$, using eq.(4.16) and expanding in normal coordinates gives, modulo irrelevant higher order terms in β,

$$\begin{aligned}
\frac{1}{\beta}\int_0^\beta d\tau\, \epsilon_\mu(x)\frac{dx^\mu(\tau)}{d\tau} &= \frac{1}{\beta}\int_0^1 d\tau\,\Big(\epsilon_\mu(x_0)+\nabla_\nu\epsilon_\mu(x_0)\sqrt{\beta}\xi^\nu(\tau)+\mathcal{O}(\beta)\Big)\sqrt{\beta}\frac{d\xi^\mu(\tau)}{d\tau} \\
&= \frac{1}{\sqrt{\beta}}\epsilon_\mu(x_0)\int_0^1 d\tau\frac{d\xi^\mu(\tau)}{d\tau}+\nabla_\nu\epsilon_\mu(x_0)\int_0^1 d\tau\xi^\nu\frac{d\xi^\mu(\tau)}{d\tau} \qquad (7.19)\\
&= \frac{1}{2}D_a\epsilon_b(x_0)\int_0^1 d\tau(\xi^a\dot{\xi}^b-\xi^b\dot{\xi}^a)\,,
\end{aligned}$$

where in the second row the first term and the symmetric part of the second vanish, being total derivatives of periodic functions. It is convenient to exponentiate the action of $Q^{\rm grav}$ and notice that it amounts to the shift $R_{ab} \to R_{ab} - D_a\epsilon_b + D_b\epsilon_a$ in the effective Routhian (4.18). The original expression (7.15) is recovered by keeping only the linear piece in ϵ. After adding the appropriate symmetrization factors and following the same procedure as for the gauge anomaly to switch to a consistent form of the anomaly, this implements the Stora–Zumino descent with respect to diffeomorphisms. The consistent form of the integrated gravitational anomaly is finally found to be

$$\mathcal{I}^{\rm grav}(\epsilon) = 2\pi i\eta\int_{M_{2n}} {\rm ch}_{\mathcal{R}}(F)\Big[\hat{A}(R)\Big]^{(1)}\,. \qquad (7.20)$$

We see from eq.(7.20) that purely gravitational anomalies, i.e. $F=0$ in eq.(7.20), can only arise in $d=4n+2$ dimensions.

The analogue of eq.(5.1) for diffeomorphisms reads

$$\delta_\epsilon\Gamma(g) = -\int d^{2n}x\,\frac{\delta\Gamma(g)}{\delta g_{\mu\nu}}(\nabla_\mu\epsilon_\nu+\nabla_\nu\epsilon_\mu) = \int d^{2n}x\sqrt{g}\,\nabla_\mu T^{\mu\nu}\epsilon_\nu = \int a^\nu\epsilon_\nu \equiv \mathcal{I}^{\rm grav}(\epsilon)\,, \qquad (7.21)$$

where $\delta_\epsilon g_{\mu\nu} = -(\nabla_\mu\epsilon_\nu+\nabla_\nu\epsilon_\mu)$ and we have defined the symmetric energy-momentum tensor

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{g}}\frac{\delta\Gamma(g)}{\delta g_{\mu\nu}}\,. \qquad (7.22)$$

A gravitational anomaly leads to a violation of the conservation of the energy-momentum tensor. It is also possible to see the consequences of a violation of the local Lorentz symmetry. In this case we see Γ as a functional of the vielbeins e_μ^a. Under a local Lorentz transformation for which $\delta_L e^a_\mu = -\alpha^a_{\ b}e^b_\mu$, we have

$$\delta_L\Gamma(e) = -\int d^{2n}x\,\frac{\delta\Gamma(e)}{\delta e_{\mu a}}\epsilon_{ab}e^b_\mu = \int d^{2n}x e\, T^{ab}\epsilon_{ab} = \int a^{ab}\epsilon_{ab} \equiv \mathcal{I}^{\rm L}(\epsilon)\,, \qquad (7.23)$$

where we have used the relation

$$e^b_\mu\frac{\delta\Gamma(e)}{\delta e_{\mu a}} = e^b_\mu 2\frac{\delta\Gamma(g)}{\delta g_{\mu\nu}}e^a_\nu = eT^{ab}\,. \qquad (7.24)$$

Since $\epsilon_{ab} = -\epsilon_{ba}$, an anomaly in a local Lorentz transformation implies a conserved, but non-symmetric, energy-momentum tensor. It is possible to show that these anomalies are not independent from those affecting diffeomorphisms, and it is always possible to add a local functional to the action to switch to a situation in which one or the other of the anomalies vanish, but not both [6]. Correspondingly, in presence of a gravitational anomaly, the energy-momentum tensor can be chosen to be symmetric or conserved at the quantum level, but not both simultaneously. The resulting theory will be inconsistent, since unphysical graviton modes will not decouple in scattering amplitudes.

We see from eqs.(7.12) and (7.20) that we can also have mixed gauge/gravitational anomalies, namely anomalies involving both gauge fields and gravitons. Contrary to the purely gravitational ones, they can arise in both $4n$ and $4n+2$ dimensions. These anomalies lead to both the non-conservation of the gauge current and of the energy momentum tensor. However, by adding suitable local counterterms to the effective action, one can in general bring the whole anomaly in either the gauge current or the energy momentum tensor. The mixed gauge/gravitational anomaly signals the obstruction in having both currents conserved at the same time.

7.1 Explicit form of Gauge Anomaly

It is useful to explicitly compute the expressions of the forms that are relevant to gauge anomalies. The starting point is the $(2n+2)$-form characterizing the chiral anomaly in $2n+2$ dimensions:

$$\Omega_{2n+2}(F) = \left(\frac{i}{2\pi}\right)^{n+1} Q_{2n+2}(F)\,, \qquad Q_{2n+2}(F) = \mathrm{tr}\, F^{n+1}\,. \tag{7.25}$$

In order to not carry the unnecessary $(i/2\pi)$ factors, we will consider in what follows the form Q, rather than Ω. As shown in section 2.3, Q is a closed form. For topologically trivial gauge field configurations we can set $F = A = 0$ in eq.(2.64). Relabelling $A' \to A$ and $F' \to F$, we have $A_t = tA$, $F_t = tdA + t^2 A^2$ and eq.(2.64) gives

$$Q^{(0)}_{2n+1}(A, F) = (n+1) \int_0^1 dt\, \mathrm{tr} \left[A\, F_t^n \right]. \tag{7.26}$$

We compute $Q^{(1)}_{2n}(v, A, F)$ using eq.(6.12). This requires to compute the Chern-Simons term (7.26) in the Δ-cohomology

$$Q_{2n+1}(\mathcal{A}, \mathcal{F}) = Q_{2n+1}(\bar{A} + \hat{v}, \bar{F}) = (n+1) \int_0^1 dt\, \mathrm{tr} \left[(\bar{A} + \hat{v})\, \mathcal{F}_t^n \right], \tag{7.27}$$

where $\mathcal{A} = \bar{A} + \hat{v}$, $\mathcal{A}_t = t\mathcal{A}$, $\mathcal{F}_t = \Delta \mathcal{A}_t + \mathcal{A}_t^2$. Using eq.(6.8), the latter can be expressed as

$$\mathcal{F}_t = t\mathcal{F} + (t^2 - t)\mathcal{A}^2 = t\bar{F} + (t^2 - t)(\bar{A} + \hat{v})^2\,. \tag{7.28}$$

The quantity we are after is found by expanding in powers of $\hat{v}$ and retaining the linear order. We have

$$\mathcal{F}_t = \bar{F}_t + (t^2 - t)[\bar{A}, \hat{v}] + \mathcal{O}(\hat{v}^2). \tag{7.29}$$

It is useful to define the symmetrized trace of generic matrix-valued forms $\omega_i = \omega_i^{\alpha_i} T^{\alpha_i}$ as

$$\mathrm{str}\,(\omega_1 \ldots \omega_p) = \frac{1}{p!}\omega_1^{\alpha_1} \wedge \ldots \wedge \omega_p^{\alpha_p} \sum_{\text{perms.}} \mathrm{tr}\,(T^{\alpha_1} \ldots T^{\alpha_p})\,. \tag{7.30}$$

In this way we find

$$\begin{aligned} Q_{2n}^{(1)}(\hat{v}, \bar{A}, \bar{F}) &= (n+1)\int_0^1 dt\, \mathrm{str}\Big[\hat{v}\bar{F}_t^n + n(t^2-t)\bar{A}\big[\bar{A},\hat{v}\big]\bar{F}_t^{n-1}\Big] \\ &= (n+1)\int_0^1 dt\, \mathrm{str}\Big[\hat{v}\Big(\bar{F}_t^n + n(t-1)\big(t\big[\bar{A},\bar{A}\big]\bar{F}_t^{n-1} - \bar{A}\big[\bar{A}_t, \bar{F}_t^{n-1}\big]\big)\Big)\Big] \\ &= (n+1)\int_0^1 dt\, \mathrm{str}\Big[\hat{v}\Big(\bar{F}_t^n + n(t-1)\big(\big(\partial_t \bar{F}_t - d\bar{A}\big)\bar{F}_t^{n-1} + \bar{A} d\bar{F}_t^{n-1}\big)\Big)\Big] \\ &= (n+1)\int_0^1 dt\, \mathrm{str}\Big[\hat{v}\Big(\bar{F}_t^n + (t-1)\partial_t \bar{F}_t^n + n(1-t)d\big(\bar{A}\bar{F}_t^{n-1}\big)\Big)\Big]. \end{aligned} \tag{7.31}$$

In the third equality we have used the identities $D_t \bar{F}_t^{n-1} = d\bar{F}_t^{n-1} + [\bar{A}_t, \bar{F}_t^{n-1}] = 0$ and $\partial_t \bar{F}_t = d\bar{A} + t[\bar{A}, \bar{A}]$. After integrating by parts, the first and second term in the last line of eq.(7.31) cancel. We can set the gauge parameters $\theta^\alpha = 0$, where $g(x,0) = 1$, so that $\hat{v}$ turns into v, $\bar{A} = A$, $\bar{F} = F$. Going back to the Ω-forms using eq.(6.22) gives

$$\Omega_{2n}^{(1)}(v, A, F) = -\Big(\frac{i}{2\pi}\Big)^{n+1} n(n+1)\int_0^1 dt\,(1-t)\,\mathrm{str}\Big[vd(AF_t^{n-1})\Big]\,. \tag{7.32}$$

The generalization of the above relations to the gravitational case is straightforward. We substitute A and F with the connection Γ and the curvature R in eq.(2.44), and consider infinitesimal diffeomorphisms. Alternatively, we substitute A and F with the spin connection ω and the curvature R, and consider infinitesimal $SO(2n)$ local rotations, when dealing with local Lorentz symmetry.

In order to give a concrete example, let us derive the contribution of a Weyl fermion with chirality $\eta = \pm 1$ to the non-Abelian gauge anomaly in 4 dimensions. Integrating in t gives the following anomalous variation of the effective action:

$$\delta_v \Gamma(A) = -\frac{\eta}{24\pi^2}\int d^4x\, \mathrm{str}\Big[vd\Big(AF - \frac{1}{2}A^3\Big)\Big]\,. \tag{7.33}$$

In model building and phenomenological applications one is mostly interested to see whether a given set of currents (global or local) is anomalous or not. This is best seen by looking at the term of the anomaly with the lowest number of gauge fields, the first term in square brackets in eq.(7.33), the full form of the anomaly being reconstructed using the Wess-Zumino consistency conditions. In 4 dimensions this term is proportional to the factor

$$D^{\alpha\beta\gamma} \equiv \frac{1}{2}\mathrm{tr}(\{T^\alpha, T^\beta\}T^\gamma) = \frac{1}{3!}\sum_{\mathrm{perms.}} \mathrm{tr}\,(T^\alpha T^\beta T^\gamma)\,. \tag{7.34}$$

When this factor vanishes, the whole gauge anomaly must vanish.

8 Gravitational Anomalies for Spin 3/2 and Self-Dual Tensors

Chiral spin 1/2 fermions are not the only fields giving rise to gravitational anomalies. In order to understand which other fields might possibly lead to anomalies, consider the explicit form of the gauge anomaly in eq.(7.32). In our notations the gauge generators are anti-hermitian, and hence $\Omega_{2n}^{(1)}$ is real in any dimension. Using eqs.(5.1) and (5.4), this implies that gauge anomalies affect only the imaginary part of the euclidean effective action Γ. Vector-like fermions do not give rise to anomalies and lead to a real effective action. So we conclude that only the imaginary part of Γ can be affected by anomalies. The same reasoning applies to gravitational anomalies (consider them as $SO(2n)$ gauge theories). From this reasoning we conclude that fields transforming in a complex representation of $SO(2n)$ might lead to anomalies. The group $SO(N)$ admits complex representations for $N = 4n+2$, and hence only in $d = 4n+2$ dimensions we might expect purely gravitational anomalies, in agreeement with our result (7.20) for spin 1/2 fermions. Chiral spin 3/2 fields also transform under complex representations of $SO(4n+2)$. Interestingly enough, (anti)self-dual tensors, $2n+1$-forms, also have complex representations in $4n+2$ dimensions. This is best seen by looking at the euclidean self-duality condition for forms: $*F_n = F_n$, where $*$ is the Hodge operation on forms. Given a form ω_p in d dimensions, $*\omega_p$ is the $(d-p)$- form given by

$$\star\,\omega_p = \frac{\sqrt{g}}{p!(d-p)!}\omega_{\mu_1\ldots\mu_p}\epsilon^{\mu_1\ldots\mu_p}{}_{\nu_1\ldots\nu_{d-p}}dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_{d-p}}\,, \tag{8.1}$$

where $\epsilon_{\mu_1\ldots\mu_d}$ is the completely antisymmetric tensor, with $\epsilon_{1\ldots d} = +1$. It is easy to see that $**\,\omega_p = (-1)^{p(d-p)}\omega_p$. Specializing for $d = 2n$ and $p = n$, gives $**\,\omega_n = (-1)^{n^2}\omega_n$. This is 1 for n even and -1 for n odd, hence in $4n$ and $4n+2$ dimensions (anti)self-dual tensors transform in real and complex representations of $SO(d)$, respectively.

It is not clear how to consistently get a QFT with (anti)self-dual tensors charged under gauge fields. Also, in most relevant theories, spin 3/2 fields are neutral under gauge symmetries. We will then consider the contribution of these fields to pure gravitational anomalies only. We very briefly outline the derivation, referring the reader to ref.[2] for further details. As for spin 1/2 fermions, it turns out that the gravitational anomalies induced by fields can be obtained starting from chiral anomalies in two dimensions higher and then apply the Stora-Zumino descent relations. The chiral anomaly can again be efficiently computed using an auxiliary supersymmetric quantum mechanical model. For spin 3/2 fermions ψ_μ, the vector index μ can be seen as a gauge field in the fundamental representation of the group, the connection being ω_μ^{ab} and the group $SO(d)$. The associated quantum mechanical model is given by eq.(4.15), with $A_{\mu\;B}^{A} \to i\omega_{\mu\;b}^{a}$ and $F_{\mu\nu}^{AB} \to iR_{\mu\nu}^{ab}$. In terms of the skew-eigenvalues x_i of the curvature two-form R_{ab} defined in eq.(2.72), the chiral anomaly in $2n$ dimensions is found to be

$$\mathcal{Z}_{3/2} = \int_{M_{2n}} (\mathrm{tr} e^{\frac{R}{2\pi}} - 1)\, \hat{A}(R) = \int_{M_{2n}} \Big(2 \sum_{j=1}^{n} \cosh x_j - 1\Big) \prod_{k=1}^{n} \frac{x_k/2}{\sinh(x_k/2)} \,, \tag{8.2}$$

where the -1 comes from the need to subtract the spin 1/2-contribution contained in ψ_μ.

Anomalies of (anti)self-dual forms are more involved. Roughly speaking, one considers the contribution of all antisymmetric fields at once. This does not change the anomaly, since the total contribution of the extra states sum up to zero. The whole series of antisymmetric fields is equivalent to a bifermion field $\psi_{\alpha\beta}$. This is understood by recalling that a matrix $A_{\alpha\beta}$ can be expanded in the basis of matrices given by antisymmetric product of gamma matrices, eq.(2.17). The analogue of the chiral matrix γ_{2n+1} in eq.(4.12) is played by the Hodge operator $*$ and the Hamiltonian is the Laplacian operator on generic p-forms. By computing the path integral associated to the resulting super quantum mechanical model one gets

$$\mathcal{Z}_A = -\frac{1}{8} \int_{M_{2n}} \prod_{k=1}^{n} \frac{x_k}{\tanh x_k} \,. \tag{8.3}$$

With these results at hand we can verify that purely gravitational anomalies do indeed cancel in a non-trivial way in a ten-dimensional gravitational theory, low-energy effective action of a string theory, called IIB theory. The spectrum of this theory includes a chiral spin 1/2 fermion, a spin 3/2 fermion with opposite chirality and a self-dual five-form antisymmetric field. The total gravitational anomaly is obtained by taking the sum

$$\mathcal{Z}_{IIB} = \mathcal{Z}_A + \mathcal{Z}_{3/2} - \mathcal{Z}_{1/2} \,, \tag{8.4}$$

evaluating each term for $n = 5$ and keeping in the expansion of the series only the form of degree twelve. This is the relevant form that, after the Stora-Zumino descent procedure, would give rise to the gravitational anomaly ten-form in ten dimensions. Remarkably, the coefficient of this form exactly vanishes in the sum (8.4), proving that the IIB theory is free of any gravitational anomaly.

9 The Green–Schwarz Mechanism

The Green–Schwarz anomaly cancellation mechanism was first discovered by Green and Schwarz in the context of string-derived effective supergravity theories in 10 dimensions [15]. It achieves in a non-trivial and interesting way the cancellation of gauge and gravitational anomalies, which is guaranteed in the full string theory by its finiteness, stemming from general principles such as modular invariance or tadpole cancellation. Thanks to this mechanism, it has been understood that field theories with an anomalous spectrum of massless fermions can be anomaly-free, and thus consistent, in certain particular circumstances. The mechanism involves antisymmetric tensor fields, and the essential idea is that the anomaly is canceled by the gauge variation of some counterterms, constructed out of these tensor fields as well as the gauge and gravitational connections and field strengths.

Before describing the Green–Schwarz mechanism and its generalization to any space-time dimension, it is necessary to introduce the notion of "reducible" and "irreducible" forms of the anomaly. As shown in eqs.(7.12) and (7.20), a generic gauge or gravitational anomaly can be written in the form $\mathcal{I} = 2\pi i \int_{M_{2n}} \Omega_{2n}^{(1)}$, where $\Omega_{2n}^{(1)}$ is the Stora–Zumino descent of a closed and gauge-invariant $(2n+2)$-form Ω_{2n+2}, function of the curvature 2-forms F and R.[11] The form $\Omega_{2n+2}(F,R)$ is said to be "irreducible" when it cannot be decomposed as a sum of products of closed and gauge-invariant forms of lower degree. Typical examples are $\mathrm{tr}\, R^{n+1}$ or $\mathrm{tr}\, F^{n+1}$ for a representation that does not admit a decomposition to lower forms. It is instead said to be "reducible" when $\Omega_{2n+2}(F,R)$ can be decomposed as $\Omega_{2n+2} = \Omega_{2k}\Omega_{2n+2-2k}$ for some $k > 0$. Examples of such a type are $\mathrm{tr}\, F^k \mathrm{tr}\, F^{n+1-k}$, $\mathrm{tr}\, F^k \mathrm{tr}\, R^{n+1-k}$ or $\mathrm{tr}\, R^k \mathrm{tr}\, R^{n+1-k}$.

The original Green–Schwarz mechanism in 10 dimensions requires the introduction of a 2-index antisymmetric tensor field, but we will describe here its generalization to $2n$ dimensions and $2l$-index antisymmetric tensor fields of the type $C_{2l}^{\mu_1 \cdots \mu_{2l}}$, with $l \geq 1$. These fields generalize the standard electromagnetic vector potential[12] and are conveniently described in terms of $2l$-forms C_{2l}, subject to the $U(1)$ gauge transformation

$$\delta C_{2l} = d\lambda_{2l-1}\,, \tag{9.1}$$

with λ_{2l-1} an arbitrary $(2l-1)$-form. The gauge-invariant field strengths

$$H_{2l+1}^{\mu_1 \cdots \mu_{2l+1}} = \partial^{\mu_1} C_{2l}^{\mu_2 \cdots \mu_{2l+1}} \pm \text{permutations}\,, \tag{9.2}$$

[11] Here and in the following we will refer to gauge symmetries in a broad sense, including in particular local Lorentz symmetries, in order to treat gauge and gravitational anomalies at once.

[12] The "electric" and "magnetic" sources of these fields in $2n$ dimensions are respectively $(2l-1)$- and $(2n-2l-3)$-dimensional extended objects.

are correspondingly described by the $(2l+1)$-forms $H_{2l+1} = dC_{2l}$.

As will become clear below, only reducible anomalies can be canceled through the Green–Schwarz mechanism. We shall therefore consider a generic reducible anomaly of the form

$$\mathcal{I} = 2\pi i \int_{M_{2n}} \Omega^{(1)}_{2n} \,, \quad \text{with} \;\; \Omega_{2n+2} = \Omega_{2k}\, \Omega_{2n+2-2k} \,. \tag{9.3}$$

Following the Stora–Zumino descent procedure, the Chern–Simons form $\Omega^{(0)}_{2n+1}$ corresponding to Ω_{2n+2} is found to be

$$\begin{aligned}\Omega^{(0)}_{2n+1} = {} & \frac{k}{n+1}\Omega^{(0)}_{2k-1}\Omega_{2n+2-2k} + \frac{n+1-k}{n+1}\Omega_{2k}\Omega^{(0)}_{2n+1-2k} \\ & + \alpha\, d\Big(\Omega^{(0)}_{2k-1}\Omega^{(0)}_{2n+1-2k}\Big)\,,\end{aligned} \tag{9.4}$$

where α is an arbitrary parameter taking into account the ambiguity in the definition of $\Omega^{(0)}_{2n+1}$. From eq.(9.4) one derives

$$\Omega^{(1)}_{2n} = \left(\frac{k}{n+1} - \alpha\right)\Omega^{(1)}_{2k-2}\Omega_{2n+2-2k} + \left(\frac{n+1-k}{n+1} + \alpha\right)\Omega_{2k}\Omega^{(1)}_{2n-2k} \,. \tag{9.5}$$

By properly choosing α, we might cancel the first or the second term in eq.(9.5), but not both at the same time. The choice of α corresponds to the fact that anomalies are defined modulo local counterterms. In particular, we see that the addition to the action of the term

$$S \supset -2\pi i \beta \int_{M_{2n}} \Omega^{(0)}_{2k-1}\Omega^{(0)}_{2n+1-2k} \tag{9.6}$$

amounts to the change $\alpha \to \alpha - \beta$ in the anomaly (9.5). Clearly, the anomaly still persists. The situation changes if one adds an antisymmetric $(2k-2)$-index tensor field C_{2k-2}. In fact, the anomaly corresponding to eq.(9.5) can be canceled by the following action:

$$\begin{aligned}S_{GS} = \int_{M_{2n}} \bigg[& \frac{1}{2}\Big|dC_{2k-2} + \sqrt{2\pi}\,\xi\,\Omega^{(0)}_{2k-1}\Big|^2 + i\frac{\sqrt{2\pi}}{\xi}\,C_{2k-2}\,\Omega_{2n-2k+2} \\ & -2\pi i\left(\frac{n+1-k}{n+1} + \alpha\right)\Omega^{(0)}_{2k-1}\Omega^{(0)}_{2n+1-2k}\bigg]\,,\end{aligned} \tag{9.7}$$

where ξ is an arbitrary dimensionful parameter. The action (9.7) is not invariant under local symmetry transformations. The modified kinetic term of the field C_{2k-2} makes it clear that the appropriate definition of its field strength H_{2k-1} is

$$H_{2k-1} = dC_{2k-2} + \sqrt{2\pi}\,\xi\,\Omega^{(0)}_{2k-1} \,. \tag{9.8}$$

This field strength can be made gauge-invariant, provided that C_{2k-2} transforms inhomogeneously under gauge transformations, in such a way as to compensate the transformations of the Chern–Simons form $\Omega^{(0)}_{2k-1}$:

$$\delta_\epsilon C_{2k-2} = -\sqrt{2\pi}\,\xi\,\Omega^{(1)}_{2k-2} \,. \tag{9.9}$$

In this way $\delta_\epsilon H_{2k-1} = 0$ and the kinetic term is invariant. However, the Wess–Zumino coupling $C_{2k-2}\Omega_{2n-2k+2}$ (and the last counterterm in eq.(9.7)) transforms non-trivially and leads to a non-vanishing variation of S that exactly compensates for the 1-loop anomaly (9.3), independently of the value of ξ:

$$\delta_\epsilon S_{GS} = -2\pi i \int_{M_{2n}} \Omega_{2n}^{(1)} \,. \tag{9.10}$$

Although the form of the anomaly (9.3) and of the last term in eq.(9.7) depend on the arbitrary parameter α, the gauge variation of S due to the transformation (9.9) is independent of α and universal. This is the essence of the generalization of the original Green-Schwarz mechanism to arbitrary dimensions and $2l$ antisymmetric forms. Although the construction might seem ad hoc, it was shown in ref.[15] that the action of certain string-derived ten-dimensional QFT – where massless antisymmetric 2-forms are present to begin with – is of the form (9.7), with the correct $\Omega_3^{(0)}$ and Ω_8 to make the theory anomaly free. Since then, the Green-Schwarz mechanism has been shown to be generally at work in string-derived low energy actions in ten or less space-time dimensions [16]. Independently of string theory, one can of course cancel reducible gauge/gravitational anomalies in QFT by adding the forms C_{2l} as in eq. (9.7).

The Green–Schwarz mechanism described above, involving a single tensor field C_{2k-2}, can cancel only reducible anomalies of the form $\Omega_{2n+2} = \Omega_{2k}\Omega_{2n+2-2k}$, with $1 \leq k \leq n$. This is clear from eq.(9.7), but also from the fact that the involved forms are physical propagating fields only for $1 \leq k \leq n$. Notice in particular that the cases $k = 0$ or $k = n+1$, corresponding to irreducible anomalies, would formally require (-1)-forms or $2n$-forms, with field strengths dual to each other, which are clearly unphysical. Indeed, the top $2n$-form has no physical degrees of freedom, since it cannot have a sensible field strength, and its equation of motion simply implies that the total charge under it should vanish; its would-be dual (-1)-form is correspondingly not existing. However, a straightforward generalization of the basic Green–Schwarz mechanism, involving several physical tensor fields $C^i_{2k_i-2}$, with $1 \leq k_i \leq n$, can cancel anomalies that are not reducible but can be decomposed into a sum of reducible ones, with $\Omega_{2n+2} = \sum_i \Omega_{2k_i}\Omega_{2n+2-2k_i}$, each tensor field being responsible for the cancellation of one of the terms in the anomaly.

Notice finally that since the anomaly (9.3) is a 1-loop effect, either the Wess–Zumino coupling or the Chern–Simons form modifying the kinetic term of the antisymmetric tensor fields in the actions (9.7) must arise at the 1-loop level, depending on n and k. One of these two terms can therefore be thought of as being induced by the heavy states associated to the physics in the UV. This was explicitly verified in string theory, where the microscopic theory is known and computable.

In four-dimensional QFT, scalar fields can be used in the Green-Schwarz mechanism. Consider for simplicity a theory with a gauge group of the form $U(1)\times G$, with G a simple group, and denote by A the $U(1)$ gauge field connection and by S the two-form field strength of G. Mixed $U(1)-S^2$ gauge and $U(1)$-gravitational anomalies can be cancelled by the following action:

$$S_{GS}=\int d^4x\bigg(\frac{1}{2}\Big(\partial_\mu\phi+f_\phi A_\mu\Big)^2+i\epsilon^{\mu\nu\rho\sigma}\frac{\phi}{f_\phi}\Big(c_1\mathrm{tr}\,S_{\mu\nu}S_{\rho\sigma}+c_2R_{\mu\nu}{}^{\alpha\beta}R_{\rho\sigma\alpha\beta}\Big)\bigg)\,, \tag{9.11}$$

where ϕ is the Green-Schwarz scalar field, f_ϕ a mass scale and c_1, c_2 are appropriate constants. For simplicity, we have omitted to write the local counterterms proportional to the product of Chern-Simons terms $\Omega_1^{(0)}\Omega_3^{(0)}$. The generalization to situations involving multiple abelian and non-abelian gauge fields is straightforward.

Notice that the Green-Schwarz scalar couples to gauge fields as axions do, though, in contrast to axions, it requires a modified kinetic term to acquire an inhomogeneous transformation under a gauge/gravitational symmetry. The latter implies that the Green–Schwarz anomaly cancellation mechanism in 4-dimensions is an Higgs mechanism where ϕ is eaten by the would-be anomalous $U(1)$ gauge field that gets a mass of order f_ϕ.[13] As is clear from eq.(9.11), the Green–Schwarz mechanism cannot be applied in renormalizable 4-dimensional theories, since operators of dimension greater than 4 necessarily appear.

10 Non-perturbative Anomalies

The gauge and gravitational anomalies that we have considered so far concern local symmetry transformations connected to the identity, and can therefore be infinitesimal. In this sense we can call them "perturbative" anomalies. There can be in general additional non-perturbative gauge and gravitational anomalies concerning symmetry transformations topologically non-trivial and disconnected from the identity, that hence exist only in a finite form and cannot be infinitesimal. The latter can occur both for gauge symmetries and for diffeomorphisms (or local Lorentz transformations). They are also called global anomalies and were first discovered by Witten [17] in an $SU(2)$ model in 4 dimensions.[14] Differently from perturbative anomalies, the non-perturbative ones cannot be directly detected through perturbative Feynman diagram computations, and this explains their name. A general discussion of gauge and gravitational non-perturbative anomalies lies

[13] On the contrary, in the original string Green-Schwarz mechanism in ten space-time dimensions, the gauge fields remain massless.

[14] We avoid the terminology "global anomaly" and use instead "non-perturbative anomaly" for a possible confusion with the anomalies involving global symmetries, that sometimes are also denoted global anomalies.

beyond the aim of this course. In the following, we will just very briefly recall a few basic features of the former in flat space, and advise the reader interested in the latter to see ref. [18].

Let us begin by examining which gauge groups G can lead to global gauge anomalies in a $2n$-dimensional flat Euclidean space-time R^{2n}. We consider gauge transformations $g(x)$ that reduce to the identity at infinity, so that they represent maps from S^{2n} (the $2n$-dimensional sphere) into the gauge group G. Such gauge transformations are classified by the $2n$-th homotopy group of G, denoted by $\pi_{2n}(G)$. If the latter is trivial, all the gauge transformations are connected to the identity and no global anomalies can arise. On the contrary, if it is not, there exist classes of topologically non-trivial gauge transformations that can potentially be anomalous. Denoting by $A^g = g^{-1}Ag + g^{-1}dg$ the gauge-transformed connection obtained from A through such a non-trivial gauge transformation g, a global anomaly can occur if the effective action — which is defined modulo physically irrelevant multiples of $2\pi i$ as in eq.(4.3) — changes under the finite transformation g by an amount $\Gamma(A^g) - \Gamma(A)$ that is not a multiple of $2\pi i$:

$$\Gamma(A^g) - \Gamma(A) \neq 2\pi i n \,, \quad n \in \mathbf{Z} \,. \tag{10.1}$$

If the situation (10.1) occurs, the quantum effective action and all the correlation functions of gauge-invariant operators it describes are not well-defined, and the theory is inconsistent [17].[15] As perturbative gauge anomalies, also non-perturbative gauge anomalies can be induced only by Weyl fermions in even-dimensional space-times and through the imaginary part of the Euclidean effective action, since Dirac fermions always allow for a manifestly gauge-invariant regularization. Computing the contribution of a Weyl fermion to the transformation (10.1) for a generic gauge group G is, however, a complicated mathematical problem. It should be clear that asking whether a theory is afflicted by non-perturbative anomalies or not is a meaningful question only when all perturbative ones cancel, the former being defined in terms of homotopy classes and hence modulo local gauge transformations.

The simplest non-trivial case where non-perturbative anomalies can arise in 4 dimensions is for $G = SU(2)$, since $\pi_4[SU(2)] = \mathbf{Z}_2$. It has been show in ref.[17] that an $SU(2)$ theory with one or any odd number of Weyl doublets is non-perturbatively inconsistent.

[15] To be precise, eq.(10.1) leads to an inconsistency only if A and A^g are connected in field space without passing infinite action barriers. Otherwise, it is possible to define a sensible quantum effective action by restricting the functional integral to topologically trivial gauge configurations only.

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