Appendix A. Basic homotopy.

Let M, N be finite dimensional manifolds. We choose a point $x_0 \in M$ and a point $y_0 \in N$; they are called the basepoints of M and N. We denote $\Gamma(M,N)$ the space of all smooth functions $f : M \to N$. (By smooth we mean continuous and r-times differentiable, with $0 \le r \le \infty$). We denote $\Gamma_*(M,N)$ the subspace of $\Gamma(M,N)$ consisting of functions that preserve basepoints, i.e. $f(x_0) = y_0$.

We say that two maps $f, g \in \Gamma(M,N)$ are homotopic (and write $f \simeq g$) if there exists a continuous map $F : M \times I \to N$ such that $F(x,0) = f(x)$, $F(x,1) = g(x)$. Intuitively, F gives a one parameter family of maps, depending continuously on t, that interpolates between f and g. Sometimes it is convenient to put into evidence the dependence on the parameter, and write $f_t = F(.,t)$; then $f_0 = f$, $f_1 = g$. In the case when M, N have basepoints and $f, g \in \Gamma_*(M,N)_*$ one requires $F(x_0,t) = y_0$ for all t. If $f_1 \simeq f_2$ are maps from N to P and $g_1 \simeq g_2$ are map from M to N, then $f_1 \circ g_1 \simeq f_2 \circ g_2$.

The relation of being homotopic is an equivalence relation. The quotient of $\Gamma(M,N)$ by this relation, i.e. the set of homotopy classes of maps from M to N, is denoted $[M,N]$. Similarly one defines $[M,N]_*$, the set of homotopy classes of basepoint-preserving maps.

The set of homotopy classes thus defined do not depend on r, the class of differentiability of the maps. In fact, from the mathematical point of view, it is most natural to assume that M and N are only topological spaces and that the maps are only continuous ($r = 0$).

Two spaces M and N are said to have the same *homotopy type* if there are maps $f : M \to N$ and $g : N \to M$ such that $f \circ g \simeq Id_M$ and $g \circ f \simeq Id_N$. It is easy to see that if M and N have the same homotopy type, then $[P,M] = [P,N]$ and $[M,Q] = [N,Q]$ for all spaces P, Q. A space N is said to be *contractible* if it is homotopy equivalent to a point or in other words if the identity map is homotopic to the constant map. Stated more explicitly, this means that there is a continuous map $F : I \times N \to N$ such that $F(0,y) = y$ and $F(1,y) = y_0$. For example, all vectorspaces are contractible. It is enough to take the origin as basepoint and consider $F(t,y) = ty$. If N is contractible, then $[M,N]_*$ is the trivial set consisting of a single element. To see this it is sufficient to note that for any map $f : M \to N$, $Id_N \circ f = f$ is homotopic to $y_0 \circ f = y_0$. So from the point of view of homotopy a contractible space is equivalent to a single point.

In the case when M is a sphere $S^M = \{x \in \mathbf{R}^{m+1} \mid x_1^2 + \cdots + x_{m+1}^2 = 1\}$ with $m \ge 1$, the sets of homotopy classes can be given a group structure. This case is so important that it deserves a special name: the space $\pi_m(N) = [S^m, N]_*$ is called m-th homotopy group of N.

We first show how the group structure is defined in the case $m = 1$ ($\pi_1(N)$ is also called the fundamental group of N). We think of S^1 as an open interval $I = [0,1]$ with the endpoints identified; the basepoint of S^1 corresponds to 0 (or 1). A basepoint preserving map $f : S^1 \to N$ is just a loop starting and ending at y_0. Given two loops f_1, f_2 we can define a third loop $f_1 \cdot f_2$ by "going first round f_1, then f_2 at double speed":

$$f_1 \cdot f_2(t) = \begin{cases} f_1(2t) & \text{for } 0 \le t \le \frac{1}{2} \\ f_2(2t-1) & \text{for } \frac{1}{2} \le t \le 1 \ . \end{cases} \tag{A.1}$$

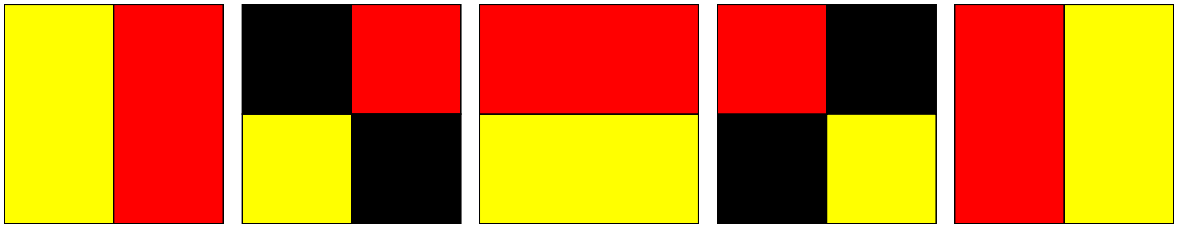
If we denote $[f] \in \pi_1(N)$ the homotopy class of the loop f, then $[f_1][f_2] = [f_1 \cdot f_2]$ defines a group multiplication in $\pi_1(N)$.

In the case $m \ge 2$, we think of S^m as the m-cube I^m with all points of the boundary identified. Note that if we call $t_1, \ldots, t_m$ the coordinates in I^m, the boundary ∂I^m of the cube consists of all points for which at least one of the coordinates is equal to 0 or 1. A map $f : I^m \to N$ such that for all $x \in \partial I^m$, $f(x) = y_0$ can be regarded as a map $f : S^m \to N$, and thus defines a homotopy class in $\pi_m(N)$. We define $f_1 \cdot f_2$ by

$$f_1 \cdot f_2(t_1, \ldots, t_m) = \begin{cases} f_1(2t_1, t_2, \ldots, t_m) & \text{for } 0 \le t_1 \le \frac{1}{2} \\ f_2(2t_1 - 1, t_2, \ldots, t_m) & \text{for } \frac{1}{2} \le t_1 \le 1 \ . \end{cases} \tag{A.2}$$

The group structure in $\pi_m(N)$ is then defined as in the case $m = 1$.

The groups $\pi_m(N)$ for $m \ge 2$ are always abelian, whereas $\pi_1(N)$ need not be abelian. The following sequence of drawings is the proof of this statement in the case $m = 2$. The first square represents the homotopy between f_1 (yellow) and f_2 (red), as given in (A.2). Black areas (including the contours of the rectangles) are points where the value of the function is y_0. By a continuous sequence of deformations one arrives at interchanging the order of f_1 and f_2 in the homotopy. It is also immediately clear why this cannot be done for $m = 1$.



The definition of $\pi_m(N)$ given above works also in the case $m=0$. The sphere S^0 consists of the two points $+1$ and -1. One of them, for example $+1$, can be taken as basepoint. A basepoint-preserving map $f: S^0 \to N$ maps $+1$ to y_0 and -1 to some point y of N. Thus there is a bijective correspondence between $\Gamma_*(M,N)$ and N. Two maps f and f' are homotopic if $y=f(-1)$ and $y'=f'(-1)$ belong to the same arcwise connected component of N. Thus $\pi_0(N)=[S^0,N]_*=\{\text{arcwise connected components of}\, N\}$. This set does not have a group structure in general.

Summarizing, the homotopy groups give some information about the topology of a manifold. $\pi_0(N)\neq 0$ if N has more than one connected component, $\pi_1(N)\neq 0$ if N is multiply connected, $\pi_m(N)\neq 0$ if N contains non-contractible m-spheres. One can prove that if M is a smooth manifold then the homotopy groups characterize its homotopy type.

If $f: N \to Q$ is a smooth map, there are natural homomorphisms $\pi_k(f): \pi_k(N) \to \pi_k(Q)$ for all k, defined as follows: $\pi_k(f)$ maps the homotopy class of a map $g: S^k \to N$ to the homotopy class of the map $f \circ g: S^k \to Q$. One can easily check that these are homomorphisms.

We will mostly need the homotopy groups of the spheres, $\pi_m(S^n)$. These can be conveniently summarized in a table:

Space	$n=1$	$n=2$	$n=3$	$n=4$
$m=1$	$\mathbf{Z}$	0	0	0
$m=2$	0	$\mathbf{Z}$	0	0
$m=3$	0	$\mathbf{Z_2}$	$\mathbf{Z}$	0

All the elements on the diagonal are **Z** by a theorem of Hopf. All the elements above the diagonal are zero. The elements below the diagonal are generally nonzero and are the tricky ones. The only one that we shall need is $\pi_2(S^3)$, which was also worked out by Hopf. It generator is the homotopy class of the Hopf map (the projection of the Hopf bundle).

Let M and N have the same dimension n. We denote $\omega=\frac{1}{n!}\omega_{i_1\cdots i_n}dy^{i_1}\wedge\cdots\wedge dy^{i_n}$ a volume-form on N. For example, if N is endowed with a riemannian metric $h=h_{\alpha\beta}dy^\alpha\otimes dy^\beta$ it is natural to consider the riemannian volume form $\omega=\sqrt{\det h}\, dy^1\wedge\cdots\wedge dy^n$. Given a map $\varphi: M\to N$ we define the winding number of φ

$$W(\varphi)=\frac{\int_M \varphi^*\omega}{\int_N \omega}=\frac{1}{\mathrm{Vol}(N)}\int_M d^n x\, \varepsilon^{\mu_1\cdots\mu_n}\partial_{\mu_1}\varphi^1\cdots\partial_{\mu_n}\varphi^n\omega_{1\cdots n}\ . \tag{A.3}$$

The geometrical meaning of this quantity can be understood as follows. Recall that a point $x\in M$ is a regular point for the map φ if the tangent map $T\varphi\big|_x$ is surjective (i.e., in coordinates, if $\det(\partial_\mu\varphi^\alpha)(x)\neq 0$). A point $y\in\mathrm{Im}\varphi\subset N$ is regular value for φ if all the points in its pre-image $\varphi^{-1}(y)$ are regular points. It can be proven that if y is any regular value of φ, then

$$W(\varphi)=\big(\#\,\text{of points in }\varphi^{-1}(y)\text{ with }\det(\partial_\mu\varphi^\alpha)>0\big)-\big(\#\,\text{of points in }\varphi^{-1}(y)\text{ with }\det(\partial_\mu\varphi^\alpha)<0\big)\ . \tag{A.4}$$

In particular the winding number is the integer topological invariant that characterizes the maps from S^n to S^n.

Appendix B. Homology and cohomology

Homology and cohomology groups are topics of algebraic topology. It is therefore most appropriate to discuss them in the context of topological spaces. However, since in this book we work only with differentiable manifolds, it is simpler to define these notions in the context of differential geometry. We will define homology using the notion of boundary of a manifold.

To define the p-th homology group of a manifold M we consider the set of all p-dimensional oriented submanifolds of M. By this we mean smooth embeddings of a p-dimensional oriented manifold in M. Obviously p is an integer smaller or equal to the dimension of M. In the case $p = 0$ an oriented manifold is just a point of M together with a sign. A formal linear combination of submanifolds with real coefficients is called a *real p-chain.* If m_1, m_2, ... are p-dimensional submanifolds of M, a p-chain is an object of the form

$$\sum_i c_i m_i \ , \tag{B.1}$$

where c_i are real numbers. Later on we shall consider a more general construction where c_i are elements of an arbitrary abelian group but for the moment we stick to the group of reals. The set of all real p-chains is denoted $C_p(M)$. It is an abelian group. (This construction is called the free abelian group generated by all p-dimensional oriented submanifolds of M).

Given an oriented submanifold m one can consider its boundary ∂m. It is a $(p-1)$-dimensional oriented submanifold of M. One can define a linear operator $\partial_p : C_p(M) \to C_{p-1}(M)$ by

$$\partial_p(\sum_i c_i m_i) = \sum_i c_i \partial m_i \ . \tag{B.2}$$

In the case $p = 0$ we make the convention that the boundary of a point is the empty set. When no confusion can arise we shall often omit the subscript p and denote the boundary operator by ∂. By construction, ∂ is a homomorphism. The boundary of a manifold of dimension p is a manifold of dimension $(p-1)$ without boundary. Therefore we have

$$\partial \circ \partial = 0 \ . \tag{B.3}$$

Let $Z_p(M) = ker \partial_p \subset C_p(M)$ and $B_p(M) = im \partial_{p-1} \subset C_p(M)$. The elements of $Z_p(M)$ are called p-cycles and the elements of $B_p(M)$ are called p-boundaries.

Because of (D.3) every boundary is a cycle, *i.e.* $B_p(M) \subset Z_p(M)$, but not every cycle is a boundary. Two cycles are said to be *homologous* if their difference is a boundary. For example if the cycles consist just of two manifolds m_1 and m_2 with coefficients 1, they are homologous provided there exists a $(p+1)$-manifold n such that its boundary is given by the union of m_1 and m_2, with the appropriate orientation. One is interested in the equivalence classes of p-cycles. The p-th homology group of M is defined by the quotient $H_p(M) = Z_p(M)/B_p(M)$.

It is remarkable that although $C_p(M)$, $Z_p(M)$ and $B_p(M)$ are infinite dimensional groups (the dimension of $C_p(M)$ being for example the number of all p-dimensional submanifolds of M), the quotient $H_p(M)$ is finite dimensional. The dimension of $H_p(M)$ is called the p-th Betti number of M, denoted $b_p(M)$.

We will not prove in general that b_p are finite, but we will make this plausible by considering a number of examples.

Let us now define the real cohomology groups. We use de Rham's definition, which is based on the use of differential forms. We denote $C^p(M)$ the space of smooth p-forms (totally antisymmetric p-tensors) on M; the exterior differential d is a map from $C^p(M)$ to $C^{p+1}(M)$, with the property that

$$d \circ d = 0 \ . \tag{B.4}$$

A p form α is said to be closed, or a *p-cocycle*, if $d\alpha = 0$, and exact, or a *p-coboundary*, if there exists a $(p+1)$-form β such that $\alpha = d\beta$. Two closed forms which differ by an exact form are said to be cohomologous. The space of all cocycles is denoted $Z^p(M)$ and the space of all coboundaries is denoted $B^p(M)$. Because of (D.4), we have $B^p(M) \subset Z^p(M)$, and we can define the p-th cohomology group of M as $H^p(M) = Z^p(M)/B^p(M)$.

Example 1. *In the case $p = 0$, $Z_0(M) = C_0(M)$ because the boundary of a point is empty and therefore one can formally say that $\partial_0 = 0$. On the other hand B_0 consists of linear combinations of points which arise by taking the boundary of a one-chain. For example, consider two zero-cycles each consisting of a single point, with coefficients 1 and -1. They are homologous only if there exist a line segment joining them, i.e. if they belong to the same connected component of M. Now choose a point p_i in each connected component of M. Every 0-cycle $m = \sum_j c_j m_j$ is homologous to a linear combination of p_i's. To see this it is enough to consider for each point m_j a line segment ℓ_j originating from m_j and ending on the point p_j lying in the same connected component of m_j. The boundary of the one-chain $\ell = \sum_j c_j \ell_j$ is $\sum_j c_j p_j - m$, Q.E.D. We conclude that b_0 is equal to the number of connected components of M.*

Example 2. *Every loop embedded in a two–sphere divides the sphere in two disks. Therefore every one-cycle on the sphere is homologous to zero, and $H_1(S^2) = 0$.*

Example 3. *Every loop embedded in a torus belongs to one of three classes: either it is the boundary of a disk, or together with one of the loops shown in fig. (bbb) it bounds an annular region. There follows that every one-chain on the torus is homologous to a linear combination of the loops shown in fig. (bbb). One says that the (homology class of) these loops generates $H_1(S^1 \times S^1)$. The first Betti number of the torus is $b_1(S^1 \times S^1) = 2$.*

Example 4. *Let M_p be a compact two-dimensional manifold without boundary of genus p. It can be obtained from a sphere by attaching p handles, i.e. removing $2p$ disks and then glueing p cylinders on the resulting boundary. Reasoning as in the previous example, the first homology of this manifold is $H_1(M_p) = \mathbf{R}^{2p}$, with generators shown in fig. (jjj).*

Example 5. *Let M be a compact, connected m-dimensional manifold without boundary. There are no $(m+1)$-chains, so $B_m(M) = 0$ and $H_m(M) = Z_m(M)$. Every cycle is a multiple of M itself, so $H_m(M) = \mathbf{R}$, the generator being M itself.*

Example 6. *There are no $(m+1)$-chains, so $H_p(M) = 0$ for $p > m$.*

Example 7. *The space $C^0(M)$ is just the space of real functions on M and the space $Z^0(M)$ is the space of locally constant functions on M. The space $B^0(M)$ is formally zero, since there are no forms of order -1. If M is connected, every locally constant function is constant, so $H^0(M) = Z^0(M) = \mathbf{R}$. If M has several connected components, a locally constant function is constant on each connected component, so $H^0(M)$ is the direct sum of b^0 copies of $\mathbf{R}$, where b^0 is the number of connected components of M.*

Example 8. *Every one-form on a sphere of dimension $m > 1$ is exact, so $H^1(S^m) = 0$.*

Example 9. *Parametrize a circle S^1 with an angle $0 \leq \varphi < 2\pi$. The form $d\varphi$ is closed and locally exact, but it is not globally exact, because ϕ is not a (single-valued) function on the circle. Given a one-form α on the circle consider the integral $f(\varphi) = \int_0^\varphi \alpha$. We have locally $\alpha = df$. If $f(2\pi) = 0$, then f is a well-defined function on the circle and α is exact. Two forms α_1 and α_2 are cohomologous if and only if $\int_{S^1} \alpha_1 = \int_{S^1} \alpha_2$. Therefore $H^1(S^1) = \mathbf{R}$*

Example 10.

Example 11.

Example 12.

The most important property of the homology and cohomology groups is that they are topological invariants, *i.e.* they are the same for topologically equivalent manifolds. The proof of this fact is not trivial and will be omitted (see ...).

Comparing the results in the examples we see that $H^p(M)$ and $H_p(M)$ are isomorphic. Is this casual? Consider the real number defined by

$$\langle \alpha | m \rangle = \int_m \alpha \; , \tag{B.5}$$

where α is a p form and m is a p dimensional submanifold of M. By linearity, this defines a bilinear pairing $C^p(M) \times C_p(M) \to \mathbf{R}$. This pairing depends only on the homology class of m and the cohomology class of

α. In fact, using Stokes' theorem, one easily gets

$$\langle \alpha + d\beta | m \rangle = \langle \alpha | m \rangle$$

and

$$\langle \alpha | m + \partial n \rangle = \langle \alpha | m \rangle$$

So we actually have a bilinear pairing $H^p(M) \times H_p(M) \to \mathbf{R}$.

One can prove that this pairing is nondegenerate, in the sense that $\langle \alpha | m \rangle = 0$ for all $\alpha \in Z^p(M)$ implies $m = \partial n$ and $\langle \alpha | m \rangle = 0$ for all $m \in Z_p(M)$ implies $\alpha = d\beta$. Therefore, $H^p(M)$ is isomorphic to the dual space $H_p(M)^*$.

This shows that there is no more information in the cohomology groups than there is in the homology groups. However, the direct sum $\oplus_p H^p(M)$ can be givem an algebra structure, with the product coming from the exterior product of forms. This whole algebra is a topological invariant, and it does not have a counterpart in homology.

One can define homology groups with coefficients in any abelian group G. In the definition of a p-chain given above one just reinterprets the coefficients c_i as elements of G instead of real numbers. The resulting homology groups are denoted $H_p(M, G)$. The most important case is $G = \mathbf{Z}$, the group of the integers. The integer homology group $H_p(M, \mathbf{Z})$ can be shown to be a finitely genarated abelian group, and has the general structure

$$\mathbf{Z} \oplus \ldots \oplus \mathbf{Z} \oplus \mathbf{Z}_{n_1} \oplus \ldots \mathbf{Z}_{n_k} \ ,$$

where there are b_p direct addends $\mathbf{Z}$ and k addends which are cyclic groups (of order $n_1, \ldots, n_k$). The direct sum of the $\mathbf{Z}$ groups forms the so-called free part, while the direct sum of the cyclic groups is called the torsion part.

The integer homology groups are the ones that contain most information. The homology groups with other coefficients can be obtained from the integer ones by using the so-called universal coefficient theorem. For example, the real homology groups are obtained by replacing every addend $\mathbf{Z}$ by an addend $\mathbf{R}$ and dropping the torsion part. Therefore, they contain less information than the integer homology groups.

One can also define cohomology groups with arbitrary coefficients. In general it is not possible to use differential forms. In the real case, one can regard a differential form as a linear map from $C_p(M)$ to $\mathbf{R}$. In general one can define $C^p(M, G)$ to be the space of all homomorphisms from $C_p(M, \mathbf{Z})$ to G. The differential d is defined in this case by the requirement that for every cochain α and chain m, $d\alpha(m) = \alpha(\partial m)$. The resulting cohomology groups $H^p(M, G)$ are again related to the corresponding homology groups. For example

$$H^p(M, \mathbf{Z}) = \mathrm{free}(H_p(M, \mathbf{Z})) \oplus \mathrm{tor}(H_{p-1}(M, \mathbf{Z})) \ . \tag{B.6}$$

It is possible to represent the integer cohomology classes by means of singular differential forms []. For our purposes it will be enough to note that the homomorphism $\mathbf{Z} \to \mathbf{R}$ gives rise to a homomorphism $H^p(M, \mathbf{Z}) \to H^p(M, \mathbf{R})$, and that the latter group can be represented in the de Rham way by differential forms. A de Rham cohomology class is in the image of this homomorphism if and only if

$$\langle \alpha | m \rangle = \int_m \alpha \in \mathbf{Z} \qquad \forall m \in Z_p(M, \mathbf{Z}) \ . \tag{B.7}$$

Finally we mention a connection between homology and homotopy groups, known as the Hurewicz theorem: if $\pi_1(M) = \ldots = \pi_r(M) = 0$, then $H_1(M) = \ldots = H_r(M) = 0$ and $H_{r+1}(M, \mathbf{Z}) = \pi_{r+1}(M)$.

Appendix C. Group actions on manifolds.

Let G be a Lie group and M a smooth manifold. One says that G acts on M from the left if there is a map $L : G \times M \to M$ such that:

$$L(e, x) = x \ \forall x \in M \ , \qquad L(g_1, L(g_2, x)) = L(g_1 g_2, x) \ .$$

(We call e the identity in the group). Similarly, G acts on M from the right if there is a map $R : M \times G \to M$ such that:

$$R(x,e) = x \ \forall x \in M \ , \qquad R(R(x,g_2),g_1) = R(x,g_2g_1) \ .$$

It is often convenient to define for each $g \in G$ a map $L_g : M \to M$ by $L_g(x) = L(g,x)$, and similarly in the case of a right action $R_g : M \to M$ is defined by $R_g(x) = R(x,g)$. The composition of these maps is governed by the rule

$$L_{g_1} \circ L_{g_2} = L_{g_1g_2} \ , \qquad R_{g_1} \circ R_{g_2} = R_{g_2g_1} \ .$$

The maps L_g are diffeomorphisms of M to itself. Here we content ourselves with showing that they are bijective, without proving smoothness. It is clear that L_g is surjective, since for any $x \in M$ the point $L_{g^{-1}}(x)$ is a pre-image of x. It is injective since this pre-image is unique. If x' is another pre-image, $L_g(x') = x$, acting on both sides with $L_{g^{-1}}$ we get $x' = L_{g^{-1}}(x)$, Q.E.D.

The map $g \mapsto L_g$ is a homomorphism from G to the diffeomorphism group of M; it is called a realization of the (abstract) group G.

In the following we will discuss mainly left actions, but all definitions can be extended in an obvious way to right actions. The action is said to be:

- *effective* if $L_g(x) = x$ for all x implies $g = e$;
- *free* if $L_g(x) = x$ for some x implies $g = e$ (this is equivalent to saying that the maps L_g have no fixed points);
- *transitive* if for any pair of points x_1, x_2 in M there is a g in G such that $L_g(x_1) = x_2$.

It is convenient to introduce some further definitions. The *orbit* through a point $x_0 \in M$ is the set O_{x_0} of all points $x \in M$ such that $x = L_g(x_0)$ for some $g \in G$. By definition the group G acts transitively on each orbit. The *stabilizer* (or *isotropy group*) of a point $x_0 \in M$ is the subgroup H_{x_0} of G such that $L_g(x_0) = x_0$ for $g \in H_{x_0}$. The stabilizers of two different points belonging to the same orbit are conjugate subgroups of G. In fact if $x = L_g(x_0)$, the stabilizer H_x consists precisely of all elements of G which are of the form ghg^{-1}, with $h \in H_{x_0}$. Thus the stabilizers of all points in the same orbit are isomorphic. By definition if the action is free the stabilizer of each point is the trivial group consisting only of the identity. The relation of belonging to the same orbit is an equivalence relation on M. Thus one can define the quotient space M/G. This is also called the space of the orbits, since every orbit of G in M corresponds to exactly one point of M/G. In general the space of orbits will not be a smooth manifold.

Let us consider some examples. If M is a vectorspace and the action of G on M is by linear transformations, then the action defines a linear representation of G (by contrast in the case when M is not a linear space L is called a nonlinear realization of the group G). In this case the action is effective if and only if the corresponding representation is faithful. A linear action can never be free or transitive, because the origin is always a singular orbit. The space of orbits has singularities corresponding to orbits with larger stabilizer. Consider for example the fundamental representation of $SO(3)$ on $\mathbf{R}^3$. All orbits are two dimensional spheres, except for the origin, which is a point. The stabilizer of the spherical orbits is $SO(2)$, while the stabilizer of the origin is $SO(3)$. The orbits can be parametrized by the radius and the space of orbits $\mathbf{R}^3/SO(3)$ is a half-line, with zero corresponding to the exceptional orbit. In this case the space of orbits is a one dimensional manifold with boundary. In the case of more complicated representations there can be many classes of orbits with different stabilizers, and the space of orbits is correspondingly more complicated.

A group acts on itself by multiplication both from the left and from the right. These two actions commute and are both free and transitive. Now let H be a proper subgroup of G (by proper it is meant that H is not G itself, and it does not consist only of the identity). The subgroup H acts on G both from the left and from the right by multiplication. We focus on the right action. It is again a free action, but it is not transitive. The orbits of H in G are called *right cosets*; the space of orbits G/H is called a *right coset space.* There is a left action $\bar{L}$ of G on G/H, defined by $\bar{L}_g(g'H) = (L_g(g'))H$, where we denote gH the coset of g and L the left action of G on itself. This action is well defined (*i.e.* it is independent of the representative g' in the coset) because the left action of G on itself and the right action of H on G commute. The coset space G/H has a preferred point which is the coset eH. We will call this point the "origin" of G/H. The action $\bar{L}$ is transitive and effective. The stabilizer of the origin is the group H.

A coset space is an example of a space of orbits which is a smooth manifold. The reason for this is that each orbit of H in G has the same stabilizer, namely the trivial group consisting only of the identity. This example can be generalized. Consider a group H acting freely from the right on a manifold P and let Q be the space of the orbits. We call $p : P \to Q$ the projection that maps each point to its orbit. We make the further technical assumption that this action admits local sections. This means that for each point of Q there is a neighbourhood $U \subset Q$ and a map $s : U \to P$ such that $p(s(q)) = q$ for $q \in U$. Under these circumstances one can prove that Q is a smooth manifold, and the space P is said to be a *principal bundle* over Q with structure group H. For example, G is a principal bundle over G/H with structure group H. The special feature of this example is that there is an action of a group on the space P (namely the left action of G on itself) commuting with the action of the structure group H. Therefore this action goes down to Q. In general there will be no group action on Q.

A manifold N is said to be *homogeneous* if a group G acts transitively on it. For example each orbit of an action of G on some space M is homogeneous. We can assume without loss of generality that this action is from the left. Let x_0 be any point of N and H_0 be the corresponding stabilizer. Then N is diffeomorphic to the coset space G/H_0. Therefore every homogeneous space is diffeomorphic to a coset space. We will prove here that there is a bijective correspondence between points of N and points of G/H_0, without showing that this correspondence is smooth. Let L denote the action of G on N and define a map $\beta : G/H_0 \to N$ by

$$\beta(gH_0) = L_g(x_0) \ .$$

First of all we have to prove that this map is well defined, *i.e.* that the r.h.s. is independent of the choice of g in the coset gH_0. If g' is another element of the coset, $g' = gh$ for some $h \in H_0$, and therefore the r.h.s. is $L_{g'}(x_0) = L_{gh}(x_0) = L_g(L_h(x_0)) = L_g(x_0)$. The map β is surjective, because L is transitive. The map β is injective. If $\beta(gH_0) = \beta(g'H_0)$, also $L_g(x_0) = L_{g'}(x_0)$ and therefore $g' = gh$ for $h \in H_0$. But then $gH_0 = g'H_0$. Q.E.D.

Appendix D. Topology and group actions.

We consider here three topics in topology that involve Lie groups: the homotopy exact sequence of a principal bundle, the cohomology of Lie groups and Lie algebras, and the cohomology of group actions

Recall that a principal bundle is a space P on which a Lie group acts (conventionally from the right) without fixed points; there is then a natural projection π from P to the quotient (or orbit space) $Q = P/G$. To discuss homotopy we choose a basepoint p_0 in P; the basepoint in Q is then $q_0 = \pi(p_0)$, while the basepoint in G is naturally the identity. There is then a natural map i from G to P which identifies G with the orbit through p_0 (with $i(e) = p_0$.

The maps π and i induce homomorphisms π_* on the homotopy groups, as discussed in Appendix A. Because $\pi \circ i$ is the constant map q_0, $\pi_* \circ i_* = 0$. Thus in the language of exact sequences,

$$\pi_k(G) \to \pi_k(P) \to \pi_k(Q)$$

is exact at $\pi_k(P)$. One can define homomorphisms $\partial : \pi_k(Q) \to \pi_{k-1}(G)$ such that

$$\ldots \pi_{k+1}(Q) \to \pi_k(G) \to \pi_k(P) \to \pi_k(Q) \to \pi_{k-1}(G) \to \ldots$$

is a long exact sequence. It is called the homotopy exact sequence of the principal bundle P. From this sequence one can for example extract information about the homotopy groups of Q from knowledge of the homotopy groups of P and G.

Let us now consider the de Rham cohomology of a Lie group G. A differential form ω on G is said to be right-invariant (or left-invariant) if $L_g^*\omega = \omega$ (resp. $R_g^*\omega = \omega$) for all $g \in G$.

One can prove the following: if G is a compact Lie group, in every cohomology class of G there is an invariant form. In fact on any compact Lie group there exists an invariant Haar measure; one can use this measure to average a form over the group. The result is an invariant form (CHECK BOOTHBY).

One can use this result to motivate a purely algebraic definition of cohomology for Lie algebras. Recall that on a Lie group one can choose global systems of left- and right-invariant basis vectors $\{L_a\}$ and $\{R_a\}$, with components

$$L_a = L_a^\alpha \frac{\partial}{\partial y^\alpha} \ ; \qquad\qquad R_a = R_a^\alpha \frac{\partial}{\partial y^\alpha} \ .$$

To construct such vectorfields one just picks an arbitrary basis of vectors tangent to the group at the identity and tranlates it to any other point using the left and right action. These vectorfields satisfy the Lie bracket relations

$$\begin{aligned}[L_a, L_b] =& c_{ab}{}^c L_c \ ; \\ [L_a, R_b] =& 0 \ ; \\ [R_a, R_b] =& - c_{ab}{}^c R_c \ ; \end{aligned}$$

The coefficients $c_{ab}{}^c$ are constant; they are (by definition) the structure constants of G. The vectorfields L_a generate the right action of G on itself and the vectorfields R_a generate the left action.

One can choose dual bases of one-forms $\{L^a\}$ and $\{R^a\}$ with components

$$L^a = L^a_\alpha dy^\alpha \ ; \qquad R^a = R^a_\alpha dy^\alpha \ .$$

satisfying

$$L^a(L_b) = L^a_\alpha L^\alpha_b = \delta^a_b \ ; \qquad R^a(R_b) = R^a_\alpha R^\alpha_b = \delta^a_b \ .$$

showing that the components of the invariant forms are the matrix inverses of the components of the invariant vectorfields. The forms L^a and R^a are called the left and right Maurer–Cartan forms. They satisfy the following Maurer–Cartan equations

$$dL^a + c_{ab}{}^c L^b \wedge L^c = 0 \ ; \qquad dR^a - c_{ab}{}^c R^b \wedge R^c = 0 \ .$$

CHECK SIGNS.

A left-invariant p-form can be written in the left-invariant basis

$$\omega = \frac{1}{p!} \omega_{a_1 \dots a_p} L^{a_1} \dots L^{a_p}$$

where $\omega_{a_1 \dots a_p}$ are constants. They are related to the natural basis components of ω by

$$\omega_{a_1 \dots a_p} = L^{\alpha_1}_{a_1} \dots L^{\alpha_p}_{a_p} \omega(e)_{\alpha_1 \dots \alpha_p} \ .$$

Using the general formula (), the exterior differential of ω is

$$d\omega(v_1, \dots, v_{p+1}) = \sum_{ij} (-1)^{i+j+1} \omega([v_i, v_j], \dots, \hat{v}_i, \dots, v_j, \dots, v_{p+1})$$

or, in components,

$$(d\omega)_{a_1 \dots a_{p+1}}) = (p+1) \left(c_{a_1 a_2}{}^b \omega_{b a_3 \dots a_{p+1}} + \dots \right)$$

CHECK SIGNS It obviously satisfies $d^2 = 0$, so it can be used to define a cohomology of the lie algebra, based only on knowledge of the structure constants. By construction this cohomology is isomorphic to the de Rham cohomology of the group.

One can generalize the Lie algebra cohomology to forms with values in a representation. The exterior differential has again the form (). In field-theoretic applications one encounters the following situation. There is a space Y on which the group G acts from the right (the same discussion can be repeated for a right action). The space L of complex-valued functions on Y carries a linear representation of G which is defined by

$$(\rho(g) f)(y) = f(yg)$$

(we use the shorthand $R_g(y) = yg$). One then considers the cohomology of the Lie algebra of G with coefficients in L. The p-cochains in this case are functions $f(v_1, \dots, v_p; y)$ depending on p elements of the Lie algebrs, linear and antisymmetric in the first p entries.

One can also construct a Lie group cohomology that corresponds to this Lie algebra cohomology. One defines the p-cochains to be functions $f(y; g_1, \dots, g_p)$, where g_k are group elements. The coboundary is defined by

$$\begin{aligned}\Delta f(y; g_1, \dots, g_{n+1}) =& f(yg_1; g_2, \dots, g_{p+1}) - f(y; g_1 g_2, g_3, \dots, g_{p+1}) + \dots \\ + (-1)^k f(y; g_1, \dots, g_k g_{k+1}, \dots, g_{p+1}) + \dots + (-1)^{p+1} f(y; g_1, \dots, g_p)\end{aligned}$$

one can verify that $\Delta^2 = 0$.

These operations occur in the theory of representations. For example one could introduce in the definition () a phase:

$$(\lambda(g)f)(y) = e^{i\omega^1(y;g)} f(L_g(y))$$

Then demanding that

$$\rho(g_1)\rho(g_2) = \rho(g_1 g_2)$$

requires that $\Delta\Omega^1 = 0 \bmod 2\pi$ or, Ω^1 has to be a one-cocycle. If Ω^1 is a one-coboundary $\Omega^1(y; g) = \Delta\Omega^0$ one can redefine

$$f'(y) = e^{i\Omega^0(y)} f(y) \ ; \qquad \rho'(g) = e^{i\Omega^0(y)} \rho(g) e^{-i\Omega^0(y)}$$

and the primed quantities satisfy (). In other words, the phase in () can be removed if the cocycle is trivial. This phenomenon occurs for example in the representation of Galilei transformations in quantum mechanics.

Another application is in the case of projective representations:

$$\rho(g_1)\rho(g_2) = e^{i\Omega^2(y;g_1,g_2)} \rho(g_1 g_2)$$

Demanding associativity

$$\rho(g_1)(\rho(g_2)\rho(g_3)) = (\rho(g_1)\rho(g_2))\rho(g_3)$$

leads to $\Delta\Omega^2 = 0 \bmod 2\pi$, *i.e.* Ω^2 has to be a two-cocycle. If it is a coboundary: $\Omega^2 = \Delta\Omega^1$, redefining

$$\rho'(g) = e^{-i\Omega^1(y,g)} \rho(g)$$

eliminates the phase from ().

We could also allow a phase in ():

$$\rho(g_1)(\rho(g_2)\rho(g_3)) = e^{i\Omega^3(y;g_1,g_2,g_3)} (\rho(g_1)\rho(g_2))\rho(g_3)$$

The consistency of four-fold products then requires that $\Delta\Omega^3 = 0 \bmod 2\pi$.