Solve the following three exercises:

Exercise 1.

Discuss the difference between regularization and renormalization, explain what "renormalization scheme" means, and finally comment on the pro and cons of cut-off and dimensional regularizations.

Exercise 2.

Consider a $U(1)$ gauge theory with two Dirac fermions ψ_1 and ψ_2 with Dirac masses m_1 and m_2 (and no Majorana mass terms) and one complex scalar ϕ in four space-time dimensions. The charges of the three fields are respectively $(0,1,-1)$.

(a) Write the most general gauge, Lorentz, parity invariant and renormalizable Lagrangian associated to the above system.

(b) Compute the one-loop wave function renormalization and anomalous dimension for the fermion field ψ_1.

(c) Compute the one-loop mass counterterm for the fermion field ψ_1.

(d) Is it natural to assume that $m_1 \ll m_2$? Motivate your answer.

Exercise 3.

The Lagrangian for massless scalar QED reads

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu\Phi|^2 - \frac{\lambda}{6}|\Phi|^4\,, \tag{1}$$

where $D_\mu\Phi = \partial_\mu\Phi - ieA_\mu\Phi$. The 1-loop β-functions for the the quartic coupling λ and the electric charge e read

$$\beta_e = \frac{e^3}{48\pi^2}\,, \qquad \beta_\lambda = \frac{1}{4\pi^2}\left(9e^4 + \frac{5}{6}\lambda^2 - 3\lambda e^2\right). \tag{2}$$

By solving the one-loop RG evolution of the above system it can be shown that there are RG flows for $e(\mu)$ and $\lambda(\mu)$ such that at some energy scale $\Lambda_{\rm IR}$ the quartic coupling vanishes: $\lambda(\Lambda_{\rm IR}) = 0$.

(a) Write (implicitly) the tree-level RG improved scalar effective potential in terms of $\rho^2 \equiv 2|\Phi|^2$. Neglect anomalous dimension corrections.

(b) Write an explicit form of the potential valid for energy scales close to $\Lambda_{\rm IR}$. Compute the extrema of this effective potential and find the minimum.

(c) Is the minimum found in (b) reliable?