Solve the following two/three exercises:

Exercise 1.

Discuss the derivation of the Lehmann-Symanzik-Zimmermann (LSZ) reduction formulas.

Exercise 2.

Consider a four-dimensional theory containing two abelian gauge fields and four Dirac fermions as follows:

$$\mathscr{L}_{\rm UV} = \sum_{k=1,2}\left(-\frac{1}{4}F_{\mu\nu,k}F_k^{\mu\nu} + \bar{\psi}_k i\not{D}_k^{(\psi)}\psi_k + \bar{\chi}_k(i\not{D}_k^{(\chi)} - m_k)\chi_k\right), \tag{1}$$

where

$$\begin{aligned} D_1^{(\psi)} &= \partial - ie_1A_1\,, \qquad & D_2^{(\psi)} = \partial - ie_2A_2\,, \\ D_1^{(\chi)} &= \partial - i(e_1A_1 + e_2A_2)\,, \qquad & D_2^{(\chi)} = \partial - i(e_1A_1 - e_2A_2)\,. \end{aligned} \tag{2}$$

Suppose that $e_1 \sim e_2 \ll 1$.

(a) Write down the most general effective IR Lagrangian obtained by integrating out the two fermions χ_1 and χ_2 for energies $E \ll m_1, m_2$. Consider up to dimension four operators.

(b) Match up to one-loop level the coefficients of the IR theory with those of the UV theory.

(c) Find a basis for the canonically normalized gauge fields. In order to simplify the algebra, take the initial diagonal gauge kinetic terms to be identical.

(d) Compute the charges of the two massless fermions ψ_1 and ψ_2 in the IR theory in the basis found in (c). Discuss the dependence of the charges on the choice of the basis found in (c).

[Hint: Recall that the contribution of a unit charge fermion to the divergent part of the one-loop vacuum polarization for a $U(1)$ gauge field is given by $i\Pi^{(1)div.}_{\mu\nu}(q) = i(\eta_{\mu\nu}q^2 - q_\mu q_\nu)\frac{e^2}{8\pi^2}\left(-\frac{4}{3}\right)\frac{1}{\varepsilon}$. Neglect renormalization scheme-dependent constant factors in the matching procedure.]

Exercise 3. *(Mandatory for TPP students only.)*

Consider the theory of Exercise 2 without the four fermions but where a massless complex scalar field ϕ with charges (p_1, p_2) under the two $U(1)$ gauge symmetries is added. Take $e_1 = e_2 \equiv e$ for simplicity. The bare Lagrangian is:

$$\mathscr{L} = -\frac{1}{4}\sum_{k=1,2} F_{\mu\nu,k}F_k^{\mu\nu} + |D_\mu\phi|^2 - \frac{\lambda}{6}(|\phi|^2)^2 \,. \tag{3}$$

(e) What are the classical symmetries of this theory?

(f) Can a mass term for ϕ be generated radiatively? Motivate your answer.

(g) Discuss (without doing a computation) a dynamical mechanism which can possibly induce a vacuum expectation value for the scalar field ϕ.

(h) Assuming the scalar field takes a vev $\langle\phi\rangle = v/\sqrt{2}$, what is the symmetry-breaking pattern of the theory? Find the masses and mass eigenstates of the two gauge fields (neglect the loop-induced kinetic mixing).