Solve the following two/three exercises:

Exercise 1.

Derive the general form of the Ward-Takahashi identities in quantum field theory.

Exercise 2.

Consider the Lagrangian density in six space-time dimensions in dimensional regularization ($\varepsilon = 6 - d$)[1]

$$\mathscr{L} = \frac{1}{2} Z_\phi \partial_\mu \phi \partial^\mu \phi - Z_g \frac{g\mu^{\varepsilon/2}}{3!}\phi^3 \tag{1}$$

(a) Discuss the classical symmetries of this theory.

(b) Calculate the counterterms Z_ϕ and Z_g at one-loop level in the MS scheme.

(c) Using (b), compute the anomalous dimension of the field ϕ and the β function of the coupling g at one-loop level.

[Hint:

$$\int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2 - X)^3} = -i\frac{a}{\varepsilon} + \textit{finite} \quad \textit{with} \quad a = \frac{1}{(4\pi)^3}]$$

Exercise 3. *(Mandatory for TPP students only.)*

Discuss briefly the strong CP problem and the axion solution. Argue why the leading potential term in the chiral Lagrangian with $n_f = 2$ flavours is

$$V(a, \pi) = -c f_\pi^3 tr(UM(a/f_a) + M^\dagger(a/f_a)U^\dagger), \tag{2}$$

where $U = \exp(i\sigma^a \pi^a / f_\pi)$ and $M(a/f_a)$ is a properly defined quark mass matrix, with a the axion field. Using eq.(2), compute the axion mass as a function of the pion and quark masses and as a function of the axion and pion decay constants f_a and f_π, with $f_a \gg f_\pi$, in the vacuum where the expectation values of all the fields vanish.
[Hint: . The eigenvalues of a mass matrix of the form

$$m = \begin{pmatrix} \alpha\varepsilon^2 & -\alpha\varepsilon \\ -\alpha\varepsilon & \beta \end{pmatrix}$$

are given by the "see-saw" formula $\lambda_1 = (\alpha - \alpha^2/\beta)\varepsilon^2 + O(\varepsilon^4)$, $\lambda_2 = \beta + O(\varepsilon^2)$*].*

[1] This theory is non-perturbatively ill-defined, but makes sense in perturbation theory.