Solve the following two/three exercises:

Exercise 1.

Discuss the quantization of non-abelian gauge theories.

Exercise 2. *(Symmetry restoration.)*

Consider two neutral scalar fields φ_1 and φ_2 with bare Lagrangian:

$$\mathscr{L} = \sum_{i=1}^{2} \left\{ \frac{1}{2}\partial_\mu \varphi_i \partial^\mu \varphi_i - \frac{m^2}{2}\varphi_i^2 - \frac{\lambda_0}{4!}\varphi_i^4 \right\} - \frac{g_0}{12}\varphi_1^2\varphi_2^2. \tag{1}$$

(a) Discuss the symmetries of this theory depending on the values of the two couplings $\lambda_0 > 0$ and $g_0 \geq 0$.

(b) Consider, first, the case $\lambda_0 = 0$ and calculate the beta function $\beta_g(g, \lambda = 0)$ of the (renormalized) coupling g at one loop, within dimensional regularization ($\varepsilon = 4 - d$) and minimal subtraction.
(Hints: (i) Be careful with the combinatorial factors of the diagrams and (ii) express the result in terms of the constant $a = 2/(4\pi)^2$:

$$\int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{1}{(p^2 - m^2)^2} \equiv i\frac{a}{\varepsilon} + \mathscr{O}(\varepsilon^0).) \tag{2}$$

(c) Determine the beta functions $\beta_g(g, \lambda)$ and $\beta_\lambda(g, \lambda)$ of the couplings g and λ in Eq. (1) on the sole basis of the result of point (b) and of the fact that the β function of the coupling constant λ of a theory with a n-component scalar field and $O(n)$ symmetry is given (in $d = 4$) by

$$\beta_{O(n)}(\lambda) = \frac{n+8}{6} a\lambda^2 + \mathscr{O}(\lambda^3). \tag{3}$$

Discuss the flow of the ratio $\rho \equiv \lambda/g$ and show that, even if originally broken, an $O(2)$ symmetry might emerge asymptotically at large distances (infrared).

Exercise 3.* *(Baryon and Lepton number in the Standard Model)*

The Standard Model Lagrangian has an accidental baryon- and lepton- number global symmetry

$$U(1)_B \times U(1)_L, \tag{4}$$

under which leptons have charge $B = 0$, $L = 1$ and quarks charge $B = 1/3$, $L = 0$. Consider one generation of SM fermions, with the inclusion of a right-handed neutrino singlet. In terms of left-handed fields, the fields are in the following $SU(3)_c \times SU(2)_L \times U(1)_Y$ representations:

$$(\mathbf{3},\mathbf{2})_{\frac{1}{6}} + (\bar{\mathbf{3}},\mathbf{1})_{-\frac{2}{3}} + (\bar{\mathbf{3}},\mathbf{1})_{\frac{1}{3}} + (\mathbf{1},\mathbf{2})_{-\frac{1}{2}} + (\mathbf{1},\mathbf{1})_1 + (\mathbf{1},\mathbf{1})_0 \tag{5}$$

(a) Calculate all the possible anomalies involving $U(1)_B$ and two SM gauge currents and show that $U(1)_B$ is anomalous.

(b) Calculate all the possible anomalies involving $U(1)_L$ and two SM gauge currents and show that $U(1)_L$ is anomalous.

(c) Define a linear combination $X = B + \eta L$ of the baryon and lepton number generators B and L. Discuss if there is a value of η for which X is anomaly-free and, in case, determine such value.

(*) Exercise 3 mandatory for TPP students only.