Solve the following three exercises:

Exercise 1.

Consider an $SU(2) \times U(1)$ gauge theory with $n_1 > 0$ and $n_2 > 0$ left-handed fermions in the fundamental representation of $SU(2)$ and non-vanishing $U(1)$ charges q_1 and q_2, respectively.

- List the possible gauge anomalies that this theory can have
- Find (if any) the values of $q_{1,2}$ and $n_{1,2}$ for which the theory is anomaly-free.

Exercise 2.

Consider two neutral scalar fields ϕ and ψ with Lagrangian

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\partial_\mu\psi\partial^\mu\psi + g\,\partial_\mu\phi\partial^\mu\psi - \frac{m^2}{2}\psi^2 - \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\left(\phi^2+\psi^2\right)^2, \tag{1}$$

characterized by a mass m and two couplings g and λ.

- Identify the symmetries of this Lagrangian for $g = 0$ and for $g \neq 0$.
- Determine the range of values of g for which $\mathscr{L}$ defines a *bona fide* quantum field theory.
- Calculate the *bare* masses (i.e., for $\lambda = 0$) of the particles associated with $\mathscr{L}$ as a function of g.
- For $\lambda \neq 0$, determine the two mass renormalization constants (counter-terms) at one-loop order in the minimal subtraction scheme.

  Hint: Recall that

$$\int \frac{d^d q_E}{(2\pi)^d}\frac{1}{q_E^2+\Delta} = \frac{1}{(4\pi)^{d/2}}\Gamma\Big(1-\frac{d}{2}\Big)\Delta^{d/2-1} = -\frac{\Delta}{8\pi^2}\frac{1}{\varepsilon} + \mathscr{O}(1) \tag{2}$$

Exercise 3.

Discuss the ideas behind the effective potential and its RG improved version.