Exercise 2. *The Axion-Photon Coupling from Chiral Perturbation Theory*

In Chapter 1 the low-energy anomalous couplings were written as

$$\mathcal{L}_{\rm anom} \supset \frac{a}{f_a}\frac{\alpha_s}{8\pi}G\tilde{G} + \frac{E}{N}\frac{a}{f_a}\frac{\alpha}{8\pi}F\tilde{F}.$$

In this exercise you will derive the leading-order QCD correction to the photon coupling in two-flavor chiral perturbation theory. Work to leading order in the light-quark masses and in $f_\pi/f_a \ll 1$, define $z \equiv m_u/m_d$, and write the final result as

$$\mathcal{L}_{a\gamma\gamma} = C_{a\gamma}\,\frac{\alpha}{8\pi}\frac{a}{f_a}F_{\mu\nu}\tilde{F}^{\mu\nu}.$$

Start from the low-energy axion Lagrangian

$$\mathcal{L} \supset \frac{\alpha_s}{8\pi}\frac{a}{f_a}G^a_{\mu\nu}\tilde{G}^{a\mu\nu} + \frac{E}{N}\frac{\alpha}{8\pi}\frac{a}{f_a}F_{\mu\nu}\tilde{F}^{\mu\nu} - m_u\bar{u}u - m_d\bar{d}d\,. \tag{1}$$

(a) **Move the axion into the quark masses.** Perform the symmetric chiral rotation

$$u \,\to\, \exp\!\left(i\gamma_5\frac{a}{4f_a}\right)u\,, \qquad d \,\to\, \exp\!\left(i\gamma_5\frac{a}{4f_a}\right)d\,. \tag{2}$$

Using the anomalous chiral-rotation discussion from Chapter 1, determine how the bosonic gluon and photon couplings transform under this axial rotation. For the purposes of this exercise, you do not need to keep the derivative axion-current terms generated by the spacetime dependence of $a(x)$. Derive the resulting terms relevant for the χPT matching. In particular, identify the new coefficient multiplying $F_{\mu\nu}\tilde{F}^{\mu\nu}$ and write the transformed quark mass matrix M_a.

(b) **Axion-pion mixing from χPT.** Using the two-flavor chiral Lagrangian from Chapter 1 with mass matrix M_a, set $\pi^\pm = 0$ and write $\Sigma = \mathrm{diag}(e^{i\varphi}, e^{-i\varphi})$ with $\varphi \equiv \pi^0/f_\pi$. Show that the potential is

$$V(\pi^0, a) = -Bf_\pi^2\left[m_u\cos\!\left(\varphi + \frac{a}{2f_a}\right) + m_d\cos\!\left(\varphi - \frac{a}{2f_a}\right)\right]. \tag{3}$$

Expand this to quadratic order around $\pi^0 = a = 0$ and derive the (π^0, a) mass matrix. Using

$$m_\pi^2 = B(m_u + m_d)\,, \tag{4}$$

show that the off-diagonal term is proportional to $m_d - m_u$, and extract the small mixing angle $\theta_{a\pi}$ to leading order in f_π/f_a.

(c) **Induced photon coupling from pion mixing.** At low energies the neutral pion couples anomalously to photons via

$$\mathcal{L}_{\pi\gamma\gamma} = -\frac{\alpha}{4\pi f_\pi}\,\pi^0 F_{\mu\nu}\tilde{F}^{\mu\nu}. \tag{5}$$

Diagonalize the (π^0, a) system to leading order in $\theta_{a\pi}$ and use the above interaction to determine the induced axion-photon term generated by mixing.

(d) **Combine the pieces.** Add the direct term from part (a) to the mixing contribution from part (c), simplify the result, and determine $C_{a\gamma}$ as a function of E/N and z. Evaluate numerically the QCD-induced subtraction term in $C_{a\gamma}$ for the traditional input $z = 0.56$ and for the more modern lattice-motivated value $z = 0.48$. Then plug in the benchmark values

$$\left(\frac{E}{N}\right)_{\text{KSVZ}} = 0, \qquad \left(\frac{E}{N}\right)_{\text{DFSZ}} = \frac{8}{3}, \tag{6}$$

and compare the resulting photon couplings.

(e) **Optional extension: include the strange quark.** In the three-flavor theory one must also include mixing with the η. Show that the two-flavor result above is promoted to

$$C_{a\gamma}^{(N_f=3)} = \left[\frac{E}{N} - \frac{2}{3}\frac{4+z+w}{1+z+w}\right], \qquad w \equiv \frac{m_u}{m_s}\,. \tag{7}$$

Using $z \simeq 0.56$ and $w \simeq 0.03$, check that the leading-order coefficient moves from the two-flavor value $\simeq 1.95$ to the more familiar three-flavor number $\simeq 1.92$.