Beyond the Standard Model (Azatov / Opferkuch)

due by 31.03.2026

Exercise 1. *Exact Tunnelling Solutions: The Triangular Barrier*

In the lectures we derived the bounce action for the scale-invariant quartic potential (the Fubini instanton). In this exercise you will compute the bounce action *exactly* for a triangular potential. This provides a useful playground in which both the thin-wall and thick-wall limits can be studied analytically.

Consider a real scalar field ϕ in four Euclidean dimensions with $O(4)$-symmetric bounce configurations $\phi = \phi(r)$, where $r = \sqrt{x_\mu x^\mu}$. The Euclidean action reduces to

$$S_E[\phi] = 2\pi^2 \int_0^\infty \mathrm{d}r\, r^3 \left[\tfrac{1}{2}\dot{\phi}^2 + V(\phi)\right] , \tag{1}$$

with $\dot{\phi} \equiv \mathrm{d}\phi/\mathrm{d}r$. The bounce equation of motion is

$$\ddot{\phi} + \frac{3}{r}\,\dot{\phi} = V'(\phi)\,, \tag{2}$$

subject to the boundary conditions

$$\lim_{r\to\infty} \phi(r) = \phi_+\,, \qquad \dot{\phi}(0) = 0\,. \tag{3}$$

The tunnelling coefficient B appearing in the decay rate $\Gamma/\mathcal{V} \propto e^{-B}$ is

$$B = S_E[\phi_{\text{bounce}}] - S_E[\phi_+]\,. \tag{4}$$

The **triangular potential** is a piecewise-linear function characterised by four parameters: a false vacuum at (ϕ_+, V_+), a true vacuum at (ϕ_-, V_-), and a barrier top at (ϕ_T, V_T), with

$$\phi_+ < \phi_T < \phi_-\,, \qquad V_- < V_+ < V_T\,. \tag{5}$$

The potential has constant slope on either side of the barrier:

$$V'(\phi) = \begin{cases} +\lambda_+ & \text{for } \phi_+ < \phi < \phi_T\,, \\ -\lambda_- & \text{for } \phi_T < \phi < \phi_-\,, \end{cases} \tag{6}$$

where the (positive) magnitudes of the gradients are

$$\lambda_+ = \frac{\Delta V_+}{\Delta \phi_+}\,, \qquad \lambda_- = \frac{\Delta V_-}{\Delta \phi_-}\,, \tag{7}$$

and we define the relative field separations and potential differences

$$\Delta\phi_+ \equiv \phi_T - \phi_+\,, \qquad \Delta\phi_- \equiv \phi_- - \phi_T\,, \qquad \Delta V_\pm \equiv V_T - V_\pm\,. \tag{8}$$

Note $\Delta\phi_\pm > 0$ and $\Delta V_\pm > 0$. We also define

$$c \equiv \frac{\lambda_-}{\lambda_+}\,. \tag{9}$$

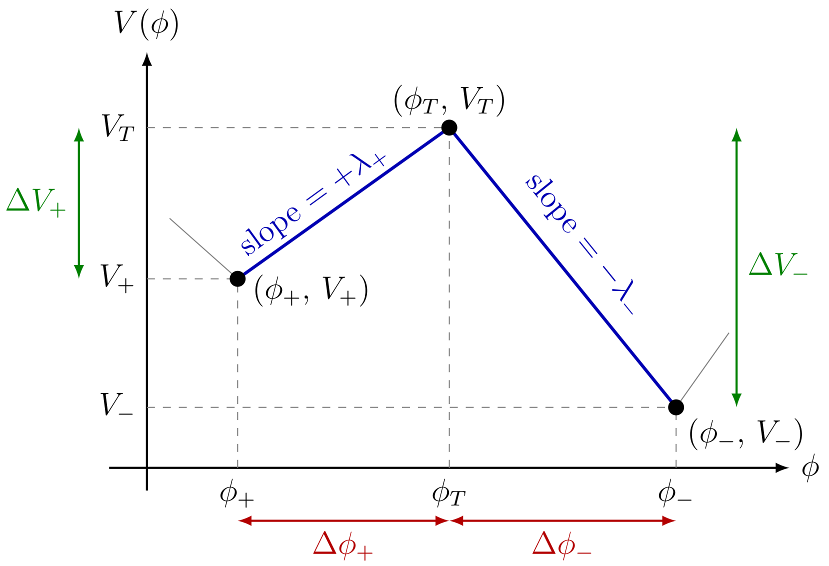

Figure 1: The triangular potential. The field starts at $\phi(0) = \phi_0$ and rolls down to ϕ_+ as $r \to \infty$.

(a) **Thin-wall warm-up.** First consider the nearly degenerate limit

$$\epsilon \equiv V_+ - V_- \ll \Delta V_+ \,. \tag{10}$$

Let $V_0(\phi)$ denote the corresponding degenerate triangular potential obtained by sending $\epsilon \to 0$ while keeping $\phi_\pm$ and ϕ_T fixed. Compute the surface tension

$$S_1 = \int_{\phi_+}^{\phi_-} \mathrm{d}\phi \, \{2[V_0(\phi) - V_+]\}^{1/2} \,, \tag{11}$$

and use Coleman's thin-wall formula

$$B_{\rm tw} = \frac{27\pi^2}{2} \frac{S_1^4}{\epsilon^3} \tag{12}$$

to obtain the tunnelling exponent in this limit.

(b) **Exact solution on linear segments.** Since $V'(\phi)$ is piecewise constant, the bounce in Eq. (2) can be solved analytically on each segment. Show that the general solution to

$$\ddot{\phi} + \frac{3}{r}\dot{\phi} = \lambda \tag{13}$$

is

$$\phi(r) = A + \frac{B}{r^2} + \frac{\lambda}{8} r^2 \,. \tag{14}$$

Explain why $B = 0$ in the innermost region. Then consider a bounce that reaches the false vacuum at a finite radius R_+, so that $\phi(R_+) = \phi_+$ and $\dot{\phi}(R_+) =$

0. Show that on the outer segment $R_T < r < R_+$ one may write

$$\phi(r) = \phi_+ + \frac{\lambda_+}{8r^2}(r^2 - R_+^2)^2 . \tag{15}$$

For a solution that begins rolling immediately, with no plateau at ϕ_-, match this to the regular inner solution at $r = R_T$ and show that

$$R_+^4 = (1 + c)\, R_T^4 , \tag{16}$$

where $\phi(R_T) = \phi_T$.

(c) **Immediate-roll exact bounce.** Assume there is no central plateau, so the field starts rolling immediately from some value ϕ_0 with $\phi_T < \phi_0 \le \phi_-$. Using the piecewise solution from part (b), compute the bounce action and show that

$$\boxed{B = \frac{32\pi^2}{3} \frac{1 + c}{(\sqrt{1+c} - 1)^4} \left(\frac{\Delta\phi_+^4}{\Delta V_+} \right) .} \tag{17}$$

Determine the condition on the potential parameters for which this immediate-roll solution exists.

(d) **Delayed roll and the thick-wall limit.** If the condition from part (c) fails, the field sits at the true vacuum for $0 \le r < R_0$ and only begins to roll at $r = R_0$. Show that matching at $r = R_T$ now gives

$$R_+^4 - R_T^4 = c\,(R_T^4 - R_0^4) . \tag{18}$$

Use this setup to study the no-barrier limit on the false-vacuum side, $\Delta V_+ \to 0$ (and therefore $\lambda_+ \to 0$), and show that the exact result reduces to

$$B_{\rm nb} = \frac{2\pi^2}{3} \frac{(\Delta\phi_+ + \Delta\phi_-)^3}{\lambda_-} \left(3\frac{\Delta\phi_+}{\Delta\phi_-} - 1 \right) . \tag{19}$$

Discuss also the limiting case $\Delta\phi_- \to 0$ (a sheer drop-off immediately after the barrier) and show that

$$B \to 2\pi^2 \frac{(\Delta\phi_+)^4}{\Delta V_-} , \tag{20}$$

in this limit.

Exercise 2. *A Vector-Like Top Partner and Vacuum Stability*

Consider extending the SM by a vector-like quark $T = (T_L, T_R)$ with the same gauge quantum numbers as the right-handed top quark, namely $(\mathbf{3}, \mathbf{1}, 2/3)$ under $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$. Let $q_L = (t_L, b_L)^T$ be the third-generation quark doublet and $\tilde{H} \equiv i\sigma_2 H^*$. Consider the Lagrangian

$$\mathcal{L} \supset \bar{T}(i\not{D} - M)T - \kappa\, \bar{q}_L \tilde{H}\, T_R + \text{h.c.} \tag{21}$$

Assume $M \gg v$. Study the effect of T in three steps: tree-level matching below $\mu = M$, one-loop matching for the Higgs quartic at $\mu = M$, and the running of λ above the threshold.

(a) **Tree-level matching.** Integrate out T at tree level by solving its classical equations of motion to first non-vanishing order in $1/M$. What is the leading operator generated in the low-energy EFT? Up to integration by parts and the light-field equations of motion, identify the corresponding SMEFT operator class.

(b) **One-loop matching for the Higgs quartic.** Along the neutral Higgs direction $H^T = (0, h/\sqrt{2})$, determine the field-dependent heavy-fermion mass and use the Coleman–Weinberg potential for a Dirac fermion,

$$V_1^{(T)}(h) = -\frac{N_c}{16\pi^2}\, m_T^4(h) \left[\ln \frac{m_T^2(h)}{\mu^2} - \frac{3}{2}\right], \tag{22}$$

expand for $h \ll M$ up to order h^4, and read off the coefficient of $h^4/4$ as the one-loop threshold correction $\delta\lambda(\mu)$ to the Higgs quartic. Evaluate this correction at the matching scale $\mu = M$ in the $\overline{\text{MS}}$ scheme. Then explain why finding $\delta\lambda(M) = 0$ does *not* mean that the heavy fermion is irrelevant for vacuum stability.

(c) **Impact on vacuum stability.** For $\mu > M$, use the leading one-loop contribution of the new Yukawa coupling to the running of λ and neglect all other terms in β_λ. Take

$$|\kappa| = 1\,, \qquad M = 1\,\text{TeV}\,, \qquad \lambda(M) \simeq 0.12\,.$$

Estimate the scale μ_I at which $\lambda(\mu_I) = 0$. Compare your answer qualitatively to the pure SM and state whether the vector-like top partner makes the Higgs potential more stable or less stable.