An Introduction to AdS/CFT

SISSA – PhD course (2024)

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(please send an email to ldonnay@sissa.it if you find typos/errors/mismatches of notations)

Contents

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References and suggested readings

- ★ Zaffaroni, "Introduction to the AdS/CFT correspondence", http://laces.web.cern.ch/Laces/LACES09/notes/dbranes/lezioniLosanna.pdf

  Large N limit (vector and matrix theories). Conformal symmetry, symmetries of AdS, operator/state correspondence, $\mathrm{AdS}_5 \times S^5$ and $\mathcal{N} = 4$ SYM. Spectrum, correlation functions, diagrams. Thermal and confining theories, transition. Plane wave limit.

- ★ Aharony, Gubser, Maldacena, Ooguri, Oz, "Large N Field Theories, String Theory and Gravity", hep-th/9905111.

  Introduction. Large N. Conformal symmetry, primaries, OPE, superconformal algebra. Conformal structures, AdS. $\mathrm{AdS}_5 \times S^5$. AdS/CFT, operator/field correspondence, correlation functions. Phases. Wilson loops. Finite temperature. More general geometries, a, c anomalies. Non-conformal deformations.

- ★ Ammon, Erdmenger, "Gauge/Gravity Duality: Foundations and Applications", http://www.nucleares.unam.mx/ alberto/apuntes/erdmenger.pdf

- Maldacena, "TASI 2003 lectures on AdS / CFT", hep-th/0309246.

- Klebanov, "TASI lectures: Introduction to the AdS / CFT correspondence", hep-th/0009139.

  One- and two-point functions. Conifold, wrapped branes and chiral operators.

- Bertolini, "Four lectures on the gauge / gravity correspondence", arXiv:hep-th/0303160.

- Polchinski, "Introduction to Gauge/Gravity Duality", arXiv:1010.6134.

- Nastase, "Introduction to AdS-CFT", arXiv:0712.0689.

  A little bit of everything. Spin chains.

- Kaplan, "Lectures on AdS/CFT from the Bottom Up", http://www.pha.jhu.edu/ jaredk/AdSCFTCourseNotesPublic.pdf

- ★ Skenderis, "Lecture notes on holographic renormalization", hep-th/0209067.

  Asymptotically AdS, conformal structures. Correlators, holographic renormalization.

1 Introduction: What is AdS/CFT?

This course is about AdS/CFT: a surprising "correspondence" between (certain)

Theories of quantum gravity in $d+1$ dimensions	$\longleftrightarrow$	Non-gravitational QFTs on the d-dimensional boundary.

In its most basic but also precise examples, the correspondence is between quantum gravities on a specific background, *anti de-Sitter* (AdS) space, and conformal QFTs (conformal field theories or CFTs). Hence here by "quantum gravity" we mean any theory which can be approximated by General Relativity in AdS spacetimes coupled with matter fields (like scalars, fermions, gauge fields or strings). This correspondence allows to make exact statements about quantum gravity in AdS by studying CFTs.

First hints: There are various indications that there should be such a correspondence. Let us mention some of them.

- A first hint comes from the black hole thermodynamics: the black hole entropy is given by the Bekenstein-Hawking formula

  $$S_{\max} = \frac{Ac^3}{4\hbar G_N} \, ,$$

  saying that the entropy scales with the horizon area A, rather than the volume. The earliest form of the "holographic principle" can be stated as the existence of a bound on the number of degrees of freedom needed to describe physics in a spatial region.

  The Bekenstein bound[1] indeed states that the maximal entropy in a region of space is given by the above formula, as otherwise one could lower the entropy by forming a black hole and this would violate the second law of thermodynamics. This suggests that the physics can be described by a theory living on the boundary.

- The study of 'asymptotic symmetries' of GR by Brown and Henneaux (1986): they showed that the asymptotic symmetry group of AdS_3 spacetimes is given by (two commuting copies) of the infinite dimensional conformal group (Virasoro algebra with a central charge given by $c = \frac{3L}{2G}$, L being the AdS radius).

[1]The Bekenstein bound [Bek81] is $S \leq 2\pi r E$, where r is the size of the system and E its energy. For given radius, the maximal possible amount of energy that can fit into a sphere is the mass of a Schwarzschild black hole, and $r_s^{D-3} \sim G_N E$. For the maximal value of the energy, the bound is given by the area of the horizon.

- 't Hooft's realization that certain gauge theories based on the group $SU(N)$, in the so-called large N limit, have an alternative description as weakly interacting string theories.

- Another argument is that quantum gravity does not have local gauge-invariant operators (only approximate ones), thus the observables should be at the boundary of spacetime.

Implications and applications: This is a strong/weak duality, meaning that we can study one side by using the other side, and vice versa. AdS/CFT is considered the finest achievements of string theory in the last decade: one can use holography to investigate strongly-coupled quantum field theories (e.g. QCD, high temperature superconductors, ...). One crucial aspect of the correspondence is the possibility of computing quantum effects in a strongly-coupled field theory using a classical gravitational theory. This has deep consequences that go far beyond string theory. Another profound implications is that (large) black holes in AdS are just CFT states; the temperature of AdS black holes correspond to the temperature of the CFT and the entropy corresponds to the number of CFT states excited at that temperature.

Originally introduced to study the quantum behaviour of scale invariant theories, the correspondence has been extended to non-conformal theories, where it gives an explanation for confinement and chiral symmetry breaking. It has also been used to study non-equilibrium phenomena in strongly coupled plasmas, and applied to condensed matter systems.

About these lecture notes: In section 2, we will first review the salient features of conformal field theories. We will then move to describing the "bulk" spacetime, namely Anti-de Sitter spacetimes in section 3. The conceptual aspects of the holographic correspondence will be exposed in section 4, including the holographic renormalization procedure in section 5. After a brief introduction on large N limits in section 6, we will turn to an explicit and most famous realization of the holographic correspondence in section 7 and present three tests (matching of spectrum, low-point correlators and conformal anamoly) in 8. We will then discuss Wilson loops in section 9. Section 11 will bring us away from the conformal case and finally we will discuss applications of gauge/gravity duality to hydrodynamics in section 12.

2 Conformal symmetry

Conformal Field Theories (CFTs) appear in many branches of theoretical physics and play a key role in all problems involving renormalization group (RG) flows, such as the theory of critical phenomena. A general Quantum Field Theory (QFT) can be thought of as an RG flow starting from a CFT in the UV and flowing to another CFT in the IR.

Let us review some basic facts about CFTs. More details can be found in Section 2 of [AGM+00], [Gin88], [Ryc16] (see also Simons-Duffin and Alday's lectures notes online) and Section 2 of [Min98].

2.1 Conformal algebra in d dimensions

Consider the flat (pseudo-Euclidean) space $\mathsf{R}^{p,q}$ (with $p+q=d$) and metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \eta_{\mu\nu}dx^\mu dx^\nu$, where $\eta_{\mu\nu}$ is the flat metric of signature (p,q) $(\mu,\nu = 0,\dots d-1)$. Under a change of coordinates $x \to x'$, we have $g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x)$.

Definition: The (global) *conformal group* $\mathrm{Conf}(p,q)$ is the subgroup of coordinate transformation that leaves the metric invariant up to a scale factor:

$$\boxed{g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x)}\,, \tag{2.1}$$

where $\Omega(x)$ is an arbitrary function of the coordinates called the scale factor. Conformal transformations are coordinate transformations that rescale lengths but preserve the angles between vectors. The conformal group includes as a subgroup the Poincaré group (since it leaves the metric invariant $g'_{\mu\nu}(x') = g_{\mu\nu}(x)$, i.e. $\Omega = 1$) and scale transformations, which have constant $\Omega = \lambda$. It is widely believed[2] (see [DKST15,DFK+16]) that unitary interacting scale-invariant theories are also invariant under the full conformal group.

Under an infinitesimal coordinate transformation $x^\mu \to x^\mu + \epsilon^\mu(x) + \mathcal{O}(\epsilon^2)$, the metric changes by

$$ds^2 \to ds^2 + (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) dx^\mu dx^\nu\ . \tag{2.2}$$

If ϵ^μ generates an isometry, then the metric is left invariant ($\delta g_{\mu\nu} = 0$) and ϵ^μ is called a Killing vector.

→ *Exercise 1:* Show that, in order to be an infinitesimal conformal transformation, ϵ^μ must satisfy

$$(\eta_{\mu\nu}\Box + (d-2)\partial_\mu\partial_\nu)\partial\cdot\epsilon = 0. \tag{2.3}$$

[2] An interesting counterexample is given by free Maxwell theory in $d \neq 4$ [ESNR11].

We can already see at this stage that something very special happens when $d = 2$, we will not focus much on this case in these lectures, see 2.6 for some comments. Looking for solutions to eq. (2.3), one finds that the conformal algebra of Minkowski space in $d > 2$ dimensions is given by the following infinitesimal transformations and their generators:

$$\begin{array}{rc|c|c} \epsilon^\mu(x) = & a^\mu & P_\mu = i\partial_\mu & d\,\text{translations} \\ & \omega^{\mu\nu}x_\nu \ (\omega^{\mu\nu} = -\omega^{\nu\mu}) & M_{\mu\nu} = \frac{i}{2}(x_\mu\partial_\nu - x_\nu\partial_\mu) & \frac{d(d-1)}{2}\,\text{Lorentz rotations} \\ & \lambda\, x^\mu & D = ix^\mu\partial_\mu & 1\,\text{dilation} \\ & b^\mu x^2 - 2x^\mu b\cdot x & K_\mu = i(x^2\partial_\mu - 2x_\mu x\cdot\partial) & d\,\text{special conformal transformations} \end{array} \tag{2.4}$$

Altogether there are $\frac{(d+1)(d+2)}{2}$ generators.

The finite form of special conformal transformations is

$$x^\mu \to \frac{x^\mu + b^\mu x^2}{1 + 2x\cdot b + b^2x^2}\ . \tag{2.5}$$

NB: In the conformal group there is also a discrete element, the inversion:[3]

$$x^\mu \to \frac{x^\mu}{x^2}\ . \tag{2.6}$$

Adding this discrete transformation, one obtains the full conformal group $O(2,d)$. Notice thus that the inversion belongs to $O(2,d)$ but not to the connected component of the conformal group, $SO(2,d)$ (namely an inversion cannot be written as $x'^\mu = x^\mu + \epsilon^\mu(x)$)[4].

The change in the action due to a change in the metric is

$$\delta S = \int d^dx\,\sqrt{g}\,T^{\mu\nu}\delta g_{\mu\nu}\ . \tag{2.7}$$

The generators are constructed (by the standard Noether procedure) from currents, and all currents are constructed with the stress tensor:

$$J_\mu^{\text{conf}} = T_{\mu\nu}\,\delta x^\nu\ , \qquad\qquad Q^{\text{conf}} = \int d^{d-1}x\,J_0^{\text{conf}}\ . \tag{2.8}$$

In particular P_μ is conserved because $\partial^\mu T_{\mu\nu} = 0$, $M_{\mu\nu}$ because additionally $T_{[\mu\nu]} = 0$, D because $T^\mu_\mu = 0$, and then also conservation of K_μ follows.

[3]The composition of inversion, translations, inversion gives special conformal transformations. Then the commutator of K_μ and P_μ gives D, after removing $M_{\mu\nu}$. Thus the conformal group can be generated by adding the inversion to the Poincaré group.

[4]It is thus possible to have CFTs which are not invariant under inversion (e.g. if they break parity).

The conformal generators satisfy the algebra

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= -2i\,\eta_{\rho[\mu}M_{\nu]\sigma} - 2i\,\eta_{\sigma[\mu}M_{\rho|\nu]} & [D, P_\mu] &= -iP_\mu \\
[M_{\mu\nu}, P_\rho] &= -2i\,\eta_{\rho[\mu}P_{\nu]} & [D, K_\mu] &= iK_\mu \\
[M_{\mu\nu}, K_\rho] &= -2i\,\eta_{\rho[\mu}K_{\nu]} & [P_\mu, K_\nu] &= 2iM_{\mu\nu} - 2i\eta_{\mu\nu}D
\end{aligned} \tag{2.9}$$

and the other ones, $[M, D]$, $[D, D]$, $[P, P]$, $[K, K]$, vanishing. One can check that the Lie algebra of the conformal group is isomorphic to the Lie algebra of the Lorentz group with one more space and one more time dimension, i.e. $\mathfrak{so}(p+1, q+1)$.

$\triangleright$ In Lorentzian signature $\mathsf{R}^{d-1,1}$, the conformal algebra is $\mathfrak{so}(d, 2)$[5]. To see this, one can define the generators J_{mn} $(m, n = 0, \ldots, d, d+1)$ by

$$J_{\mu\nu} = M_{\mu\nu}\ , \qquad J_{\mu d} = \frac{K_\mu - P_\mu}{2}\ , \qquad J_{\mu(d+1)} = \frac{K_\mu + P_\mu}{2}\ , \qquad J_{d(d+1)} = D\ , \tag{2.10}$$

and check that they satisfy the following algebra:

$$[J_{mn}, J_{rs}] = -2i\,\tilde{\eta}_{r[m}J_{n]s} - 2i\,\tilde{\eta}_{s[m}J_{rn]} \tag{2.11}$$

with $\tilde{\eta}$ =diag$(-1, 1, \ldots, 1, 1, -1)$. Similarly, in the Lorentzian case if we decompose

$$SO(d, 2) \to SO(d-1, 1) \times SO(1, 1) \tag{2.12}$$

then $M_{\mu\nu}$ generates $SO(d-1, 1)$ (Lorentz group in d dimensions) and D generates $SO(1, 1) \cong \mathsf{R}$.

$\triangleright$ For Euclidean space $\mathsf{R}^{d,0}$, one can similarly identify the conformal algebra with $\mathfrak{so}(d+1, 1)$.

Conformal compactifications: A special conformal transformation with parameter b^μ maps the point $x^\mu = -b^\mu/b^2$ to infinity (alternatively, the inversion maps points such that $x^2 = 0$ to infinity so it is also not globally well-defined). Therefore, in order to define the finite conformal transformations globally, one considers the so-called conformal compactifications of $\mathsf{R}^{d-1,1}$ or R^d.

For the Lorentzian case, the conformal compactification yields $S^{d-1} \times \mathsf{R}$ (one has to add all points that satisfy $x^2 = 0$, in this case the entire light-cone of the origin). Indeed the maximal compact subgroup of $SO(2, d)$ is $SO(d) \times SO(2)$, and a covering of it (or its algebra)[6] acts in the obvious way on $S^{d-1} \times \mathsf{R}$. The vacuum of a CFT is invariant under all generators.

[5] Thus the conformal group is some covering of $SO(2, d)$. Adding the inversion, one gets some covering of $O(2, d)$.

[6] The relevant covering of $SO(2)$ is R, with the same algebra, and it acts on R by translations.

It is often useful to study the CFT in Euclidean signature, and as we saw above the Euclidean conformal group is $SO(d+1,1)$. In this case the compactification of R^d amounts to adding one point called infinity (this is enough since $x^2 = \delta_{\mu\nu}x^\mu x^\nu = 0$ is satisfied only for $x = 0$) and we get S^d, acted upon by the maximal compact subgroup $SO(d+1)$.

Weyl transformations: If the trace of the stress tensor is zero $T^\mu_\mu = 0$, then the theory is also invariant under *Weyl transformations*

$$g_{\mu\nu} \to e^{\phi(x)} g_{\mu\nu} \tag{2.13}$$

for arbitrary scalar ϕ. These are not only coordinate transformations, but rather local rescalings of the metric (the curvature invariants change, but the Weyl tensor is left invariant), producing a new metric within the same conformal class. In the quantum theory this symmetry has a (calculable) anomaly.

2.2 Representations, primary fields

Fields in a CFT transform in irreducible representations of the conformal algebra. To construct representations of the conformal group, we can use the method of induced representations: First, we look at transformation properties of the fields at the origin $x = 0$, then we use P_μ to obtain the field to an arbitrary point x. The subgroup that leaves the origin invariant (stabilizer group) is spanned by Lorentz transformations, the dilation operator and the SCT operator. We define a *primary field* as a field whose actions of these operators at the origin is given by

$$\begin{aligned} &[M_{\mu\nu}, \Phi(0)] = \Sigma_{\mu\nu}\Phi(0)\,, \\ &[D, \Phi(0)] = -i\Delta\Phi(0) \\ &[K_\mu, \Phi(0)] = 0\,. \end{aligned} \tag{2.14}$$

$\Sigma_{\mu\nu}$ are the matrices of a finite-dimensional representation of the Lorentz group, determining the spin of the field $\Phi(0)$. The second equation implies that, under dilations,

$$x^\mu \to \lambda\, x^\mu\ . \tag{2.15}$$

the field transforms as

$$\phi(x) \to \lambda^\Delta\, \phi(\lambda x)\ . \tag{2.16}$$

Δ is called the scaling dimension of the (quasi-)primary[7] field. The operators P_μ (*i.e.* derivatives) raise the eigenvalue of D, while K_μ lower it. Hence P and K act like ladder operators

[7] The distinction between primary and quasi-primary fields is only relevant in $d = 2$; see section 2.6.

for dilation eigenvalues. In unitary field theories there is a lower bound on the dimension of a field, depending on its Lorentz representation, called *unitarity bound* (see section 2.5). This implies that any conformal unitary representation must contain operators of lowest dimension (these representations are the "lowest weight representations"). Primary operators are those operators with the lowest dimension, since by definition they are annihilated by K_μ at the origin $x = 0$.

The representation is infinite-dimensional, and all other fields are constructed with the action of P_μ on primary fields: they are called *descendants*. We thus have two types of local operators: primary operators (which cannot be written as derivatives of other local operators), and descendants (which can be written as linear combinations of derivatives of other local operators).

This was for the transformation properties of Φ at the origin (representation of the little group). From there one can obtain the commutation relations for a conformal primary $\Phi(x)$ of scaling dimension Δ at any x by using $\Phi(x) = e^{-iPx}\Phi(0)e^{iPx}$ and the conformal algebra,

$$\begin{aligned}
[P_\mu, \Phi(x)] &= i\partial_\mu \Phi(x) \\
[M_{\mu\nu}, \Phi(x)] &= \left[i(x_\mu\partial_\nu - x_\nu\partial_\mu) + \Sigma_{\mu\nu}\right]\Phi(x) \\
[D, \Phi(x)] &= i\left[x^\mu\partial_\mu - \Delta\right]\Phi(x) \\
[K_\mu, \Phi(x)] &= \left[i(x^2\partial_\mu - 2x_\mu x^\nu\partial_\nu + 2x_\mu\Delta) - 2x^\nu\Sigma_{\mu\nu}\right]\Phi(x)\ .
\end{aligned} \tag{2.17}$$

→ *Exercise 2* Derive $[D, \Phi(x)]$.

2.3 OPE and correlation functions

Conformal symmetry strongly constrains the form of quantum field theory correlation functions. One-point functions are zero (except for the identity operator). The 2-point functions are completely fixed, up to a rescaling (redefinition of the operators). For primary scalar ($s = 0$) operators $\mathcal{O}_1, \mathcal{O}_2$ of scaling dimension Δ_1, Δ_2, the two-point function is zero, unless $\Delta_1 = \Delta_2 \equiv \Delta$. In that case it is given by

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{1}{|x_1 - x_2|^{2\Delta}}, \tag{2.18}$$

where the operators have been normalized and diagonalized. Similar expressions hold for higher spin primaries. The 2-point functions of descendants are obtained by taking derivatives. In particular for two fields in the same "conformal family":

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{c_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}\ , \tag{2.19}$$

with $\Delta_1 - \Delta_2 \in \mathsf{Z}$.

The 3-point function of primary operators $\mathcal{O}_i$ ($i = 1, 2, 3$) depends only on a single "structure constant" C_{123} (determined by the field content). For scalar primaries normalized to have canonical 2-point function, it is given by

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle = \frac{C_{123}}{|x_1 - x_2|^{\Delta_1+\Delta_2-\Delta_3} |x_2 - x_3|^{\Delta_2+\Delta_3-\Delta_1} |x_3 - x_1|^{\Delta_3+\Delta_1-\Delta_2}} \,, \tag{2.20}$$

and similarly for higher spin. The 3-point functions involving descendants are again obtained by taking derivatives.

Higher point functions are functions of the conformal invariants constructed out of the x_i, not determined by the conformal symmetry alone. For instance, the 4-point function of a scalar operator of dimension Δ take the more general form (setting $x_{ij} \equiv |x_i - x_j|$)

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \mathcal{O}(x_4) \rangle = \frac{F(\eta, \xi)}{x_{12}^{2\Delta} x_{34}^{2\Delta}} \,, \tag{2.21}$$

with $F(\eta, \xi)$ a function of the two "conformal cross ratios"

$$\eta = \frac{x_{12} x_{34}}{x_{13} x_{24}} \,, \qquad \xi = \frac{x_{14} x_{23}}{x_{13} x_{24}} \,. \tag{2.22}$$

It turns out that the correlation functions of local operators in a CFT are all completely fixed, once we know the spectrum of primaries and their 3-point functions. This is for two reasons.

- The 3-point functions involving descendants are fixed by the 3-point functions of primaries: just act with derivatives.
- Correlators can be decomposed using the OPE.

The **operator product expansion** (OPE) is a general property of local QFTs, but it is particularly powerful in CFTs. It claims that the product of two operators at nearby points can be rewritten as a series of operators at one point only:

$$\mathcal{O}_i(x)\, \mathcal{O}_j(y) = \sum_k c_{ij}^k(x - y)\, \mathcal{O}_k(y) \tag{2.23}$$

and conformal symmetry fixes

$$c_{ij}^k(x - y) = \frac{c_{ij}^k}{|x - y|^{\Delta_i+\Delta_j-\Delta_k}} \,. \tag{2.24}$$

Notice that the sum includes descendants and higher-spin operators (for which index contractions are implicit)! This is an operator equation, valid inside any correlation function. (In CFTs, the series that one obtains for correlation functions are convergent until the circle around y hits another operator.) The OPE has a finite radius of convergence inside correlation functions; this follows from the state-operator map (see below) with an appropriate choice of origin for radial quantization.

The coefficients of the OPE algebra are fixed by the 3-point functions of primaries. Indeed, on the one hand, we have

$$\begin{aligned} \langle \Phi_i(x)\Phi_j(y)\Phi_k(z)\rangle &= \langle \Phi_i(x) \sum_\ell \frac{c_{jk}^\ell}{|y-z|^{\Delta_j+\Delta_k-\Delta_\ell}} \phi_\ell(z)\rangle \\ &= \sum_\ell \frac{c_{jk}^\ell}{|y-z|^{\Delta_j+\Delta_k-\Delta_\ell}} \frac{c_{i\ell}}{|x-z|^{\Delta_i+\Delta_\ell}} \end{aligned} \tag{2.25}$$

The only operators with non-vanishing 2-point function with Φ_i are Φ_i and its descendants, *i.e.* ϕ_ℓ must be in the conformal family of Φ_i. Then $\Delta_\ell = \Delta_i + \text{integer}$ and $c_{ii} = 1$ for the primary. Then, for $y \to z$:

$$\langle \Phi_i(x)\Phi_j(y)\Phi_k(z)\rangle = \frac{c_{jk}^i}{|x-z|^{2\Delta_i}|y-z|^{\Delta_j+\Delta_k-\Delta_i}} + O\Big(\frac{|y-z|}{|x-y|}\Big) . \tag{2.26}$$

On the other hand we can expand the 3-point function of primaries for $y \to z$:

$$\frac{C_{ijk}}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}|x-z|^{\Delta_i+\Delta_k-\Delta_j}|y-z|^{\Delta_j+\Delta_k-\Delta_i}} = \frac{C_{ijk}}{|x-z|^{2\Delta_i}|y-z|^{\Delta_j+\Delta_k-\Delta_i}} + O\Big(\frac{|y-z|}{|x-y|}\Big) . \tag{2.27}$$

It follows that the coefficients for primaries in the OPE are the same (with normalized operators) as ones of the 3-point functions: $c_{ijk} = C_{ijk}$. All other coefficients can be fixed as well.

2.4 Radial quantization and state-operator correspondence

In radial quantization, one foliates the d-dimensional spacetime in terms of S^{d-1} spheres of various radii centered at the origin. The R^d metric in spherical coordinates is

$$ds^2 = dr^2 + r^2 d\Omega_{d-1}^2 . \tag{2.28}$$

States living on these spheres are not classified by their 4-momenta, as with the Poincaré group, but instead by their scaling dimension Δ ($D|\Delta\rangle = i\Delta|\Delta\rangle$). To express the evolution operator, we define $\tau = \log r$ $(r = e^\tau)$ and, dividing by r^2, we get a map to the cylinder:

$$ds^2_{plane} = dr^2 + r^2 d\Omega^2_{d-1} \qquad \mathsf{R}^d \qquad \overset{\text{Weyl}}{\rightarrow} \qquad S^{d-1} \times \mathsf{R} \qquad ds^2_{cyl} = d\tau^2 + d\Omega^2_{d-1}\,. \tag{2.29}$$

This coordinate τ on the cylinder is the coordinate for time evolution and the evolution operator is $U = e^{iD\tau}$. It acts on a eigenstate $|\Delta\rangle$ as

$$U|\Delta\rangle = e^{-\Delta\tau}|\Delta\rangle = r^{-\Delta}|\Delta\rangle. \tag{2.30}$$

A rescaling of r corresponds to a shift in τ: moving towards the origin $r \to 0$ (infinity $r \to \infty$) in radial quantization corresponds to approaching the past (future) infinity $\tau \to -\infty$ $(\tau \to +\infty)$. Since the Hamiltonian generates time translations, the dimension of the operator in R^d equals the energy of the state on S^{d-1}:

$$\Delta = E_{\text{cylinder}}\ . \tag{2.31}$$

The **state-operator correspondence** in CFT states that all states in the theory can be created by operators which act locally in a small neighborhood of the origin:

$$\text{states on } S^{d-1} \ = \ \text{local operators on } \mathsf{R}^d\ . \tag{2.32}$$

With no insertions, the state created on S^{d-1} should be the conformal vacuum state $|0\rangle$ (i.e. unique ground state invariant under all global conformal transformations). Given an operator $\mathcal{O}$, the corresponding state $|\mathcal{O}\rangle$ is the one created on S^{d-1} around $\mathcal{O}(0)$:

$$|\mathcal{O}\rangle = \mathcal{O}(0)|0\rangle\ . \tag{2.33}$$

When inserting a primary operator at the origin, we get a state with scaling dimension Δ that is annihilated by K. The other way around: given some state with scaling dimension Δ that is annihilated by K, we can construct an associated local primary operator.

As a functional $\Psi\big(\phi(\omega)\big)$ of field configurations on S^{d-1}, it is given by the path-integral on the ball with $\mathcal{O}$ at the origin and boundary conditions $\phi(\omega)$ on the boundary:

$$\Psi\big(\phi(\omega)\big) = \int_{\Phi=\phi \text{ on } S^{d-1}} \mathcal{D}\Phi\, \mathcal{O}(0)\, e^{-S[\Phi]}\ . \tag{2.34}$$

On the contrary, if we think of a state as a functional of field configurations, as we shrink the ball to zero size using conformal invariance we obtain a local operator. In fact, we have defined a local operator if we know how to compute its correlation functions. To compute the path-integral with insertions of $\mathcal{O}$, we cut small balls around the insertion points and we define their contribution to the path-integral as $\Psi\big(\phi(\omega)\big)$, where $\phi(\omega)$ is the field configuration on the boundary of the ball.

2.5 Unitarity bounds

In this section, we will see that unitarity provide constraints on the scaling dimension Δ of CFT operators [8]. Theories that violate the unitarity bounds will be non-unitary in Lorentzian signature, and will violate reflection positivity in Euclidean signature.

The conformal group in Lorentzian signature is $SO(d,2)$, and in a unitary theory the generators are Hermitian:

$$M_{\mu\nu}^\dagger = M_{\mu\nu} \ , \qquad P_\mu^\dagger = P_\mu \ , \qquad K_\mu^\dagger = K_\mu \ , \qquad D^\dagger = D \ . \tag{2.35}$$

The representation in terms of fields was given before. We are looking for unitary representations in Lorentzian signature, however it is useful to work in radial quantization, therefore we rotate to Euclidean signature and the conformal algebra is $\mathfrak{so}(d+1,1)$. The generators J_{ab} were defined in (2.10). To obtain $\mathfrak{so}(d+1,1)$, set

$$M'_{\mu\nu} = J_{\mu\nu} \ , \quad D' = iJ_{-1,0} \ , \quad P'_\mu = J_{\mu,-1} + iJ_{\mu,0} \ , \quad K'_\mu = J_{\mu,-1} - iJ_{\mu,0} \ . \tag{2.36}$$

These generators satisfy

$$M'^\dagger_{\mu\nu} = M'_{\mu\nu} \ , \qquad P'^\dagger_\mu = K'_\mu \ , \qquad K'^\dagger_\mu = P'_\mu \ , \qquad D'^\dagger = -D' \ . \tag{2.37}$$

These relations can be understood as follows. In radial quantization, the spacelike foliation is in terms of S^{d-1}, and it is preserved by $M'_{\mu\nu}$ which is then Hermitian. It is not preserved by P'_μ, which is not Hermitian. In Euclidean time D' is anti-Hermitian. Thus, we construct representations of $\mathfrak{so}(d+1,1)$ such that they are unitary in Lorentzian signature.

Consider the state $|\Phi\rangle$ corresponding to $\Phi(0)$; unitarity implies that the norm of the state must be non-negative:

$$\left|P'_\mu|\Phi\rangle\right|^2 \geq 0 \qquad \text{and} \qquad = 0 \quad \text{for} \quad P'_\mu|\Phi\rangle = 0 \ . \tag{2.38}$$

This implies

$$\sum_\mu \langle\Phi|K'_\mu P'^\mu|\Phi\rangle = 2\langle\Phi|\Delta - i\underbrace{M'^{\,\mu}_{\mu}}_{=0}|\Phi\rangle \geq 0 \ . \tag{2.39}$$

Hence unitarity implies that $\Delta \geq 0$, and $\Delta = 0$ if and only if $D_\mu\Phi = 0$, *i.e.*, if Φ is the identity operator.

For a scalar operator, it is useful to go one further level up:

$$\left|P'_\mu P'^\mu|\Phi\rangle\right|^2 \geq 0 \qquad \Rightarrow \qquad \Delta\Big(\Delta - \frac{d-2}{2}\Big) \geq 0 \ . \tag{2.40}$$

[8] See [Min98] for further details related to this section.

→ *Exercise 3:* Derive the bound (2.40).

We have derived a necessary condition for unitarity, it is in fact a sufficient condition for a scalar operator (namely one can shown that going to higher levels does not bring any new constraint).

Any Poincaré-invariant local quantum field theory has a symmetric conserved stress tensor $T_{\mu\nu}$.[9] In a CFT, the dimension of $T_{\mu\nu}$ is fixed to be $\Delta = d$. Similarly, whenever there are continuous global symmetries, there are conserved currents J_μ with dimension $\Delta = d - 1$. The scaling dimensions of the other operators are not fixed by conformal symmetry, and receive quantum corrections.

We have the following unitarity bounds:

- For scalar fields:
$$\boxed{\Delta \geq \frac{d-2}{2}} \tag{2.41}$$
and there is equality if and only if the field obeys free field equations ($\partial^2\Phi = 0$).

- For vector operators $\mathcal{O}_\mu$: $\Delta \geq d - 1$ and there is equality if and only if $\partial^\mu \mathcal{O}_\mu = 0$. Similarly, for spin-2 symmetric operators $\mathcal{O}_{\mu\nu}$: $\Delta \geq d$ and there is equality if and only if $\partial^\mu \mathcal{O}_{\mu\nu} = 0$. In particular, conserved currents have canonical dimension and are not renormalized. More generally, for spin $s \geq 1$ operators, the bound is
$$\boxed{\Delta \geq d + s - 2} \tag{2.42}$$

- For spinors,
$$\boxed{\Delta \geq \frac{d-1}{2}}\,, \tag{2.43}$$
and $\Delta = \frac{d-1}{2}$ if and only if $\partial\!\!\!/\Psi = 0$ (free fermion field).

2.6 Comments on $2d$ CFTs

Conformal algebra in $d = 2$

As one could notice from (2.3), the case $d = 2$ stands alone: infinitesimal conformal transformations satisfy
$$\partial_1 \epsilon_1 = \partial_2 \epsilon_2 \,, \qquad \partial_1 \epsilon_2 = -\partial_2 \epsilon_1 \,, \tag{2.44}$$

[9]Invariance under translations and locality give, by Noether theorem, a conserved local stress tensor. Then invariance under Lorentz rotations assures that it can be improved to a symmetric tensor.

which are the Cauchy-Riemann differential equations of complex analysis in Euclidean space-time. It is natural to introduce the holomorphic and anti-holomorphic functions $\epsilon(z) = \epsilon^1 + i\epsilon^2$ $\bar{\epsilon}(\bar{z}) = \epsilon^1 - i\epsilon^2$, using complex coordinates $z = x^1 + ix^2$, $\bar{z} = x^1 - ix^2$. One can expand

$$\epsilon(z) = -\sum_{n\in\mathsf{Z}} \epsilon_n z^{n+1} , \qquad \bar{\epsilon}(\bar{z}) = -\sum_{n\in\mathsf{Z}} \bar{\epsilon}_n \bar{z}^{n+1}; \tag{2.45}$$

the infinitesimal conformal transformations are given by $z \to z' = z+\epsilon(z)$, $\bar{z} \to \bar{z}' = \bar{z}+\bar{\epsilon}(\bar{z})$. The corresponding generators are

$$\ell_n = -z^{n+1}\partial_z , \qquad \bar{\ell}_n = -\bar{z}^{n+1}\partial_{\bar{z}}, \tag{2.46}$$

which satisfy the commutation relations (Witt algebra)

$$[\ell_n, \ell_m] = (m-n)\ell_{m+n} , \qquad [\bar{\ell}_n, \bar{\ell}_m] = (m-n)\bar{\ell}_{m+n} , \qquad [\ell_n, \bar{\ell}_m] = 0 \, . \tag{2.47}$$

The key features about the $2d$ conformal algebra is thus the fact that it is *infinite-dimensional*. The subset generated by $\ell_0, \ell_1, \ell_{-1}$ and their complex conjugate generates the finite-dimensional subalgebra $\mathfrak{sl}(2,\mathsf{R}) \oplus \mathfrak{sl}(2,\mathsf{R})$ (called the *global* subalgebra, because the generators are globally well-defined and invertible on the Riemann sphere). The transformations for all other $n \in \mathsf{Z}$ are referred to as *local* conformal transformations, which have no analogs in $d > 2$. The finite form of the global transformations gives

$$z \to \frac{az+b}{cz+d} , \qquad \bar{z} \to \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}} \tag{2.48}$$

where $a, b, c, d \in \mathsf{C}$ and $ad - bc = 1$. The global group (or group of projective conformal transformations) is thus $SL(2,\mathsf{C})/\mathsf{Z}_2 \sim SO(3,1)$.

Conformal primary in $d = 2$ and Virasoro algebra

A field is called a primary field of weights $(h, \bar{h})$ if it transforms under a two-dimensional conformal transformation $z \to f(z)$, $\bar{z} \to \bar{f}(\bar{z})$ as

$$\Phi(z,\bar{z}) \to \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \Phi(f(z), f(\bar{z})). \tag{2.49}$$

All primaries are quasi-primaries, but the converse is not always true. Notice that the conformal dimension is related to the weights as $\Delta = h + \bar{h}$. Conservation and tracelessness of the $2d$ stress tensor ($T_{z\bar{z}} = T_{\bar{z}z} = 0$) imply that its only two non-vanishing components are holomorphic/anti-holomorphic:

$$T_{zz}(z) \equiv T(z) , \qquad T_{\bar{z}\bar{z}}(\bar{z}) \equiv \bar{T}(\bar{z}) \; . \tag{2.50}$$

One can see that their OPE with a primary field takes the form

$$T(z)\Phi(w,\bar{w}) = \frac{h}{(z-w)^2}\Phi(w,\bar{w}) + \frac{1}{(z-w)}\partial_w\Phi(w,\bar{w}) \ , \tag{2.51}$$

and similarly for $\bar{T}\Phi$.

One can expand the stress tensor into Laurent modes

$$T(z) = \sum_{n\in\mathsf{Z}} z^{n-2}L_n \ , \qquad \bar{T}(\bar{z}) = \sum_{n\in\mathsf{Z}} \bar{z}^{n-2}\bar{L}_n \tag{2.52}$$

and compute their commutators, they give rise to the famous (2 copies of the) Virasoro algebra:

$$\begin{aligned} [L_n, L_m] &= (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n,0} \ , \qquad [L_n,\bar{L}_m] = 0 \\ [\bar{L}_n,\bar{L}_m] &= (m-n)\bar{L}_{m+n} + \frac{\bar{c}}{12}(m^3-m)\delta_{m+n,0}. \end{aligned} \tag{2.53}$$

This algebra is the quantum version of the Witt algebra (2.47); the two algebras differ by the presence of the central charges $c, \bar{c}$, which signals the presence of a conformal anomaly. See [Gin88] for more details on the interesting singular case of $2d$ CFTs.

2.7 Embedding space formalism

We have seen in section 2.1 that the conformal algebra is in fact isomorphic to $so(p+1, q+1)$, the algebra of Lorentz transformations in $\mathsf{R}^{p+1,q+1}$ Minkowski space. This suggests that the d-dimensional conformal group should act in a much more natural way on $\mathsf{R}^{p+1,q+1}$. The embedding space formalism traces back to this idea of Dirac, who noticed that the conformal group can be realized as the group of *linear* isometries on the *embedding space* $\mathsf{R}^{p+1,q+1}$. The latter has coordinates X^A, $(A = 0, \ldots, d-1, d+1, d+2)$ where X^{d+1} is the extra spacelike direction and X^{d+2} the extra timelike direction. Using light-cone coordinates

$$X^{\pm} = X^{d+2} \pm X^{d+1} \tag{2.54}$$

the metric in $\mathsf{R}^{p+1,q+1}$ reads

$$ds^2 = \sum_{\mu=0}^{d-1}(dX^\mu)^2 - dX^+dX^- \, . \tag{2.55}$$

From the embedding space, a d-dimensional space is obtained by imposing

$$X^2 = \eta_{AB}X^AX^B = 0 \, , \tag{2.56}$$

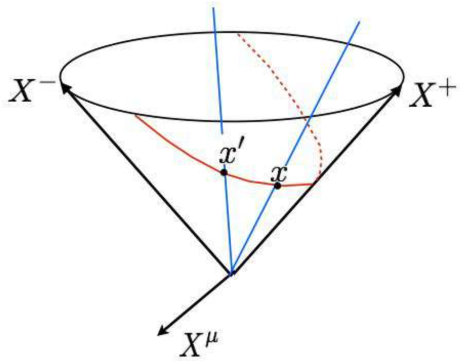


Figure 1: Figure taken from [Ryc16]. The red curve is the Poincaré section. Light rays are depicted in blue.

and also requiring that the overall scale of X is arbitrary, namely[10]

$$X^A \sim \lambda X^A \,, \qquad \lambda \in \mathsf{R}^+ \,. \tag{2.57}$$

With these conditions, X^A are coordinates for a d-dimensional *projective* (because of (2.57)) *null cone* embedded in $\mathsf{R}^{p+1,q+1}$

$$S^p \times S^q \cong \frac{\{X \in \mathsf{R}^{p+1,q+1} \text{ s.t. } X^2 = 0\}}{X \sim \lambda X} \,. \tag{2.58}$$

This defines the conformal compactification of $\mathsf{R}^{p,q}$, which is thus equivalent to the projective null cone. One can identify a copy of $\mathsf{R}^{p,q}$ by making the gauge choice $X^+ = 1$, this defines the 'Poincaré section', parametrized as

$$X_P^A = (X^+, X^-, X^\mu)_P = (1, x^2, x^\mu) \,. \tag{2.59}$$

Hence a point x in the physical space is put into correspondence with a null ray in the embedding space. Notice that this section does not cover all light rays; it misses the points 'at infinity' that one needs to add in order to obtain the conformally compactified spacetime.

A linear $SO(p+1, q+1)$ transformation on $\mathsf{R}^{p+1,q+1}$ maps null rays into null rays, and via the above equation this defines a map of the $S^p \times S^q$ into itself. When restricted to the Poincaré section, one can check that this map realizes a conformal transformation. Every conformal transformation can be realized this way.

Let us now say how (spinning) fields on the embedding space define primaries on the physical space. Rank s tensor fields $\Phi^{\Delta}_{A_1 \dots A_s}(X)$ on the light cone $X^2 = 0$ that are

- homogeneous of degree $-\Delta$, namely $\Phi^{\Delta}_{A_1 \dots A_s}(\lambda X) = \lambda^{-\Delta} \Phi^{\Delta}_{A_1 \dots A_s}(X)$
- symmetric and traceless[11]

[10] Quotient by R

[11] This is because we most often care about symmetric and traceless primary fields.

- transverse: $X^{A_i}\Phi^{\Delta}_{A_1\ldots A_i\ldots A_s}=0\ \forall i$

define a tensor field on $\mathsf{R}^{p,q}$ via the projection

$$\phi_{\mu_1\ldots\mu_s}(x)=\frac{\partial X_P^{A_1}}{\partial x^{\mu_1}}\cdots\frac{\partial X_P^{A_s}}{\partial x^{\mu_s}}\Phi^{\Delta}_{A_1\ldots A_s}(X_P)\,. \tag{2.60}$$

One can check that (2.60) transforms as a primary field of conformal dimension Δ. One can then compute correlation functions directly in the embedding space, where the constraints of conformal symmetry are just homogeneity and $SO(p+1,q+1)$ Lorentz invariance. Physical correlators are simply obtained by restricting to the section of the lightcone associated with the physical space of interest.

→ *Exercise 4:* Write down the most general Lorentz-invariant expression for the 2pt-function of two scalar fields homogeneous of degree Δ on the light cone. By projecting it on the Poincaré section show that it reduces to the usual expression of the two-point function of primary fields.

2.8 Superconformal algebras

The bosonic Poincaré algebra is extended, by the addition of fermionic generators, into superalgebras. If we also add conformal generators, we obtain superconformal algebras. They are constrained by the following requirements:

- they contain both the conformal algebra $\mathfrak{so}(d,2)$ and the Poincaré supersymmetry superalgebra;
- the fermionic generators transform in the spinor representation of $\mathfrak{so}(d,2)$.

These requirement are quite restrictive. In fact, they only exist in some dimensions – $d\leq 6$ – and for some number of supersymmetries (classified by Nahm [Nah78], see also [Min98] and [VP99]).

We can think of a superalgebra as being generated by matrices T that act on a superspace $\mathcal{X}$:

$$T\mathcal{X}=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}x\\ \theta\end{pmatrix}\,. \tag{2.61}$$

The superspace is Z_2 graded by the fermion number, and so are the homeomorphisms. Thus A,D are bosonic, while B,C are fermionic and anticommute. The structure of a superalgebra is

$$[T_b, T_b] = T_b \ , \qquad [T_b, T_f] = T_f \ , \qquad \{T_f, T_f\} = T_b \ . \tag{2.62}$$

Let us look at superconformal algebras.

In addition to the conformal generators P_μ, $M_{\mu\nu}$, D, K_μ and the supersymmetry generators Q^a_α, $\bar{Q}_{a\dot{\alpha}}$ ($a = 1, \dots, \mathcal{N}$), closure of the superconformal algebra requires further fermionic supercharges (one for each susy generator), denoted by S^a_α and $\bar{S}_{a\dot{\alpha}}$. While Poincare supercharges Q^a_α, $\bar{Q}_{a\dot{\alpha}}$ correspond to the fermionic superpartners of P_μ, the special conformal supercharges S^a_α, $\bar{S}_{a\dot{\alpha}}$ are the fermionic superpartners of K_μ (there are as many as the Q's, arising from the commutators of K with Q). One also needs R-symmetry generators, the R's here below, forming a Lie algebra (and arising from the anti-commutator of Q and S).

Both types of supercharges are constructed out of the supersymmetry current $S_{\mu\alpha}$. More specifically the currents for Q_α and S_α are

$$S_{\mu\alpha} \qquad \text{and} \qquad \gamma^\rho_{\alpha\beta} S^\beta_\mu x_\rho \tag{2.63}$$

respectively. Conservation follows from $\partial^\mu S_{\mu\alpha} = \gamma^\mu_{\alpha\beta} S^\beta_\mu = 0$. The second quantity is called "gamma trace".

The superconformal algebra is, schematically:

$$\begin{aligned} &[D, Q] = -\frac{i}{2} Q \qquad &&[D, S] = \frac{i}{2} S \qquad &&[K, Q] \simeq S \qquad &&[P, S] \simeq Q \\ &\{Q, \bar{Q}\} \simeq P &&\{S, \bar{S}\} \simeq K &&\{Q, S\} \simeq M + D + R \ . \end{aligned} \tag{2.64}$$

In particular the R-symmetry is part of the superconformal algebra, not just an outer automorphism as for supersymmetry, and in particular it *must* be there. The exact form of the the commutation relations is different for different dimensions (since the spinorial representations of the conformal group behave differently) and for different R-symmetry groups.

List of superconformal algebras: In a generic superalgebra that contains $\mathfrak{so}(d,2)$, the fermionic generators are in the vector representation. The requirement that they are in the spinor representation is quite restrictive in $d \geq 3$, leading the following short list:

$$\begin{array}{llllll} d = 3: & \mathfrak{osp}(\mathcal{N}|4) & \supset \ \mathfrak{so}(\mathcal{N}) \times \mathfrak{sp}(4) & \simeq \ \mathfrak{so}(\mathcal{N}) \times \mathfrak{so}(3,2) & \\ d = 4: & \mathfrak{su}(2,2|\mathcal{N}) & \supset \ \mathfrak{su}(2,2) \times \mathfrak{u}(\mathcal{N}) & \simeq \ \mathfrak{so}(4,2) \times \mathfrak{u}(\mathcal{N}) & \mathcal{N} \neq 4 \\ & \mathfrak{psu}(2,2|4) & \supset \ \mathfrak{su}(2,2) \times \mathfrak{su}(4) & \simeq \ \mathfrak{so}(4,2) \times \mathfrak{su}(4) & \mathcal{N} = 4 \\ d = 5: & \mathfrak{f}(4) & \supset \ \mathfrak{so}(5,2) \times \mathfrak{su}(2) & & \mathcal{N} = 1 \\ d = 6: & \mathfrak{osp}(6,2|\mathcal{N}) & \supset \ \mathfrak{so}(6,2) \times \mathfrak{sp}(2\mathcal{N}) & & \end{array} \tag{2.65}$$

For free field theories with no gravity (which do not include fields with spin bigger than one), the maximal number of supersymmetries is 16. It is believed to be the same for interacting theories. In the case of SCFTs, one can rigorously prove [CDI19] that the existence of a stress tensor multiplet (containing $T_{\mu\nu}$, $S_{\mu\alpha}$ and R_μ from which the superconformal charges are constructed) and the requirement that the theory is not free, limits[12]

$$N_Q \leq 16 \tag{2.66}$$

in $d \geq 3$, where N_Q is the total number of Poincaré supercharges (the relation between N_Q and $\mathcal{N}$ depends on the dimension d). Therefore, the maximal number of fermionic generators in a superconformal algebra is 32. Theories with such a superconformal algebra are known in $d = 3, 4, 6$.

$$\begin{array}{llllll} d=3: & OSp(8|4) & \supset & SO(8)\times Sp(4) & \simeq & SO(8)\times SO(3,2) \\ d=4: & PSU(2,2|4) & \supset & SU(2,2)\times SU(4) & \simeq & SO(4,2)\times SO(6) \\ d=6: & OSp(8^*|4) & \supset & SO^*(8)\times USp(4) & \simeq & SO(6,2)\times SO(5)\ . \end{array} \tag{2.67}$$

Superconformal and chiral primaries: Primary fields of the superconformal algebras are defined to be annihilated by both K_μ and S_α:

$$\text{superconformal primary}: \qquad [K_\mu, \Phi(0)] = [S_\alpha, \Phi(0)\} = [\bar{S}_{\dot{\alpha}}, \Phi(0)\} = 0\ . \tag{2.68}$$

We also have special kinds of descendants of the superconformal primary operator, the *superdescendants* Φ' defined by

$$\Phi' = [Q, \Phi\}, \tag{2.69}$$

whose dimension is $\Delta_\Phi + \frac{1}{2}$.

→ *Exercise 5:* Check that superdescendants are conformal primary operators (hint: use Jacobi identity).

A superconformal multiplet can include multiple conformal primaries (primaries of the conformal group alone), obtained by acting with Q_α. There can also be special representations, called *chiral primary operators*, which, in addition to the above superconformal primary constraints, are annihilated also by *some* of the Q^a_α's:

$$\text{chiral primary}: \qquad [K, \Phi] = [S, \Phi] = [Q^a_\alpha, \Phi] = 0 \tag{2.70}$$

[12] From the same requirement, one also excludes $\mathfrak{su}(2,2|4)$ as a superconformal algebra in $d = 4$, allowing $\mathfrak{psu}(2,2|4)$ instead.

for at least one $a \in \{1, \ldots \mathcal{N}\}$ and one $\alpha \in \{1, 2\}$. They are BPS operators. The multiplet formed by chiral primary operators is smaller (but still infinite) than the multiplets formed by non-chiral superconformal primaries. A special property of chiral primaries is that their conformal dimension Δ is fixed by the R-symmetry representation and cannot receive any quantum corrections[13].

Example: $4d$ $\mathcal{N} = 1$ superconformal theories. The R-symmetry group is $U(1)$. In that case, a chiral field (annihilated by $\bar{Q}$) which is a primary is also a chiral primary. Then its conformal dimension is fixed by its R charge:

$$\Delta = \frac{3}{2} R \, . \tag{2.71}$$

[13] This follows from vanishing of some $\{Q, S\}$ anticommutators.

3 Anti-de Sitter space

In this section, we will review the space AdS and its conformal structure. AdS_{d+1} is a maximally symmetric[14] Lorentzian manifold with constant negative scalar curvature. The effect of the negative curvature is to create a "conformal boundary". To give the idea, take the Euclidean case. To compactify R^d it is enough to add a "point at infinity", indeed Euclidean CFTs are naturally defined on S^d. Instead, the $(d+1)$-dimensional hyperbolic space (the Euclidean version of AdS) is conformally equivalent to a disk D_{d+1}, which has a boundary S^d.

One of the most important geometrical properties of AdS spacetime is the relationship between its conformal compactification to the one of flat spacetime. We will thus start by reviewing the conformal structure of Minkowski space[15].

3.1 Conformal structure of flat spacetime

Penrose diagram of flat spacetime

A very convenient way to understand the conformal and causal structure of a given spacetime, is to use Penrose diagrams. One performs a Weyl transformation of spacetime such that the transformed spacetime is "compact" (finite): the Penrose diagram is a diagram of the latter. A Weyl transformation preserves the signature and angles, in particular light-rays (light-like geodesics) remain at 45°, and time-like and space-like directions remain such. Thus a Penrose diagram correctly reproduces the causal structure of spacetime.[16]

Let us start with flat Minkowski spacetime $\mathsf{R}^{d-1,1}$ in polar coordinates

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{d-2}^2 \qquad \text{with} \qquad -\infty < t < +\infty, 0 \le r < +\infty \,. \tag{3.1}$$

where $d\Omega_{d-2}^2$ is the unit sphere metric. The "diagram" of this spacetime is non-compact. Let's first define the null coordinates (called retarded and advanced time)

$$u = t - r \,, \qquad v = t + r \,, \tag{3.2}$$

[14] This means that it has the maximal amount of spacetime symmetries, namely $\frac{1}{2}(d+1)(d+2)$, which is the same number as in $d+1$ flat spacetimes, with $d+1$ translations, d boosts and $\frac{1}{2}d(d-1)$ rotations.

[15] This is not devoted to interest on its own, in the light of understanding the putative flat version of the AdS/CFT correspondence.

[16] For a conformal theory, the Penrose diagram is a faithful representation of spacetime because the theory is invariant under Weyl transformations. For a non-conformal theory, instead, it is a distorted representation since distances are changed.

this gives ($v \geq u$)

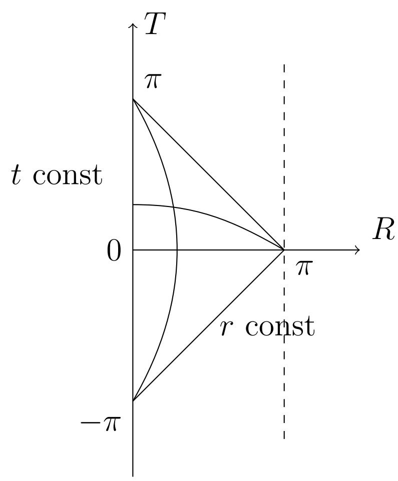
$$ds^2 = -dudv + \frac{(v-u)^2}{4} d\Omega^2 \,. \tag{3.3}$$

Next, we define $u = \tan p$, $v = \tan q$, where now p, q have a *finite* range of variation ($-\frac{\pi}{2} < p \leq q < \frac{\pi}{2}$), and $T = p + q$, $R = q - p$ ($-\pi < T + R < \pi$, $-\pi < T - R < \pi$, $R \geq 0$). The line element reads

$$ds^2 = \Psi^{-2} d\tilde{s}^2 \,, \qquad d\tilde{s}^2 = \left(- dT^2 + dR^2 + \sin^2 R \, d\Omega^2_{d-2} \right) \tag{3.4}$$

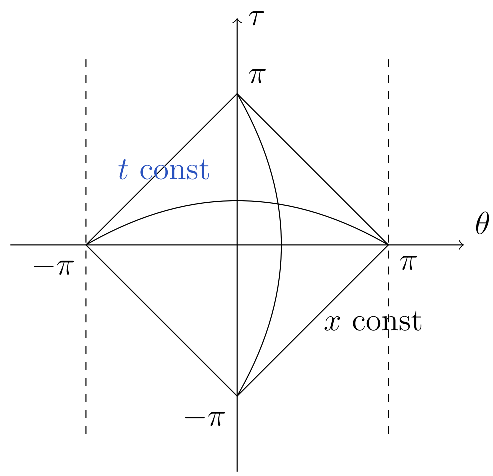
with the conformal factor $\Psi = 2\cos\left(\frac{T+R}{2}\right)\cos\left(\frac{T-R}{2}\right)$. We thus see that, while the conformal factor blows up at the boundary, the conformally rescaled (or 'unphysical') metric $d\tilde{s}^2$ is now perfectly regular. It shares the same conformal properties as the physical metric ds^2. The metric $d\tilde{s}^2$ is sometimes referred to as the 'Einstein static universe'.

We get the following Penrose diagram of flat spacetime. In this figure, all of Minkowski space is pulled into a finite region by a conformal transformation that diverges at the boundaries. Distances are not faithfully represented, but the causal structure is preserved.



with $0 \leq \theta \leq \pi$. Each point represents a whole sphere S^{d-2} of radius $\sin R$, which shrinks on the axis $R = 0$ and at $R = \pi$ (this point is called 'spatial infinity' i^0).

Another way of representing the Penrose diagram of Minkowski is as follows, where now each point is a half-sphere.



The two corners $(\tau, \theta) = (0, \pm\pi)$ correspond to spatial infinity. By identifying those two points, we can consider Minkowski space to be conformally embedded into the Einstein cylinder. This amounts to analytically extend the metric $d\tilde{s}^2$ in (3.4) to cover the manifold $S^{d-1} \times \mathsf{R}$ where $-\infty < T < \infty$ and R and the angular coordinates are regarded as coordinates on S^{d-1}. Hence the whole of Minkowski spacetime is conformal to a region of the Einstein static universe; see Figure 2. The conformal manifold into which Minkowski space-time is embedded (here the Einstein cylinder) is not fixed by the metric. Obviously, had we chosen a different conformal factor, we would not have obtained the metric of the Einstein cylinder but a different one. Hence, although the conformal boundary is unique, the conformal extension beyond the boundary is not.

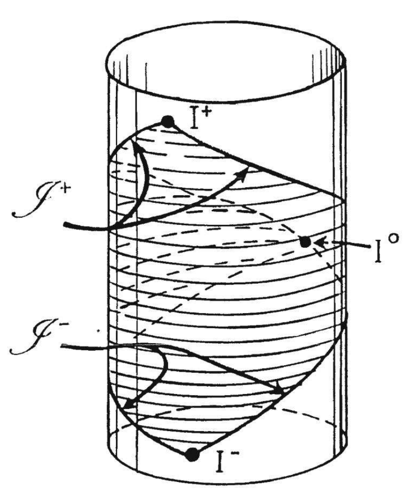


Figure 2: Figure taken from Penrose. The cylinder represents Einstein universe, with only one time and one space dimension depicted. The shaded region corresponds to the conformal compactification of Minkowski spacetime.

3.2 Asymptotically simple spacetimes

An asymptotically simple spacetime is defined by the requirement to allow the attachment of a smooth conformal boundary. A more precise definition is:

Definition: A smooth spacetime $(\mathscr{M}, g)$ is called *asymptotically simple* if there exists another smooth Lorentz manifold $(\hat{\mathscr{M}}, \hat{g})$ such that

$$\begin{aligned} (i)\ \hat{g} = \Omega^2 g\,, \qquad &\Omega > 0 \quad \text{on}\,\mathscr{M} \\ \Omega = 0 \quad \text{and} \quad &d\Omega \neq 0 \quad \text{on}\,\partial\mathscr{M} \end{aligned} \tag{3.5}$$

for some smooth scalar function Ω (called the boundary defining function) and (ii) every null geodesic in $\mathscr{M}$ has future and past endpoints at infinity[17].

Definition: An *asymptotically flat, AdS, dS* spacetime is an asymptotically simple spacetime which obeys Einstein's equations with, respectively zero, negative, positive cosmological constant in an open neighbourhood of the boundary.

Under some assumptions, one can show [Geroch] that the conformal boundary ($\mathscr{I}$) of these spacetimes will be null (for flat), timelike (for AdS) and spacelike (for dS), respectively.

3.3 Anti-de Sitter spacetime

Anti-de Sitter (AdS_{d+1}) is the most symmetric[18] Lorentzian space with (constant) negative curvature. It is a solution of Einstein equations with negative cosmological constant Λ:

$$S = \frac{1}{16\pi G_N} \int d^{d+1}x\ \sqrt{|g|}\big(R - 2\Lambda\big) \qquad\qquad (\text{take } \Lambda < 0) \tag{3.6}$$

leads to the vacuum Einstein equation

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = -\Lambda\, g_{\mu\nu} \qquad \Rightarrow \qquad R = \frac{d+1}{d-1}\, 2\Lambda\ . \tag{3.7}$$

As a metric space, AdS_{d+1} can be described by embedding an hyperboloid into $\mathsf{R}^{d,2}$:

$$-X_{-1}^2 - X_0^2 + X_1^2 + \ldots + X_d^2 = -L^2 \qquad \text{in} \qquad \mathsf{R}^{d,2}\ . \tag{3.8}$$

L is the AdS curvature radius. Both the ambient space and the equation have $SO(d, 2)$ isometry, thus the resulting space has that isometry too (recall that $SO(d, 2)$ is the conformal group in d dimensions in Lorentzian signature). The metric is the one induced by

$$ds^2 = -dX_{-1}^2 - dX_0^2 + dX_1^2 + \ldots + dX_d^2\ . \tag{3.9}$$

[17] This requirement can be relaxed in order to include black holes.

[18] That means that it has the maximum number of spacetime symmetries, namely $\frac{1}{2}(d+1)(d+2)$.

As said above, the isometry of AdS_{d+1} is the conformal group in d dimensions, $SO(d,2)$, but it is not always manifest in the metric description in terms of certain choice of coordinates, as we will see below.

Global coordinates: We can solve the equation (parametrize the solutions) by

$$X_{-1} = L\cosh\rho\,\sin\tau\ ,\quad X_0 = L\cosh\rho\,\cos\tau\ ,\quad X_i = L\,y_i\sinh\rho\quad\text{with}\quad \sum_{i=1}^{d} y_i^2 = 1\ , \tag{3.10}$$

with y_i $(i = 1,\ldots,d)$ the angular coordinates, $\rho \in \mathsf{R}_+$, $\tau \in [0, 2\pi]$. The induced metric is

$$ds^2 = L^2\big(-\cosh^2\rho\,d\tau^2 + d\rho^2 + \sinh^2\rho\,d\Omega_{d-1}^2\big)\ , \tag{3.11}$$

where Ω_{d-1} is the line element of a $d-1$-sphere. Although the ambient space had two "time directions", the embedded AdS has one and it is a standard Lorentzian spacetime.

What we have written is AdS in *global coordinates* (τ,ρ,Ω_i), which cover the entire hyperboloid once. ∂_τ is a timelike Killing vector so τ may be called the global time coordinate. However $\tau \in [0, 2\pi]$, and thus it seems that there are closed time-like curves (which violates causality). To solve the problem, we "unfold" τ:[19] we take the universal covering space with $\tau \in \mathsf{R}$. This is what we call global AdS. In global coordinates, the subgroup $SO(d) \times SO(2)$ is manifest. The universal cover of $SO(2)$ is the Killing vector ∂_τ, which is the Hamiltonian on S^{d-1} in field theory. $SO(d)$ gives the rotations of S^{d-1}.

To draw the Penrose diagram of AdS, it is convenient to redefine $\sinh\rho = \tan r$ with $r \in [0, \frac{\pi}{2}[$:

$$ds^2 = \frac{L^2}{\cos^2 r}\big(-d\tau^2 + dr^2 + \sin^2 r\,d\Omega_{d-1}^2\big)\ . \tag{3.12}$$

After a conformal rescaling, we find (like in the flat case) the Einstein static universe metric, but only one-half of it, since now $r \in [0, \frac{\pi}{2}[$ (rather than $r \in [0,\pi[$). In these coordinates, we can thus picture AdS as the interior of a cylinder $S^{d-1} \times \mathsf{R}$, whose boundary is located at $r = \frac{\pi}{2}$[20]. Recall that the boundary metric is only determined up to multiplication of an

[19] It is possible to unfold τ from S^1 to R because, in this metric, the circle S^1 never shrinks. Otherwise, unfolding would introduce singularities at points where S^1 shrank.

The fundamental group of $SO^+(p,q)$ (the connected component containing the identity) is the product $\pi_1\big(SO(p)\big) \times \pi_1\big(SO(q)\big)$. In particular $SO(d,2)$ has a covering extension by Z, which acts on the unfolded manifold. Such a covering extension has no finite-dimensional faithful representations, and so it cannot be represented as a matrix group. The case of $SO(1,2) \cong SL(2,\mathsf{R})/\mathsf{Z}_2$ is discussed in details in [Raw12].

[20] There are also two disjoints points, future and past timelike infinities i^+ and i^-.

arbitrary Weyl factor that depends on how one approaches infinity and one can obtain as boundary any conformally flat manifold.

Notice that the boundary of conformally compactified AdS_{d+1} is equal to the conformal compactification of $\mathsf{R}^{d-1,1}$.

Poincaré patch coordinates: There is another useful parametrization of AdS_{d+1} in terms of coordinates $(u, t, \vec{x})$ $(u > 0, t \in \mathsf{R}, \vec{x} \in \mathsf{R}^{d-1})$ defined as

$$\begin{aligned} X_0 &= \frac{1}{2u}(1 + u^2(L^2 + \vec{x}^2 - t^2))\,, \qquad X_{-1} = Lut\,, \\ X^i &= Lux^i \ (i = 1, \ldots, d-1)\,, \\ X^d &= \frac{1}{2u}(1 - u^2(L^2 - \vec{x}^2 + t^2))\,. \end{aligned} \tag{3.13}$$

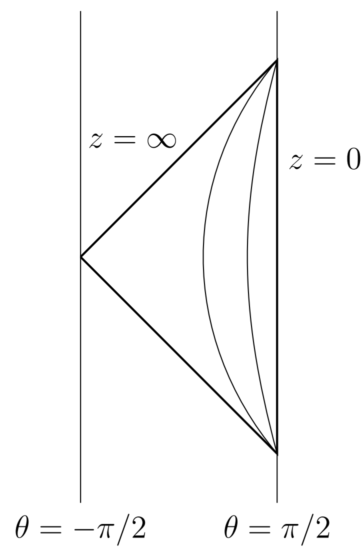
In these coordinates the metric is

$$ds^2 = L^2 \left(\frac{du^2}{u^2} + u^2(-dt^2 + d\vec{x}^2) \right)\,, \tag{3.14}$$

or, using $z = 1/u$,

$$ds^2 = \frac{L^2}{z^2} \left(dz^2 - dt^2 + d\vec{x}^2 \right)\,. \tag{3.15}$$

These local coordinates are called Poincaré coordinates and only cover a portion of AdS (because $z > 0$). AdS_2 can be conformally mapped into $\mathsf{R} \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ and the Poincaré coordinates cover the triangular region depicted below:



The boundary of the Poincaré patch is at $z = 0$, it is flat space $\mathsf{R}^{d-1,1}$.

At $z = \infty$ there is an horizon, because the Killing vector ∂_t has zero norm. It is only a coordinate singularity, not a curvature singularity, and in fact the patch can be embedded into the global coordinates where there is no horizon. In a loose sense, it is an horizon because signals cannot come back (this is obvious from the Penrose diagram in global coordinates), indeed it is called an "apparent (or Poincaré) horizon".

In the Poincaré patch, the subgroup $SO(d-1,1) \times SO(1,1)$ is manifest. The first one is the Lorentz group of the boundary, while $SO(1,1)$ corresponds to dilations on the boundary and it is realized as

$$(t, \vec{x}, z) \to \lambda(t, \vec{x}, z) , \qquad \lambda > 0 . \tag{3.16}$$

In AdS/CFT, this is identified with the dilatation D in the conformal symmetry group of $\mathsf{R}^{1,d-1}$.

→ *Exercise 6:* Check that the Ricci scalar of the metric (3.15) is given by

$$R = -\frac{d(d+1)}{L^2} \tag{3.17}$$

which is indeed a negative curvature of radius L, itself related to the cosmological constant as

$$\Lambda = -\frac{d(d-1)}{2L^2} . \tag{3.18}$$

Euclidean AdS: For the AdS/CFT correspondence, we will also need the AdS in Euclidean signature. Since AdS has the global time coordinate τ and the metric (3.11) is static with respect to τ, QFT on AdS (with appropriate boundary condition near spatial infinity) allows the Wick rotation $\tau \to \tau_E = i\tau$. In terms of the original coordinates for the hyperboloid, that Wick rotation is expressed as $X_{-1} \to X_E = -iX_{-1}$ and the space becomes

$$ds_E^2 = -dX_0^2 + dX_E^2 + d\vec{X}^2 , \qquad X_0^2 - X_E^2 - \vec{X}^2 = L^2 . \tag{3.19}$$

Notice that we could have obtained the same space by Wick rotation the time coordinate t of the Poincaré coordinates (3.15) as $t \to t_E = -it$, even though those coordinates only cover half of the hyperboloid.

Euclidean AdS in global coordinates thus reads

$$ds_E^2 = L^2 \Big(\cosh^2 \rho \, d\tau_E^2 + d\rho^2 + \sinh^2 \rho \, d\Omega_{d-1}^2 \Big) . \tag{3.20}$$

The isometry group of Euclidean AdS is given by $SO(d+1,1)$. In Poincaré coordinates (which in this case cover the entire spacetime), it can be written

$$ds_E^2 = \frac{L^2}{z^2} (dz^2 + dx_1^2 + \cdots + dx_p^2) . \tag{3.21}$$

Alternatively, Euclidean AdS_{d+1} can be identified with the unit open ball B_{d+1}

$$ds^2 = 4\sum_{i=0}^{d} \frac{dy_i^2}{(1-|y|^2)^2}\,, \tag{3.22}$$

where $\sum_{i=0}^{d} y_i^2 < 1$. Compactification of B_{d+1} leads to the closed unit ball $\bar{B}_{d+1}$ defined by $\sum_{i=0}^{d} y_i^2 \leq 1$. Its boundary is the sphere S^d defined by $\sum_{i=0}^{d} y_i^2 = 1$.

3.4 Particles in AdS

One can show that timelike geodesics never reach the boundary, they oscillate with period 2π in global time[21]. This is why people say that AdS 'acts like a box' that confines massive particles. On the other hand, light rays in AdS start and end at the conformal boundary travelling for a global time interval equal to π.

Consider a scalar field propagating in AdS_d+1 written in global coordinates (τ, r, Ω_{d-1}). The field equation

$$(\Box - m^2)\phi = 0 \tag{3.23}$$

has stationary wave solutions

$$\phi = e^{i\omega\tau} G(r) Y_\ell(\Omega_{d-1})\,, \tag{3.24}$$

where $Y_\ell(\Omega_{d-1})$ is a spherical harmonic, which is an eigenstate of the Laplacian on S^{d-1} with eigenvalue $\ell(\ell+d-2)$ $(\ell = 0, 1, 2, \dots)$, and

$$G(r) = (\sin r)^\ell (\cos r)^{\Delta_\pm}\, {}_2F_1(a, b, c; \sin r) \tag{3.25}$$

with

$$\begin{aligned} a &= \frac{1}{2}(\ell + \Delta_\pm - \omega L), \\ b &= \frac{1}{2}(\ell + \Delta_\pm + \omega L), \\ c &= \ell + \frac{d}{2} \end{aligned} \tag{3.26}$$

and

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{d^2 + 4(mL)^2}\,. \tag{3.27}$$

[21] By choosing a specific time coordinate we have chosen a particular center in AdS and, *according to this time* objects move in orbits. Any given orbit can be transformed to a particle at rest by performing a conformal transformation.

We still have to look at the behavior of this solution close to the boundary. We want to require energy-momentum conservation in AdS, namely the flux of energy momentum-tensor through the boundary should be zero. That amounts to the restriction

$$\int_{S^{d-1}} d\Omega_{d-1}\sqrt{g}\, n_i T^i_0\, |_{r=\pi/2} = 0\,. \tag{3.28}$$

By using the energy-momentum tensor for ϕ, one gets

$$\tan^{d-1} r[(1-4\beta)\partial_r + 4\beta \tan r]\phi^2|_{r\to\pi/2} = 0\,. \tag{3.29}$$

→ *Exercise 7:* Using properties of hypergeometric functions around $\sin r \sim 1$, show that the above condition is satisfied if a, b arguments in ${}_2F_1(a, b, c; \sin r)$ are integer-valued.

The above constraint on a, b is satisfied if

$$|\omega|L = \Delta_\pm + \ell + 2n\,, \qquad n = 0, 1, 2, \ldots \tag{3.30}$$

In order words, we have just seen that imposing that there is no energy flux through the AdS boundary leads to the quantization of energy. Now, condition (3.30) is possible only when $\Delta_\pm$ are real. Finally, from (3.27) one thus gets

$$\boxed{-\frac{d^2}{4} \leq m^2L^2}\,. \tag{3.31}$$

This is the so-called *Breitenlohner-Freedman (BF) bound*. Hence particles in AdS can have negative mass squared and be stable (energy is positive). It can be seen as due to the fact that the curvature gives a positive contribution to the energy of a scalar field propagating in AdS. Below the BF bound the AdS background becomes unstable. To stabilize the theory bulk interactions must be introduced, which deform the AdS metric. If there are no interactions between the particles in AdS, then the Hilbert space is a Fock space and the energy of a multi-particle state is just the sum of the energies of each particle[22].

[22] Turning on small interactions leads to small energy shifts of the multi-particle states. One can observe that this structure is very similar to the space of local operators in large N CFTs if we identify single-particle states with single-trace operators.

4 The holographic dictionary

In this section, we present the conceptual core of the holographic correspondence, without referring to specific realizations of it (we will do that later). In the previous sections, we have seen that there are two kinds of theories with $SO(d, 2)$ symmetry: conformal field theories in d dimensions and gravitational theories in AdS_{d+1}. But in what sense the CFT could be equivalent to the gravitational theory in AdS? The two look completely different, they even have different dimension.

4.1 Formulating the correspondence

Let us focus on the Euclidean case, where Euclidean AdS = hyperbolic plane. Moreover focus on the Poincaré patch of AdS_{d+1} (upper half-plane)

$$ds^2 = L^2 \frac{d\vec{x}^2 + dz^2}{z^2} \,, \tag{4.1}$$

whose boundary is R^d. The dual CFT will live on this d-dimensional boundary.

Since AdS has a boundary, to define the gravitational theory we need to specify boundary conditions for the fields $\phi(\vec{x}, z)$. On the other hand, in a CFT, important observables are the correlation functions of local operators $\mathcal{O}(\vec{x})$, whose generating functional $W[J]$ is constructed out of sources J:

$$Z[J]_{\mathrm{CFT}} = e^{-W[J]} = \left\langle e^{\int d^d x \, J(x) \mathcal{O}(x)} \right\rangle_{\mathrm{CFT}} \tag{4.2}$$

As usual the connected correlation functions are obtained from the generating function:

$$\langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_n) \rangle_{\mathrm{CFT},c} = \frac{\delta^n W}{\delta J(x_1) \ldots \delta J^n(x_n)} |_{J=0} . \tag{4.3}$$

In its strongest from, the AdS/CFT correspondence equates the partition function of the bulk theory of gravity (e.g. type IIB string theory) with the generating functional of the CFT:

$$\boxed{\mathcal{Z}_{\mathrm{bulk}}[\phi_0] = \left\langle e^{\int d^d x \, \phi_0(\vec{x}) \, \mathcal{O}(\vec{x})} \right\rangle_{\mathrm{CFT}} = Z[\phi_0]_{\mathrm{CFT}}} \tag{4.4}$$

where the CFT source J is identified as the boundary value of the bulk field in AdS, denoted $\phi_0(\vec{x})$,

$$\phi(\vec{x}, z) \overset{z \to 0}{\sim} z^{d-\Delta} \phi_0(\vec{x}) \,. \tag{4.5}$$

On the LHS is the partition function of supergravity or string theory, function of the boundary values of the bulk fields. On the RHS is the generating functional of correlators in the CFT for operators $\mathcal{O}(\vec{x})$ whose sources are given by $\phi_0(\vec{x})$. This identification requires a correspondence

$$\text{field in the bulk} \qquad \leftrightarrow \qquad \text{operator in the boundary CFT} . \tag{4.6}$$

A field that is the derivative of another field corresponds to a descendant in the CFT, thus we will only describe primary operators.

The partition function of string theory is a very complicated (and unknown) object.[23] When gravity is weakly coupled, we can approximate the string partition function by the classical action:

$$\mathcal{Z}_{\text{bulk}} \simeq e^{-S_{\text{class}}} \tag{4.7}$$

Then the partition function is dominated by the saddle points, *i.e.* the classical solutions to the EOMs.

Thus, in the classical limit the correspondence states that:

The classical gravity action is the generating functional
of connected correlators in the CFT.

The classical EOMs are second order, we thus need to specify two boundary conditions in order to find a unique solution: we impose Dirichlet boundary conditions on the boundary, and regularity at the horizon working with Poincaré coordinates (regularity at the center of the AdS ball in the Euclidean version). This fixes the classical solutions. Let us stress that we used the AdS EOM. Thus the correspondence relates an on-shell theory in $d+1$ to an on-shell theory in d dimensions.

4.2 Field-operator map: boundary asymptotics

Consider a scalar field ϕ of mass m^2 in AdS_{d+1}, dual to some operator $\mathcal{O}$ in the dual CFT. We work in Poincaré coordinates[24] $(\alpha, \beta = 0, \dots d)$

$$ds^2 = g_{mn} dx^m dx^n = \frac{L^2}{z^2}(dz^2 + \delta_{\alpha\beta} dx^\alpha dx^\beta) . \tag{4.8}$$

At this stage, it is enough to consider a toy model action of the form

$$S[\phi] = \frac{C}{2} \int d^d x \, dz \, \sqrt{g} \Big[g^{mn} \partial_m \phi \partial_n \phi + m^2 \phi^2 + O(\phi^3) \Big] , \tag{4.9}$$

[23] The quantity $\log \mathcal{Z}_{\text{bulk}}$ can be interpreted as a string theory S-matrix element of the state ϕ_0.

[24] In this section we'll always work in Euclidean space.

with C some normalization. The EOMS read

$$0 = -\Box_g \phi + m^2 \phi \tag{4.10}$$

with

$$\Box_g = -\frac{1}{\sqrt{g}}\partial_m\left(\sqrt{g}\, g^{mn}\partial_n\phi\right) = \frac{1}{L^2}(z^2\partial_z^2 - (d-1)z\partial_z + z^2\delta_{\alpha\beta}\partial^\alpha\partial^\beta). \tag{4.11}$$

It turns out convenient to go to Fourier space and consider a plane wave ansatz $\phi(x,z) = e^{ip\cdot x}\phi_p(z)$. Then

$$0 = z^{d+1}\partial_z\left(z^{1-d}\partial_z\phi_p\right) - p^2 z^2\phi_p - m^2L^2\phi_p \;, \tag{4.12}$$

with $\delta_{\alpha\beta}p^\alpha p^\beta = p^2$. There are two independent solutions, characterized by their asymptotic behavior as $z \to 0$. They can be written exactly in terms of Bessel functions.[25] Near the boundary, $z \sim 0$, the term with momentum can be neglected, and the solutions are power-law:

$$\phi_p(z) \;\sim\; z^{\Delta_\pm} \;, \tag{4.13}$$

where $\Delta_\pm$ are the roots of $m^2L^2 = \Delta(\Delta - d)$, namely

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2L^2} \;. \tag{4.14}$$

Near the boundary, we can thus expand the field as

$$\phi(x,z) \sim \phi_{(0)}(x)z^{\Delta_-} + \phi_{(1)}(x)z^{\Delta_+} + \cdots \tag{4.15}$$

where the $\ldots$ stand for subleading terms in z. The first solution ($\sim z^{\Delta_-}$) dominates and can diverge at the boundary; the associated mode $\phi_{(0)}$ is called the *non-normalizable* mode:

$$\phi_{(0)}(x) \equiv \lim_{z\to 0}\phi(x,z)z^{-\Delta_-} = \lim_{z\to 0}\phi(x,z)z^{\Delta_+ - d} \,. \tag{4.16}$$

On the other hand, the solution with Δ_+ always decays. We will identify the *normalizable* AdS mode $\phi_{(1)}$ as the VEV for a dual CFT scalar operator $\mathcal{O}$ of conformal dimension

$$\Delta \equiv \Delta_+ = d - \Delta_- \;, \tag{4.17}$$

we will see later that is indeed a conformal dimension. The non-normalizable mode $\phi_{(0)}$ will be identified as *source* for this operator.

$\to$ *Exercise 8:* We say that a solution is normalizable if the action evaluated on this solution is finite. Check that near the boundary where $\phi \sim z^\Delta$, the action is indeed finite when

[25] The two solutions are $\phi = c_1 z^{d/2} I_a(pz) + c_2 z^{d/2} K_a(pz)$ with $a = 12\sqrt{d^2 + 4m^2L^2}$.

integrating from $z = 0$ to $z = \epsilon$ provided that $\Delta \geq d/2$. This justifies why Δ_+ leads to the normalizable mode.

In summary, we associate a bulk scalar field of squared mass m^2 to a boundary operator of dimension Δ; notice the key relationship[26]

$$\boxed{m^2 L^2 = \Delta(\Delta - d)} \, . \tag{4.18}$$

We thus see that $m^2L^2 \geq 0$ only for $\Delta \geq d$. One still gets a real acceptable dimension if m^2L^2 is negative, as we have seen before, provided we respect the BF bound

$$-\frac{d^2}{4} \leq m^2 L^2 \, . \tag{4.19}$$

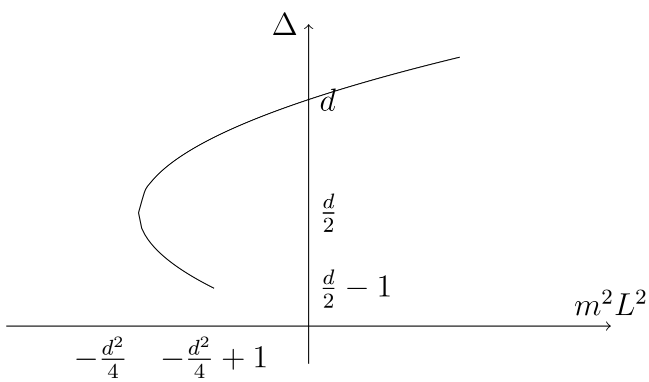
On the other hand, we have the unitarity bound on the dimension of a scalar operator $\Delta \geq \frac{d}{2} - 1$.

Now, remember that we decided to choose Δ as the largest solution Δ_+. But in the mass range

$$-\frac{d^2}{4} \leq m^2 L^2 \leq -\frac{d^2}{4} + 1 \, , \tag{4.20}$$

also Δ_- would be an acceptable dimension (i.e. above the unitarity bound). In this mass range, double quantization is thus possible, depending on the two possible boundary conditions (while for higher m only $\Delta = \Delta_+$ is allowed and the boundary condition is imposed on the $\phi \sim z^{\Delta_-}$ mode). The two different AdS theories with a given m^2 will thus correspond to two different CFTs, one with an operator of dimension Δ_+ and source of dimension Δ_- and the other with the role source $\leftrightarrow$ VEV exchanged. Notice that this is the only way to get operators of dimension $< \frac{d}{2}$.

The picture that we get is thus:



[26]This is valid for a scalar field, for other spin fields see e.g. Zaffaroni's lectures.

$$
\begin{array}{llll}
-\frac{d^2}{4} \le m^2L^2 < 0 & \Delta < d & \text{relevant} & \left(-\frac{d^2}{4} \le m^2L^2 \le -\frac{d^2}{4}+1 \quad \text{double quantization} \right) \\
m^2L^2 = 0 & \Delta = d & \text{marginal} & \\
0 < m^2L^2 & \Delta > d & \text{irrelevant} &
\end{array}
$$

The field $\leftrightarrow$ operator map (holographic dictionary) is not given a priori. In many cases it can be determined based on the mass/dimension, the spin, and some other quantum numbers. Some operators are easy to determine. The boundary value of the bulk metric g_{mn} is the boundary metric $g_{\mu\nu}$, which is the source for the stress tensor $T_{\mu\nu}$. Thus

$$
g_{mn} \qquad \leftrightarrow \qquad T_{\mu\nu} \ . \tag{4.21}
$$

An analysis of gravitational waves in AdS_{d+1} shows that the dual operator has dimension $\Delta = d$.

Suppose we have a gauge field A_m in the bulk. The dual operator must be a vector J^μ, coupled to the source as

$$
\int d^dx\, A_\mu J^\mu \ .
$$

The bulk theory must be invariant under gauge transformations, $\delta A_m = D_m\lambda$, thus (in the absence of anomalies[27]) the boundary coupling should be invariant as well:

$$
0 = \delta \int d^dx\, A_\mu J^\mu = -\int d^dx\, \lambda\, D_\mu J^\mu \ . \tag{4.22}
$$

Thus J_μ should be a conserved current in the boundary theory. From the wave equation in AdS, one obtains that J_μ has dimension $\Delta = d-1$. Thus

$$
\begin{array}{ccc}
\text{bulk gauge symmetry} & \leftrightarrow & \text{boundary global symmetry} \\
A_m & \leftrightarrow & J_\mu \ .
\end{array} \tag{4.23}
$$

We could go on and discover that gravitinos correspond to supersymmetry currents $S_{\alpha\mu}$, Abelian p-form potentials correspond to Abelian higher-form symmetries [GKSW15]. And study the mass/dimension formula for arbitrary spin.

[27] In fact, the bulk theory should only be invariant under gauge transformations that vanish at infinity. Under gauge transformations that are non-trivial at infinity, the effective action can have an anomalous variation which is constrained, by the Wess-Zumino consistency conditions, to be a local functionals of the field strengths. This reproduces global 't Hooft anomalies on the boundary. Another argument is that the dimension of J^μ is fixed.

4.3 Correlation functions

In the previous section, we have seen that AdS/CFT establishes a one-to-one correspondence (dictionary) between bulk fields ϕ and field theory operators $\mathcal{O}$. As said in section (4.1), the correspondence states that connected correlation functions are computed by derivatives of the classical on-shell action.

4.3.1 Two-point functions

Let's start with the computation of two-point functions. Even if it looks simple, we will see that there are tricky aspects regarding this computation which are related to the treatment of potential divergences at the boundary. Only the part of the action quadratic in the relevant field perturbation is needed[28], so it will be sufficient to consider again the action $S[\phi]$ given by (4.9), where the mass of the scalar field is related to the conformal dimension via (4.18). The procedure will go as follows: we find a solution of the EOMs (4.10) subject to the boundary condition (4.15) for any source $\phi_{(0)}(x)$. Then we insert this solution into the action; by construction this on-shell action reduces to a boundary term of the form

$$S[\phi] = -\frac{C}{2} \int d^d x \sqrt{g} g^{zz}\, \phi(x,z) \partial_z \phi(x,z)|_{z=\epsilon} \,. \tag{4.24}$$

The integrand above should be evaluated both at $z = \infty$ and $z = 0$. We will see below, however, that imposing regularity in the interior ensures that the integrand vanishes as $z \to \infty$. The issue is that, at $z = 0$, the expression $\sqrt{g} g^{zz} = (L/z)^{d-1}$ diverges. We will regularise $S[\phi]$ by introducing a cut-off ϵ and impose the boundary conditions at $z = \epsilon$.

We will work in momentum space (it turns out more convenient in order to compute 2-point functions) using the Fourier transform

$$\phi(x,z) = \int \frac{d^d p}{(2\pi)^d} e^{ip\cdot x} \phi(p,z) \,, \tag{4.25}$$

with p the momentum. We look for a solution that satisfies (4.12) with $p^2 = \delta_{\alpha\beta} p^\alpha p^\beta$, $|p| = \sqrt{p^2}$. Setting $u \equiv z|p|$, the two solutions are expressed in terms of modified Bessel functions:[29]

$$\phi(z,p) = c_1\, z^{d/2}\, I_\nu(u) + c_2\, z^{d/2}\, K_\nu(u) \tag{4.29}$$

[28] We consider the probe limit in which we neglect the contributions of the scalar field to the energy-momentum tensor as they will appear in interaction vertices in Witten diagrams which do not enter here.

[29] The Bessel function $I_\alpha(u)$ can be defined by a series expansion:

$$I_\alpha(u) = \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(m+\alpha+1)} \left(\frac{u}{2}\right)^{2m+\alpha} \,. \tag{4.26}$$

where $\nu \equiv \Delta - \frac{d}{2} = \sqrt{\frac{d^2}{4} + m^2 L^2}$ (recall $\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2 L^2}$).

We should also impose regularity in the interior of AdS (from which we get a single solution). We therefore omit the first solution (by setting $c_1 = 0$) since $I_\nu(u)$ is exponentially diverging in the interior of AdS ($u \to +\infty$). In contrast, $K_\nu(z)$ decays exponentially and is thus regular in the interior, leading to a finite contribution in the on-shell action as $z \to \infty$. Moreover, for $\nu \neq 0$, $K_\nu(z) \sim z^{-\nu}$ at the boundary, leading to the desired asymptotic behavior for the field as $z \to 0$,

$$\phi(z,p) \sim z^{d/2-\nu} c_2 = z^{d-\Delta} c_2, \tag{4.30}$$

if c_2 is related to the Fourier modes of the sources, denoted by $\phi_{(0)}(p)$. The matching of $\phi(z,p)$ to $\phi_{(0)}(p)$ at $z = \epsilon$ leads to the following normalization:

$$\phi(z,p) = \epsilon^{d-\Delta} \frac{z^{d/2}\, K_\nu(z|p|)}{\epsilon^{d/2}\, K_\nu(\epsilon|p|)}\, \phi_{(0)}(p)\ . \tag{4.31}$$

Now, plugging (4.25) into the boundary on-shell action (4.24) leads to (the integral over $d^d x$ gives $(2\pi)^d \delta^d(\vec{p} + \vec{q})$)

$$S[\phi] = -\frac{C L^{d-1}}{2\epsilon^{d-1}(2\pi)^d} \int d^d p\, d^d q\, \delta^d(p+q)\, \phi(z,p) \partial_z \phi(z,q)|_{z=\epsilon}\ . \tag{4.32}$$

We can then use (4.31) to obtain an action which will only depend on $\phi_{(0)}$, and from there one gets the two-point functions for the dual CFT operators in momentum space[30]

$$\begin{aligned} \langle \mathcal{O}(p)\, \mathcal{O}(q) \rangle_\epsilon &= -(2\pi)^{2d} \frac{\delta^2 S[\phi_{(0)}]}{\delta\phi_{(0)}(-p)\, \delta\phi_{(0)}(-q)}\Big|_{\phi_{(0)}=0} \\ &= -\frac{(2\pi)^d C L^{d-1} \delta^d(p+q)}{\epsilon^{2\Delta-d}} \left(\frac{d}{2} + \frac{\epsilon|p| K'_\nu(\epsilon|p|)}{K_\nu(\epsilon|p|)} \right)\ . \end{aligned} \tag{4.34}$$

The Bessel function $K_\alpha(u)$ can be defined by $K_\alpha(u) = \frac{\pi}{2} \frac{I_{-\alpha}(u) - I_\alpha(u)}{\sin(\alpha\pi)}$ for $\alpha \notin \mathsf{Z}$, and by a limit otherwise. They have the following asymptotic behavior. At the boundary $u \to 0$:

$$I_\alpha(u) \sim \frac{1}{\Gamma(\alpha+1)} \left(\frac{u}{2}\right)^\alpha\ , \qquad K_\alpha(u) \sim \begin{cases} -\log\left(\frac{u}{2}\right) - \gamma & \alpha = 0\ , \\ \frac{\Gamma(\alpha)}{2} \left(\frac{2}{u}\right)^\alpha & \alpha > 0\ . \end{cases} \tag{4.27}$$

At the horizon $u \to +\infty$:

$$I_\alpha(u) \sim \frac{e^u}{\sqrt{2\pi u}}\ , \qquad K_\alpha(u) \sim \sqrt{\frac{\pi}{2u}}\, e^{-u}\ . \tag{4.28}$$

[30]Notice that our conventions are such that

$$\int d^d x \mathcal{O}(x) \phi_{(0)}(x) = \int \frac{d^d p}{(2\pi)^d} \phi_{(0)}(-p) \mathcal{O}(p). \tag{4.33}$$

We should expand this for $\epsilon \to 0$. The form of the expansion depends on whether ν is a positive integer or not. Here we discuss the case where ν is a positive integer (this includes the case of CFT operators of dimension $\Delta = \nu + d/2$).

The expansion of the Bessel function for $u \to 0$ and $\nu \in \mathsf{N}$ is of the form[31]

$$K_\nu(u) \overset{u\to 0}{\sim} u^{-\nu}(a_0 + a_1 u^2 + \ldots) + u^\nu \ln u (b_0 + b_1 u^2 + \ldots)\,, \tag{4.35}$$

with some coefficients functions of ν. We will only need the quotient

$$\frac{2\nu b_0}{a_0} = \frac{(-1)^{\nu+1}}{2^{2(\nu-1)}\Gamma(\nu)^2}. \tag{4.36}$$

Using that expansion in (4.34) we get

$$\begin{aligned}\langle \mathcal{O}(p)\, \mathcal{O}(q)\rangle_\epsilon = &(2\pi)^d C L^{d-1} \delta^d(p+q) \\ &\times \left(\frac{\beta_0 + \beta_1 \epsilon^2 |p|^2 + \cdots + \beta_\nu (\epsilon |p|)^{2(\nu-1)}}{\epsilon^{2\Delta - d}} - \frac{2\nu b_0}{a_0} |p|^{2\nu} \ln(\epsilon |p|)(1 + \mathcal{O}(\epsilon^2)) \right)\,,\end{aligned} \tag{4.37}$$

where the coefficients β_i are ratios of a_k, b_k. The first terms in the bracket correspond to contact terms: their Fourier transform gives terms of the form $\Box^m \delta^d(x-y)$ (see Exercise below).

→ *Exercise 9:* Show that, for β is a positive even integer, the integral

$$I_\beta(\vec{x}) = \int d^d p\, e^{i\vec{p}\cdot\vec{x}}\, |p|^\beta \tag{4.38}$$

can be written as

$$I_{2n}(\vec{x}) = (-\Box)^n I_0 = (2\pi)^d (-\Box)^n \delta^d(\vec{x}) \qquad \text{for } n \in \mathsf{Z}_{\geq 0}$$

and thus it is a contact term.

As we will see in the next section, such contact terms can be removed by local counter-terms and thus be neglected. We thus focus on the last terms, where in the $\epsilon \to 0$ limit only the term $\ln(\epsilon|p|)$ contributes. Using (12.34) we obtain the non-local result for the correlator

$$\langle \mathcal{O}(p)\, \mathcal{O}(q)\rangle_\epsilon = -(2\pi)^d C L^{d-1} \frac{(-1)^{\nu+1}}{2^{2(\nu-1)}\Gamma(\nu)^2} \delta^d(p+q) |p|^{2\nu} \ln(\epsilon |p|)\,. \tag{4.39}$$

[31]For non-integer ν, one has to replace $u^\nu \ln u$ by u^ν in the expansion.

Transforming the non-local contribution back to position space yields the ϵ-independent final result[32]

$$\langle \mathcal{O}(x)\, \mathcal{O}(y) \rangle = C L^{d-1} \frac{\Gamma(\Delta)}{\Gamma(\Delta - d/2)} \frac{2\Delta - d}{\pi^{d/2} |x-y|^{2\Delta}} \,, \tag{4.41}$$

matching the expected form of a conformal correlator. For a given theory, we only have to determine C.

Remark: One can carry out the computation for the case ν non-integer. In that case, the power-law behaviour in the two-point function comes from the result of the integral (4.38) for β non-integer[33].

4.3.2 Bulk-to-boundary propagator

The goal of this section is to introduce an important object, which is the bulk-to-boundary propagator. As we will see, it is one of the key ingredients in Witten diagrams. Before going to higher-point, we will use it to revisit the computation of the two-point function. The set-up here is again Euclidean AdS in Poincaré coordinates ($z, x \in \mathsf{R}^d$) but now staying in position space (because it leaves conformal invariance manifest).

Let us go back to the problem of constructing a classical solution of $(\Box - m^2)\phi(x,z) = 0$ (where as usual the relation (4.18) holds) with given boundary conditions

$$\phi(x,z) \overset{z\to 0}{\sim} \phi_{(0)}(x) z^{d-\Delta} + \cdots \tag{4.43}$$

As in classical electrodynamics, we can reformulate this problem by using the integral kernel K_Δ, which we will refer to as the *bulk-to-boundary propagator*,

$$\phi(x,z) = \int_{\partial AdS} d^d y \, K_\Delta(x,y,z)\, \phi_{(0)}(y) \tag{4.44}$$

which satisfies

$$\big(-\Box + m^2 \big) K_\Delta = 0 \,, \qquad \lim_{z\to 0} (K_\Delta(z,x;y) z^{\Delta - d}) = \delta^d(x-y) \,. \tag{4.45}$$

[32] We want to compute the Fourier transforms $\int d^d p \, e^{i\vec{p}\cdot\vec{x}} \, p^{2n} \log|p|$. We first consider the case $n=0$. A scaling argument fixes the behavior $|x|^{-d}$ up to a contact term. The contact term is $-(2\pi)^d \log \frac{|x|}{\mu} \, \delta^d(\vec{x})$. Since, as a distribution, this has to be multiplied by functions that vanish at zero, we neglect the contact term. We find

$$\int d^d p \, e^{i\vec{p}\cdot\vec{x}} \, \log|p| = -2^{d-1} \pi^{d/2} \Gamma\big(\tfrac{d}{2}\big) \frac{1}{|x|^d} \,. \tag{4.40}$$

Other values of n are obtained by acting with $(-\Box)^n$, and give a behavior $|x|^{-d-2n}$.

[33] In that case

$$I_\beta(\vec{x}) = \frac{2^{\beta+d} \pi^{d/2} \Gamma\big(\frac{\beta+d}{2}\big)}{\Gamma\big(-\frac{\beta}{2}\big)} \frac{1}{|x|^{\beta+d}} \qquad \text{for } \beta \notin 2\mathsf{Z}_{\geq 0} \,. \tag{4.42}$$

The Green's function we are looking for is thus a solution on Euclidean AdS (the ball), solution whose boundary value is a δ-function at a point on the boundary (setting for now $y = 0$ this point is at $x = 0$). We first notice that[34]

$$cz^{\Delta}$$

is a solution to the Klein-Gordon equation (c being a constant). This solution is regular everywhere except at $z = \infty$. If we regard $z = \infty$ as the point that compactifies the boundary of AdS from R^d to S^d, we can think of that solution as the one with a δ-function at the point at infinity on S^d. Then we move that point to a finite point, the origin ($x = 0, z = 0$) by an isometry of AdS:

$$z \to \frac{z}{z^2 + x^2} \, , \qquad\qquad x \to \frac{x}{z^2 + x^2} \, . \tag{4.46}$$

We obtain

$$\boxed{K_{\Delta}(x, z; y) = c\left(\frac{z}{z^2 + (x - y)^2}\right)^{\Delta}}\, , \tag{4.47}$$

where the normalization will be fixed below. Note that K is regular in the interior.

→ *Exercise 10:* Show that, in order to reproduce the boundary condition (4.43), the normalization is fixed to

$$c = \Big(\int d^d y \, (1 + y^2)^{-\Delta} \Big)^{-1} = \frac{\Gamma(\Delta)}{\pi^{d/2}\Gamma(\Delta - \frac{d}{2})}. \tag{4.48}$$

Hence, for a given source ϕ_0, the solution to the source-free eom is

$$\phi(x, z) = \frac{\Gamma(\Delta)}{\pi^{d/2}\Gamma(\Delta - \frac{d}{2})} \int_{\partial \mathrm{AdS}} d^d y \left(\frac{z}{z^2 + (x - y)^2}\right)^{\Delta} \phi_{(0)}(y) \, . \tag{4.49}$$

Next recall (see (4.15))

$$\phi(x, z) \to \phi_{(0)}(x)\big(z^{d-\Delta} + \dots\big) + \phi_{(1)}(x)\big(z^{\Delta} + \dots\big) \tag{4.50}$$

where ... are a series expansion in z and depend on x as well. Expanding K, we thus get

$$\phi_{(1)}(x) = \frac{\Gamma(\Delta)}{\pi^{d/2}\Gamma(\Delta - \frac{d}{2})} \int d^d y \, \frac{\phi_{(0)}(y)}{|x - y|^{2\Delta}} \, , \tag{4.51}$$

namely we have expressed $\phi_{(1)}$ as a non-local functional of $\phi_{(0)}$. Thus regularity, which is a local condition at the horizon, appears as a non-local relation at the boundary.

[34] cf. Witten's argument

As we did before, we can plug the solution (4.49) into the action; the latter reduces to a boundary term:

$$S = \frac{1}{2} \int d^d x \, \sqrt{g} g^{zz} \phi(x,z) \partial_z \phi(x,z) \Big|_{z=\epsilon} . \tag{4.52}$$

Inserting ϕ we find many diverging terms, such as $\phi^2_{(0)}(x)$ or other local functions of $\phi_{(0)}(x)$ (we will see holographic renormalization). Those are contact terms that can be removed by local counter-terms. The leading *non-diverging* term, which is also non-local, is

$$S \sim \int d^d x \, \phi_{(0)}(x) \, \phi_{(1)}(x) \sim \int d^d x \, d^d y \, \frac{\phi_{(0)}(x) \, \phi_{(0)}(y)}{|x-y|^{2\Delta}} . \tag{4.53}$$

From that, we can infer the 1-point function:

$$\langle \mathcal{O}(x) \rangle = \frac{\delta S}{\delta \phi_{(0)}(x)} \Big|_{\phi_{(0)}=0} = \phi_{(1)}(x) \Big|_{\phi_{(0)}=0} . \tag{4.54}$$

In AdS this is zero, because with no source also $\phi_{(1)} = 0$. This matches the fact that in CFTs the 1-point functions vanish (because they are dimensionful). However, it is important to recall that the normalizable mode at infinity is the VEV of the operator (indeed it scales with z^Δ). Thus, in general (it is particularly important in non-conformal examples), the two modes are the source and the VEV of $\mathcal{O}$.

Then the 2-point function is

$$\langle \mathcal{O}(x) \, \mathcal{O}(y) \rangle = \frac{\delta^2 S}{\delta \phi_{(0)}(x) \, \delta \phi_{(0)}(y)} \Big|_{\phi_{(0)}=0} = \frac{1}{|x-y|^{2\Delta}} . \tag{4.55}$$

We have reproduced the 2-point functions in another way. We have been sloppy with eliminating divergences. In fact, it turns out that this sloppy way to compute the 2-point function gives the wrong normalization (while the one in momentum space is correct). We will see in section 5 how to perform holographic renormalization properly. Higher-point functions turn out to come out correctly.

4.3.3 Higher-point functions

We now sketch the computation for n-point correlators for a CFT with a gravitational dual given by an effective action in AdS_d.

For simplicity, we consider a set of scalar fields ϕ_i with mass m_i interacting with a local Lagrangian $\mathcal{L}_{\text{AdS}}$. We call $\phi_{0,i}$ the boundary value of the fields at the $z = 0$:

$$\phi_i(x,z) \;\to\; z^{d-\Delta_i} \phi_{0,i}(x) \qquad \text{for } z \to 0 \tag{4.56}$$

and $m_i^2 = \Delta_i(\Delta_i - d)$ ($L \equiv 1$ here). As we have seen, $\phi_{0,i}$ is identified on the field theory side with the source for the operators $\mathcal{O}_i$ of dimension Δ_i.

The CFT connected generating function is the on-shell bulk action evaluated on the classical solutions with prescribed boundary conditions (and regular in the interior). A connected n-point function is thus

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle_c = \frac{\partial^n S}{\partial \phi_{0,1} \dots \partial \phi_{0,n}} \Big|_{\phi_{0,i}=0} . \tag{4.57}$$

Since on-shell fields vanish with sources turned off, only terms in the action with at most n fields can contribute.

To compute higher-point functions, we need interaction terms in the Lagrangian. Consider a massive scalar whose linearized dynamics is given by a coupling to a scalar source $J(x,z)$

$$S = \int d^d x\, dz\, \sqrt{g} \Big[\frac{1}{2} g^{mn} \partial_m \phi \partial_n \phi + \frac{1}{2} m^2 \phi^2 - \phi J \Big] \tag{4.58}$$

In the same spirit as above, the field is given in response to the source by the *bulk-to-bulk* propagator:

$$\phi_\Delta(x,z) = \lambda \int d^d x'\, dz' \sqrt{g}\, G_\Delta(x,z;x',z')\, J(x',z') \,, \tag{4.59}$$

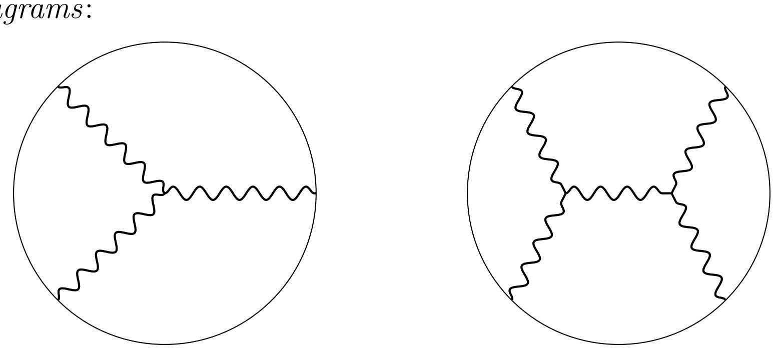
where G satisfies

$$\big(- \Box_{(x,z)} + m^2 \big) G_\Delta(x,z;x',z') = \frac{1}{\sqrt{g}} \delta^d(x-x') \delta(z-z') \,. \tag{4.60}$$

(This Green function vanishes at the boundary.) It turns out that the scalar Green function is a hypergeometric function; see [hep-th/0201253] eqn. (6.12) [DF02]. Note that G should respect the isometries of AdS and so can only depend on the distance between the points (z,x) and (z',x'). One can obtain the expression for the bulk-to-boundary propagator K_Δ by putting one of the points in G_Δ to the boundary; more precisely

$$K_\Delta(x,z;x') = \lim_{z' \to 0} \frac{2\Delta - d}{z'^\Delta} G_\Delta(x,z;x',z') \,. \tag{4.61}$$

The bulk-to-bulk and bulk-to-boudary propagators are the key ingredients in the so-called called *Witten diagrams*:



Because we are solving classical EOMs, these are tree-level[35] diagrams, so they are finite in number.

The "Feynman rules" for Witten diagrams are the following:

- The external sources $\phi_{(0)}(x)$ of composite gauge invariant operators $\mathcal{O}$ in the CFT are located at the conformal boundary (the circle).
- Propagators departing from the external sources go to either another boundary point or to an interior point (bulk-to-boundary)
- The structure of the interior interaction points is governed by the interaction terms in the bulk action.
- The bulk-to-bulk propagator connects two interior interaction points.

For 3-point functions, we only have *one* graph with a cubic vertex and 3 bulk-to-boundary propagators:

$$\begin{aligned}\langle \mathcal{O}_1(x_i)\,\mathcal{O}_2(x_2)\,\mathcal{O}_3(x_3)\rangle &= -\lambda \int d^{d+1}x\,\sqrt{g}\,K_{\Delta_1}(x;x_1)\,K_{\Delta_2}(x;x_2)\,K_{\Delta_3}(x;x_3) \\ &= \frac{A\,\lambda}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_1-x_3|^{\Delta_1+\Delta_3-\Delta_2}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}}\,.\end{aligned} \tag{4.62}$$

The dependence on x_i is eventually fixed by conformal invariance. But to compute A one really needs to do the integral.

The computation of higher-point functions is much more complicated, for two reasons:

- There are various graphs that one has to compute.
- Some of the graphs involve standard bulk-to-bulk propagators, which however involve supergravity fields with spin, in general.

We will not look at those.[36]

[35] Loop diagrams correspond to quantum effects in the bulk, and produce contributions that are suppressed by powers of $1/N^2$. We neglect quantum corrections here, but they have been studied in the literature.

[36] There has been some recent development, see for instance [RZ17, RZ18].

4.4 Another formulation of the dictionary

The relationship (4.4) is one version of the AdS/CFT dictionary, the one which is used most often. There is another version, which is sometimes referred to as the 'extrapolate dictionary' and consisting of computing the bulk correlators and extracting their leading behaviour to give the field theory correlators of the dual operators:

$$\boxed{\lim_{z\to 0} z^{-n\Delta}\langle\phi(x_1,z)\dots\phi(x_n,z)\rangle_{\text{bulk}} = \langle\mathcal{O}(x_1)\dots\mathcal{O}(x_n)\rangle_{\text{CFT}}} \tag{4.63}$$

This dictionary has been shown to be equivalent to (4.4); see [HS11].

5 Holographic renormalization

This section is based on the review by K. Skenderis [Ske02].

In QFT there are UV divergences that need to be removed to obtain sensible finite answers. This is renormalization. One could expect that the same problem arises in gravity doing AdS/CFT, and we saw in section 4 that it is indeed the case. Our way to deal with them was simply to discard divergences, but this is sloppy (since one might lose finite contributions). The correct way to do that (in QFT too) is to subtract divergences by means of local covariant counter-terms.

5.1 Holographic renormalization method

Holographic renormalization is a general method which allows for a careful treatment of divergences associated with the boundary behavior of bulk fields. It is used to make explicit checks of AdS/CFT, in particular to compute correlation functions and anomalies (see section 5.2). We will explain the general method step by step and illustrate it with the example of a massive scalar. We will use the so-called Fefferman-Graham coordinates for AdS space

$$ds^2 = g_{mn} dx^m dx^n = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}\delta_{\alpha\beta} dx^\alpha dx^\beta \,, \tag{5.1}$$

where $\rho = 1/z^2$ and here $L \equiv 1$.

Step 1. Asymptotic solution. The first step is to construct the bulk solution for prescribed, but arbitrary, Dirichlet boundary conditions. One can use the diagrammatic method that we discussed (Witten diagrams).

Suppressing spacetime and internal indices, let us denote bulk fields collectively by $\mathcal{F}(x, \rho)$. Near the boundary, there is an asymptotic expansion (near the boundary at $\rho = 0$)

$$\mathcal{F}(x,\rho) = \rho^m \Big(f_{(0)}(x) + \rho\, f_{(2)}(x) + \ldots + \rho^n \big(f_{(2n)}(x) + \log\rho\, \tilde{f}_{(2n)}(x) + \ldots \big) \Big) \,. \tag{5.2}$$

The two asymptotic solutions have behavior ρ^m and ρ^{m+n}. Here (contrary to before) we assume $n \in \mathsf{Z}_{\geq 0}$, then we get the logarithmic term.

Here:

- $f_{(0)}$ is the source for the dual operator, and it is fixed (it is the boundary condition).
- the EOMs fix $f_{(2)}, \ldots, f_{(2n-2)}$ and $\tilde{f}_{(2n)}$ *algebraically* in terms of $f_{(0)}$ and its derivatives (i.e. they are local functions of $f_{(0)}$).

- $f_{(2n)}$ is not fixed, because it is an independent solution (that coefficient is the Dirichlet boundary condition for a solution which is independent from the one that starts at ρ^m). It can be fixed by asking for regularity in the interior.
- The log term is needed to get a solution. As we will see, the presence of $\tilde{f}_{(2n)}$ is related to conformal anomalies.

Example: Let us again consider the case of a massive scalar field in AdS. The action is given by

$$S = \frac{1}{2} \int d^{d+1}x \sqrt{g} \left(g^{mn} \partial_m \Phi \partial_n \Phi + m^2 \Phi^2 \right) . \tag{5.3}$$

We want to obtain the asymptotic solutions of the EOM

$$(-\Box_g + m^2)\Phi = -\frac{1}{\sqrt{g}} \partial_m (\sqrt{g} g^{mn} \partial_n \Phi) + m^2 \Phi = 0. \tag{5.4}$$

In general, the scalar field couples to Einstein equation through its stress tensor and one needs to solve the coupled system gravity-scalar field equations. Luckily, in this case, the equations decouple near the boundary and one can simply study (5.4) in a fixed gravitational background. Let's look for a solution which goes near the boundary as

$$\Phi(x,\rho) \overset{\rho \to 0}{=} \rho^{\frac{d-\Delta}{2}} \underbrace{\left(\phi_{(0)}(x) + \rho\, \phi_{(2)}(x) + \rho^2\, \phi_{(4)}(x) + \dots \right)}_{\equiv \phi(x,\rho)} . \tag{5.5}$$

Inserting this in (5.4), one gets

$$0 = (m^2 - \Delta(\Delta - d))\phi(x,\rho) - \rho \left(\Box_0 + 2(d - 2\Delta + 2)\partial_\rho + 4\rho \partial_\rho^2 \right) \phi(x,\rho) , \tag{5.6}$$

where $\Box_0 = \delta^{\alpha\beta}\partial_\alpha\partial_\beta$ is the boundary Laplace operator.

→ *Exercise 11:* Solving order by order in ρ, recover the famous relation between the mass and the conformal weight of the dual operator. Then show that $\phi_{(2)}$ is determined algebraically by $\phi_{(0)}$ via

$$\phi_{(2)}(x) = \frac{1}{2(2\Delta - d - 2)} \Box_0 \phi_{(0)} . \tag{5.7}$$

Going on, one can find

$$\phi_{(4)}(x) = \frac{1}{4(2\Delta - d - 4)} \Box_0 \phi_{(2)} , \qquad \dots \qquad \phi_{(2n)}(x) = \frac{1}{2n(2\Delta - d - 2n)} \Box_0 \phi_{(2n-2)} . \tag{5.8}$$

This procedure stops, however, when the denominator vanishes, i.e. when $2\Delta - d - 2n = 0$. This can happen for integer Δ in even d or for half integer Δ in odd d. In those cases, one needs to introduce the logarithmic term in (5.5) at order ρ^n to obtain a solution. Let's consider for concreteness $n = 1$, i.e. $\Delta = d/2 + 1$. The new asymptotic expansion is then

$$\phi(x, \rho) = \phi_{(0)} + \rho(\phi_{(2)} + \log \rho\, \psi_{(2)}) + \dots , \tag{5.9}$$

leading now to

$$\psi_{(2)} = -\frac{1}{4}\Box_0 \phi_{(0)} , \tag{5.10}$$

and thus we find that $\phi_{(2)}$ is *not* determined by the field equations[37]. In the general case $\Delta = d/2 + k$ ($k \in \mathsf{Z}$), a similar computation yields

$$\psi_{(2\Delta-d)} = -\frac{1}{2^{2k}\Gamma(k)\Gamma(k+1)}(\Box_0)^k \phi_{(0)} , \tag{5.11}$$

and $\phi_{(2\Delta-d)}$ is *not* determined by the bulk field equations.

Step 2. Regularization. We evaluate the on-shell action on the asymptotic solution. To regularize, we cut the integrals at $\rho = \epsilon$. By doing the integrals, we get a finite number of boundary terms that diverge as $\epsilon \to 0$:

$$S_{\text{reg}}[f_{(0)}; \epsilon] = \int_{\rho=\epsilon} d^d x \sqrt{g_{(0)}} \Big(\epsilon^{-\nu} a_{(0)} + \epsilon^{-\nu+1} a_{(2)} + \dots - \log \epsilon\, a_{(2\nu)} + \mathcal{O}(\epsilon^0) \Big) , \tag{5.12}$$

with ν a positive number that only depends on the scale dimension Δ and again, the log only comes for even d (they give the conformal anomaly [HS98]). Here $a_{(2k)}$ are local functionals of the source $f_{(0)}$, and do not depend on $f_{(2n)}$ (which is not fixed by the boundary conditions). All coefficients in (5.12) are covariant, so they can be cancelled by substracting covariant counterterms, this is what will be done in the next step.

Example: Let us regularize the massive scalar field action (5.3), which on-shell reduces to

$$S_{reg} = -\frac{1}{2}\int_{\rho=\epsilon} d^d x \sqrt{g}\, g^{\rho\rho} \Phi \partial_\rho \Phi . \tag{5.13}$$

Inserting the asymptotic expansion, we get

$$\begin{aligned} S_{reg} &= -\int_{\rho=\epsilon} d^d x \epsilon^{-\Delta+\frac{d}{2}} \left(\frac{1}{2}(d-\Delta)\phi(x,\epsilon)^2 + \epsilon \phi(x,\epsilon)\partial_\epsilon \phi(x,\epsilon) \right) , \\ &= \int_{\rho=\epsilon} d^d x \left(\epsilon^{-\Delta+\frac{d}{2}} a_{(0)} + \epsilon^{-\Delta+\frac{d}{2}+1} a_{(2)} + \dots - \log \epsilon\, a_{(2\Delta-d)} \right) \end{aligned} \tag{5.14}$$

[37] $\phi_{(2)}$ can be related to $\phi_{(0)}$ in a non-local way by using infinitely many derivatives.

where

$$a_{(0)} = -\frac{1}{2}(d-\Delta)\phi_{(0)}^2 \,, \qquad a_{(2)} = -(d-\Delta+1)\phi_{(0)}\phi_{(2)} = -\frac{d-\Delta+1}{2(2\Delta-d-2)}\phi_{(0)}\Box_0\phi_{(0)},$$
$$a_{(2\Delta-d)} = -\frac{d}{2^{2k+1}\Gamma(k)\Gamma(k+1)}\phi_{(0)}(\Box_0)^k\phi_{(0)} \,, \tag{5.15}$$

illustrating that the coefficients $a_{(2\nu)}$ of the divergent terms are indeed local functions of the source $\phi_{(0)}$.

Step 3. Counterterms. One defines a covariant "counterterm" action of the fields $\mathcal{F}(x,\epsilon)$ (and their derivatives) and the induced metric on the hyperplane given by $\rho = \epsilon$, in such a way that it cancels the divergent terms:

$$S_{\rm ct}[\mathcal{F}(x,\epsilon);\epsilon] = -\text{divergent terms of } S_{\rm reg}[f_{(0)};\epsilon]\,. \tag{5.16}$$

In practice one has to invert the asymptotic expansion (5.2) and find a formula for

$$f_{(0)} = f_{(0)}\big(\mathcal{F}(x,\epsilon),\epsilon\big) \tag{5.17}$$

then plug it in $a_{(2k)}(f_{(0)})$ and finally insert the latter coefficients in (5.12).

Example: To obtain the counterterm, we need to invert the series (5.5). Up to second order we find

$$\begin{aligned}\phi_{(0)} &= \epsilon^{-\frac{d-\Delta}{2}}\left[\Phi(x,\epsilon) - \frac{\Box_\gamma\Phi(x,\epsilon)}{2(2\Delta-d-2)}\right]\\ \phi_{(2)} &= \epsilon^{-\frac{d-\Delta}{2}-1}\frac{\Box_\gamma\Phi(x,\epsilon)}{2(2\Delta-d-2)}\,,\end{aligned} \tag{5.18}$$

where $\Box_\gamma = \gamma^{\alpha\beta}\partial_\alpha\partial_\beta$ is the Laplacian of the induced metric $\gamma_{\alpha\beta} = \frac{1}{\epsilon}\delta_{\alpha\beta}$ at $\rho = \epsilon$ and $\Box_\gamma = \epsilon\,\Box_0$ was used. These terms are enough to rewrite $a_{(0)}$ and $a_{(2)}$ in terms of $\Phi(x,\epsilon)$. One can then construct the counterterm action:

$$S_{\rm ct}[\Phi(x,\epsilon);\epsilon] = \int d^dx\sqrt{\gamma}\left(\frac{d-\Delta}{2}\Phi^2 + \frac{\Phi\Box_\gamma\Phi}{2(2\Delta-d-2)}\right) + \dots \tag{5.19}$$

where the dots mean higher derivative terms. Notice that when $\Delta = \frac{d}{2}+1$ the coefficient of the second term is replaced by $-\frac{1}{4}\log\epsilon$. Similarly, when $\Delta = \frac{d}{2}+k$, there is a k-derivative logarithmic counterterm.

Step 4. Renormalized on-shell action. We first define a subtracted action at the cutoff

$$S_{\text{sub}}\big[\mathcal{F}(x,\epsilon),\epsilon\big] \equiv S_{\text{reg}}[f_{(0)};\epsilon] + S_{\text{ct}}\big[\mathcal{F}(x,\epsilon);\epsilon\big]\ . \tag{5.20}$$

This has a finite limit as $\epsilon \to 0$, and the renormalized action is a functional of the sources defined by this limit, namely

$$\boxed{S_{\text{ren}}[f_{(0)}] = \lim_{\epsilon\to 0} S_{\text{sub}}\big[\mathcal{F}(x,\epsilon),\epsilon\big]}\,. \tag{5.21}$$

Notice that the first step is important as the variations required to obtain correlation functions are performed *before* taking the $\epsilon \to 0$ limit.

Example: The renormalized action is given by the limit $\epsilon \to 0$ of the sum of (5.14) and (5.19). Notice that one still has the freedom to add finite counterterms; this corresponds to the "scheme dependence" in the field theory (scheme = choice of counterterms).

Exact 1-point functions. The 1-point function of the operator $\mathcal{O}_{\mathcal{F}}$ is the presence of source (hence the subscript s) is defined as

$$\langle \mathcal{O}_{\mathcal{F}}(x)\rangle_s = \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon^{d/2-m}\sqrt{\gamma}}\frac{\delta S_{\text{sub}}}{\delta \mathcal{F}(x,\epsilon)}\right) \equiv \frac{1}{\sqrt{g_{(0)}}}\frac{\delta S_{\text{ren}}}{\delta f_{(0)}(x)}\ . \tag{5.22}$$

We can schematically write it as in the last expression (notice we do not set the source to zero), but we should really take the variation first and the limit after. The limit gives

$$\langle \mathcal{O}_{\mathcal{F}}\rangle_s \sim f_{(2n)} + C(f_{(0)})\ . \tag{5.23}$$

The coefficient in front of $f_{(2n)}$ depends on the theory, but it is scheme independent. $C(f_{(0)})$ is a local function of $f_{(0)}$, so it gives contact terms. Its exact form is scheme dependent (and usually can be removed by suitable finite counterterms).

Example: The 1-point function (5.22) for our massive scalar case is thus (since $m = \frac{d-\Delta}{2}$)

$$\langle \mathcal{O}_{\Phi}(x)\rangle_s = \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon^{\Delta/2}\sqrt{\gamma}}\frac{\delta S_{\text{sub}}}{\delta \Phi(x,\epsilon)}\right). \tag{5.24}$$

For concreteness let's consider the case $\Delta = \frac{d}{2}+1$, where the asymptotic is (5.9).

$$\begin{aligned}\delta S_{\text{sub}} &= \delta S_{reg} + \delta S_{ct}\\ &= \int_{\rho\geq\epsilon} d^{d+1}x\sqrt{g}\,\delta\Phi(-\Box_G + m^2)\Phi + \int_{\rho=\epsilon} d^dx\sqrt{\gamma}\,\delta\Phi(-2\epsilon\partial_\epsilon\Phi + (d-\Delta)\Phi - \frac{1}{2}\log\epsilon\,\Box_\gamma\Phi)\,.\end{aligned} \tag{5.25}$$

Using the bulk field EOMs, we get

$$\frac{1}{\sqrt{\gamma}}\frac{\delta S_{sub}}{\delta\Phi} = -2\epsilon\partial_\epsilon\Phi + (d-\Delta)\Phi - \frac{1}{2}\log\epsilon\Box_\gamma\Phi. \tag{5.26}$$

Inserting this in (5.24) and replacing Φ, we find that the divergent terms cancel as they should, and that the finite part is

$$\langle\mathcal{O}_\Phi(x)\rangle_s = -2(\phi_{(2)} + \psi_{(2)})\,. \tag{5.27}$$

As promised, the 1-pt function depends on the part of the asymptotic solution that is not determined by the near-boundary analysis (the free data $\phi_{(2)}$). $\psi_{(2)}$ is a local function of the sources, see (5.10); those are the contributions we called $C(\phi_{(0)})$ in (5.23) (they give rise to contact terms in the correlation functions). Actually, this term is scheme dependent; indeed, adding the finite counterterm

$$S_{new} = -\frac{1}{4}\int d^dx\,\phi_{(0)}\Box_0\phi_{(0)} \tag{5.28}$$

in the action we can remove completely the factor of $\psi_{(2)}$ from the VEV.

Finally, note that the computation for general Δ gives

$$\langle\mathcal{O}_\Phi(x)\rangle_s = -(2\Delta-d)\phi_{(2\Delta-d)} + C(\phi_{(0)})\,. \tag{5.29}$$

***n*-point functions.** To compute n-point functions, we need the exact (as opposed to asymptotic) solutions to the EOMs with prescribed boundary conditions. Regularity in the interior fixes $f_{(2n)}$ as a non-local function of $f_{(0)}$. In general the exact solution cannot be found. But to compute an n-point function we only need a perturbative expression of $f_{(2n)}$ to order $n-1$ in $f_{(0)}$.

n-point functions can be computed using

$$\begin{aligned}\Big\langle\mathcal{O}(x_1)\ldots\mathcal{O}(x_n)\Big\rangle &= \frac{1}{\sqrt{g_{(0)}(x_1)\ldots g_{(0)}(x_n)}}\;\frac{\delta^n S_{\rm ren}}{\delta f_{(0)}(x_1)\ldots\delta f_{(0)}(x_n)}\Big|_{f_{(0)}=0}\\ &= \frac{1}{\sqrt{g_{(0)}(x_2)\ldots g_{(0)}(x_n)}}\;\frac{\delta\langle\mathcal{O}(x_1)\rangle_s}{\delta f_{(0)}(x_2)\ldots\delta f_{(0)}(x_n)}\Big|_{f_{(0)}=0}\,.\end{aligned} \tag{5.30}$$

In other words, once we have the 1-point function with sources, we use that one to compute higher-point functions.

Example: See section 5.7 of [Ske02] for details.

5.2 Metric and the Weyl anomaly

We can apply the above method to the metric. The starting point will be the Einstein-Hilbert action (in Euclidean signature) with a Gibbons-Hawking boundary term:

$$S = -\frac{1}{16\pi G}\int d^{d+1}x\sqrt{g}\left(R + \frac{d(d-1)}{L^2}\right) - \frac{1}{8\pi G}\int d^dx\sqrt{\gamma}\,K, \tag{5.31}$$

with K the extrinsic curvature and γ the induced metric on the boundary.

As we did for the scalar field, we will now want a boundary expansion of the metric as it deviates from the AdS spacetimes. This expansion is given by the famous Fefferman-Graham theorem [FG85].

Fefferman-Graham expansion: The Fefferman-Graham theorem [FG85] says that it is always possible to bring an asymptotically AdS metric to the Fefferman-Graham form (we set the AdS radius $L = 1$)

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}g_{\alpha\beta}(x,\rho)dx^\alpha dx^\beta \qquad\qquad g_{\alpha\beta}(x,\rho) \text{ smooth at } \rho \to 0 \tag{5.32}$$

where

$$g_{\alpha\beta}(x,\rho) \overset{\rho\to 0}{=} g_{(0)\alpha\beta}(x) + \rho g_{(2)\alpha\beta}(x) + \ldots + \rho^{d/2}g_{(d)\alpha\beta}(x) + \underbrace{\rho^{d/2}\log\rho\, h_{(d)\alpha\beta}(x)}_{\text{if } d \text{ even}} + \mathcal{O}(\rho^{\frac{d}{2}+1})\,. \tag{5.33}$$

Using Einstein's equations, explicit computation shows that coefficients multiplying *odd* powers of z vanish up to z^d. Moreover, the logarithmic term appears only in even d.

Euclidean AdS spacetimes is recovered for $g_{\alpha\beta}(x,\rho) = \delta_{\alpha\beta}$. In this expansion $g_{\alpha\beta}(x,\rho)$ is allowed to depend on ρ and may generate a boundary curvature. One can pick the boundary metric $g_{(0)\alpha\beta}$ to be a representative of the conformal equivalence class, e.g. choose it to be the flat metric.

The tensors $g_{(k)\alpha\beta}(x)$ are constructed from the boundary metric $g_{(0)\alpha\beta}$, its curvature and covariant derivative.

$\rightarrow$ *Exercise 12:* Inserting the AdS metric into the Einstein equation, show that

$$g_{(2)\alpha\beta} = \frac{1}{d-2}\big(R_{\alpha\beta} - \frac{1}{2(d-1)}Rg_{(0)\alpha\beta}\big)\,, \tag{5.34}$$

which is a conformally covariant tensor.

Holographic renormalization for gravity: Einstein's equations can be solved order by order in ρ. The equations fix $g_{(2)}$, …, $g_{(d-2)}$, $h_{(d)}$ (present only for d even) as well as part of $g_{(d)}$ in terms of $g_{(0)}$. In particular the equations fix the divergence and the trace of $g_{(d)}$. Those terms are fixed *algebraically* in terms of $g_{(0)}$ and its derivatives, with no need to solve any differential equation. In other words, they are local functions of $g_{(d)}$. As we will see, all these terms are part of the non-normalizable mode, and $g_{(0)}$ is the source for the boundary stress tensor.

The other components are free (and fixed by regularity in the interior).[38] In other words, the other components of $g_{(d)}$ are non-local functions of $g_{(0)}$. The mode $g_{(d)}$ is the normalizable mode, corresponding to the VEV of the stress tensor. Thus, the divergence and trace of the stress tensor are fixed local functions of the boundary metric: this the *conformal anomaly*.

Conformal anomalies: CFTs have a Weyl anomaly: conformal invariance is broken when the theory is coupled to an external metric (and it cannot be restored by modifying details of the coupling to the background field):

$$\langle T^\mu_\mu \rangle \neq 0 \qquad \text{if} \qquad g_{\mu\nu} \neq 0 \, . \tag{5.35}$$

Conformal anomalies are non-trivial contributions to the trace of the energy-momentum tensor involving the curvature. In odd dimension there is no Weyl anomaly since it is impossible to construct scalars of odd dimension using just the curvature. In even dimension one can prove that the only expression which is a local invariant of the metric, with the correct dimension (since the energy-momentum tensor has scaling dimension d, the anomaly in d dimensions should be a scalar of dimension d), and that cannot be removed by local counterterms is (see e.g. [DS93])

$$\langle T^\mu_\mu \rangle = \# \, E_{(d)} + \# \, I_{(d)} \tag{5.36}$$

where $E_{(d)}$ is the d-dimensional Euler density[39] and $I_{(d)}$ is a conformal (Weyl) invariant[40]. The space of conformal invariants of dimension d increases with dimension.

[38]This is strictly speaking true with no matter. With matter this is still true if the fields correspond to marginal or relevant operators. Fields corresponding to irrelevant operators (large mass) should have infinitesimal sources in order not to destroy AdS.

[39]The Euler density, up to normalization, is: $E_{(2n)} = \frac{1}{2^n} R_{i_1 j_1 k_1 l_1} \dots R_{i_n j_n k_n l_n} \epsilon^{i_1 j_1 \dots i_n j_n} \epsilon^{k_1 l_1 \dots k_n l_n}$. Its integral on a $2n$ compact manifold gives the Euler characteristic of that manifold, which is a topological invariant. In fact $\delta E / \delta g$ is a total derivative.

For instance in $d=2$, the Euler number is $\chi = \frac{1}{4\pi} \int d^2x \sqrt{g} \, R = \int d^2x \sqrt{g} \, \frac{1}{8\pi} R_{\mu\nu\rho\sigma} \epsilon^{\mu\nu} \epsilon^{\rho\sigma}$.

[40]These terms are referred to as type A and type B anomaly, respectively.

In $d = 2$ there are no conformal invariants of the correct dimension, and we only have the Euler number. Indeed:

$$\langle T^\mu_\mu \rangle = -\frac{c}{24\pi} R \tag{5.37}$$

where c is the central charge.
In $d = 4$ there is one conformal invariant (of correct dimension):

$$\langle T^\mu_\mu \rangle = -\frac{1}{16\pi^2}(aE_4 + cI_4) \tag{5.38}$$

with[41]

$$\begin{aligned} E_4 &= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \\ I_4 &= -R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 2R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2 = -C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} \end{aligned} \tag{5.39}$$

and $C_{\mu\nu\rho\sigma}$ is the Weyl tensor (traceless part of the Riemann tensor; note that the Weyl tensor vanishes for $d \leq 3$). Then a, c are called *4d central charges*.

Many explicit calculation methods (e.g. heat-kernel approach) can be used to calculate the coefficients in field theory at *weak* coupling. However they receive quantum corrections and are very hard to determine at strong coupling. To lowest order in perturbation theory, they are determined by the number of scalar N_s, fermionic N_f and vector fields N_v present in the field theory under consideration:

$$c = \frac{1}{120}(N_s + 6N_f + 12N_v)\,, \qquad a = \frac{1}{360}(N_s + 11N_f + 62N_v)\,. \tag{5.40}$$

For a theory with $\mathcal{N} = 1$ supersymmetry with N_Φ chiral multiplets and N_V vector multiplets, they can be expressed as

$$c = \frac{1}{24}(N_\Phi + 3N_V)\,, \qquad a = \frac{1}{48}(N_\Phi + 9N_V)\,. \tag{5.41}$$

Remarkably, holographic renormalization allows to compute anomalies. Applying the procedure to the metric one can compute the 1-point function of the stress tensor:

$$\begin{aligned} \langle T_{\alpha\beta}(x) \rangle &= -\frac{2}{\sqrt{g_{(0)}}} \frac{\delta S_{\text{ren}}}{\delta g^{\alpha\beta}_{(0)}(x)} \\ &= \lim_{\epsilon \to 0} -\frac{2}{\sqrt{g(\epsilon, x)}} \frac{\delta S_{\text{sub}}}{\delta g^{\alpha\beta}(x)}\,. \end{aligned} \tag{5.42}$$

One finds (see [dHSS01, HS98] for details):

$$\langle T_{\alpha\beta} \rangle = \frac{dL^{d-1}}{16\pi G} g_{(d)\alpha\beta} + X_{\alpha\beta}(g_{(n)})\,, \tag{5.43}$$

[41] E_4 thus satisfies $\int d^4x \sqrt{g} E_4 = 4\pi\chi$.

where $X_{\alpha\beta}(g_{(n)})$ is a function of $g_{(n)}$ for $n < d$ whose form depends on the spacetime dimension. As we discussed, $g_{(d)}$ is partially fixed by the EOMs: they fix the divergence and the trace of $g_{(d)}$ (with sources). This info turns out to be enough in order to compute the divergence and the trace of the stress tensor.

d=2: One does find an expression of the form (5.37) with

$$c = \frac{3L}{2G} \,, \tag{5.44}$$

where the value of the central charge matches previous computations from the asymptotic symmetry algebra (Virasoro) of AdS_3 gravity.

d=4: The expression is of the form (5.38) with

$$a = c \sim \, L^3 \, \sim \, \Lambda^{-3/2} \,. \tag{5.45}$$

Thus, theories with a weakly-coupled holographic dual have $a = c$ at leading order. One can generate $a \neq c$ with higher-derivative corrections, but in order to stay within weakly-coupled gravity the deviation has to be small. As we will see, the conformal anomalies $a = c$ from IIB supergravity on $AdS_5 \times S^5$ have been computed (strong coupling result), and they match (at leading order N^2) the ones of the dual theory side ($\mathcal{N} = 4$ SYM), which are one-loop exact.

6 Large N limits in gauge theory: a precursor of the duality

See also Section 3 of Marco Serone's lecture notes for "QFT II".

Certain classes of theories simplify when we take a large number of fields, or a large gauge group, or large central charges. The large N limit is a particular perturbative analysis of quantum field theories with an internal symmetry group such as $SO(N)$, $U(N)$ or $SU(N)$. One performs an expansion of in $1/N$ (or $1/N^2$), which is treated as a small parameter. This technique is used in QCG (even though the gauge group is $SO(3)$, and also used a lot in condensed matter physics.

As first pointed out by 't Hooft, the expansion of non-Abelian gauge theory in $1/N$ rearranges the Feynman diagrams in such a a way that they correspond to a string theory expansion with string coupling $1/N$. This suggest an equivalence of these theories, at least at large N.

6.1 $4d$ YM at large N

Yang-Mills in $4d$ has no dimensionless coupling: the gauge coupling is classically marginal but quantum mechanically it is dimensionally transmuted into $\Lambda_{\rm QCD}$ (the strong coupling scale) which is a mass scale. Thus, there is no small parameter to expand on to understand the physics at energies around $\Lambda_{\rm QCD}$. However, if we consider YM for the gauge group $SU(N)$, described by the action

$$S_{\rm YM} = -\frac{1}{2g_{\rm YM}^2}\int d^4x \,{\rm tr}\, F_{\mu\nu}F^{\mu\nu} \,, \tag{6.1}$$

we can play with the parameter N and take the limit $N \to \infty$.

How do we scale $g_{\rm YM}$ as we send $N \to \infty$? Recall that $g_{\rm YM}$ depends on the energy scale μ as (i.e. the one-loop beta function is)

$$\beta(g_{\rm YM}) \equiv \frac{\partial g_{\rm YM}}{\partial \log\mu} = -\frac{11}{3}N\frac{g_{\rm YM}^3}{16\pi^2} + \mathcal{O}(g^5) \,. \tag{6.2}$$

The minus sign tells us that the gauge coupling gets stronger as we flow to longer length scales, while it is weaker at short distance scales thus at high energy (asymptotic freedom). In an asymptotically free theory, it is natural to keep $\Lambda_{\rm QCD}$ fixed. The latter is given by

$$\Lambda_{\rm QCD} = \Lambda_{\rm UV}\exp\left(-\frac{3}{22}\frac{16\pi^2}{g_{\rm YM}^2 N}\right) \,. \tag{6.3}$$

In order to have a parametric separation between the physical scale Λ_{QCD} and the UV cut-off, we should take $N \to \infty$ while keep fixed the 't Hooft coupling

$$\boxed{\lambda \equiv g_{\text{YM}}^2 N}. \tag{6.4}$$

That is the 't Hooft limit. Then

$$\frac{\partial \lambda}{\partial \log \mu} = -\frac{22}{3}\frac{\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3) \ . \tag{6.5}$$

Hence the limit $N \to \infty$ with λ fixed exists and is non-trivial since the corresponding field theory is not free as we can see from (6.5). In particular, the effective coupling constant in the large N limit is not g_{YM} which goes to zero but rather λ. The same computation is true if we add matter fields in the adjoint representation, at least as long as the theory is still asymptotically free.

In order to illustrate the relation between a field theory expansion in the large N limit and a string theory expansion, let us consider a toy model involving only a scalar field Φ in the **adjoint** representation of the gauge group, $\Phi = \Phi^a T_a$; writing the indices of the matrices T_a, we have

$$\Phi^i_j \equiv \Phi^a (T_a)^i_j. \tag{6.6}$$

The generators T_a themselves are taken to be in the fundamental representation. In this toy model, we will assume that the interaction vertices mimic the ones of YM theory: the cubic vertex is proportional to g_{YM} and the quartic vertex to g_{YM}^2 (in canonical normalisation). The Lagrangian is of the form

$$\mathcal{L} = -\frac{1}{2}\text{Tr}(\partial_\mu \Phi \partial^\mu \Phi) + g_{YM}\text{Tr}(\Phi^3) + g_{YM}^2 \text{Tr}(\Phi^4) \ , \tag{6.7}$$

Redefining the fields ($\Phi \to \Phi / g_{YM}$), one can bring all dependence on g_{YM} in front:

$$\mathcal{L} = \frac{1}{g_{\text{YM}}^2}\left[-\frac{1}{2}\text{Tr}(\partial_\mu \Phi \partial^\mu \Phi) + \text{Tr}(\Phi^3) + \text{Tr}(\Phi^4)\right] \ . \tag{6.8}$$

We ask what is the large N limit. Naively,

$$\frac{1}{g_{\text{YM}}^2} = \frac{N}{\lambda} \tag{6.9}$$

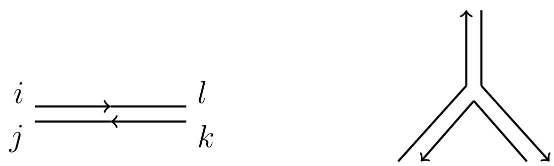
sending $N \to \infty$ with λ fixed gives a divergent coefficient, but this is tricky because also the number of components in the fields diverges. In fact, a subtle cancellation mechanism between these two infinities will take place.

For a theory of $U(N)$ or $SU(N)$ matrices, the propagators are[42]

$$\langle \Phi^i_j(x)\Phi^k_l(y)\rangle_{U(N)} \propto \delta^i_l\delta^k_j \frac{g^2_{YM}}{(x-y)^2}\,, \qquad\qquad \langle \Phi^i_j(x)\Phi^k_l(y)\rangle_{SU(N)} \propto \delta^i_l\delta^k_j - \frac{1}{N}\delta^i_j\delta^k_l \tag{6.10}$$

respectively[43]. Since for $SU(N)$ the second term is subleading, we can safely neglect it a leading order.

To draw Feynman diagrams, we employ a double-line notation to keep track of the matrix indices, in which an adjoint field Φ^a is represented as a formal product of a fundamental and an anti-fundamental field:



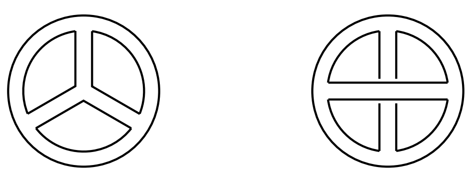
The arrow on each line points from an upper to a lower index and are of opposite direction to indicate that they are associated to complex conjugate representations.

From (6.10), we see that each propagator gives a factor $g^2_{\rm YM} = \lambda/N$ and from (6.8) that each vertex gives a factor $1/g^2_{\rm YM} = N/\lambda$.

Let's count the power of N and λ associated with each diagram. Each closed line represents a sum over matrix indices, therefore gives a factor of N. Thus, each diagram with V vertices, P propagators and L closed lines contributes with

$$N^{V-P+L}\lambda^{P-V}\,. \tag{6.11}$$

We can transform a diagram into a two-dimensional surface, completing the closed lines with non-intersecting faces. *E.g.*,



[42] The second one is derived imposing that the propagator annihilates $M^i_j = \delta^i_j$.

[43] This comes from the $\mathfrak{u}(N)$ completeness relation

$$\sum_{a=1}^{N^2} (T_a)^i{}_j (T_a)^k_l = \delta^i_l\delta^k_j,$$

while for $\mathfrak{su}(N)$,

$$\sum_{a=1}^{N^2-1} (T_a)^i{}_j (T_a)^k_l = \delta^i_l\delta^k_j - \frac{1}{N}\delta^i_j\delta^k_l\,.$$

.

We have

$$\#\ \text{Prop} \;\rightarrow\; \#\ \text{Edges}\,(E)\ , \qquad\qquad \#\ \text{Closed lines} \;\rightarrow\; \#\ \text{Faces}\,(F)\ . \tag{6.12}$$

Since the double-lines are oriented, the resulting surface is oriented. Consider first the case of connected vacuum diagrams. The resulting surface is a compact, closed, oriented surface. We can write

$$N^{V-E+F}\lambda^{E-V} = N^{\chi}\lambda^{E-V}, \tag{6.13}$$

with χ is precisely the Euler characteristic $\chi = V - E + F = 2 - 2g$ related to the genus g of the $2d$ surface.

Thus, the perturbative expansion of any diagram in field theory may be written as a double expansion of the form

$$iW = \ln Z = \sum_{g=0}^{\infty} N^{2-2g} \sum_{i=0}^{\infty} c_{g,i}\lambda^{i} = \sum_{g=0}^{\infty} N^{2-2g}\ \ f_g(\lambda)\,, \tag{6.14}$$

with f_g some polynomial in λ. Therefore, in a large N limit in which $N \to \infty$ with $\lambda = g^2 N$ fixed ('t Hooft limit), any computation will be dominated by the surfaces of minimal genus. All these *planar diagrams* will give a contribution of order N^2, while other diagrams are suppressed by powers of $1/N^2$. Notice that the above argument is also true for any gauge theory coupled to adjoint matter fields, like $\mathcal{N} = 4$ SYM theory.

The form of the expansion (6.14) is the same as one finds in a perturbative theory with closed oriented strings, if we identify $1/N$ as the string coupling constant g_s. The analogy of (6.14) with perturbative string theory is one of the strongest motivations for believing that field theories and string theories are related, and it suggests that this relation would be more visible in the large N limit where the dual string theory may be weakly coupled. However, since the analysis was based on perturbation theory which generally does not converge, it is far from a rigorous derivation of such a relation, but rather an indication that it might apply, at least for some field theories.

While we have derived the behavior (6.14) only for vacuum diagrams, it actually holds for any correlation function involving the so-called "single-trace operators":[44] namely gauge-invariant operators that cannot be written as products of other gauge-invariant operators. For instance it can be of the form

$$\mathcal{O}_j(x) \equiv \frac{1}{N}\,\text{Tr}\left(\prod_{i=1}^{j}\Phi_i(x)\right)\ . \tag{6.15}$$

[44]For multi-trace operators, the connected 2-point function in general scales with some positive power of N. Therefore the operators have to be normalized dividing by some powers of N, and then the correlators are suppressed.

To compute correlation function, we simply add them with source terms to the Lagrangian:

$$\mathcal{L} \to \mathcal{L} + N \sum_j g_j \mathcal{O}_j \ , \tag{6.16}$$

so if we keep g/N fixed, we have the same large N scaling as before, thus the new partition function is dominated by planar diagrams. If we look at connected diagrams instead, we find

$$\langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle_c = \frac{1}{i^{n-1} N^n} \frac{\delta^n W}{\delta g_1 \ldots \delta g_n} |_{g_i=0} \sim N^{2-n} \ . \tag{6.17}$$

Restricting to connected diagrams, the operators are correctly normalized (2-point functions are of order 1) and the 3-point function goes like $1/N$. Thus, we can interpret the operators $\mathcal{O}_j$ as vertex operator insertions on the string worldsheet, again if $\frac{1}{N}$ is the string coupling g_s.

The argument generalizes in many ways.

If we add matter in the fundamental representation of $U(N)$, we have propagators with a single line. This gives rise to diagrams with boundaries. This suggests that at large N one gets a string theory with open strings (and D-branes).

For $SO(N)$ and $USp(N)$ gauge group, the adjoint representation can be represented by the product of two fundamental representations, and the fundamental is real. So there is a double-line notation with no arrows, leading to non-orientable surfaces. This suggests that one could get non-orientable strings.

The previous arguments suggest that certain gauge theories at large N should behave as string theories. Notice however that it does not tell us *which* string theory is dual to a particular gauge theory.

6.2 Extra dimension

Attempts to directly construct a string theory equivalent to a $4d$ gauge theory are plagued with the well-known problems of string theory in four dimensions (or generally below the critical dimension). In particular, consistency of the theory requires to add fields on the worldsheet beyond the four embedding coordinates of the string. In the standard quantization of four dimensional string theory an additional field called the Liouville field arises, which may be interpreted as a fifth space-time dimension. Polyakov has suggested that such a five dimensional string theory could be related to four dimensional gauge theories if the couplings of the Liouville field to the other fields take some specific forms. As we will see, the

AdS/CFT correspondence realizes this idea, but with five additional dimensions (in addition to the radial coordinate on AdS which can be thought of as a generalization of the Liouville field), leading to a standard (critical) ten dimensional string theory.

7 4d $\mathcal{N} = 4$ SYM and strings on $\mathrm{AdS}_5 \times S^5$

We have argued that d-dimensional matrix QFTs at large N should/could be described by a string theory (theory of gravity) in $d+1$ dimensions. In order to find examples, it is natural to start with cases with maximal symmetry.

- On the QFT side, we consider conformal theories (the conformal group is larger than Poincaré). We have argued that this should correspond to strings in AdS_{d+1}.

  Indeed the conformal group is $SO(d,2)$, which is also the isometry group of AdS_{d+1}.

- On the QFT side, we consider supersymmetric theories. They have larger symmetry, and quantum effects are more under control. Supersymmetry is particularly crucial to have control on the string theory side. Thus we are led to SCFTs.

- We should start with the maximal possible number of supercharges, which is 32. Theories are known only in $d = 3, 4, 6$.

  Only in $d = 4$ the theory has Lagrangian description with fully manifest superconformal symmetry. Thus we are led to

$$4\text{d } \mathcal{N} = 4 \text{ SYM}.$$

- The superconformal algebra is

$$PSU(2,2|4) \supset SO(4,2) \times SO(6) . \tag{7.1}$$

What could the dual string theory be? The isometry group leads to the space $\mathrm{AdS}_5 \times S^5$. Luckily, the perturbative superstring is consistent precisely in 10 dimensions, and it turns out that one particular superstring theory — type IIB — has an $\mathrm{AdS}_5 \times S^5$ vacuum solution preserving 32 (all) supercharges.

Indeed

$$4\text{d } \mathcal{N} = 4 \text{ SYM} \qquad \leftrightarrow \qquad \text{type IIB on } \mathrm{AdS}_5 \times S^5$$

is the golden (and most studied) example of the AdS/CFT correspondence. In its stronger form, the correspondence says that these two theories are dynamically equivalent. The two free parameters on the AdS side are the string coupling g_s and the dimensionless ratio L^2/α', where $\alpha' = l_s^2$ (l_s being the string length; the string tension is $(2\pi\alpha')^{-1}$). Note that only the ratio L^2/α' is important rather than the two length scales L (the AdS radius) and l_s separately. The parameters in the 'CFT side' are the rank of the gauge group N and the coupling constant g^2_{YM}. The correspondence identifies the parameters as

$$\boxed{g^2_{YM} = 2\pi g_s} \quad \text{and} \quad \boxed{2\,\underbrace{g^2_{YM}N}_{\lambda} = \frac{L^4}{\alpha'^2}}\,. \tag{7.2}$$

In practice, it is very difficult to do explicit calculations for generic values of the parameters. Hence we take certain limits on both side, having in mind the idea of obtaining new insights into the non-perturbative behavior of one theory from the computable weak coupling behavior of the other.

Strongest form: The duality holds for any parameters (7.2).

Strong form: Since we best understand string theory in the perturbative regime, it is useful to consider the weak coupling $g_s \ll 1$, while keeping $L/\sqrt{\alpha'}$ constant. At leading order in g_s, the AdS side then reduces to *classical* string theory (i.e. only with tree level diagrams). In the field theory side, this means $g_{YM} \ll 1$ while λ stays finite, namely we are taking the 't Hooft limit (large N for fixed λ).

Weak form: On top of that, since we are after strongly coupled field theories, one can consider the limit $\lambda \to \infty$, which corresponds to $\sqrt{\alpha'}/L \to 0$. The string length is then very small compared to the curvature radius: this is the point-particle limit of type IIB string theory, which is given by type IIB *supergravity* (hence weak coupling on the AdS side).

7.1 $4d$ $\mathcal{N} = 4$ SYM

See also Section 6.2 of Matteo Bertolini's lecture notes for "Supersymmetry I".

$\mathcal{N} = 4$ SYM theories in $4d$ are *maximally* supersymmetric: in $4d$, the largest amount of supersymmetry with a particle multiplet representation of spin $s \leq 1$ is $\mathcal{N} = 4$, corresponding to 16 preserved Poincaré supercharges. Indeed, theories with more supersymmetric generators will involve a $s = 2$ field and thus gravity[45]. Any multiplet has to include $s = 1$ particles, thus all $\mathcal{N} = 4$ susy theories must be constructed *only* from the vector multiplet. Thus all particles are massless and there will be no central charges.

The massless $\mathcal{N} = 4$ vector multiplet

$$V_{\mathcal{N}=4} = (A_\mu, \lambda_\alpha^{i=1\ldots,4}, \phi^{I=1,\ldots,6}) \tag{7.3}$$

[45]Each supercharge Q^a_α, $\bar{Q}_{a\dot{\alpha}}$ changes the spin of a state by 1/2. All massless states with helicities between -1 and +1 are thus generated by action with no more than $\mathcal{N} = 4$ different supercharges.

contains a gauge field A_μ, 4 Weyl spinors λ^i_α and 6 real scalars ϕ^I. The R-symmetry group is

$$R\text{-symmetry:} \qquad SU(4) \cong SO(6)\ .$$

Then $A_\mu = \mathbf{1}$ is the singlet representation of $SU(4)$, $\phi_I = \mathbf{6}$ is the fundamental of $SO(6)$ = antisymmetric of $SU(4)$, while $\lambda_i = \mathbf{4}$ is the fundamental of $SU(4)$ = Weyl spinor of $SO(6)$.

Lagrangian from $\mathcal{N}=1$ superspace

On can use $\mathcal{N}=1$ superspace formalism to write down the theory. In order to obtain the full $\mathcal{N}=4$ supersymmetry with R-symmetry group $SU(4)$, the coupling constants and the superpotential of the $\mathcal{N}=1$ formulation have to preserve certain constraints. In this formulation, we need three chiral superfields Φ_i $(i=1,2,3)$ as well as the gauge superfield of field strength $\mathcal{W}_\alpha$:

$$V_{\mathcal{N}=4} = V_{\mathcal{N}=1} \text{ plus three chiral } \Phi_{1,2,3} \text{ in the adjoint.} \tag{7.4}$$

With a vector multiplet we can write a gauge theory: in $\mathcal{N}=4$ the only parameters are the group G, the gauge coupling $g_{\rm YM}$ (if G is simple) and a theta (instanton) angle $\theta_{\rm YM}$[46]. The unique field theory action with $\mathcal{N}=4$ supersymmetry is given by, (with $\mathrm{Tr}(T_aT_b)=\delta_{ab}$),

$$\mathcal{L} = \mathrm{Tr}\left[\int d^4\theta \sum_i \overline{\Phi}_i e^{2g_{\rm YM}V}\Phi_i + \frac{1}{32\pi}\mathrm{Im}\left(\tau\int d^2\theta\, \mathcal{W}^\alpha\mathcal{W}_\alpha\right) - \left(\sqrt{2}g_{\rm YM}\int d^2\theta\, \Phi_1[\Phi_2,\Phi_3] + \text{h.c.}\right)\right]\ . \tag{7.5}$$

There τ is the complexified gauge coupling

$$\tau = \frac{\theta_{\rm YM}}{2\pi} + i\frac{4\pi}{g^2_{\rm YM}}\ . \tag{7.6}$$

A lengthy computation in superspace leads to a Lagrangian schematically of the form (check factors)

$$\mathcal{L} \sim \mathrm{Tr}\Big[-\frac{1}{2g^2_{\rm YM}}F_{\mu\nu}F^{\mu\nu} - \sum_I D_\mu\phi_I D^\mu\phi_I + \bar\lambda^i\bar\sigma^\mu D_\mu\lambda_i + g^2_{\rm YM}\sum_{I,J}[\phi_I,\phi_J]^2 + g_{\rm YM}\bar\lambda^i\Gamma^I\phi_I\lambda_i + \frac{\theta_{\rm YM}}{16\pi^2}\widetilde{F}_{\mu\nu}F^{\mu\nu}\Big]\ , \tag{7.7}$$

with $\widetilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ and Γ^I denotes the structure constant of the $SU(4)$ R-symmetry group (couple two $\mathbf{4}$ representations to one $\mathbf{6}$ representation of $\mathfrak{su}(4)$). The Lagrangian (7.7) is unvariant under supersymmetry transformations, whose form we don't explicit here. Notice that the latter and the Lagrangian can be derived, alternatively, from a dimensional

[46] Note that the θ parameter breaks CP invariance and may be set to zero.

reduction of $\mathcal{N}=1$ SYM in $10d$.

Some properties

- Since the coupling constant is dimensionless and all fields are massless, the action of $\mathcal{N}=4$ SYM is scale invariant at classical level.

- Remarkably, the theory is also scale invariant after quantisation. The one-loop beta-function is given by[47]

$$\frac{\partial g_{\rm YM}}{\partial \log\mu} = -\frac{g_{\rm YM}^3}{16\pi^2}\Big(\frac{11}{3}c(\text{Adj}) - \frac{2}{3}\sum c(\text{Weyl}) - \frac{1}{6}\sum c(\text{scalars})\Big) \ . \tag{7.8}$$

  Here all fields are in the adjoint representation, so all Casimirs are equal and

$$\frac{11}{3} - \frac{2}{3}4 - \frac{1}{6}6 = 0 \ .$$

  In fact the beta-function was checked to be zero up to three loops and believed to vanish to all orders in perturbation theory as well as non-perturbatively: the theory is conformal for all values of τ and there is a complex line of IR fixed points. In other words τ is an exactly marginal deformation.
  The superconformal group is $PSU(2,2|4)$, obtained by adding to the conformal group $SO(2,4)$ the global $SU(4)_R$ generators, the four Weyl supercharges Q^a_α $(a=1,\ldots,4)$ and the four so-called conformal supercharges $S^a_{\dot\alpha}$. That amounts to 32 real supersymmetry generators, 16 supercharges and 16 conformal supersym metries.

- The theory enjoys electric magnetic duality [MO77] in which

$$\tau \ \rightarrow \ -\frac{1}{\tau} \ . \tag{7.9}$$

  Combining this with the invariance under $\tau\to\tau+1$ (shift of $\theta_{\rm YM}$ angle by 2π), one gets invariance under the *S-duality* group $SL(2,\mathsf{Z})$ acting on τ as on the upper half-plane:[48]

$$\tau \rightarrow \frac{a\tau+b}{c\tau+d} \, ; \qquad\qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathsf{Z}) \ . \tag{7.10}$$

  Remarkably, this leads to a *weak-strong* duality as

$$g_{\rm YM} \rightarrow \frac{4\pi}{g_{\rm YM}} \ . \tag{7.11}$$

[47] Here c is the quadratic Casimir of the representation, defined by $\mathrm{Tr}\, T^aT^b = c\,\delta^{ab}$. Notice that $c(\text{Adj})$ is usually called $C_2(G)$.

[48] The element $C=-\mathbb{1}\in SL(2,\mathsf{Z})$ does not act on τ: it corresponds to charge conjugation.

7.2 Type IIB strings on $\mathrm{AdS}_5 \times S^5$

Perturbative bosonic strings require a spacetime of dimension 26 to be consistent at the quantum level, however they have a tachyon (instability of the flat spacetime solution). Superstrings solve the problem. Consistency requires a spacetime of dimension 10, then there is no tachyon in the spectrum, therefore $\mathsf{R}^{9,1}$ is a consistent vacuum.

As in the bosonic case, the spectrum of the closed superstring around $\mathsf{R}^{9,1}$ contains the graviton plus other massless fields of spin < 2, and then an infinite tower of massive modes of masses

$$m_n^2 \sim \frac{n}{\alpha'} \; . \tag{7.12}$$

It turns out that the superstring gives spacetime (target) supersymmetry, thus the spectrum is organized into supermultiplets. There are two closed superstring theories with 32 supercharges: a *non-chiral* called type IIA ($\mathcal{N} = (1,1)$ in 10d) and a *chiral* theory called type IIB ($\mathcal{N} = (2,0)$). For both theories, all massless modes are in one multiplet: the graviton multiplet.

At low energies, the massive string modes decouple, and one is left with an effective theory for the graviton multiplet: *supergravity*. The supergravity EOMs are obtained by requiring that the worldsheet theory remains conformal, and can be described by a Lagrangian[49]. This leads to type IIA and IIB supergravity. Of course, the supergravity approximation as long as

$$L^2 \gg \alpha' \; , \qquad g_s \ll 1 \; , \qquad E \ll \frac{1}{\sqrt{\alpha'}} \; . \tag{7.13}$$

The first condition expresses the fact that the length scale L associated with a field configuration in supergravity must be big compared to the string size. It is equivalent to saying that the scale of the curvature tensor $R_{MNPQ} \sim 1/L^2$ must be small so that higher derivative corrections to the basic SUGRA action are negligible. The second condition expresses the perturbative regime (the string is weakly interacting), and the last one the fact that we are at low energy.

Type IIB supergravity It turns out that IIB supergravity admits a solution $\mathrm{AdS}_5 \times S^5$, so we focus on this type. Due to its field content the theory is chiral and violates parity. The graviton multiplet contains

$$\mathcal{G} = \big(g_{MN}, \phi, B_{MN}, C_{(0)}, C_{MN}, C_{MNRS}, \Psi_{\alpha M}, \psi_\alpha\big) \; . \tag{7.14}$$

- g_{MN}: the metric.

[49] With a caveat as we will see below.

- ϕ: the dilaton.
- $B_{(2)}$: the NS 2-form with field strength $H_{(3)} = dB_{(2)}$.
- $C_{(0)}$: a RR 0-form potential, *i.e.* an axion, with $C_{(0)} \cong C_{(0)} + 2\pi$ and "field strength" $F_1 = dC_{(0)}$.
- $C_{(2)}$: a RR 2-form potential, with $\widetilde{F}_{(3)} = dC_{(2)} - C_{(0)}H_{(3)}$.
- $C_{(4)}$: a RR 4-form potential with $\widetilde{F}_{(5)} = F_{(5)} - \frac{1}{2}C_{(2)} \wedge H_{(3)} + \frac{1}{2}B_{(2)} \wedge F_{(3)}$ and susy imposes that it is self-dual
$$*\widetilde{F}_{(5)} = \widetilde{F}_{(5)} \ . \tag{7.15}$$
- $\Psi_{M\alpha}$, ψ_α: the gravitino and a chiral fermion.

As understood by Green and Schwarz, the gauge-invariant field strengths are modified because the gauge transformations are. The modified large gauge transformation of $C_{(0)}$ is

$$C_{(0)} \to C_{(0)} + 2\pi \ , \qquad\qquad C_{(2)} \to C_{(2)} + B_{(2)} \ . \tag{7.16}$$

In fact this is the element $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ of $SL(2,\mathsf{Z})$. The modified gauge transformations of $B_{(2)}$ and $C_{(2)}$ are

$$\begin{aligned} C_{(2)} &\to C_{(2)} + d\lambda_1 \\ C_{(4)} &\to C_{(4)} + \tfrac{1}{2}\lambda_1 \wedge H_{(3)} \end{aligned} \qquad \text{and} \qquad \begin{aligned} B_{(2)} &\to B_{(2)} + d\widetilde{\lambda}_1 \\ C_{(4)} &\to C_{(4)} - \tfrac{1}{2}\widetilde{\lambda}_1 \wedge dC_{(2)} \ . \end{aligned} \tag{7.17}$$

The bosonic part of the type IIB supergravity action in string frame is

$$\begin{aligned} S^{\text{s.f.}}_{\text{IIB, bos}} = \frac{1}{2\tilde{\kappa}_{10}^2}\Bigg[\int d^{10}x \sqrt{-g}\Big(e^{-2\phi}\left(R + 4|\partial\phi|^2 - \frac{1}{2}|H_{(3)}|^2\right) - \frac{1}{2}|F_1|^2 - \frac{1}{2}|\widetilde{F}_{(3)}|^2 - \frac{1}{4}|\widetilde{F}_{(5)}|^2\Big) \\ - \frac{1}{2}\int C_{(4)} \wedge H_{(3)} \wedge F_{(3)}\Bigg] \ , \end{aligned} \tag{7.18}$$

with $2\tilde{\kappa}_{10}^2 = (2\pi)^7\alpha'^4$ (related to the Newton constant by $\kappa_{10}^2 = 8\pi G_{10} = \tilde{\kappa}_{10}^2 g_s^2$), R the Ricci scalar associated to the target spacetime metric g and the string coupling is related to the expectation value of the dilaton by $g_s = e^{\phi}$ (which can be arbitrary). However one has to supplement the EOMs with the self-duality constraint $*\widetilde{F}_{(5)} = \widetilde{F}_{(5)}$ (not derived from the action[50]). We will not need the fermionic part, which is completely fixed by supersymmetry.

[50]That's why it is said that type IIB sugra has EOMs and not really a Lagrangian.

The full action is invariant under 32 supersymmetries given by two Majorana-Weyl parameters in $10d$.

(7.18) is thus the effective action of type IIB string theory at low energy, restricted to massless modes and two-derivatives. Both parameters α' and g_s can correct the effective action: if we integrate the massive fields, we get higher derivative corrections weighted by powers of α', while quantum corrections will be weighted by powers of g_s. Any physical quantity has an expansion of the form

$$\sum_{g=0}^{\infty} g_s^{2g-2} f_g \left(\frac{\alpha'}{L^2}\right) . \tag{7.19}$$

S-duality Note that IIB supergravity eom are invariant under $SL(2, \mathsf{R})$ transformations ($ad - bc = 1$)

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{7.20}$$

with the complex scalar

$$\tau = C_{(0)} + ie^{-\phi} . \tag{7.21}$$

In the full string theory, the invariance is broken down to $SL(2, \mathsf{Z})$ by instanton effects.[51] As a particular case of the above symmetry, we have S-duality (take $C_{(0)} = 0$) as it maps $g_s \to 1/g_s$.

7.2.1 Dp-branes as extremal p-branes

A Dp-brane is a BPS solution of $10d$ supergravity, i.e. it preserves half of the Poincaré supercharges Q_α of the background. It has a $(p+1)$-dimensional flat hypersurface with Poincaré invariance group $\mathsf{R}^{p+1} \times SO(p, 1)$. The transverse space is then of dimension $D - p - 1$.

A Dp-brane in $10d$ has symmetries $\mathsf{R}^{p+1} \times SO(p, 1) \times SO(9 - p)$. An ansatz which solves the type IIB supergravity EOMs is

$$\begin{aligned} ds^2 &= H_p(r)^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + H_p(r)^{1/2} dy^i dy^i, \\ e^\phi &= g_s H_p(r)^{(3-p)/4}, \\ C_{(p+1)} &= \left(H_p(r)^{-1} - 1\right) dx^0 \wedge \cdots \wedge dx^p, \\ B_{(2)} &= 0, \end{aligned} \tag{7.22}$$

[51] D(-1) string instantons have an action that depends on $C_{(0)}$.

where x^μ ($\mu = 0, \ldots, p$) are the coordinates on the brane worldvolume and y^i ($i = p+1, \ldots, 9$) the coordinates transverse to the brane (and $r^2 = \sum_i y_i^2$). Plugging this ansatz, the EOMs imply that

$$H_p(r) = 1 + \left(\frac{L_p}{r}\right)^{7-p} . \tag{7.23}$$

The characteristic length L_p can be determined from the charge of the Dp-brane solution. As in electromagnetism, it is useful to measure the charge by the flux on a sphere surrounding the source. Hence for an extended $p+1$-dimensional object, the charge is calculated by integrating the R-R flux through the $(8-p)$-dimensional sphere at infinity, which surrounds the pointlike charge in the $(9-p)$-dimensional transverse space,

$$Q = \frac{1}{2\kappa_{10}^2} \int_{S^{8-p}} {}^*F_{(p+2)} . \tag{7.24}$$

with $*$ the ten-dimensional Hodge dual. The charge is the given by Calculating the rhs and setting $Q = N$ units of flux, one gets

$$L_p^{7-p} = (4\pi)^{(5-p)/2} \Gamma(\frac{7-p}{2}) g_s N \alpha'^{(7-p)/2} . \tag{7.25}$$

In type IIB superstring theory, Dp-branes with p *odd* are stable; these solutions are referred to as 'BPS' because their mass is proportional to their charge Q. They are sometimes called extremal p-brane; the horizon of an extremal p-brane collapses on the singularity. However, for the special case of a D3-brane, the metric is actually completely regular.

7.2.2 Low-energy effective action of D-branes

In perturbative string theory, D-branes arise as hyperplanes where open strings can end. The endpoint of a string is charged and thus couples to a gauge field A (of field strength F) living on the D-brane. The dynamics of A is subject to constraints which can be rewritten in terms of EOMs of a Dp-brane action.

For a single Dp-brane in Einstein frame, the bosonic part of action is the Dirac-Born-Infeld (DBI) action, of the form

$$S_{\mathrm{DBI}} = -\tau_p \int d^{p+1}\xi \, e^{-\phi} \sqrt{-\det\left(\gamma_{ab} + \mathcal{F}_{ab}\right)} \tag{7.26}$$

where ξ^a ($a = 0, \ldots p$) denote the coordinates[52] of the worldvolume, the tension of the brane is

$$\tau_p = (2\pi)^{-p} \alpha'^{-(p+1)/2} \tag{7.27}$$

[52] We can choose coordinates aligned with the first $p+1$ coordinates, $x^a = \xi^a$.

and

$$\gamma_{ab} = \partial_a X^M \partial_b X^N g_{MN} \tag{7.28}$$

is the pullback of the bulk metric g_{MN} on the brane. The combination $\mathcal{F}_{ab} = 2\pi\alpha' F_{ab} + B_{ab}$ includes the (pull-back of the) Kalb-Ramond field B and the field strength F_{ab} of a $U(1)$ gauge field living on the brane. Let us consider a simple example where the dilaton is constant $g_s = e^{\phi}$, $B = F = 0$. Then the DBI action reduces to

$$S_{\rm DBI} = -\frac{\tau_p}{g_s} \int d^{p+1}x \, \sqrt{-\det\left(\gamma_{ab}\right)} \tag{7.29}$$

hence the brane tends to minimise its volume. One can view the prefactor $\frac{\tau_p}{g_s}$ as a tension and think of the DBI action as a generalization of the worldsheet action of strings to higher dimensions. However, it is important to keep in mind that, unlike fundamental strings, D-branes are non-perturbative objects.

→ *Exercise 13:* Consider a Dp-brane living in flat space with $B = 0$ and a constant dilaton $g_s = e^{\phi}$. Show that[53], at lowest order in α',

$$S_{\rm DBI} = -(2\pi\alpha')^2 \frac{\tau_p}{4 g_s} \int d^{p+1}x \; F_{ab}F^{ab} \,. \tag{7.31}$$

Thus, at quadratic order, the DBI action for a single Dp-brane reproduces the $U(1)$ YM kinetic term. The YM coupling can be read off as

$$g^2_{\rm YM} = \frac{g_s}{\tau_p (2\pi\alpha')^2} = (2\pi)^{p-2} \alpha'^{\frac{p-3}{2}} g_s. \tag{7.32}$$

Remarks The exact non-Abelian form of the DBI action is not known. However at two-derivative level its form is fixed by symmetries and supersymmetry.

There are also non-trivial couplings to the R-R forms C_q (q even here in IIB). The latter define charges for D-branes. The Dp-brane full action $S = S_{\rm DBI} + S_{\rm WZ}$ involves the Wess-Zumino (WZ):

$$S_{\rm WZ} = \mu_p \int_{{\rm D}p} e^{\mathcal{F}} \wedge \sum_q C_q \,, \qquad \mu_p = \frac{\tau_p}{g_s} \,, \tag{7.33}$$

where only the term in the expansion of $e^{\mathcal{F}} \wedge \sum_q C_q$ with $p+1$ indices should be kept. This action describes the interaction of the R-R forms with the NS-NS field B. Expanding this

[53] Hint: use the expansion of $\det(\mathbb{1} + M) = \exp\log\det(\mathbb{1} + M) = \exp {\rm Tr} \log(\mathbb{1} + M)$:

$$\det(\mathbb{1} + M) = 1 + {\rm Tr}\, M + \frac{({\rm Tr}\, M)^2 - {\rm Tr}\, M^2}{2} + \ldots \tag{7.30}$$

term gives

$$\int_{\mathrm{D}p} \left(C_{p+1} + \mathcal{F} \wedge C_{p-1} + \frac{1}{2} \mathcal{F} \wedge \mathcal{F} \wedge C_{p-3} + \dots \right) .$$

Thus this term reproduces the electric coupling to the potential C_{p+1}, a theta term where C_{p-3} plays the role of the theta angle, as well as other terms.

7.3 The $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence

We can give a physical proof of the correspondence [Mal99], focusing on the weak form. We will thus study the physics at low energies

$$E \ll \alpha'^{-1/2} . \tag{7.34}$$

Then only the massless modes can be excited since other stringy excitations have energies of order $\alpha'^{-1/2}$. The argument of the correspondence relies on the combination of two different ways at looking at D-branes: the *open string* and the *closed string* perspectives.

When applied to a stack of N D3-branes in flat spacetime, these two perspectives allow us to motivate the correspondence. The stack of N D3-branes will extend along the spacetime directions x^0, x^1, x^2, x^3 and is transversal to the other six spatial directions $x^4, \dots, x^9$. Without loss of generality, we may describe the embedding of the stack of D3-branes into $10d$ spacetime by $x^4 = \dots = x^9 = 0$. This configuration of N coincident D3-branes embedded in $\mathsf{R}^{9,1}$ breaks half of the 32 Poincaré supercharges of IIB supergravity, leaving us with 16 supercharges.

Open string perspective ($g_s N \ll 1$)

In this picture, D-branes are boundary conditions for open strings: a flat Dp-brane is an $\mathsf{R}^{p,1}$ submanifold of spacetime where open strings can end. This is the point of view adopted in section 7.2.2. Since we have to treat strings as small perturbations, this point of view if reliable if the coupling constant for open and closed strings is small $g_s \ll 1$. Moreover, since we are neglecting massive string excitations (low energies (7.34)), the dynamics of the open strings is described by a supersymmetric gauge theory living on the worldvolume of the D-branes. The gauge field A_μ corresponds to open string excitations parallel to the D-brane while open string excitations transversal to the D-brane are scalar fields from the worldvolume point of view. Thus, D3-branes carry 6 real scalars. Therefore, the massless open string excitations may be grouped into a $4d$ $\mathcal{N}=4$ supermultiplet consisting of the gauge field A_μ, 6 real scalars ϕ as well as fermionic superpartners.

Let's consider the stack of N D3-branes. Then the effective coupling constant is given by $g_s N$ and the open string perspective is reliable for $g_s N \ll 1$. Open strings connecting different D-branes give rise to massive vector multiplets, the mass being proportional to the distance between branes. When N D-branes coincide, however, the masses of these vector multiplets vanish, this gives rise to an enhance $U(N)$ symmetry. Hence, *the world-volume theory on N D3-branes is $\mathcal{N}=4$ SYM with gauge group $U(N)$*, the only $4d$ two-derivative gauge theory with 16 Poincaré supercharges.

The D3-brane system interacts with the bulk fields which live in flat $10d$. The total effective action for all massless string modes is

$$S = S_{\text{closed}} + S_{\text{open}} + S_{\text{int}} \; . \tag{7.35}$$

S_{closed} is IIB supergravity + higher-derivative corrections, S_{open} contains the open string modes and S_{int} the interactions between open and closed string modes. S_{closed} schematically reads (see (7.18), dropping the field strengths of the R-R forms)

$$\begin{aligned} S_{\text{closed}} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}e^{-2\phi}\left(R+4|\partial\phi|^2\right)+\ldots \\ &\sim -\frac{1}{2}\int d^{10}x|\partial h|^2 + \mathcal{O}(\kappa), \end{aligned} \tag{7.36}$$

at lowest order in metric fluctuation $g=\eta+\kappa h$.

Let's now look at what happens when we take the limit $\alpha'\to 0$. In this limit, $\kappa_{10}\propto\alpha'^2\to 0$, so S_{closed} reduces to the action of *free* supergravity in $\mathsf{R}^{9,1}$. On the other hand, as we have seen, S_{open} reduces at lowest order in α' to the bosonic part of the action of $\mathcal{N}=4$ SYM theory provided that, from (7.32), we identify

$$g_{\text{YM}}^2 = 2\pi g_s \; . \tag{7.37}$$

All other terms of S_{open} are of order α' or higher. Finally, one can also see that S_{int} vanishes in the limit $\alpha'\to 0$, i.e. open and closed strings decouple.

In summary, in the naive $\alpha'\to 0$ limit, the open and closed strings decouple[54]. The dynamics of open strings rise rise to $\mathcal{N}=4$ SYM and the closed strings are effectively described by supergravity in $10d$ flat space.

[54]Notice that there is a more precise way to take this "decoupling limit"; see [Mal99] for more details.

Closed string perspective ($g_s N \gg 1$)

As the string coupling increases, the perturbative picture of string theory breaks down, but at low energy (large characteristic length scale L), supergravity is a good approximation. We deal now with the other perspective on D-branes, the one considered in section 7.2.1, where they are viewed as solitonic solutions of supergravity. For N coincident D-branes, $L^4/\alpha'^2 \propto g_s N$ (see below (7.39)) and thus the closed string perspective is reliable for $g_s N \gg 1$. In this picture, D-branes are seen as massive charged objects *sourcing* various fields of type IIB supergravity, and also type IIB string theory.

The system of N D3-branes deforms the background as (7.22) taking $p = 3$ (dropping the (3) subscripts and using spherical coordinates):

$$\begin{aligned} ds^2 &= H(r)^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H(r)^{1/2}\big(dr^2 + r^2 d\Omega_5^2\big), \\ e^\phi &= g_s\,, \qquad C_{(4)} = \big(1 - H(r)^{-1}\big)\, dx^0 \wedge \cdots \wedge dx^3 + \ldots, \end{aligned} \tag{7.38}$$

where $\mu, \nu = 0, 1, 2, 3$ and $i, j = 4, \ldots, 9$. The $\ldots$ in $C_{(4)}$ stand for terms which ensure self-duality of $F_{(5)}$, but we will not need those. Moreover,

$$H(r) = 1 + \left(\frac{L}{r}\right)^4, \qquad L = (4\pi g_s N \alpha'^2)^{1/4}\,. \tag{7.39}$$

This solution preserves $SO(3,1) \times SO(6)$ isometries of $\mathsf{R}^{9,1}$ and half of the supercharges. Closed strings propagate in this background.

The background consists of two different regions: the near-horizon region (small r) and the asymptotically flat region (large r). If $r \gg L$, then $H(r) \sim 1$ and (7.3) reduces to $10d$ flat spacetime. On the other hand, $r \ll L$ corresponds to the "throat", in which $H(r) \sim L^4/r^4$ and the metric zoomed there becomes

$$ds^2 \sim \frac{r^2}{L^2} ds_{3,1}^2 + L^2 \frac{dr^2}{r^2} + L^2 d\Omega_5^2 = L^2\left(\frac{ds_{3,1}^2 + dz^2}{z^2} + d\Omega_5^2\right) \tag{7.40}$$

where we have introduced $z = L^2/r$. We recognize $\mathrm{AdS}_5 \times S^5$. The radius of the sphere S^5 and of Ads_5 are equal and given by L. The AdS_5 Ricci scalar is $R = -20/L^2$ while the S^5 Ricci scalar is $R = 20/L^2$, which thus sum up to zero for $\mathrm{AdS}_5 \times S^5$.
So, we have found two different types of closed strings: closed strings propagating in flat $10d$ spacetime and closed strings propagating in the near-horizon region.

Because of the warp factor, there is a redshift from the "throat" around $r = 0$ to an observer at infinity. Consider indeed a string excitation with energy $\sqrt{\alpha'}E_r$ measured in string units at a fixed radial position r. For an observer at infinity, it will measure the energy

$$\sqrt{\alpha'}E_\infty = \sqrt{-g_{00}}\sqrt{\alpha'}E_r = H(r)^{-1/4}\sqrt{\alpha'}E_r \sim \frac{r}{L}\sqrt{\alpha'}E_r \to 0 \tag{7.41}$$

for fixed $\sqrt{\alpha'}E_r$ but $r \ll L$.

There are thus two types of low energy excitations. We can have excitations of very low energy away from $r = 0$, or we can have excitations that have arbitrary energy around $r = 0$ — but have very low energy from infinity. In the IR these two types are decoupled: excitations inside the throat cannot escape because of the redshift (gravitational potential); excitations at infinity have vanishing cross-section on the brane (the cross section scales as $\sigma \sim \omega^3 R^8$).[55])

Combining the two perspectives

The upshot of all this is the following: in both open/closed string perspectives, we found two decoupled effective theories in the low-energy limits:

- 4d $\mathcal{N} = 4$ SYM theory on flat $4d$ spacetime and type IIB supergravity on flat $10d$ spacetime;
- type IIB supergravity on $AdS_5 \times S^5$ and type IIB supergravity on flat $10d$ spacetime.

These two perspectives should be equivalent descriptions of the same physics, and since one of the two systems is free gravity in both cases, we are led to identify the two other theories:

$$\text{4d } \mathcal{N} = 4 \text{ SYM} \qquad \leftrightarrow \qquad \text{IIB string theory on AdS}_5 \times S^5.$$

Since S^5 is compact, one can perform a dimensional reduction (see later) and think about the gravity theory as an effective $5d$ theory with an infinite number of fields and an AdS_5 vacuum.

Recall that one can approximate IIB string theory by IIB supergravity as long as

$$\frac{1}{L^2} \sim \frac{1}{(g_s N)^{1/2}\alpha'} \ll \frac{1}{\alpha'}\,, \qquad 2\pi g_s = g_{\text{YM}}^2 \ll 1\,. \tag{7.42}$$

[55] The cross-section computation is summarized in Section 1.3.3 of [AGM+00]. The result, for this particular type of black branes, is that the cross section for ℓ-wave at $\omega \to 0$ is $\sigma^\ell_{\text{abs}} \sim \omega^{3+4\ell}R^{8+4\ell}$. The result is specific to this black brane, for instance the s-wave cross section of Schwarzschild for $\omega \ll T_H$ approaches the horizon area.

This corresponds to[56]

$$N \gg 1 \,, \qquad \lambda = g_{\rm YM}^2 N \gg 1 \,. \tag{7.43}$$

Thus, the large N limit of the QFT at large 't Hooft coupling λ is described by classical gravity! α' corrections, *i.e.* higher derivative corrections to supergravity coming from integrating out the massive string modes, correspond to $\lambda^{-1/2}$ corrections. The CFT expansion in $1/N$ corresponds to the loop expansion in string theory; both are organized as a Riemman surface genus expansion.

$$\alpha' \text{ corrections} \quad \leftrightarrow \quad \lambda^{-1/2} \text{ corrections} \,, \qquad g_s \text{ corrections} \quad \leftrightarrow \quad \frac{1}{N} \text{ corrections} \,. \tag{7.44}$$

The missing $U(1)$ We stated that we had $\mathcal{N} = 4$ SYM with gauge group $SU(N)$, not $U(N)$. On the field theory side, we thus should also have a free decoupled $U(1)$[57]. This lives somehow at the boundary between AdS and the bulk. Indeed IIB supergravity on $\mathrm{AdS}_5 \times S^5$ has a topological sector

$$S_5 = \int C_{(2)} \wedge dB_2 \tag{7.45}$$

whose dynamical dof's are at the boundary of AdS_5, and include a free Abelian gauge field. This is called the *singleton sector*.

On the decoupling limit procedure The method of taking a near-horizon limit in order to obtain the gravity dual of the gauge theory living on a set of D-branes can be applied beyond the specific context studied above. This method works whenever the limit allows to decouple consistently the brane and bulk physics. It can be for instance used with other less supersymmetric examples by placing branes in curved geometries or considering complicated set of intersecting branes.

[56]More precisely, one takes $N \to \infty$ keeping λ fixed, possibly large. This assures that R, and thus the geometry, is fixed in the limit.

[57]See page 58 of [AGM+00].

8 Tests of the AdS_5/CFT_4 correspondence

Giving a proof of the correspondence described in section 7 would require a full non-perturbative control of quantised string theory in curved spacetime background. Having not such understanding at the time being, the best thing one can do is to provide non-trivial tests of the correspondence. Again, we will focus on tests for the weak form of the duality, for which both N and λ are large, i.e. classical supergravity on the gravity side.

8.1 Matching the spectrum

There is a one-to-one map between operators in $\mathcal{N}=4$ SYM theory and the spectrum of type IIB string theory (again we will focus on the weak form, hence supergravity) on $\mathrm{AdS}_5 \times S^5$. As a consequence of the symmetry matching on both sides, the map should relate field theory operators to supergravity states which transform in the same representation of the superconformal algebra $\mathfrak{su}(2,2|4)$ or its bosonic subalgebra $\mathfrak{so}(6) \oplus \mathfrak{so}(4,2)$. We will focus on the matching of the spectrum for chiral primary (or 1/2 BPS) operators.

8.1.1 CFT side

The spectrum of operators in the field theory side includes all the gauge invariant quantities that can be formed by the fields of the theory (built out of the gauge field A_μ, the scalars ϕ^i, fermions and their covariant derivatives). Let us focus on local operators. For $\mathcal{N}=4$ SYM with gauge group $SU(N)$, properties of the adjoint representation of $SU(N)$ determine that such operators necessarily involve (a product of) traces of products of fields. We divide these local operators into *single-trace* and multiple-trace operators. Single-trace operators (here for scalars in the adjoint representation $\phi^i = \phi^{ia} T_a$) are of the form[58]

$$\mathcal{O}(x) = \mathrm{Tr}(\phi^{i_1} \dots \phi^{i_k})(x)\,, \tag{8.1}$$

and multiple-trace operators are products of them. Single-trace operators will be the focus of our interest from now on as, in the large N limit, correlation functions involving multiple-trace operators are suppressed by powers of N compared to those of single-trace operators.

Now, we have a superconformal theory, so it is very useful to classify operators into *chiral primary* operators and non-chiral primary operators. As we said in section 2.8, chiral primary operators are annihilated by some of the supercharges and sit in 'short representations' of the superconformal algebra.

[58] The trace is taken over color indices.

Short multiplets Short multiplets contain less states than a generic multiplet, saturate a unitarity bound, and the dimension of operators is fixed by Lorentz and R-symmetry quantum numbers.

One can analyze short multiplets of $\mathcal{N} = 4$ SYM from the generalization of the chiral multiplets of $\mathcal{N} = 1$ susy. In $\mathcal{N} = 1$, recall that a multiplet is chiral when it is annihilated by half of the Q's (say $\overline{Q}$). The corresponding superfield depends only on θ (and not on $\bar{\theta}$) therefore it contains less operators (is shorter). A chiral multiplet satisfies the unitarity bound

$$\Delta = \frac{3}{2}R\,, \tag{8.2}$$

with R the R-charge. In particular Δ and R can both be renormalized, but their ratio is not.

The analogous case in $\mathcal{N} = 4$ is a multiplet annihilated by half of the supercharges. Hence such a multiplet has fields with spin range from $\lambda - 2$ to $\lambda + 2$, λ being the spin of the lowest dimension operator (because 8 Q's, if you act with more than 8 supercharges, either you will annihilate the state or create a descendant). It turns out that the lowest state is built only out of scalars fields (hence we will have max spin 2), like (2.70).

In fact, chiral primary operators involve only symmetric combinations of the form

$$\mathcal{O}(x) = \mathrm{Str}(\phi^{\{i_1}\phi^{i_2}\dots\phi^{i_k\}})(x), \tag{8.3}$$

where Str stands for the symmetrised color trace of the gauge algebra, which for scalars in the adjoint representation $\phi^i = \phi^{ia}T_a$ is given by the sum over all permutations

$$\mathrm{Str}(T_{a_1}\dots T_{a_n}) = \sum_{\text{all permutations }\sigma} \mathrm{Tr}(T_{\sigma(a_1)}\dots T_{\sigma(a_n)})\,, \tag{8.4}$$

ensuring that (8.3) is totally symmetric. Moreover, the curly brackets for the field indices means that all traces are removed; this ensures that the resulting operators correspond to an irrep of the superconformal algebra. We can understand why only symmetric traceless combinations are in $\mathcal{N} = 4$ chiral multiplets using the $\mathcal{N} = 1$ subalgebra (we wrote the $\mathcal{N} = 4$ Lagrangian in $\mathcal{N} = 1$ notation in (7.5)). Indeed, if we consider the operators $\mathrm{Tr}\,\Phi_{I_1}\dots\Phi_{I_k}$, only those that are completely symmetrized are chiral primaries. This is because the EOMs (or the F-term relations) imply

$$[\Phi_I, \Phi_J] \sim \epsilon_{IJK} D^2 \overline{\Phi}_K\,. \tag{8.5}$$

Therefore, operators that are not completely symmetrized are superconformal *descendants*, and not primaries. Now, if we write the symmetrized operators $\mathrm{Tr}\,\Phi_{(I_1}\dots\Phi_{I_k)}$ in terms of real scalars ϕ_i, we discover that they are automatically traceless. Indeed, take for concreteness

$$\begin{aligned} \Phi_1\Phi_2 &= \phi_1\phi_3 - \phi_2\phi_4 + i\phi_1\phi_4 + i\phi_2\phi_3 \\ \Phi_1\Phi_1 &= \phi_1\phi_1 - \phi_2\phi_2 + 2i\phi_{(1}\phi_{2)}\ . \end{aligned} \tag{8.6}$$

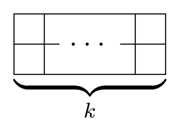
In the first line each operator on the RHS is traceless (it gives zero when contracted with δ^{ij}); in the second line, both the real and the imaginary parts are traceless.

Unitary representations of the superconformal algebra $\mathfrak{su}(2,2|4)$ are labelled by the following quantum numbers: the spin $s_\pm$ of the Lorentz algebra, the scale dimension Δ for dilatations, and the three Dynkin labels $[r_1, r_2, r_3]$ for $\mathfrak{su}(4)$ R-symmetry algebra.

Since the dimension of the scalars Φ^i is $\Delta = 1$ in $4d$, chiral primary operators (8.3) have a dimension $\Delta = k$, with k the number of scalar fields present.

Let's look at the $\mathfrak{su}(4)$ representations chiral primary operators (8.3) belong to. Since the scalars Φ^i transform in the $[0,1,0]$ representation of $\mathfrak{su}(4)$ and that a chiral primary operator (8.3) is built from a k-fold symmetric product of Φ^i, it has to be in $[0,k,0]$[59], which is of dimension $\frac{1}{12}(k+1)(k+2)^2(k+3)$.

With Dynkin diagrams:[60]

$\mathrm{STr}(\phi_{\{i_1}\dots\phi_{i_k\}})$ $\qquad$ $[0,k,0]$ of $\mathfrak{su}(4)$ $\qquad$  .

We call A_k (with $k \geq 2$)[61]. One can prove that the A_k are short multiplets with protected dimensions and that they are the *only* single-trace short multiples of of $\mathcal{N} = 4$ SYM. Hence all single-trace operators *not* in one of the A_k are renormalized.

A special case of protected operators are the conserved currents, and then their superpartners. In $\mathcal{N} = 4$ SYM there are various conserved currents:[62] the stress tensor $T_{\mu\nu}$, the supersymmetry currents $S^a_{\mu\alpha}$, the $SU(4)_R$ symmetry currents $R^a_{b\,\mu}$. They all belong to the same supermultiplet, called 'supercurrent multiplet'

$$\Big(\,\mathrm{Tr}\,\phi_{\{i}\phi_{j\}}\,,\ \dots\ ,\ R^a_{b\,\mu}\,,\ S^a_{\mu\alpha}\,,\ T_{\mu\nu}\Big)$$

[59] A more general short multiplet with lowest scalar state transforming in the $[p,k,p]$ with dimension $\Delta = k + 2p$ plays a role in multi-trace operators.

[60] For $SU(N)$, the representation $[d_1, \dots, d_{N-1}]$ has a Young diagram with d_j columns of height j.

[61] Notice that depending on the gauge group the spectrum changes. For $SU(N)$ we have all $k \geq 2$. Only $U(1)$ can have $k = 1$ (and that is a free field) the chiral multiplets obtained by acting with susy operators on the above lowest state. For $SO(N)$ we have all even k.

[62] The charges P_μ, $M_{\mu\nu}$, D and K_μ are constructed with $T_{\mu\nu}$. The supercharges Q^a_α and S^a_α are constructed with $S^a_{\mu\alpha}$. Conservation of all charges follows from $\partial^\mu T_{\mu\nu} = T_{[\mu\nu]} = T^\mu_\mu = 0$ and $\partial^\mu S_{\mu\alpha} = \gamma^\mu_{\alpha\beta} S^\beta_\mu = 0$.

whose lowest component is a scalar in the **20**.[63] The dimension of these fields (as well as the one of their superpartners) is not renormalized. The supercurrent multiplet corresponds to the chiral multiplet A_2.

Example. Consider the quadratic operators ($k = 2$) made from the six scalars $\phi_{i=1,\dots,6}$ of $\mathcal{N} = 4$ SYM:

$$\mathrm{Tr}(\phi^i \phi^j) \ .$$

These are automatically symmetric in i and j (because of the cyclicity of the trace), there are 21 independent operators which fall in two $SU(4)_R$ representations:

$$\begin{array}{lll} \mathbf{1} & K = \sum_i \mathrm{Tr}(\phi^i \phi^i) & \Delta = 2 + \mathcal{O}(g_{\mathrm{YM}}) \\ \mathbf{20} & \mathrm{STr}(\phi^{\{i}\phi^{j\}}) = \mathrm{Tr}(\phi^i \phi^j) - \frac{\delta_{ij}}{6} \sum_k \mathrm{Tr}(\phi^k \phi^k) & \Delta = 2 \ . \end{array} \tag{8.7}$$

The first operator is a singlet in terms of $SU(4)_R$ representations; it is the lowest component of a long, unprotected multiplet called the *Konishi multiplet*. Its dimension is renormalized (at one-loop and beyond). The second one sits in the representation $[0, 2, 0]$ (historically called $\mathbf{20'}$ of $SU(4)_R$) of dimension 20. It belongs to the short multiplet A_2; it is the lowest component of A_2, and its dimension is thus not renormalized.

8.1.2 Gravity side

To read off the spectrum from gravity, we need the effective $5d$ supergravity theory in AdS_5, that follows from the $10d$ IIB supergravity on $\mathrm{AdS}_5 \times S^5$ from Kaluza-Klein (KK) reduction.

In the standard KK reduction on S^1 of radius L, one expands all fields in Fourier modes on S^1:

$$\phi(x_\mu, y) = \sum\nolimits_k \phi_k(x_\mu)\, e^{iky/L} \qquad\qquad k \in \mathsf{Z} \ . \tag{8.8}$$

Then one plugs into the EOMs:

$$-\Box_{d+1}\, \phi(x_\mu, y) = \big(- \Box_d - \partial_y^2 \big)\, \phi(x_\mu, y) = \sum_k e^{iky/L} \left(- \Box_d + \frac{k^2}{L^2} \right) \phi_k(x_\mu) \ . \tag{8.9}$$

Thus one gets an infinite tower or KK modes, with square masses k^2/L^2. (One can then write an action that reproduces the EOMs).

[63] The multiplet contains more scalars, and then the spin $\frac{1}{2}$ operators. In particular there are the $\Delta = 3$ scalars $\mathrm{Tr}\, \lambda^a \lambda^b + \phi^3$ in the **10** (which is a symmetric bi-spinor, or an antisymmetric self-dual rank-3 tensor), and the $\Delta = 4$ scalar $\mathcal{L} = \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} + \dots$.

We should do a similar KK reduction on S^5. The bosonic massless modes of $10d$ type IIB string theory are ($M, N, \ldots$ denoting $10d$ indices)

$$\left(g_{MN}\,,\,\phi\,,\,B_{MN}\,,\,C_{(0)}\,,\,C_{2\,MN}\,,\,C_{4\,MNPQ}\right)\,. \tag{8.10}$$

We should diagonalize the Laplacian on S^5 using spherical harmonics. For a scalar:

$$\phi(x,z,\Omega_5)=\sum_{I=0}^{\infty}\phi_I(x,z)\,Y^I(\Omega_5) \tag{8.11}$$

where (x^μ, z) ($\mu = 0,1,2,3$) are the coordinates on AdS_5 and Y^I are the eigenfunctions of the scalar Laplacian on S^5:

$$\Box_{S^5}Y^I=-\frac{1}{L^2}\ell(\ell+4)Y^I. \tag{8.12}$$

Each KK mode $\Phi_I(x,z)$ is a scalar field on AdS_5. For scalars, Y^I are labelled by the rank of the totally symmetric traceless representation of $SO(6)$. The Y^I transform in the representation $[0,\ell,0]$ of $SU(4)$ (or equivalently in the $[\ell,0,0]$ of $SO(6)$). In the same way as fields on a circle received a mass contribution from the momentum mode on the circle, fields compactified on S^5 receive a contribution to the mass. From the eigenvalues of the Laplacian on S^5, we can find the following relations between mass and scaling dimensions for various spins.

Inserting (8.11) into the $10d$ supergravity EOMs linearized around the $\mathrm{AdS}_5\times S^5$ background and diagonalizing the equations to give EOMs of free fields. This determines the masses and coupling of the AdS_5 fields ϕ^I. This is a complicated computation that we will not do it in details.

As an example, we consider fluctuations which are dual to 1/2 BPS operators. The sugra theory contains in particular a self-dual five-form $F_{(5)}$. It enters the $10d$ EOMs for the graviton via

$$R_{MN}=\frac{1}{3!}F_{MABCD}F_N^{ABDC} \tag{8.13}$$

In the $\mathrm{AdS}_5\times S^5$ background solution, $F_{(5)}$ takes particularly simple values: along the legs of AdS space, it is proportional to the volume form of AdS_5 while along the legs of the sphere, it is proportional to the volume form of S^5:

$$\bar F_{m_1\ldots m_5}=\frac{4}{L}\sqrt{-g_{AdS_5}}\varepsilon_{m_1\ldots m_5}\,,\qquad \bar F_{\alpha_1\ldots\alpha_5}=\frac{4}{L}\sqrt{g_{S^5}}\varepsilon_{\alpha_1\ldots\alpha_5}\,, \tag{8.14}$$

with m_i AdS indices and α_i sphere indices. Consider now fluctuations of the metric and the five-form around that background (of metric $\bar g$), namely

$$g_{MN}=\bar g_{MN}+h_{MN}\,,\qquad F=\bar F+\delta F\,. \tag{8.15}$$

In what follows, we focus on the scalar modes dual to 1/2 BPS operators and consider the following form of the fluctuations of the S^5 parts:

$$h_{\alpha\beta} = h_{(\alpha\beta)} + \frac{h^\gamma_\gamma}{5}\bar{g}_{\alpha\beta}\,, \qquad \delta F_{\alpha\beta\gamma\sigma\rho} = \nabla_{[\alpha} a_{\beta\gamma\sigma\rho]}\,. \tag{8.16}$$

Their KK expansion is

$$h^\alpha_\alpha(x,z,\Omega_5) = \sum_{I=0}^{\infty} h^{\alpha I}_\alpha(x,z)\, Y^I(\Omega_5)\,, \qquad a_{\beta\gamma\sigma\rho}(x,z,\Omega_5) = \sum_{I=0}^{\infty} b^I(x,z)\, \epsilon_{\beta\gamma\sigma\rho\epsilon}\nabla^\epsilon Y^I(\Omega_5)\,. \tag{8.17}$$

It turns out that out of those fluctuations, a certain scalar can be isolated and found to satisfy the free field equations. Indeed, to linear order in the fluctuations, the EOMs give the two coupled equations for b^I and h^I can be diagonalized and decouple. This lengthy procedure leads to the equation

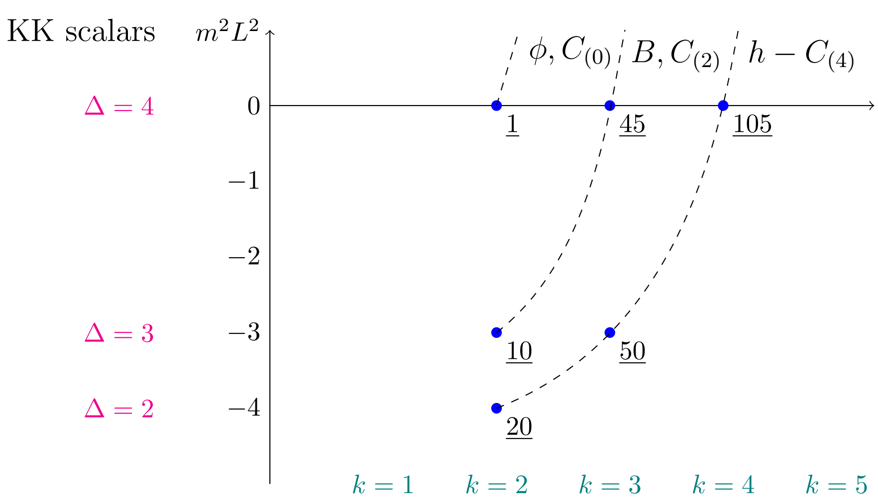
$$\Box_{AdS_5} s^I(z,x) = \frac{1}{L^2}\ell(\ell-4)s^I(z,x). \tag{8.18}$$

for a certain combination s^I given by

$$s^I \propto h^{\alpha I}_\alpha - 10(\ell+4)b^I\,. \tag{8.19}$$

Equations (8.18) is a Klein-Gordon equation for a free scalar of mass $m^2L^2 = \ell(\ell-4)$. Anticipating for the spectrum matching, we will already identify $\ell = \Delta$. The scalar s^Δ sits in the representation $[0,\Delta,0]$ of $SU(4)$ and its holographic dual will be the 1/2 BPS operator (8.1) for $k = \Delta$.

The full reduction has been done in [KRvN85, GM85] and the spectrum was organized in supersymmetry multiplets. We plot only the scalar modes, and for dimensions up to $\Delta = 4$.



For each mode we indicate the $SU(4)$ representation.[64] The mass/dimension is on the vertical axis. Modes in the same vertical line are scalars in the same supermultiplet A'_k. One can show that the entire spectrum consists of a series of short multiplets A'_k of the superconformal algebra, labelled by an integer $k \geq 2$. Modes connected by a dashed line are different KK modes of the same $10d$ field.

- The lowest scalar in each multiplet A'_k is in the k-fold symmetric traceless representation of $SO(6)$, namely

$$[0, k, 0] \qquad \text{with } m^2 = \frac{k(k-4)}{L^2} \quad \leftrightarrow \quad \Delta = k \ .$$

- The next scalar is in representation

$$[2, k-2, 0] \qquad \text{with } \Delta = k+1 \ .$$

  This scalar corresponds to two-form fields.

- The next scalar is in representation

$$[0, k-2, 0] \qquad \text{with } \Delta = k+2 \ .$$

The multiplet contains spins up to 2 (because no $10d$ field has higher spin).

Let us make some remarks:

- In the standard KK reduction the massive modes in each tower are separated from the massless modes by a gap of order $1/L$. Thus one can decouple massive modes by taking the limit $L \to 0$.

  Due to the curvature, in AdS there is no separation: all KK modes have a mass of the same order as the zero-modes. In fact, we even have fields with different masses in the same supermultiplet!

- Even if we cannot decouple the multiplets A'_k with $k \geq 3$ by taking the radius of S^5 small, we can write an effective action that reproduces the EOMs for A'_2. This action is 5d $\mathcal{N} = 8$ $SO(6)$ maximal gauged supergravity.

[64] $\underline{10}$ is a self-dual antisymmetric 3-tensor equal to a symmetric 2-spinor, while $\underline{20}$ is the symmetric traceless 2-tensor.

A'_2 is coupled to $A'_{k\geq 3}$, however if we set the latter to zero, they are not sourced by A'_2. In practice, the fields $A'_{k\geq 3}$ never appear linearly in the Lagrangian.[65] The reduction to A'_2 is called a *consistent truncation*: every solution of 5d $\mathcal{N} = 8$ gauged SUGRA can be uplifted to a 10d solution of IIB SUGRA.

8.1.3 Matching both sides

KK modes: There is a perfect match between short multiplets of 4d $\mathcal{N} = 4$ SYM and multiplets in AdS_5 upon KK reduction of IIB SUGRA on S^5. In both cases, there is precisely one short multiplet of the superconformal algebra for each $k \geq 2$.

The multiplet A'_2 contains the graviton, the gravitino and the $SO(6)$ gauge fields: it corresponds to the supercurrent multiplet on the field theory side (this is the manifestation that gauge symmetries in bulk = global symmetries at boundary).

The multiplets A'_k correspond to the field theory multiplets A_k with lowest component

$$\mathrm{STr}(\phi_{\{i_1} \dots \phi_{i_k\}}) \ .$$

Using susy, mass/dimension relations and $SU(4)$ quantum numbers, the KK scalars in the above figure are identified as follows:

$SU(4)$ rep		operator	mult./dim.
20	$\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}$	$\mathrm{Tr}(\phi_{\{i}\phi_{j\}})$	A_2 $\Delta = 2$
10	$\square\square$	$\mathrm{Tr}\,\lambda_a\lambda_b + \phi^3$	A_2 $\Delta = 3$
1		$F_{\mu\nu}(F^{\mu\nu} + i\widetilde{F}^{\mu\nu})$	A_2 $\Delta = 4$
50	$\begin{smallmatrix}\square&\square&\square\\\square&\square&\square\end{smallmatrix}$	$\mathrm{STr}(\phi_{\{i}\phi_j\phi_{k\}})$	A_3 $\Delta = 3$
45	$\begin{smallmatrix}\square&\square&\square\\\square&\square&\end{smallmatrix}$	$\mathrm{Tr}\,\lambda_a\lambda_b\phi_i + \phi^4$	A_3 $\Delta = 4$
105	$\begin{smallmatrix}\square&\square&\square&\square\\\square&\square&\square&\square\end{smallmatrix}$	$\mathrm{STr}(\phi_{\{i}\phi_j\phi_k\phi_{l\}})$	A_4 $\Delta = 4$

Notice that some of the KK modes have negative mass. The corresponding mode is stable if $m^2L^2 \geq -4$ (BF bound). Here, all operators have $\Delta \geq 2$ hence the BF bound is satisfied. Stability is a consequence of supersymmetry.

String modes: So far we have only talked about KK modes. In the bulk there are also string modes, what about them? In the supergravity approximation (i.e. λ large in the field

[65] One way for this to happen is if the (classical) gravitational theory has a global symmetry, and the fields $A'_{k\geq 3}$ are charged while A'_2 is neutral.

theory side) the their masses go like

$$m^2 = \frac{\Delta(\Delta-4)}{L^2} \sim \frac{1}{\alpha'} = \frac{\sqrt{4\pi\lambda}}{L^2} \qquad \Rightarrow \qquad \Delta \sim \lambda^{1/4} \,. \tag{8.20}$$

We thus have the following prediction about $\mathcal{N}=4$ SYM: at strong coupling, all unprotected multiplets (and in particular all operators with spin >2) have divergent dimensions that go like[66] $\lambda^{1/4}$. The only operators with finite dimension are those in the protected chiral multiplets A_k. We thus have a very large separation between a set of operators with maximum spin two and all the other operators in the theory. This separation is necessary for every CFT with a weakly coupled dual.

Let us finish by saying that the exact correspondence between KK modes and protected operators is a peculiarity of $\mathcal{N}=4$ SYM and it does not extend to theories with less supersymmetry. For instance, in examples of $\mathcal{N}=1$ dual pairs there are KK modes corresponding to non-protected operators with finite dimension.

8.2 (Low point) correlation functions

We will focus on the three-point function of 1/2 BPS operators, which are protected operators under quantum corrections. In addition, there are many examples of *correlators* of 1/2 BPS operators which do not depend on the coupling λ. The key idea is to compute their three-point function in $\mathcal{N}=4$ SYM first perturbatively at weak coupling, and then at strong coupling using AdS/CFT. Due to non-renormalisation theorem, the two calculations should give the same result. An important step is to make sure that the operators are normalised in the same way in both sides, this is achieved via the two-point function.

The steps are the following:

- compute two-point functions on both sides to fix the normalisation:
- calculate the three-point function in the SYM side at lowest order in the coupling
- check that this result is not renormalised at higher order
- calculate the three-point function on the gravity side. The most difficult part is to compute the coupling, which comes from the KK reduction.
- check they match and be happy.

[66] This is true for operators of fixed length as $N \to \infty$. One could also consider operators whose length increases with N, like giant gravitons.

8.2.1 Field theory side

Let's write the 1/2 BPS operators of $\mathcal{N}=4$ SYM of dimension $\Delta=k$ as

$$\mathcal{O}_k^I = C^I_{i_1\ldots i_k}\,\mathrm{Tr}(\phi^{i_1}\ldots\phi^{i_k}), \tag{8.21}$$

where the C^I are totally symmetric traceless rank k tensors of $SO(6)$. The $\mathcal{N}=4$ SYM action gives the scalar propagators

$$\langle\phi^{ia}(x)\phi^{jb}(y)\rangle = \frac{\delta^{ij}\delta^{ab}}{(2\pi)^2(x-y)^2}\,. \tag{8.22}$$

To lowest order in the coupling $\mathcal{O}(g^0_{YM})$ in the large N limit (only planar diagram), the two-point function are given by a Feynman diagram where the composite operators correspond to k legs each. The two operators are linked by k scalar propagators. We thus have

$$\begin{aligned}\langle\mathcal{O}_k^I(x)\mathcal{O}_k^J(y)\rangle &= C^I_{i_1\ldots i_k}C^J_{j_1\ldots j_k}\langle\mathrm{Tr}(\phi^{i_1}(x)\ldots\phi^{i_k}(x))\,\mathrm{Tr}(\phi^{j_1}(y)\ldots\phi^{j_k}(y))\rangle\\ &= C^I_{i_1\ldots i_k}C^J_{j_1\ldots j_k}\frac{N^k(\delta^{i_1j_1}\ldots\delta^{i_kj_k}+\text{permutations})}{(2\pi)^{2k}(x-y)^2}\\ &= \frac{kN^k\delta^{IJ}}{(2\pi)^{2k}(x-y)^{2k}}\,,\end{aligned} \tag{8.23}$$

where the last equality holds at leading order N where only cyclic permutations are taken into account. Orthonormality of the C tensors has been also used.

Similarly, the three-point function is given by (to lowest order and a large N):

$$\langle\mathcal{O}^I_{k_1}(x)\mathcal{O}^J_{k_2}(y)\mathcal{O}^K_{k_3}(z)\rangle = \frac{k_1k_2k_3N^{\Sigma/2}\langle C^IC^JC^K\rangle}{N(2\pi)^\Sigma|x-y|^{2\alpha_3}|y-z|^{2\alpha_1}|x-z|^{2\alpha_2}}\,, \tag{8.24}$$

where we defined $\Sigma=k_1+k_2+k_3$, $\alpha_i=\Sigma/2-k_i$ and $\langle C^IC^JC^K\rangle$ denotes a uniquely defined $SO(6)$ tensor contraction of indices determined by the Feynman graph.

It will be important to normalize the operators as

$$\widetilde{\mathcal{O}}^I_k \equiv \frac{(2\pi)^k}{N^{k/2}\sqrt{k}}\mathcal{O}^I_K, \tag{8.25}$$

so that the two-point function is normalised to one:

$$\langle\widetilde{\mathcal{O}}^I_k(x)\widetilde{\mathcal{O}}^J_k(y)\rangle = \frac{\delta^{IJ}}{(x-y)^{2k}}\,, \tag{8.26}$$

and the three-point reads

$$\langle\widetilde{\mathcal{O}}^I_{k_1}(x)\widetilde{\mathcal{O}}^J_{k_2}(y)\widetilde{\mathcal{O}}^K_{k_3}(z)\rangle = \frac{\sqrt{k_1k_2k_3}\langle C^IC^JC^K\rangle}{N|x-y|^{2\alpha_3}|y-z|^{2\alpha_1}|x-z|^{2\alpha_2}}\,. \tag{8.27}$$

Non-renormalization theorem: It is possible to show that higher order quantum corrections of order $\mathcal{O}(\lambda)$ do not change the two nor three-point function (see e.g. sec. 6.1.3 of [AE15]).

8.2.2 Gravity side

Let us consider now three-point functions of scalar field in AdS_{d+1}: the Witten tree diagram is specified by 3 boundary points x, y, z, 3 bulk-to-boundary propagators (4.47)

$$K_k(w, \vec{x}; y) = \frac{\Gamma(k)}{\pi^{d/2}\Gamma(k - \frac{d}{2})} \left(\frac{w_0}{w_0^2 + (\vec{w} - \vec{x})^2} \right)^k = \frac{\Gamma(k)}{\pi^{d/2}\Gamma(k - \frac{d}{2})} \left(\frac{w}{w^2 + (x - y)^2} \right)^k \tag{8.28}$$

and by a bulk coupling associated to the cubic vertex.

In the literature, one often uses the compact notation where bulk points are denoted by $(d+1)$-dimensional variables $w = (w_0, \vec{w})$ with w_0 the radial coordinate, and the boundary points are denoted $\vec{x}, \vec{y}, \vec{z}$. We also write $(w - \vec{x})^2 \equiv w_0^2 + (\vec{w} - \vec{x})^2$; then

$$K_k(w, \vec{x}) = \frac{\Gamma(k)}{\pi^{d/2}\Gamma(k - \frac{d}{2})} \left(\frac{w_0}{(w - \vec{x})^2} \right)^k \tag{8.29}$$

We thus have to compute

$$A(\vec{x}, \vec{y}, \vec{z}) \equiv \int dw_0 d^d\vec{w} \frac{1}{w_0^{d+1}} \left(\frac{w_0}{(w - \vec{x})^2} \right)^{k_1} \left(\frac{w_0}{(w - \vec{y})^2} \right)^{k_2} \left(\frac{w_0}{(w - \vec{z})^2} \right)^{k_3} . \tag{8.30}$$

To do it, we use the "inversion trick": re-express the $(d+1)$-dimensional integration variables as

$$w_m = \frac{w'_m}{(w')^2}, \tag{8.31}$$

and set similarly

$$\vec{x} = \frac{\vec{x}'}{(\vec{x}')^2} , \qquad \vec{y} = \frac{\vec{y}'}{(\vec{y}')^2} , \qquad \vec{z} = \frac{\vec{z}'}{(\vec{z}')^2} . \tag{8.32}$$

→ *Exercise 14:* show that this leads to

$$K_k(w, \vec{x}) = (\vec{x}')^{2k} K_k(w', \vec{x}') . \tag{8.33}$$

The inversion being an isometry of AdS, the volume element is invariant, and hence under inversions the three-point function transforms as

$$A(\vec{x}, \vec{y}, \vec{z}) = (\vec{x}')^{2k_1} (\vec{y}')^{2k_2} (\vec{z}')^{2k_3} A(\vec{x}', \vec{y}', \vec{z}') . \tag{8.34}$$

Translation invariance then allows us to set one argument to zero, say $\vec{z}=0$, hence

$$A(\vec{x},\vec{y},\vec{z}) = A(\vec{x}-\vec{z},\vec{y}-\vec{z},0) \equiv A(\vec{u},\vec{v},0)\,. \tag{8.35}$$

Thanks to the inversion, we are thus left with a much simpler integral, with only two denominators

$$A(\vec{u},\vec{v},0) = \frac{1}{|\vec{u}|^{2k_1}|\vec{v}|^{2k_2}} \int \frac{d^{d+1}\vec{w}'}{w_0'^{d+1}} \left(\frac{w_0'}{(w'-\vec{u}')^2}\right)^{k_1} \left(\frac{w_0'}{(w'-\vec{v})^2}\right)^{k_2} w_0'^{k_3}\,. \tag{8.36}$$

This can be evaluated in closed form using Feynman parameter methods. It finally leads to[67]

$$A(\vec{x},\vec{y},\vec{z}) = \frac{a}{(\vec{x}-\vec{y})^{k_1+k_2-k_3}(\vec{y}-\vec{z})^{k_2+k_3-k_1}(\vec{z}-\vec{x})^{k_3+k_1-k_2}}\,, \tag{8.37}$$

with a a function of Gamma functions given e.g. in (6.35) of [AE15].

Now that we have done the integral, the remaining thing is to calculate the coupling with which it enters in the three-point function. For that we need the KK reduction of the type IIB sugra action on S^5; the steps of how to obtain the coupling are outlined in section 6.1.4 of [AE15].

Combining everything together, and paying attention to the normalisation matching of the two-point function, the end result is found to be

$$\boxed{\langle \widetilde{\mathcal{O}}^I_{k_1}(x)\widetilde{\mathcal{O}}^J_{k_2}(y)\widetilde{\mathcal{O}}^K_{k_3}(z)\rangle = \frac{\sqrt{k_1k_2k_3}\langle C^IC^JC^K\rangle}{N|x-y|^{2\alpha_3}|y-z|^{2\alpha_1}|x-z|^{2\alpha_2}}}\,, \tag{8.38}$$

in perfect agreement with the field theory side. Recall that it was essential to consider observables which do not depend on the coupling. Impressive further tests of the correspondence were obtained beyond non-renormalised operators, where the results do depend of the couplings, in the "integrability" approach, which requires the strong form of the correspondence (beyond the scope of these lectures).

8.3 Conformal anomaly

Non-renormalisation theorems also protect the coefficient of the conformal anomaly in $\mathcal{N}=4$ SYM theory, which is one-loop exact and independent of λ.

[67] A useful formula to reinstate the original variables is $(\vec{u}'-\vec{v}')^2 = \frac{(\vec{x}-\vec{y})^2}{(\vec{x}-\vec{z})^2(\vec{y}-\vec{z})^2}$.

8.3.1 Field theory side

As we have seen previously, the conformal anomaly of a $4d$ field theory is of the form (5.38), where the a and c numbers are expressed, in a $\mathcal{N}=1$ supersymmetric theory as (5.41). For $\mathcal{N}=4$ $SU(N)$ SYM with $N_\Phi = 3(N^2-1)$, $N_V = N^2-1$ this implies

$$c = a = \frac{1}{4}(N^2-1) \approx \frac{1}{4}N^2 \tag{8.39}$$

in the large N limit. The $c=a$ equality is very special of this theory, it does not hold in general. Given that, the generic form of the conformal anomaly reads, in large N,

$$\langle T^\mu_\mu(x)\rangle = \frac{N^2}{32\pi^2}\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right) . \tag{8.40}$$

It can be shown to be one-loop exact, so it is independent of f λ to all orders in perturbation theory. We can thus compare it directly to the strong coupling result obtained by mapping to AdS.

8.3.2 Gravity side

From the gravity side, the conformal anomaly is computed from the action of $(d+1)$-AdS gravity (at the end we will specify to $d=4$)

$$S = -\frac{1}{16\pi G}\int d^{d+1}x\sqrt{g}\left(R + \frac{d(d-1)}{L^2}\right) - \frac{1}{8\pi G}\int d^dx\sqrt{\gamma}\,K, \tag{8.41}$$

and the asymptotically AdS metric

$$ds^2 = L^2\left(\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}g_{\alpha\beta}(x,\rho)dx^\alpha dx^\beta\right) \qquad \lim_{\rho\to 0} g_{\alpha\beta}(x,\rho) = g^{(0)}_{\alpha\beta}(x) . \tag{8.42}$$

The question we'd like to answer is can we find a $(d+1)$-dimensional diffeo which reduces to Weyl transformation on the boundary? Such a diffeo is known as a Penrose-Brown-Henneaux (PBH) transformation

$$\rho = \rho'(1-2\sigma(x')) , \qquad x^\mu = x'^\mu + a^\mu(x',\rho'). \tag{8.43}$$

Preserving $g'_{\rho\rho} = g_{\rho\rho}$ and $g'_{\rho\mu} = g_{\rho\mu} = 0$ imposes the constraints

$$\partial_\rho a^\mu = \frac{L^2}{2}g^{\mu\nu}\partial_\nu\sigma \Rightarrow a^\mu(x,\rho) = \frac{L^2}{2}\int_0^\rho d\hat\rho g^{\mu\nu}(x,\hat\rho)\partial_\nu\sigma(x). \tag{8.44}$$

Under this diffeo, the metric transforms as

$$g_{\mu\nu} \to g_{\mu\nu} + 2\sigma\left(1 - \rho\partial_\rho\right) g_{\mu\nu} + \nabla_\mu a_\nu + \nabla_\nu a_\mu\,. \tag{8.45}$$

We indeed have a Weyl transformation at the boundary $\rho \to 0$ since then

$$\delta g_{\mu\nu}(x) = 2\sigma(x) g^{(0)}_{\mu\nu}(x) \tag{8.46}$$

and the variation of the action gives the expected boundary value of the trace of the energy-momentum tensor

$$\delta S = \frac{1}{2}\int d^d x \sqrt{g^{(0)}}\langle T_{\mu\nu}\rangle \delta g^{(0)}_{\mu\nu}\,. \tag{8.47}$$

Let's now recall the Fefferman-Graham expansion, which reads (in even d)

$$g_{\alpha\beta}(x,\rho) \overset{\rho\to 0}{=} g_{(0)\alpha\beta}(x) + \rho g_{(2)\alpha\beta}(x) + \rho^2 g_{(4)\alpha\beta}(x) + \rho^{d/2}\log\rho\, h_{(d)\alpha\beta}(x) + \dots \tag{8.48}$$

and using the holographic renormalization method. Inserting the metric expansion in the action gives

$$S = -\frac{1}{16\pi G}\int d^{d+1}x\sqrt{\det g^{(0)}}(\epsilon^{-d/2}a_{(0)} + \epsilon^{-d/2+1}a_{(2)} + \dots - \ln\epsilon a_{(d)}) + (finite)\,. \tag{8.49}$$

Explicit computation leads to

$$\begin{aligned} a_{(0)} &= \frac{2(d-1)}{L} \\ a_{(2)} &= \frac{L}{2(d-1)}R \\ a_{(4)} &= \frac{L^3}{2(d-2)^2}(R_{\mu\nu}R^{\mu\nu} - \frac{1}{d-1}R^2)\,. \end{aligned} \tag{8.50}$$

In the minimal substraction scheme, the divergence coming from $a_{(0)}$ can be cured by adding the counterterm

$$S_{ct} = \frac{(d-1)}{8\pi G\, L}\int d^d x\sqrt{\gamma}\,. \tag{8.51}$$

which cancels the infinite volume of AdS. Moreover generally, the counterterm action needs more counterterm, with higher powers in R

$$S_{ct} = \frac{1}{16\pi G}\int d^{d+1}x\sqrt{\det g^{(0)}}(\epsilon^{-d/2}a_{(0)} + \epsilon^{-d/2+1}a_{(2)} + \dots - \ln\epsilon a_{(d)})\,. \tag{8.52}$$

In $d = 4$, we only need the three terms given by (8.50). The counterterm action is no longer invariant under the PBH action, this will give rise to finite contributions to the trace of the energy-momentum tensor in the limit $\epsilon \to 0$. Close to the boundary, the PHB transformation amounts to

$$\delta(S+S_{ct}) = 2\int d^d x \sigma(x) \left(\epsilon \frac{\delta}{\delta\epsilon} - g^{(0)\mu\nu}\frac{\delta}{\delta g^{(0)\mu\nu}}\right)(S+S_{ct})\,. \tag{8.53}$$

→ *Exercise 15:* Look at $d=4$ and take $\sigma(x) = (\text{constant} - 2b\cdot x)$, show that the terms involving $\sqrt{\det g^{(0)}}\epsilon^{-2}a_{(0)}$ and $\sqrt{\det g^{(0)}}\epsilon^{-1}a_{(2)}$ are invariant under the residual PBH transformation (8.53).

→ *Exercise 16:* For the last term involving $\sqrt{\det g^{(0)}}\ln\epsilon a_{(4)}$ show that

$$\delta S_{ct} = 2\int d^4 x\sigma(x)\epsilon\frac{\delta}{\delta\epsilon}S_{ct} = -\frac{L^3}{64\pi G_5}\int d^4x\sigma(x)\sqrt{\det g^{(0)}}\left(R_{\mu\nu}R^{\mu\nu} - \frac{R^2}{3}\right)\,. \tag{8.54}$$

The holographic trace anomaly is obtained from the variation of the renormalized on-shell action trace for the energy-momentum tensor

$$g^{(0)\mu\nu}\langle T_{\mu\nu}\rangle = \frac{L^3}{64\pi G_5}\left(R_{\mu\nu}R^{\mu\nu} - \frac{R^2}{3}\right) \tag{8.55}$$

with the $5d$ Newton constant

$$G_5 = \frac{G_{10}}{\text{vol}(S^5)}\,. \tag{8.56}$$

Now since $\text{vol}(S^5) = \pi^3 L^5$ and $G_{10} = \frac{1}{8\pi}\kappa_{10}^2$ with $2\kappa_{10}^2 = (2\pi)^7\alpha'^4 g_s^2$, we get

$$\boxed{\langle T^\nu_\nu\rangle = \frac{N^2}{32\pi^2}\left(R_{\mu\nu}R^{\mu\nu} - \frac{R^2}{3}\right)} \tag{8.57}$$

which exactly matches the field theory result.

9 Wilson loops

In gauge theories, a natural observable is the Wilson loop, defined for every closed contour $\mathcal{C}$ in spacetime and representation $\mathcal{R}$ of the gauge group:

$$W_{\mathcal{R}}(\mathcal{C}) = \mathrm{Tr}_{\mathcal{R}}\ \mathrm{P}e^{i\int_{\mathcal{C}} A_\mu^a T^a dx^\mu}\ . \tag{9.1}$$

T^a are the generators in representation $\mathcal{R}$.

Path-ordered exponential. For an Abelian gauge field, the open Wilson line (where the contour $\mathcal{C}$ is taken to be open) is simply constructed as

$$W_q = \exp\left\{iq\int_{\mathcal{C}} A_\mu dx^\mu\right\} = \exp\left\{iq\int_0^1 d\tau\, A_\tau(\tau)\right\}\ . \tag{9.2}$$

Here q is the charge parametrizing representations of $U(1)$, $\tau \in [0,1]$ is a coordinate along the path, $x^\mu(\tau)$ is the path and

$$A_\tau = A_\mu \frac{dx^\mu}{d\tau} \tag{9.3}$$

is the gauge field tangent to the path. Under a gauge transformation $A_\mu \to A_\mu + \partial_\mu\lambda$ we have

$$e^{iq\int_0^1 A_\tau d\tau} \to e^{iq\int_0^1 A_\tau d\tau + iq\int_0^1 \partial_\tau\lambda\, d\tau} = e^{iq\lambda(1)}e^{-iq\lambda(0)}W_q \tag{9.4}$$

where we have integrated by parts. We see that, even though we made a gauge transformation $\lambda(x)$ at all points in spacetime, only the endpoints of the Wilson line transform, one as a particle of charge q and the other one as of charge $-q$.

The non-Abelian case is more complicated. Under a gauge transformation $U = e^{i\lambda}$ we have

$$A_\mu \to U(A_\mu + i\partial_\mu)U^{-1} \qquad \Rightarrow \qquad \delta A_\mu = D_\mu\lambda = \partial_\mu\lambda - i[A_\mu,\lambda]\ . \tag{9.5}$$

If we define the Wilson line as the exponential of the integral, then it will have a complicated gauge transformation (and it will not be usable to construct gauge invariants). Instead we should use the path-ordered exponential:

$$\begin{aligned} W = \mathrm{P}e^{i\int_0^1 d\tau\, A_\tau} &= \mathbb{1} + \sum_{n=1}^\infty \frac{i^n}{n!}\int_0^1 d\tau_1 \cdots \int_0^1 d\tau_n\, \mathrm{P}\big[A_\tau(\tau_1)\ldots A_\tau(\tau_n)\big] \\ &= \mathbb{1} + \sum_{n=1}^\infty i^n \int_0^1 d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{n-1}} d\tau_n\, A_\tau(\tau_1)\ldots A_\tau(\tau_n)\ . \end{aligned} \tag{9.6}$$

Here P orders the operators from the last one to the first one (the operator with smallest τ is the first one to act). We can then reduce to a fundamental domain (second line) and multiply by the number $n!$ of permutations.

Let us define as I_n the quantity in the summation: $\sum_n I_n$. We compute its (infinitesimal) gauge variation:

$$\delta I_n = i^n \sum_{j=1}^{n} \int_0^1 d\tau_1 \cdots \int_0^{\tau_{n-1}} d\tau_n \, A_\tau(\tau_1) \dots \Big(\partial_\tau \lambda - i[A_\tau, \lambda]\Big)(\tau_j) \dots A_\tau(\tau_n) \, . \tag{9.7}$$

This gives two types of terms. The commutator gives

$$\delta I_n^{[,]} = i^{n-1} \sum_{j=1}^{n} \int_0^1 d\tau_1 \cdots \int_0^{\tau_{n-1}} d\tau_n \, A_\tau(\tau_1) \dots [A_\tau, \lambda](\tau_j) \dots A_\tau(\tau_n) \, . \tag{9.8}$$

The derivative gives

$$\delta I_n^{\partial} = i^n \sum_{j=1}^{n} \int_0^1 d\tau_1 \, A_\tau(\tau_1) \cdots \int_0^{\tau_{j-1}} d\tau_j \, \partial_\tau \lambda(\tau_j) \int_0^{\tau_j} d\tau_{j+1} \, A_\tau(\tau_{j+1}) \cdots \int_0^{\tau_{n-1}} d\tau_n \, A_\tau(\tau_n) \, . \tag{9.9}$$

We perform the integral in $d\tau_j$ by parts. We get:

$$\begin{aligned}
\delta I_n^{\partial} = & -i^n \sum_{j=1}^{n-1} \int_0^1 d\tau_1 \, A_\tau(\tau_1) \cdots \int_0^{\tau_{j-1}} d\tau_j \, \lambda(\tau_j) \, A_\tau(\tau_j) \int_0^{\tau_j} d\tau_{j+2} \, A_\tau(\tau_{j+2}) \dots \\
& + i^n \sum_{j=2}^{n} \int_0^1 d\tau_1 \, A_\tau(\tau_1) \cdots \int_0^{\tau_{j-2}} d\tau_{j-1} \, A_\tau(\tau_{j-1}) \, \lambda(\tau_{j-1}) \int_0^{\tau_{j-1}} d\tau_{j+1} \, A_\tau(\tau_{j+1}) \dots \\
& + i^n \lambda(1) \int_0^1 d\tau_2 \cdots \int_0^{\tau_{n-1}} d\tau_n \, A(\tau_2) \dots A(\tau_n) \\
& - i^n \int_0^1 d\tau_1 \cdots \int_0^{\tau_{n-2}} d\tau_{n-1} \, A_\tau(\tau_1) \dots A_\tau(\tau_{n-1}) \, \lambda(0) \, .
\end{aligned} \tag{9.10}$$

The first line comes from moving the derivative. The second line comes from the boundary term evaluated at the upper bound τ_j for $j > 1$. The third line is the case $j = 1$. For $j < n$ there is no contribution from the lower bound 0 because an integration follows, however there is such a contribution for $j = n$ and this is the fourth line. We can put the first and second line into a commutator, and write

$$\delta I_n^{\partial} = -\delta I_{n-1}^{[,]} + i \, \lambda(1) \, I_{n-1} - i \, I_{n-1} \, \lambda(0) \, . \tag{9.11}$$

Summing all terms we have

$$\delta W = i \, \lambda(1) \, W - i \, W \, \lambda(0) \, . \tag{9.12}$$

Once again, the end of the line transforms as a point particle in representation $\mathcal{R}$ while the beginning transforms as a particle in $\overline{\mathcal{R}}$. If we take a closed loop, starting and ending at x_0, we have

$$\delta W = i\big[\lambda(x_0), W\big] \qquad \Rightarrow \qquad \delta \operatorname{Tr} W = 0 \ . \tag{9.13}$$

Taking the trace, we have constructed a gauge invariant.

Physical interpretation. The Wilson loop has this interpretation. We introduce external massive sources (quarks) in representation $\mathcal{R}$. The loop represents the propagation of a quark-antiquark pair along the loop $\mathcal{C}$, from creation to annihilation, and it measures the free energy. For a rectangular Wilson loop in Euclidean space with length ℓ and height T, we have

$$W_{\mathcal{R}}(\mathcal{C}) \simeq e^{-T\,E_I(\ell)} \qquad\qquad (\text{for large } T, \ell) \tag{9.14}$$

where E_I is the interaction energy of the pair at distance ℓ.

The Wilson loop is a signal for confinement, if it grows as the exponential of the area inside the loop. In a confining theory the quark-antiquark pair has a binding energy that grows linearly with distance,

$$E = m_q + m_{\bar{q}} + E_I(\ell) \ , \qquad\qquad E_I(\ell) \sim \tau\ell \ , \tag{9.15}$$

since the interaction is given by a confined flux tube of tension τ. Therefore

$$W_{\mathcal{R}}(\mathcal{C}) \sim e^{-\tau T \ell} \sim e^{-\tau\, A(\mathcal{C})} \ , \tag{9.16}$$

where $A(\mathcal{C})$ is the area of the worldsheet swept by the flux tube. In this picture, the Wilson loop of pure gauge theory captures the confinement of non-dynamical external massive quarks, entirely due to gauge dynamics.

Wilson loops in AdS/CFT. We can define an analogous quantity in AdS. If we have a line source on the boundary, we can attach a string. Depending on the specific AdS/CFT realization, we will have different boundary line operators. The natural action for the string is the Nambu-Goto action

$$S = \int d^2\sigma \sqrt{\hat{g}} \ , \tag{9.17}$$

so this gives the AdS definition of an observable

$$W(\mathcal{C}) = \text{minimal area surface with boundary } \mathcal{C} \ . \tag{9.18}$$

Since the metric diverges at the boundary $z = 0$,

$$ds^2 = \frac{dx_\mu^2 + dz^2}{z^2} \ , \tag{9.19}$$

it is energetically favorable for the string to bend inside AdS.

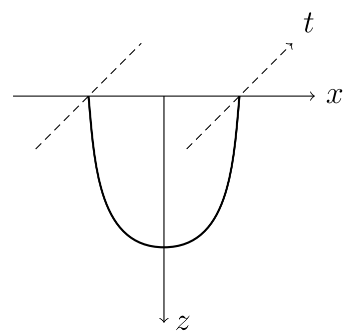
We can parametrize the string by (τ, σ), and the embedding by $X(\tau, \sigma)$. The (Euclidean) action is

$$S = \frac{1}{2\pi\alpha'} \int d\tau \, d\sigma \, \sqrt{\det_{ab} \partial_a X^M \partial_b X^N g_{MN}(X)} \ . \tag{9.20}$$

Let us consider a time-invariant configuration of two static sources at distance ℓ. We have

$$t = \tau \ , \qquad x = \sigma \ , \qquad z = z(\sigma) = z(x) \ . \tag{9.21}$$

The shape is



Then the action is

$$S = \frac{T}{2\pi\alpha'} \int_{-\ell/2}^{\ell/2} dx \, \frac{\sqrt{1 + (z')^2}}{z^2} \ . \tag{9.22}$$

Since the action does not depend explicitly on x, there is a conserved quantity:

$$\frac{\delta \mathcal{L}}{\delta z'} z' - \mathcal{L} = -\frac{1}{z^2 \sqrt{1 + (z')^2}} = \text{const} \ . \tag{9.23}$$

We can evaluate the constant at the turning point, where $x = 0$ (by symmetry), $z = z(0)$ and $z' = 0$. The constant is $-1/z(0)^2$. We thus find a differential equation

$$z' = \frac{dz}{dx} = -\sqrt{\frac{z(0)^4}{z^4} - 1} \ . \tag{9.24}$$

This can be solved (we make a change of variables $z(0)/z = y$):

$$x = \int_{z(x)}^{z(0)} \frac{dz}{\sqrt{\frac{z(0)^4}{z^4} - 1}} = z(0) \int_1^{z(0)/z(x)} \frac{dy}{y^2 \sqrt{y^4 - 1}} \ . \tag{9.25}$$

If we go to the boundary, where $x = \ell/2$ and $z = 0$, we find a relation between ℓ and the turning point:[68]

$$\frac{\ell}{2} = z(0) \int_1^\infty \frac{dy}{y^2\sqrt{y^4-1}} = \#\, z(0)\ . \tag{9.26}$$

The on-shell action is

$$S = \frac{2T}{2\pi\alpha'\, z(0)} \int_1^\infty \frac{y^2 dy}{\sqrt{y^4-1}}\ . \tag{9.27}$$

The integral is linearly divergent. This is because the energy includes the two infinitely massive quarks, $E = m_q + m_{\bar{q}} + E_I$. Their bare contribution is given by two straight lines at $x = \pm\ell/2$, which gives a linear divergence. Removing the divergence:

$$S_{\rm reg} = \frac{2T}{2\pi\alpha'\, z(0)} \int_1^\infty \Big(\frac{y^2}{\sqrt{y^4-1}} - 1\Big) dy = \#\, \frac{T}{\alpha'\,\ell}\ . \tag{9.28}$$

- The result is consistent with conformal invariance: the potential energy must scale as $\frac{1}{\ell}$.

  We will discuss the different behavior in confining models.

- The further apart are the quarks, the more the string penetrates in the interior of AdS. The turning point of the string is $\frac{1}{z(0)} \sim \frac{1}{\ell}$.

  This is consistent with the interpretation that

  $$\frac{1}{z} \sim E_{\rm process} \tag{9.29}$$

  is an energy scale, *i.e.* processes at energy scale E take place at $\frac{1}{z} \sim E$ in AdS.

[68]The constant is $\sqrt{\pi}\,\Gamma\left(\frac{3}{4}\right)/\,\Gamma\left(\frac{1}{4}\right)$. The constant in (9.28) is $\big[4\sqrt{\pi}\,\Gamma\left(\frac{1}{4}\right) + \pi\Gamma\left(-\frac{1}{4}\right)\big]/\big[\sqrt{2}\,\Gamma\left(\frac{1}{4}\right)^3\big]$.

10 Non-conformal theories and confinement

AdS/CFT applies to non-conformal theories as well, and thus it can describe phenomena like confinement, mass gap, discrete spectra, symmetry breaking, RG flows, etc... This makes it extremely interesting.

We already know one way to obtain non-conformal theories. We know how to turn on sources in the CFT, and if the source is for a *relevant* operator, this will induce an RG flow to some other fixed point.

If the CFT has a moduli space (usually only for supersymmetric CFTs), then another way is to turn on a VEV for some operator without sources.

In the description of non-conformal theories, the metric has Poincaré invariance but it is not AdS:

$$ds^2 = e^{2A(z)}\big(dz^2 + dx_\mu dx^\mu\big) \tag{10.1}$$

where $e^{2A(z)}$ is called the *warp factor*. If at the boundary ($z \to 0$) the metric is asymptotically AdS, namely

$$e^{2A(z)} \;\to\; \frac{1}{z^2} \qquad \text{as } z \to 0 \;, \tag{10.2}$$

and it is everywhere regular, then we can apply the rules of AdS/CFT. The bulk fields are still associated to primary operators with definite dimension in the far UV, but not in the complete theory at finite energy scales. They are *bound states* of the gauge theory (such as mesons and glueballs). Heavier objects, like baryons, usually appear as solitonic objects like wrapped branes.

10.1 RG flows

Domain wall flows Let us study some simple and universal properties of RG flows seen from the point of view of AdS. We consider a toy model of $5d$ gravity with a single scalar field and a general potential. This model may be part of an action obtained from a dim reduction of $10d$ or $11d$ supergravity, but here we will view it as a genuinely $5d$ model with gauge+gravity duality, which brings us beyond the AdS/CFT correspondence. The idea is to obtain an RG equation as a gradient flow equivalent to the sugra eom. This will provide a first-order differential equation needed for formulating an RG equation.

$$S = \int d^{d+1}x\, \sqrt{-g}\left[\frac{R}{16\pi G} - \frac{1}{2}\partial_m\phi\partial^m\phi - V(\phi)\right] \tag{10.3}$$

where we write $d^{d+1}x = d^d x dr$. We choose the potential such that it has one or more stationary points with $V'(\phi) = 0$. From the EOM for ϕ and g_{mn},

$$\begin{aligned} &\frac{1}{\sqrt{-g}}\partial_m(\sqrt{-g}g^{mn}\partial_n\phi) - V'(\phi) = 0 \\ &R_{mn} - \frac{R}{2}g_{mn} = 8\pi G(\partial_m\phi\partial_n\phi - \frac{1}{2}g_{mn}\partial_l\phi\partial^l\phi - g_{mn}V(\phi)) \equiv 8\pi G T_{mn}\,, \end{aligned} \tag{10.4}$$

one sees that at the stationary points ϕ_i, there is a trivial solution of the scalar EOM with constant $\phi(r) = \phi_i$. The Einstein equations then give

$$R_{mn} - \frac{R}{2}g_{mn} = -8\pi G g_{mn} V(\phi_i). \tag{10.5}$$

This is the same as the Einstein equations in AdS if we identify

$$\Lambda_i = -8\pi G V(\phi_i) = -\frac{d(d-1)}{L_i^2}. \tag{10.6}$$

Hence, constant scalar fields with AdS_{d+1} geometry of scale L_i are critical solutions which correspond to (different) conformal theories at RG fixed points on the field theory side.

A more general ansatz than (10.3) for solving the EOMs involves a metric with *warp factor* $A(r)$,

$$ds^2 = dr^2 + e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu\,, \qquad \phi = \phi(r)\,. \tag{10.7}$$

It's called a 'domain wall ansatz'. For a linear function $A(r) = r/L$, we recover the AdS metric (redefine $e^{y/L} = L/z$)

$$ds^2 = dr^2 + e^{2r/L}dx_\mu dx^\mu = L^2\frac{dz^2 + dx_\mu dx^\mu}{z^2}\,. \tag{10.8}$$

Together with a constant scalar we recover the dual of a CFT, as expected at an RG fixed point. Here we'll consider solutions which have linear $A(r)$ and constant ϕ near the boundary (at $r \to \infty$) and in the deep interior (at $r \to -\infty$). This is conjectured to be dual to an RG flow from a UV fixed point to an IR fixed point. It is natural to identify[69] r with the field theory RG scale via

$$\mu = \mu_0 \exp(r/L)\,, \tag{10.9}$$

so that in the UV (AdS boundary) $\mu \to \infty$ and in the deep interior $\mu \to 0$.

→ *Exercise 17:* Compute the components G^t_t and G^r_r of the Riemann tensor for the metric (10.7) and by considering the difference $G^t_t - G^r_r$; show that one can extract the following bound

$$A'' = \frac{8\pi G}{d-1}(T^t_t - T^r_r) = -\frac{8\pi G}{d-1}(\phi')^2\,, \qquad \Rightarrow A'' \le 0\,. \tag{10.10}$$

[69] This is scheme dependent.

The above condition is consistent with the null energy condition $T_{mn}\xi^m\xi^n \geq 0$ for ξ^m a null vector. The latter is one of the weakest of the classical energy conditions and it is expected to classically hold in all physically relevant gravity solutions (for a fluid, it means $p + \rho \geq 0$).[70] Combining Einstein and scalar EOMs give

$$\begin{aligned}\phi'' + dA'\phi' &= \frac{dV(\phi)}{d\phi} \\ (\phi')^2 - 2V(\phi) &= \frac{1}{8\pi G} d(d-1)(A')^2 \,.\end{aligned} \tag{10.11}$$

At each critical point of the potential, the A' equation yields $A(r) = \pm(r + r_0)/L_i/$ The integration constant r_0 and the sign have no significance since they can be changed by scaling of the coordinates x and changing $r \to -r$. Thus we can choose $A(r) = r/L_i$ which brings us to the form (10.7).

The domain wall equations (10.11) are a nonlinear second order system with no apparent method of analytic solution. There is nevertheless an interesting procedure that does give exact solutions; by introducing an auxiliary function, the *superpotential*[71] $W(\phi)$,

$$V(\phi) = \frac{1}{2}\left(\frac{dW}{d\phi}\right)^2 - \frac{d}{2(d-1)}W^2 \,. \tag{10.12}$$

Then it may be shown that any solution to the *first order gradient flow* equations

$$\sqrt{8\pi G}\frac{d\phi}{dr} = \frac{dW}{d\phi} \,, \qquad A' = -\frac{\sqrt{8\pi G}}{(d-1)}W \tag{10.13}$$

is also a solution to the equations of motion (10.11).

The goal is to find a solution which interpolates between two stationary points, namely a domain wall solution interpolating between and AdS space of radius L_{UV} for $r \to \infty$ and another AdS space of radius L_{IR} for $r \to -\infty$ where $A'' \leq 0$ implies that $L_{UV} \geq L_{IR}$. At the same time, the scalar should flow from a constant ϕ_{UV} in the UV to a constant ϕ_{IR} in the IR, with $\phi_{UV} \geq \phi_{IR}$. Such a domain wall solution is expected to be dual to a field theory RG flow between two fixed points.

[70] In the context of quantum theories, it is easy to construct counter-examples even in flat space. The condition that has been proven to be satisfied by all CFTs in flat space is the "averaged null energy condition" [FLPW16, HKT17], which is integrated along a null direction. Classically, the ANEC is necessary to avoid time machines and violations of the second law of thermodynamics.

[71] In sugra it does correspond to the contribution of the scalar field to the F-term potential.

Holographic c-theorem The existence of a holographic dual to an RG flow has striking consequences for CFTs at strong coupling. Holographic interpolating flows allow for a holographic proof of the c-theorem for higher dimensional QFTs in even dimensions that have a gravity dual.

In the 4d superconformal algebra, there are 2 central charges a, c, that control the conformal anomalies. It was conjectured by Cardy [Car88] and later proven by Komargodski-Schwimmer [KS11] that a is always decreasing along (unitary) RG flows from one CFT to another.[72] This has remained a conjecture for a long time, but it is easy to prove holographically.

Let us consider the case of $d = 4$. As discussed, we know the conformal anomaly of $\mathcal{N} = 4$ SYM in the large N limit. Consider now the holographic interpolating flow with metric (10.7) where r is interpreted as an energy scale. At fixed points, it has to coincide with the AdS metric, thus

$$A(r)|_{FP} = r/L \, , \qquad A'(r) = 1/L. \tag{10.14}$$

This suggests a generalized expression for the trace anomaly, which is obtained by replacing the AdS radius L by $1/A'(r)$,

$$\langle T^{\nu}_{\nu} \rangle = \frac{C(r)}{64\pi} \left(R_{\mu\nu} R^{\mu\nu} - \frac{R^2}{3} \right) \, , \qquad C(r) = \frac{1}{G_5 A'(r)^3} \, . \tag{10.15}$$

Then from the eom of the interpolating flow we have, using $A'' \leq 0$,

$$C'(r) = -\frac{3A''(r)}{G_5 A'(r)^4} \geq 0 \, . \tag{10.16}$$

We thus have constructed a monotonically decreasing $c-$function when moving to the IR at $r \to \infty$ which reproduces the conformal anomaly at the fixed points

$$C_i = \frac{L_i}{G_5} \, . \tag{10.17}$$

We can discuss the holographic c-theorem in an even better way by considering a gravity action now for several scalars,

$$S_5 = \int d^5x \, \sqrt{-g} \left[\frac{R}{16\pi G_5} - \frac{1}{2} G_{IJ} \partial_m \phi_I \partial^m \phi_J - V(\phi^I) \right] \tag{10.18}$$

[72] On the other hand, one can show in counter-examples that any other combination of a and c does not satisfy such a theorem [Car88]. KS have also constructed a function along the flow which is monotonic and agrees with a at the end-points. This is usually called a c-function. However, as opposed to Zamolodchikov's c-function in 2d, the one of KS in 4d is not such that the RG flow is its gradient flow.

where G_{IJ} is a positive definite matrix on the space of scalar fields (in order to obtain a unitary theory). Again this is in agreement with the null energy condition, which in this case is

$$T_r^r - T_t^t = G_{IJ}\partial^m \phi^I \partial_m \phi^J \geq 0\,. \tag{10.19}$$

→ *Exercise 18:* Given a superpotential $W(\phi^I)$ that satisfies the partial differential equation

$$V(\phi) = \frac{1}{2}\left(\frac{dW}{d\phi}\right)^2 - \frac{d}{2(d-1)}W^2\,, \tag{10.20}$$

show that the first order flow equations

$$\sqrt{8\pi G}\frac{d\phi^I}{dr} = G^{IJ}\frac{\partial W}{\partial \phi^J}\,, \qquad \frac{dA}{dr} = -\frac{\sqrt{8\pi G}}{3}W \tag{10.21}$$

automatically give a solution of the second order equations of motion of (10.18).

This implies

$$A''(r) = -\frac{8\pi G_5}{3}G^{IJ}\frac{d\phi^I}{dr}\frac{d\phi^J}{dr}\,. \tag{10.22}$$

Using this we can write the $c-$theorem for a function of the scalars ϕ^i which have the interpretation of sources or generalised couplings on the field theory side

$$C = \frac{\pi}{G_5 A'(r)^3} = -\frac{27\pi}{G_5(8\pi G_5)^{3/2}}\frac{1}{W^3}\,. \tag{10.23}$$

We now define a gravity β function as $\beta^I \equiv \frac{d\phi^I}{dr}$. Using this, we find

$$-\beta^I \partial_I C = -\frac{d\phi^I}{dr}\partial_I C = -27\cdot 8\frac{1}{W^4}\frac{d\phi^I}{dr}\frac{d\phi^J}{dr} \leq 0, \tag{10.24}$$

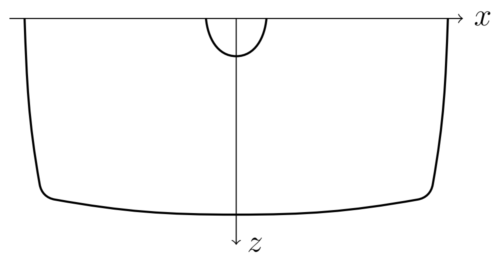
which coincides with the field theory result. We thus have a decreasing function along the RG flows towards the IR and, at the fixed points, C coindices with the conformaly anomaly coefficients.

10.2 Confining theories

Let us describe the qualitative features of the gravitational dual to a confining theory.

The warp factor $e^{2A(z)}$ is bounded above zero.

This follows from the Wilson loop. When the quarks are close to each other, the suspended string is in the asymptotically AdS region and $E_I \sim 1/\ell$.



When the quarks are further apart, the string is essentially the sum of two vertical pieces (the masses m_q, $m_{\bar{q}}$ to be subtracted) and an horizontal piece around z_h where $e^{2A(z_h)}$ has a minimum.

$$\text{For large } \ell: \qquad E(\ell) \simeq m_q + m_{\bar{q}} + \tau e^{2A(z_h)} \ell \tag{10.25}$$

where τ is the tension of the string $\quad\Rightarrow\quad$ linear confinement.

In a theory with mass gap and discrete spectrum we expect poles in the two-point functions, corresponding to physical states:

$$\langle \mathcal{O}_\phi(k)\, \mathcal{O}_\phi(-k) \rangle = \sum_i \frac{A_i}{k^2 + M_i^2} \ . \tag{10.26}$$

We are in Euclidean signature, and the poles are at $k^2 = -M_i^2$.

Let us consider a minimally-coupled scalar. At quadratic order the EOM is

$$\partial_z \big(e^{(d-1)A(z)} \partial_z \phi \big) - e^{(d-1)A(z)} k^2 \phi = e^{(d+1)A(z)} m^2 \phi \ . \tag{10.27}$$

At small z the metric is asymptotically AdS, therefore

$$z \to 0: \qquad \phi = z^{d-\Delta} \Big(A(k) + O(z) \Big) + z^\Delta \Big(B(k) + O(z) \Big) \tag{10.28}$$

with

$$L^2 m^2 = \Delta(\Delta - d) \ . \tag{10.29}$$

In standard quantization, A is the source and B is the VEV. In particular B is always normalizable. The corrections $O(z)$ are fixed by the EOMs, while regularity in the interior fixes B as a (non-local) function of A. By normalizing the field with the value of the source:

$$\phi_k(z) = \phi_k^0 \left[z^{d-\Delta} \big(1 + O(z) \big) + \frac{B(k)}{A(k)} z^\Delta \big(1 + O(z) \big) \right] \ . \tag{10.30}$$

(Here k is boundary momentum.) The two-point function has poles if and only if $A(k) = 0$:

if there are normalizable *and* regular solutions to the EOMs (*quasi-normal modes*).

There cannot be solutions for[73] $k^2 \geq 0$, but there can be for $k^2 < 0$. They correspond to the bound-states $M_i^2 = -k^2$.

We have reduced the problem to that of finding normalizable regular solutions to the EOMs. Let us redefine

$$\phi(x_\mu, z) = e^{-\frac{d-1}{2}A(z)}\,\psi(z)\,e^{ik_\mu x^\mu} \qquad \text{with} \qquad k^2 = -M^2 \,. \tag{10.31}$$

Then the radial equation becomes a Schrödinger-like equation:

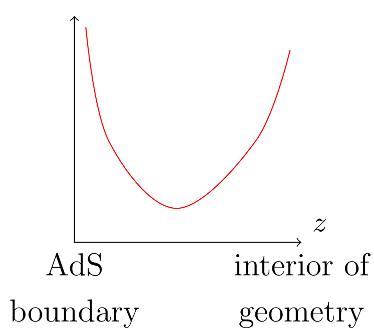
$$-\psi'' + \left(\frac{d-1}{2}A'' + \frac{(d-1)^2}{4}(A')^2 + m^2 e^{2A}\right)\psi = E\,\psi \qquad\qquad E = M^2 > 0\,. \tag{10.32}$$

We should find positive-energy solutions in a potential. The boundary conditions follow from the ones of the original problem. Normalizability

$$\int \sqrt{g}|\phi|^2 = \int e^{2A}|\psi|^2 \overset{z\to 0}{\sim} \int \frac{|\psi|^2}{z^2} \tag{10.33}$$

implies that $\psi \to 0$ at the boundary. The conditions at $z \to \infty$ are imposed by regularity.

- For $\mathrm{AdS}_{d=1}$, the potential[74] $\propto \frac{1}{z^2}$. One gets a continuous spectrum of scattering states (non-normalizable modes) starting from zero, appropriate for a conformal theory.
- The typical confining solution generates a potential that is constant or diverges for $z \to +\infty$, such as



This gives a discrete spectrum M_i^2 of glueballs above zero.

[73]Because for $k^2 \geq 0$ the Euclidean on-shell action is positive definite, while for a normalizable regular solution the on-shell action, which reduces to a boundary term, vanishes.

[74]The exact expression is $V_{\text{eff}}(z) = \left(m^2 + \frac{d^2-1}{4}\right)\frac{1}{z^2}$.

11 Theories at finite temperature

A simple way to obtain non-conformal theories is to turn on a temperature. This also breaks supersymmetry. The aim of this section is to see how theories at finite temperature appear in AdS/CFT.

11.1 Thermodynamics on $\mathbb{R}^3$

From 4d $\mathcal{N} = 4$ SYM to 3d YM

Let us study 4d $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ at finite temperature T. This can be obtained by first going to Euclidean signature, and then compactifying the Euclidean time to a circle of length $\beta = \frac{1}{T}$.

The reason is the following. The path-integral on a "strip" of length β with fixed boundary conditions $\Phi_{i,f}$ computes the propagation from an initial state $|\Phi_i\rangle$ to a final state $|\Phi_f\rangle$:

$$\int_{\Phi_i}^{\Phi_f} \mathcal{D}\varphi\, e^{-S} = \langle \Phi_f | e^{-\beta H} | \Phi_i \rangle \tag{11.1}$$

because H is the generator of time translations. The path-integral on a circle is obtained by identifying $\Phi_i = \Phi_f$ and integrating over them. This produces a trace:

$$\int_{\text{periodic on } S^1} \mathcal{D}\varphi \quad e^{-S} = \operatorname{Tr} \, e^{-\beta H} \; . \tag{11.2}$$

If we insert operators (and compute correlators) on the cylinder, we compute $\operatorname{Tr} e^{-\beta H} \mathcal{O}$. But $e^{-\beta H}$ (up to normalization) is the density matrix of a thermal state, and so we are computing matrix elements of $\mathcal{O}$ in a thermal state.

The fermions have anti-periodic boundary conditions along S^1:[75]

$$\psi(y) = -\psi(y+\beta) \; , \qquad \psi = \sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi_k \, e^{2\pi i k y/\beta} \qquad \Rightarrow \qquad m^2_{\psi_k} = 4\pi^2 \frac{k^2}{\beta^2} > 0 \; . \tag{11.3}$$

Thus all fermionic modes get a mass at tree level. Conformal invariance is broken by the compactification (the temperature set a scale, and it also breaks Lorentz) and supersymmetry is broken by the boundary conditions. Then the scalars get a mass at one-loop (loops below the fermionic mass are no longer canceled).

[75] That is because non-vanishing correlators have an even number of fermions. If we take one and bring it all along the circle, it crosses an odd number of fermions and so it changes sign.

For $\beta \to 0$ all fermionic and scalar modes get a mass and decouple: one is left with pure YM in three dimensions. From

$$\frac{1}{g_4^2}\int d^4x\, F_{\mu\nu}^2 = \frac{\beta}{g_4^2}\int d^3x\, F_{\mu\nu}^2 \qquad \Rightarrow \qquad \frac{1}{g_3^2} = \frac{\beta}{g_4^2}\ . \tag{11.4}$$

To have a smooth IR physics, we should send

$$\beta \to 0\ , \qquad g_4 \to 0 \qquad \text{with} \qquad g_3 \text{ fixed}\ . \tag{11.5}$$

In this limit we obtain a non-supersymmetric and non-conformal YM theory in 3d: it confines, it has a mass gap and a discrete spectrum of massive glueballs.

Gravity dual thermodynamics

The model can be studied using a weakly-coupled gravity dual, a black brane in asymptotically AdS spacetimes. The starting point is a system of N Euclidean D3-branes on $\mathsf{R}^3 \times S^1$. In the Lorentzian version they form a *black three-brane*: a sort of Schwarzschild black hole that extends along R^3 and is charged under $F_{(5)}$. The background metric is

$$ds^2 = H(r)^{-1/2}(-f(r)dt^2 + d\vec{x}^2) + H(r)^{(1/2}\left(\frac{dr^2}{f(r)} + r^2 d\Omega_5^2\right) \tag{11.6}$$

with $H(r) = 1 + L^4/r^4$ with $L^4 = 4\pi g_s N \alpha'^2$ and the 'blackening factor'

$$f(r) = 1 - \left(\frac{r_h}{r}\right)^4\ . \tag{11.7}$$

(There is also a flux for $F_{(5)}$ that we do not write) The near-horizon metric $r/L \ll 1$ is, using $z \equiv L^2/r$, and introducing Euclidean time $\tau = it$,

$$ds^2 = L^2\left\{\frac{dz^2}{z^2\left(1 - \frac{z^4}{z_h^4}\right)} + \frac{1}{z^2}\left[\left(1 - \frac{z^4}{z_h^4}\right)d\tau^2 + d\vec{x}^2\right] + d\Omega_5^2\right\}. \tag{11.8}$$

Lorentz invariance along the boundary directions is broken by temperature. At the boundary, we still have $\mathrm{AdS}_5 \times S^5$ asymptotically. But at $z = z_h$ the Lorentzian metric has a true horizon. The Euclidean metric is simply capped at $z = z_h$.

We can expand around $z = z_h$:

$$z = z_h(1 - \rho^2)\ . \tag{11.9}$$

Then expanding around $\rho = 0$:

$$ds^2 \simeq L^2\left\{d\rho^2 + \frac{4\rho^2}{z_h^2}d\tau^2 + \frac{1}{z_h^2}d\vec{x}^2 + d\Omega_5^2\right\}\ . \tag{11.10}$$

The metric looks like R^2 $(\times \mathsf{R}^3 \times S^5)$ in polar coordinates around $\rho = 0$, but it has a conical singularity unless τ is a periodic variable on a circle of length

$$\beta = \pi z_h \qquad \Rightarrow \qquad T = \frac{1}{\pi z_h} \,. \tag{11.11}$$

This is a way to compute the temperature of the black hole (or a black brane as here).

This space is simply connected, thus there is a unique possible spin structure, in which the fermions are antiperiodic on S^1. A translation of β along S^1 is a standard 2π rotation in R^2 around $\rho = 0$, and the spin-$\frac{1}{2}$ wavefunction changes sign under such a rotation.

We see that the black three-brane solution has all correct features to be the gravitational dual to the strongly coupled $4d$ $\mathcal{N} = 4$ SYM at finite temperature (on $\mathsf{R}^3 \times S^1$). Let us see what we can infer.

Entropy. Let's compute the entropy of the black brane. The Bekenstein-Hawking entropy formula gives

$$S_{BH} = \frac{A}{4G}; \tag{11.12}$$

the horizon area is given by

$$A = \int d^3x \sqrt{g_{11} g_{22} g_{33}|_{z=z_h}} \mathrm{Vol}(S^5) = \frac{L^6}{z^6} \pi^3 L^5 = \pi^6 L^8 T^3 \mathrm{Vol}(\mathsf{R}^3) \,. \tag{11.13}$$

Thus, using the value of G_{10}, we get the entropy of the $5d$ Schwarzschild AdS black brane

$$S_{BH} = \frac{\pi^2}{2} N^2 T^3 \mathrm{Vol}(\mathsf{R}^3) \tag{11.14}$$

and identify it with the entropy of the strongly coupled $\mathcal{N} = 4$ SYM plasma (where it is impossible to perform the same computation directly as it would require to sum an infinite number of Feynman diagrams). The free energy is given by[76]

$$F = -\frac{\pi^2}{8} N^2 T^4 \mathrm{Vol}(\mathsf{R}^3) \,, \qquad S_{BH} = -\frac{\partial F}{\partial T} \tag{11.15}$$

Energy density of the quark-gluon plasma If we compare the result (11.15) with the corresponding weak coupling perturbative result for $\mathcal{N} = 4$ SYM at finite temperature. Using a heat kernel approach, it was found that to leading order,

$$F_{weak\,coupling} = -\frac{\pi^2}{6} N^2 T^4 \mathrm{Vol}(\mathsf{R}^3) = \frac{4}{3} F_{strong\,coupling} \,. \tag{11.16}$$

[76]It can be computed using the gravitational on-shell action.

For the energy density $\epsilon = E/\text{Vol}(\mathsf{R}^3)$ obtained from the energy $E = F - TS$ we have

$$\epsilon = \frac{3\pi^2}{8} N^2 T^4 \tag{11.17}$$

and thus, denoting by ϵ_0 the energy density at vanishing coupling,

$$\frac{\epsilon}{\epsilon_0} = \frac{3}{4} \,. \tag{11.18}$$

This ratio was also calculated within lattice gauge theory for QCD, for instance for $N = 3$ colours and $N_f = 2$ flavours, and also extrapolating to large N. Within lattice gauge theory, it is possible to calculate ϵ as a function of the temperature: it rises rapidly around the deconfinement transition and then reaches a plateau at an almost constant value at $\frac{\epsilon}{\epsilon_0} \sim 0.8$ to 0.85. The lattice gauge theory for QCD and gauge/gravity duality result are impressively close to each other: this leads to the expectation that there are universal mechanisms at work, here that gauge/gravity duality may provide useful statements about the deconfined phase of QCD.

Confinement. The warp factor attains a minimum at the horizon (for strings along R^3):

$$\text{At } z = z_h: \qquad e^{2A(z_h)} = \frac{1}{z_h^2} \,. \tag{11.19}$$

Then the theory has stable strings with tension

$$\tau = e^{2A(z_h)}/2\pi\alpha' = \frac{\pi}{2}\sqrt{4\pi g_4^2 N}\, T^2 \,. \tag{11.20}$$

The fact that it goes like $\lambda^{1/2}$ is a hallmark of strong coupling.

Glueball spectrum. The masses of bound states can be extracted from the behavior of correlation functions of gauge-invariant operators at large distances:[77]

$$\langle \mathcal{O}(\vec{x})\, \mathcal{O}(0) \rangle \sim \sum_i A_i \, \frac{e^{-M_i x}}{x^{(d-1)/2}} \,. \tag{11.22}$$

[77]The Green's function of $-\Box\, G(\vec{x}) = \delta^d(\vec{x})$ is proportional to $1/x^{d-2}$. Instead the Green's function of

$$\big(-\Box + M^2\big) G(\vec{x}) = \delta^d(\vec{x}) \qquad \text{is proportional to} \qquad \frac{1}{x^{(d-2)/2}}\, K_{\frac{d-2}{2}}(kx) \sim e^{-Mx}\, \frac{1}{x^{(d-1)/2}} \,. \tag{11.21}$$

By a Fourier transform, in momentum space this is

$$\langle \mathcal{O}(\vec{k})\, \mathcal{O}(-\vec{k})\rangle \sim \sum_i \frac{A_i}{k^2 + M_i^2} \;. \tag{11.23}$$

Thus we can extract M_i from the normalizable solutions to the EOMs.

If we consider zero-modes on S^5, we can decompose the field into modes with fixed momentum $\vec{k}$ along R^3 and n along S^1:

$$\phi(z,t,x) = \phi(z)\, e^{int/\beta}\, e^{i\vec{k}\cdot\vec{x}} \;. \tag{11.24}$$

For simplicity, let us further restrict to zero-modes on S^1 as well: $n = 0$. Then, factoring out the volumes of S^5, R^3 and S^1, the action

$$S = \int dz\, dt\, d^3x\, d\Omega_5\, \sqrt{g}\Big(\partial_M\phi\partial^M\phi + m^2\phi^2\Big) \tag{11.25}$$

becomes

$$S \sim \int_0^{z_h} \frac{dz}{z^5}\bigg[z^2\Big(1 - \frac{z^4}{z_h^4}\Big)(\partial_z\phi)^2 + k^2z^2\phi^2 + m^2\phi^2\bigg] \;. \tag{11.26}$$

The EOM from this action is

$$-\partial_z\bigg(\frac{1}{z^3}\Big(1 - \frac{z^4}{z_h^4}\Big)\,\partial_z\phi\bigg) + \frac{k^2}{z^3}\phi + \frac{m^2}{z^5}\phi = 0 \;. \tag{11.27}$$

To bring it to the form of a Schrödinger equation, first we change radial coordinate to

$$\tau(z) = \int_0^z \frac{dy}{\sqrt{1 - \frac{y^4}{z_h^4}}} \qquad \to \qquad d\tau = \Big(1 - \frac{z^4}{z_h^4}\Big)^{-\frac12} dz \;. \tag{11.28}$$

Then we redefine the field as

$$\phi = e^{-A/2}\psi \qquad\qquad \text{with} \qquad e^{A(\tau)} = \frac{1}{z^3}\sqrt{1 - \frac{z^4}{z_h^4}} \;. \tag{11.29}$$

This gives the equation

$$-\psi'' + \bigg(\frac{A''}{2} + \Big(\frac{A'}{2}\Big)^2 + \frac{m^2}{z^2}\bigg)\,\psi = E\,\psi \qquad\qquad E = -k^2 = M^2 \tag{11.30}$$

where everything is a function of τ.

Exercise. Study qualitatively the spectrum (for $n = 0$).
One finds

$$\frac{A''}{2} + \Big(\frac{A'}{2}\Big)^2 = \frac{15 - 18z^4/z_h^4 - z^8/z_h^8}{4z^2\big(1 - z^4/z_h^4\big)} \;.$$

11.2 Thermal phase transitions

We will discuss now how the Hawking-Page phase transition in quantum gravity can be mapped into a confinement-deconfinement transition in gauge theory.

Let us consider a field theory defined on a spacetime manifold $\mathsf{R} \times S^3$, namely where now the spatial dimensions are compactified. At finite T, the time direction gets compactified too and the manifold becomes $S^1 \times S^3$. There are now two dimensioful quantitites, $\beta = 1/T$ and $\beta' = 1/l$, with l the radius of S^3, and the physics will depend on the quotient β/β'.

The holographic prescription at large N and large (but fixed) $g_{YM}^2 N$ involves extremizing the classical gravity action subject to asymptotic boundary conditions. This is the saddle-point approximation to the path-integral of gravity.[78] There can be more than one saddle point: this is a general feature of boundary value problems in differential equations. In this case we are supposed to sum

$$e^{-S_{\text{gravity}}}$$

over the various classical configurations. The solution that globally minimizes S_{gravity} dominates the saddle-point approximation. When there are two or more competing solutions, *e.g.*

$$Z = e^{-S_1} + e^{-S_2} + \dots \ , \tag{11.31}$$

there can be *phase transitions*.

Let us study an example in AdS$_5$. We take the standard action

$$S = -\frac{1}{16\pi G_{\text{N}}} \int d^5x \, \sqrt{g} \left(\mathcal{R} + \frac{12}{L^2} \right) . \tag{11.32}$$

• One can embed the (Euclidean) Schwarzschild black hole in AdS$_5$:

$$ds^2 = f \, d\tau^2 + \frac{1}{f} \, dr^2 + r^2 d\Omega_3^2 \ , \qquad\qquad f = 1 + \frac{r^2}{L^2} - \frac{\mu}{r^2} \ . \tag{11.33}$$

Assuming $\mu > 0$, it follows that f has a positive root $r^2 = r_+^2$ and a negative root $r^2 = -r_-^2$ (with $r_-^2 - r_+^2 = L^2 > 0$):

$$f = \frac{(r^2 - r_+^2)(r^2 + r_-^2)}{r^2 L^2} \ , \tag{11.34}$$

[78] The path-integral of gravity is ill-defined because gravity is non-renormalizable. However string amplitudes render on-shell quantities well-defined.

then the Euclidean solution is defined for $r \geq r_+$. By the same argument as before,[79] the geometry is smooth at $r = r_+$ iff

$$t \cong t + \beta \qquad \text{with} \qquad \beta = \frac{1}{T} = \frac{2\pi L^2 r_+}{2r_+^2 + L^2} \,. \tag{11.35}$$

We can think of this as fixing $\mu = r_+^2 r_-^2 = r_+^2(L^2 + r_+^2) > 0$ in terms of the temperature T.

Let us call this space

$$X_2 = \text{Euclidean Schwarzschild BH in AdS}_5 \,.$$

This space has topology $D_2 \times S^3$, with boundary $S^1 \times S^3$. The latter is the relevant space for a QFT on S^3 at finite temperature.

This space is simply connected, and so it has a unique spin structure in which the fermions are anti-periodic in τ.

- There is another (Euclidean) solution with the same boundary conditions:

$$ds^2 = f\, d\tau^2 + \frac{1}{f}\, dr^2 + r^2 d\Omega_3^2 \,, \qquad\qquad f = 1 + \frac{r^2}{L^2} \,. \tag{11.36}$$

(The same as before, but with $\mu = 0$). In fact this is global AdS_5,[80] but with compactification of the Euclidean time:

$$\tau \cong \tau + \beta \tag{11.37}$$

with any β. This space is called *thermal AdS*. We call it

$$X_1 = \text{Thermal AdS} \,.$$

This space has a topology completely different from the previous one: $S^1 \times D_4$, but with same boundary $S^1 \times S^3$.

This space is not simply connected, so it admits two spin structures: periodic or anti-periodic fermions along S^1. With anti-periodic boundary conditions it describes AdS at finite

[79]Define $r - r_+ = \rho^2$, and use $r_+^2 + r_-^2 = 2r_+^2 + L^2$.

[80]With the coordinate change $r = L \tan\theta$ and $\tau = L\tau$, one gets

$$ds^2 = \frac{L^2}{\cos^2\theta}\big(d\tau^2 + d\theta^2 + \sin^2\theta\, d\Omega_3^2\big) \,.$$

On the other hand, with the coordinate change $r = L\sinh\rho$ and $\tau = L\tau$, one gets

$$ds^2 = L^2\big(\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2\big) \,.$$

temperature, *i.e.* a thermal gas of gravitons in AdS. With periodic boundary conditions, instead, supersymmetry remains unbroken and the path-integral computes

$$\mathrm{Tr}\,(-1)^F\, e^{-\beta H} \dots$$

which is called a Witten index.

If we impose periodic (supersymmetric) conditions on S^1 at the boundary, *i.e.* we compute $\mathrm{Tr}\,(-1)^F e^{-\beta H}$ in the QFT, only the space X_1 is relevant (has the correct boundary conditions). But if we impose anti-periodic (thermal) conditions, *i.e.* we study the QFT on S^3 at finite temperature, both X_1 and X_2 are relevant.[81] Both are saddle-point contributions to the path-integral, and the dominant one is the one with smaller on-shell action.

This problem has been studied by Hawking and Page [HP83] first, as a question in 4d quantum gravity, and later adapted by Witten [Wit98] to AdS/CFT.

One computes the on-shell actions. Both $S[X_1]$ and $S[X_2]$ are divergent and should be regularized, however $S[X_2]-S[X_1]$ is finite (one sets a cutoff $r \leq L_0$ and then takes $L_0 \to \infty$). One imposes that X_1 has the same S^1 radius as X_2. One finds

$$S[X_2] - S[X_1] = \frac{\pi^2 r_+^3 (L^2 - r_+^2)}{4G_N(2r_+^2 + L^2)} \tag{11.38}$$

This expression changes sign at $r_+ = L$, therefore there is a phase transition between thermal AdS_5 and a large black hole.

- $r_+ > L$, *i.e.* $T > \dfrac{3}{2\pi L}$: large black hole, X_2 Confinement
- $r_+ < L$, *i.e.* $T < \dfrac{3}{2\pi L}$: thermal AdS_5, X_1 Deconfinement

On the boundary, this phase transition is interpreted as a *confinement/de-confinement* transition, from the behavior of the Wilson loop. If T is roughly smaller than the scale $\frac{1}{L}$ set by the sphere, we don't see T and the physics is that of a conformal theory. If T is larger than $\frac{1}{L}$, we do see the temperature, scalars and fermions get massive and we are left with 3d YM on S^3 which confines.

[81] The BH solution has a minimal temperature $T_{\min} = \sqrt{2}/\pi L$ (attained at $2r_+^2 = L^2$), and for all larger values of the temperature there are two solutions: the small BH and the large BH. Thus for $T < T_{\min}$ only X_1 can contribute, while for $T > T_{\min}$ there are X_1, X_2^{small} and X_2^{large}. It turns out that X_2^{small} has larger action and so is always thermodynamically unfavored [HP83].

12 Linear response and hydrodynamics

We will discuss real-time correlators in AdS/CFT, transport coefficients, and how AdS/CFT can be applied to systems oriented towards condensed matter problems. Good reviews are [Her09, Har09].

Strongly coupled fluids are difficult to describe by standard methods. The gauge/gravity duality has famously allowed to make predictions of universal values of certain transport coefficients.

12.1 Field theory retarded Green's functions

The idea of linear response is to consider small space- and time-dependent perturbations about the equilibrium state of a physical system.

Green's functions The basic object in linear response theory is the retarded Green's function: it relates linear fluctuations of sources to the corresponding expectation values. This allows to compute transport coefficients from two-point correlation functions.

Suppose we have a system described by the (possibly time-dependent) Hamiltonian $\widetilde{H}(t)$. We perturb the system by external sources $\phi_I(t,\vec{x})$ coupled to a set of operators $\mathcal{O}_I(t,\vec{x})$. The Hamiltonian is modified by a term

$$\delta H = -\int d^dx\, \phi_I(t,\vec{x})\, \mathcal{O}^I(t,\vec{x}) \,. \tag{12.1}$$

Under the assumption that we keep fixed the states in the far past, we ask what is the effect on the expectation values $\langle \mathcal{O}^I(x)\rangle$. To linear order, the causal effect of the perturbation is controlled by the *retarded Green's function* G_R

$$\begin{aligned} \delta\langle \mathcal{O}^I(x)\rangle &= \int d^dx\, G_R^{IJ}(x,y)\phi_J(y) + \mathcal{O}(\phi^2)\,, \\ G_R^{IJ}(x,x') &= i\,\Theta(t-t')\left\langle \left[\mathcal{O}^I(x)\,,\,\mathcal{O}^J(x')\right]\right\rangle , \end{aligned} \tag{12.2}$$

where Θ is the Heaviside step function. The retarded Green's function is non-vanishing only in the forward light-cone and thus provides a cauality structure, namely $\delta\langle\mathcal{O}^I(x)\rangle$ is influenced only by sources $\phi^I(t',\vec{x}')$ with $t'<t$. Using translational invariance, in momentum space we have

$$G_R^{IJ}(\omega,k) = \int d^dx\, e^{-ik\cdot x}\, G_R^{IJ}(x,0) \,. \tag{12.3}$$

Then the causal effect of a perturbation is simply

$$\boxed{\delta\langle \mathcal{O}_I(k)\rangle = G_R^{IJ}(k)\,\phi_J(k) + \mathcal{O}(\phi^2)}\;. \tag{12.4}$$

Expression (12.2) can be understood in the following way. It is convenient to do the computation in the Schrödinger picture. The evolution of states is controlled by the Schrödinger equation

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = \widetilde{H}(t)\,|\psi(t)\rangle\;, \tag{12.5}$$

where we use $\widetilde{\ }$ for operators in the Schrödinger picture. This is solved by the unitary time-evolution operator

$$U(t,t_0) = \mathrm{T}\,e^{-i\int_{t_0}^t \widetilde{H}(t')\,dt'} \qquad \text{such that} \qquad \frac{\partial}{\partial t}U(t,t_0) = -i\,\widetilde{H}(t)\,U(t,t_0)\;. \tag{12.6}$$

The expectation value of an operator $\mathcal{O}(t,\vec{x})_i$ is

$$\langle \mathcal{O}_i(t,\vec{x})\rangle = \mathrm{Tr}\,\rho(t)\,\widetilde{\mathcal{O}}_i(t,\vec{x}) = \mathrm{Tr}\,U(t,t_0)\,\rho_0\,U(t,t_0)^{-1}\,\widetilde{\mathcal{O}}_i(t,\vec{x}) = \mathrm{Tr}\,\rho_0\,\mathcal{O}_i(t,\vec{x})\;, \tag{12.7}$$

where ρ is a density matrix (evolving with $i\partial_t\rho = [\widetilde{H},\rho]$) and ρ_0 is the density matrix at the far-past time t_0. Operators in the Heisenberg picture are related to those in the Schrödinger picture by the unitary transformation

$$\mathcal{O}(t,\vec{x}) = U(t)^{-1}\,\widetilde{\mathcal{O}}(t,\vec{x})\,U(t)\;. \tag{12.8}$$

The Heisenberg picture is the one used in QFT.[82]

Now we separate

$$\widetilde{H}(t) = \widetilde{H}_0(t) + \delta\widetilde{H}(t)\;.$$

The evolution operator for $\widetilde{H}$ is (as can be checked by computing ∂_t)

$$U(t) = U_0\,\cdot\,\mathrm{T}\exp\Big\{-i\int^t U_0(t')^{-1}\,\delta\widetilde{H}(t')\,U_0(t')\,dt'\Big\} = U_0\,\cdot\,\mathrm{T}\,e^{-i\int^t \delta H(t')\,dt'}\;, \tag{12.9}$$

where $U_0(t,t_0)$ is the time evolution of $\widetilde{H}_0$. Expanding at first order we find

$$\begin{aligned}\delta\langle\mathcal{O}_i(t,\vec{x})\rangle &= -i\,\mathrm{Tr}\,\rho_0\int^t dt'\,\big[U_0(t)^{-1}\,\widetilde{\mathcal{O}}_i(t,\vec{x})\,U_0(t)\,,\,\delta H(t')\big]\\ &= i\int^t dt'\int d^{d-1}\vec{x}'\,\Big\langle\big[\mathcal{O}_i(t,\vec{x})\,,\,\mathcal{O}_j(t',\vec{x}')\big]\Big\rangle\,\phi_j(t',\vec{x}')\;.\end{aligned} \tag{12.10}$$

[82]In the Schrödinger picture states evolve with time, while operators $\widetilde{\mathcal{O}}$ do not (but can have an explicit dependence on time, like $\widetilde{H}$). In the Heisenberg picture, instead, states remain constant while operators $\mathcal{O}$ evolve. Things get a bit confusing, though, if the Hamiltonian depends explicitly on time, since $\widetilde{H}(t)\neq H(t)$ but the evolution operator is constructed with $\widetilde{H}$.

Notice that time integration is only over $t' < t$.

Remarks Retarded Green's functions have interesting properties.

- If the operators $\{\mathcal{O}_i\}$ are Hermitian, it easily follows

$$G_R^{ij}(t,\vec{x})^* = G_R^{ij}(t,\vec{x}) \;, \qquad\qquad G_R^{ij}(\omega,k)^* = G_R^{ij}(-\omega,-k) \;. \tag{12.11}$$

- If the system is invariant under time reversal, and the operators $\{\mathcal{O}_i\}$ transform as $T\mathcal{O}_i(t,\vec{x})T^{-1} = \epsilon_i \mathcal{O}_i(-t,\vec{x})$ with $\epsilon_i = \pm 1$, then

$$G_R^{ij}(t,\vec{x})^* = \epsilon_i\,\epsilon_j\,G_R^{ji}(t,-\vec{x}) \;. \tag{12.12}$$

  This follows because T is an anti-unitary operator:

$$\Big\langle \big[\mathcal{O}_i(t,\vec{x}),\mathcal{O}_j(0,0)\big]\Big\rangle = \Big\langle T\big[\mathcal{O}_i(t,\vec{x}),\mathcal{O}_j(0,0)\big]T^{-1}\Big\rangle^* = -\epsilon_i\epsilon_j\Big\langle \big[\mathcal{O}_j(t,-\vec{x}),\mathcal{O}_i(0,0)\big]\Big\rangle \;.$$

  Combined with Hermiticity of $\{\mathcal{O}_i\}$ one gets

$$G_R^{ij}(\omega,k) = \epsilon_i\,\epsilon_j\,G_R^{ji}(\omega,-k) \;. \tag{12.13}$$

  If time reversal is broken, for instance by a magnetic field B, the property is still true except that one of the two sides is evaluated in the flipped background (*e.g.* $B \to -B$). At zero momentum k the relation is called Onsager relation.

- Retarded Green's functions are causal, in the sense that a perturbation only affects later times and so $G_R^{ij}(t,\vec{x})$ vanishes for $t<0$. Consider

$$G_R(t,k) = \int d\omega\, e^{-i\omega t}\, G_R(\omega,k) \;.$$

  For $t<0$ we can close the contour in the upper half-plane, and the fact that we get zero means that $G_R(\omega,k)$ is analytic (no poles) in the upper half-plane.

  From this fact, if moreover $G_R(\omega)$ vanishes for $|\omega|\to\infty$, one obtaines the Kramers-Kronig relations[83]

$$\mathsf{Re}\, G_R(\omega) = P\int_{-\infty}^{\infty}\frac{d\omega'}{\pi}\,\frac{\mathsf{Im}\, G_R(\omega)}{\omega'-\omega} \;, \qquad \mathsf{Im}\, G_R(\omega) = -P\int_{-\infty}^{\infty}\frac{d\omega'}{\pi}\,\frac{\mathsf{Re}\, G_R(\omega)}{\omega'-\omega} \;. \tag{12.15}$$

[83] They follow from

$$G_R(z) = \oint_\Gamma \frac{d\zeta}{2\pi i}\,\frac{G_R(\zeta)}{\zeta - z} \qquad\qquad \text{for } \mathsf{Im}\, z > 0 \tag{12.14}$$

where Γ is within $\mathsf{Im}\,\zeta > 0$, right above the real axis and then closed in the upper half-plane. Then one takes the limit $z \to \omega + i0$.

- Retarded Green's function satisfy positivity properties. One can show that

$$-i\omega\big(G_R^{ij}(\omega) - G_R^{ji}(\omega)^*\big) \text{ is def } \geq 0\,. \tag{12.16}$$

The anti-Hermitian part of the retarded Green's function is called the *spectral function*. If there is a unique operator involved, the spectral function satisfies

$$\omega\ \mathsf{Im}\, G_R(\omega) \geq 0\,. \tag{12.17}$$

Transport coefficients Retarded Green's functions are directly related to *transport coefficients*. Realistic fluids include additional contributions to the energy-momentum tensor (and conserved current) of an ideal fluid. To first order in the so-called derivative expansion, we have

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + p(g^{\mu\nu} + u^\mu u^\nu) - \sigma^{\mu\nu} + \dots\,, \tag{12.18}$$

where u^μ denotes the relativistic velocity of the fluid ($u_\mu u^\mu = -1$), ε is the energy density and p the pressure. $\sigma^{\mu\nu}$ encodes as transport coefficients the so-called *shear viscosity* η (coefficient of the traceless part) and *bulk viscosity* ζ (coefficient of the trace part).

For metric perturbations $g = \delta + h(t)$ which are time-dependent but homogeneous in space, the expression for $\sigma^{\mu\nu}$ is simply

$$\sigma_{xy} = \eta \partial_t h_{xy}\,. \tag{12.19}$$

Thus from the linear response theory (12.3), we find that the (zero spatial momentum) retarded Green's functions associated to the above metric perturbation is

$$\begin{aligned} G_R^{xy,xy}(\omega,\vec{0}) &= \int dt d^3x\, e^{-i\omega t}\Theta(t)\langle[T^{xy}(t,\vec{x}), T^{xy}(0,\vec{0})]\rangle \\ &= -i\eta\omega + \mathcal{O}(\omega^2)\,. \end{aligned} \tag{12.20}$$

We thus read off the relationship[84]

$$\eta = -\lim_{\omega\to 0} \frac{1}{\omega}\,\mathrm{Im}\, G_R^{xy,xy}(\omega,\vec{0})\,. \tag{12.21}$$

You can also take the example of Ohm's law, which defines the electric conductivity. It states that for an electric field that is constant in space ($k = 0$) but oscillating in time with frequency ω, the spatial part of the charge current response is given by

$$J_i(\omega) = \sigma_{ij}(\omega)\, E_j(\omega)\,. \tag{12.22}$$

[84] It is an example of a Green-Kubo relation.

(Here ij are spatial indices.) Here $\sigma_{ij}(\omega)$ is called *optical conductivity*.

In our language, ϕ_i is an external vector potential A_μ while $\mathcal{O}_i$ is the conserved current J_μ. We take a gauge where $A_t = 0$, then $E_i = -\partial_t A_i$. Making a Fourier decomposition $A_i \sim e^{-i\omega t}$, we have

$$E_j = i\omega A_j \ .$$

We see that the optical conductivity is related to the current-current correlator:

$$\sigma_{ij}(\omega) = \frac{G_R^{ij}(\omega, 0)}{i\omega} \ . \tag{12.23}$$

By considering correlators of the current and the stress tensor, one can similarly construct a matrix of transport coefficients that include also the heat conductivity and the thermoelectric coefficients.

This is good new for AdS/CFT, because two-point functions is precisely one of the things that we can easily compute with AdS/CFT in strongly-coupled theories.

12.2 Holographic Green's functions

Let us now discuss how to implement real time Green's functions on the gravity side. We already know how to compute holographically two-point functions, but now we need to look at the case of Lorentzian signature and impose boundary conditions compatible with causality.

Let us consider again a $4d$ boundary and metric (11.6) with $H(r) = L^4/r^4$. We introduce $u = r_h^2/r$, hence

$$ds^2 = \frac{(\pi T L)^2}{u}(-f(u)dt^2 + d\vec{x}^2) + \frac{L^2}{4u^2 f(u)}du^2 + L^2 d\Omega_5^2 \, , \tag{12.24}$$

with $f(u) = 1 - u^2$. This is an asymptotically AdS space with boundary at $u = 0$ and horizon at $u = 1$. Near the boundary, the Fourier modes of a scalar field will have the asymptotic behaviour that we have already encountered in the past,

$$\phi(u, k) \sim \phi_{(0)}(k) u^{(d-\Delta)/2}(1 + \mathcal{O}(u)) + \phi_{(+)}(k) u^{\Delta/2}(1 + \mathcal{O}(u)) \, . \tag{12.25}$$

The difference is now how to impose the correct boundary conditions to get the retarded Green function. One condition is given by fixing $\phi_{(0)}(k)$ at the boundary $u = 0$. Near the horizon at $u = 1$, the solutions scale as

$$\phi_k(u) \sim (1 - u)^\kappa \tag{12.26}$$

In the Euclidean signature, κ is real, $\kappa = \pm\omega/(4\pi T)$, thus only the solution $\kappa = +\omega/(4\pi T)$ is regular at the horizon (assuming $\omega > 0$). But in Lorentzian signature, we have

$$\kappa = \pm\frac{i\omega}{4\pi T} \tag{12.27}$$

and thus *both* solutions are regular. The exercise below will tell you that the − sign solution corresponds to an *infalling* boundary conditions, while the + sign solution corresponds to an *outgoing* boundary conditions. We want to ensure causality, so we will impose the infalling boundary condition since then only boundary sources located in the past may influence the bulk physics.

→ *Exercise 19* Restore the time dependence from the Fourier mode and consider $e^{-i\omega t}(1-u)^{\pm i\omega/(4\pi T)}$. Then, introducing the variable $\tilde{r} \equiv (\ln(1-u))/(4\pi T)$, show that the solution with $\kappa = -\frac{i\omega}{4\pi T}$ behaves as $\sim e^{-i\omega(t+\tilde{r})}$, which corresponds to a wave moving *towards* the horizon.

We can also recall from holographic renormalization that the one-point function in presence of source is given by

$$\langle\mathcal{O}(x)\rangle_s = L^{d-1}(2\Delta - d)\phi_{(+)}(k) + \mathcal{C}(\phi_{(0)}) \tag{12.28}$$

which is consistent with the interpretation of $\phi_{(+)}$ as encoding the response of the expectation value of the operator to the fluctuation caused by the source $\phi_{(0)}$.

Shear viscosity from AdS/CFT To obtain η at strong coupling, we compute the retarded Green's function (12.20) from the correlator $\langle T^{xy}T^{xy}\rangle$. For this we need to look at the propagation of the dual graviton mode h_{xy} in AdS. Let us start from the EH action in $5d$; the part which is quadratic in h_{xy} is

$$S[h_{xy}] = \frac{N^2}{8\pi^2 L^3}\int d^4x dr\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu h_{xy}\partial_\nu h_{xy}\right), \tag{12.29}$$

where we neglect local counterterms. This action gives rise to the linearised EOM

$$\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu h_{xy}) = 0$$

, which is exactly the same form as the EOM for a scalar field, so we can replace $h_{xy} \to \phi$. The background metric is again AdS-Schwarzschild metric (12.24). To compute holographically the retarded Green's functions, we write the solution as

$$\phi(u,k) = \phi_{(0)}(k)\cdot\phi_k(u)\,, \tag{12.30}$$

where $\phi_k(u)$ satisfies the infalling wave boundary condition, i.e. $\phi_k(u) \sim (1-u)^{-i\omega/(4\pi T)}$. Then the retarded Green function can be obtained from the on-shell action. In the limit $|\vec{q}| \to 0$, the result is

$$G_R(\omega) = -\frac{\pi N^2 T^3}{8} i\omega \,. \tag{12.31}$$

According to the relation , we thus obtain

$$\eta = \frac{\pi}{8} N^2 T^3 \,. \tag{12.32}$$

Recalling (11.15), the entropy density is

$$s = \frac{S}{\mathrm{Vol}(\mathsf{R}^3)} = \frac{\pi^2}{2} N^2 T^3 ; \tag{12.33}$$

so finally

$$\boxed{\frac{\eta}{s} = \frac{1}{4\pi}} \,. \tag{12.34}$$

This result is very important for the following reasons.

- It is universal, in the sense that all gravity duals involving the EH action give the same result, irrespective of whether other fields (gauge or scalars) are added to the gravity action, or whether conformal symmetry or supersymmmetry is broken or not. It is also independent of the spacetime dimension.
- Experimental results at the Brookhaven's Relativistic Heavy Ion Collider (RHIC) and LHC show that the measured value of $\frac{\eta}{s}$ for the quark-gluon plasma is in good agreement with (12.34) (while other liquids such as water or liquid helium have a value of $\frac{\eta}{s}$ which is larger by orders of magnitude). This was the first example of a successful measurement of an observable calculated using gauge/gravity duality.

The following sections are extra material.

13 The conifold

In this lecture we want to study another exact example of AdS/CFT, with only $\mathcal{N} = 1$ supersymmetry. This is a generalization of type IIB on $\mathrm{AdS}_5 \times S^5$, in which the manifold S^5 is substituted by another one, $T^{1,1}$, called the "base of the conifold". The physics is much richer: anomalies, dimensional transmutation, confinement, chiral symmetry breaking, domain walls separating inequivalent vacua appear. Good reviews are [Kle00, HKO02].

The starting point of Maldacena's proof is D3-branes on flat space, namely on

$$\mathsf{R}^{3,1} \times \mathsf{R}^6 \ .$$

We can construct other examples by taking a more general geometry:

$$\mathsf{R}^{3,1} \times \mathcal{M}_6 \ ,$$

in other words the D3-branes are at a point of $\mathcal{M}_6$. In order to have a stable vacuum, we'd better preserve some supersymmetry: the minimal one is $\mathcal{N} = 1$ in 4d. The condition for SUSY[85] is that on $\mathcal{M}_6$ there is a covariantly constant spinor ζ, because the gravitino variation is

$$\delta_\zeta \Psi_\mu = \nabla_\mu \zeta = 0 \ . \qquad (13.1)$$

Manifolds $\mathcal{M}_6$ satisfying the condition are very special: they are called *Calabi-Yau manifolds*.

- They are complex and Kähler: the tensors

$$J_{ij} = \zeta^\dagger \Gamma_{ij} \zeta \ , \qquad\qquad J^i{}_j = g^{ik} J_{kj} \qquad (13.2)$$

  are a (closed) Kähler form and a (integrable) complex structure.

- The metric is Ricci-flat:[86]

$$R_{ij} = 0 \ . \qquad (13.3)$$

- They have trivial canonical bundle: the complex tensor

$$\Omega_{ijk} = \zeta^\mathsf{T} \Gamma_{ijk} \zeta \qquad (13.4)$$

  is closed and covariantly-constant.

[85] We assume that we do not turn on fluxes, besides the $F_{(5)}$ generated by the D3-branes. Then the dilatino variation automatically vanishes, $\delta\lambda = 0$.

[86] From $[\nabla_i, \nabla_j]\zeta = 0$ one gets $R_{ij}{}^{kl}\Gamma_{kl}\zeta = 0$. Then use $\Gamma^j\Gamma^{kl} = \Gamma^{jkl} + \delta^{jk}\Gamma^l - \delta^{jl}\Gamma^k$ and the Bianchi identity $R_{i[jkl]} = 0$ to get $R_{ij}\Gamma^j\zeta = 0$, finally use that the gamma matrices are independent.

- The holonomy group is reduced from $SO(6)$ to $SU(3)$: a chiral spinor is in the **4** of $SU(4)$, and it breaks it to $SU(3)$.

In Maldacena's argument we take a near-horizon limit to the branes, *i.e.* we focus around the branes. If the D3-branes sit at a smooth point on $\mathcal{M}_6$, the near-horizon leads to the same $\mathrm{AdS}_5 \times S^5$ as before.

In order to get something new, we should place the D3-branes at a singular point on $\mathcal{M}_6$. Focusing, the singularity looks like a *conical singularity*:

$$ds^2(\mathcal{M}_6) = dr^2 + r^2\, ds^2(X_5) \; . \tag{13.5}$$

The SUSY condition for X_5 is that it is Sasaki-Einstein (in particular positively curved).

The near-horizon geometry to D3-branes at a CY singularity is

$$\mathrm{AdS}_5 \times X_5$$

with N units of 5-form flux on AdS_5 and X_5.

13.1 Conifold geometry and SCFT

The conifold (a conical Calabi-Yau threefold) is described by one complex equation in four variables:

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0 \; . \tag{13.6}$$

The equation is invariant under rescaling, thus the geometry is a cone. The base is called $T^{1,1}$, which is a coset space:

$$T^{1,1} = \frac{SU(2) \times SU(2)}{U(1)} \tag{13.7}$$

and has $SU(2) \times SU(2) \times U(1)$ isometry. It is obtained by intersecting the equation with

$$\sum_{a=1}^{4} |z_a|^2 = 1 \; , \tag{13.8}$$

which has $SO(4)\times U(1)$ invariance. Since there is a unique Abelian isometry, that is identified with the superconformal R-symmetry $U(1)_R$.

The Ricci-flat metric on the conifold is known (a rare fact) because of the large isometry. It follows from the metric on $T^{1,1}$:

$$ds^2(T^{1,1}) = \frac{1}{6} \sum_{i=1}^{2} \big(d\theta_i^2 + \sin^2\theta_i\, d\varphi_i^2 \big) + \frac{1}{9} \Big(d\psi - \sum_{i=1}^{2} \cos\theta_i\, d\varphi_i \Big) \; . \tag{13.9}$$

The first term represents $S^2 \times S^2$, while the second one with $\psi \cong \psi + 4\pi$ represents a $U(1)$ bundle over it. Therefore

$$T^{1,1} = U(1) \text{ bundle over } S^2 \times S^2 \;\cong\; S^2 \times S^3 \,. \tag{13.10}$$

The fact that the topology is $S^2 \times S^3$ is a geometric fact.[87]

After putting N D3-branes at the tip of the conifold and taking the near-horizon limit, we get the following 10d solution of IIB supergravity:[88]

$$\begin{aligned} & ds^2 = R^2\big(ds^2(\mathrm{AdS}_5) + ds^2(X_5)\big) \qquad && R^4 = 4\pi g_s N \alpha'^2 \, \frac{\mathrm{Vol}(S^5)}{\mathrm{Vol}(X_5)} \\ & F_{(5)} = \tfrac{(4\pi^2\alpha')^2 g_s}{\mathrm{Vol}(X_5)} \, (1 + *) \, N \, d\mathrm{vol}_{X_5} \,. \end{aligned} \tag{13.11}$$

The dual field theory has been identified by Klebanov and Witten [KW98]. To identify the dual field theory we use the following argument. In general, the theory on the D3-branes is a gauge theory with matter fields that parametrize their motion in the orthogonal directions. Recall that for a single D3-brane on flat space, we have a $U(1)$ theory with three neutral complex scalars $\Phi_{1,2,3}$ parametrizing the orthogonal $\mathbb{C}^3$. For N branes we have a $U(N)$ theory, the neutral scalars become adjoint, and interactions are fixed by SUSY. The $U(1)$ and the traces of Φ_i decouple in the IR.

In the case of the conifold we can rewrite the equation as follows:

$$Z = \sum_{a=1}^{4} i\sigma_a z_a = \begin{pmatrix} z_4 + i z_3 & i z_1 + z_2 \\ i z_1 - z_2 & z_4 - i z_3 \end{pmatrix} \,, \qquad \det Z = 0 \tag{13.12}$$

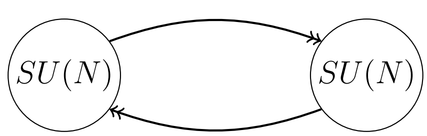
where $\sigma_{a=1,2,3}$ are the Pauli matrices and $\sigma_4 = -i\mathbb{1}$. The equation is solved by imposing that Z_{ij}, as a matrix, is the product of two vectors:

$$Z_{ij} = A_i B_j \tag{13.13}$$

with unconstrained $A_{i=1,2}$, $B_{j=1,2}$. If we parametrize the conifold by A_i, B_j there is no equation, but there is a redundancy:

$$(A_i, B_j) \;\to\; (e^{i\theta} A_i, e^{-i\theta} B_j) \tag{13.14}$$

which is implemented by a $U(1)$ gauge symmetry. Led by the $\mathcal{N} = 4$ example, we should expect another decoupled $U(1)$. Indeed we can propose the following theory:



$$W = \epsilon^{ij}\epsilon^{kl} \,\mathrm{Tr}\, A_i B_k A_j B_l \,.$$

[87] Oriented S^3 bundles over S^2 are topologically parametrized by $\pi_1\big(SO(4)\big) = \mathbb{Z}_2$.

[88] The volume of S^5 is π^3, the volume of $T^{1,1}$ is $16\pi^3/27$.

This is a "quiver diagram": nodes are gauge groups, and arrows are chiral multiplets in the bifundamental representation: fundamental with respect to the group at the tail, antifundamental under the group at the head.

For a single D3-brane we have a $U(1) \times U(1)$ gauge theory with 4 chiral multiplets A_i, B_j, where A_i have charges $(1, -1)$ while B_j have charges $(-1, 1)$.

For multiple D3-branes the groups are $U(N)$. However one $U(1)$ is decoupled and free, while the other one becomes free in the IR. We thus have gauge group $SU(N) \times SU(N)$. This theory indeed has $SU(2) \times SU(2) \times U(1)_B \times U(1)_R$ symmetry, where $U(1)_B$ is a baryonic symmetry that gives charge $+1$ to A_i and -1 to B_j. Led by the $\mathcal{N} = 4$ example, we should also expect a superpotential.

Cancelation of the $U(1)_R$ anomaly fixes the dimensions of A_i, B_j at the fixed point. We impose

$$\operatorname*{Tr}_{\text{fermions}} T^a_{SU(N)} \, T^b_{SU(N)} \, R = 0 \; . \tag{13.15}$$

We use that $\operatorname{Tr} T^a T^b$ equals N for the adjoint representation, and $\frac{1}{2}$ for the (anti)fundamental. Calling R_A the R-charge of A_i, B_j:

$$N + \frac{1}{2}(R_A - 1)4N = 0 \qquad \Rightarrow \qquad R_A = \frac{1}{2} \, , \qquad \Delta = \frac{3}{4} \; .$$

The chiral multiplets do not have canonical dimension, thus the fixed point is necessarily strongly-coupled. There is a unique superpotential which is compatible with the symmetries:

$$W = \epsilon^{ij} \epsilon^{kl} \operatorname{Tr} A_i B_k A_j B_l = \operatorname{Tr}(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1) \; . \tag{13.16}$$

In fact, this superpotential is necessary to give a 2d line of exactly marginal deformations, as observed in SUGRA.

How does $U(1)_B$ appear in supergravity? The internal manifold has topology $S^2 \times S^3$, therefore the KK reduction of $C_{(4)}$ gives a gauge field in AdS$_5$. This is the bulk gauge field dual to the baryonic current.

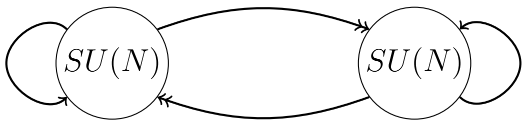
13.2 RG flow to $T^{1,1}$

As a nice check, we can reproduce the conifold field theory from an RG flow.

Start with N D3-branes on the orbifold geometry

$$\mathsf{R}^{3,1} \times \mathsf{C}^2/\mathsf{Z}_2 \times \mathsf{C} \; .$$

The near-horizon geometry is AdS$_5 \times S^5/\mathsf{Z}_2$. Since the original geometry is an orbifold of flat space, the perturbative open string is well-defined and one can compute the spectrum exactly as one does for flat D3-branes. The theory on the D3-branes has 4d $\mathcal{N}=2$ supersymmetry and it is



$$W = \operatorname{Tr}\left(\Phi \sum_{i=1}^{2} A_i B_i - \widetilde{\Phi} \sum_{i=1}^{2} B_i A_i\right) .$$

We can add a relevant deformation:

$$W_{\text{def}} = \frac{m}{2} \operatorname{Tr}\left(\Phi^2 - \widetilde{\Phi}^2\right) . \tag{13.17}$$

The fields Φ, $\widetilde{\Phi}$ are massive and can be integrated out. One obtains precisely the conifold theory.

We can understand that geometrically. The perturbation W_{def} is odd under the exchange of the two gauge groups, and it is odd under Z_2. Thus it corresponds to a twisted mode. Such twisted mode corresponds to a resolution of the orbifold singularity into the conifold. The full RG flow has been constructed in supergravity.

It is interesting to examine how the central charges change under this RG flow. Recall that, al leading order in N:

$$a \simeq c \simeq \frac{9}{32} \operatorname*{Tr}_{\text{fermions}} R^3 . \tag{13.18}$$

In the $\mathcal{N}=2$ orbifold theory we have

$$c \simeq \frac{9}{32}\left(2N^2 + 6\left(-\frac{1}{3}\right)^3\right) = \frac{N^2}{2} . \tag{13.19}$$

In the $\mathcal{N}=1$ KW theory we have

$$c \simeq \frac{9}{32}\left(2N^2 + 4\left(-\frac{1}{2}\right)^3\right) = \frac{27}{64} N^2 . \tag{13.20}$$

Therefore

$$\frac{c_{\text{IR}}}{c_{\text{UV}}} = \frac{27}{32} . \tag{13.21}$$

According to the c-theorem, this is smaller than 1.

We can compare with the holographic computation. We have seen that the central charge is proportional to the Newton constant in AdS_5, which is inversely proportional to the radius of X_5:

$$\frac{c_{\text{IR}}}{c_{\text{UV}}} = \frac{\text{Vol}(S^5/\mathsf{Z}_2)}{\text{Vol}(T^{1,1})} = \frac{27}{32} . \tag{13.22}$$

13.3 Spectrum of chiral primaries

The analysis of single-trace chiral primaries for the conifold theory is very similar to the one for $\mathcal{N}=4$ SYM.

On the gravity side, we should KK reduce IIB supergravity on $T^{1,1}$ to get an effective theory in AdS$_5$. The lowest scalar in each multiplet comes from a mixture of modes of the warp factor $g_{\mu\mu}$ and $C_{(4)}$. The wavefunctions are in the representation

$$\left(\frac{k}{2},\frac{k}{2}\right)_k$$

of the isometry group $SU(2)\times SU(2)\times U(1)_R$ — and are neutral under $U(1)_B$. The mass of the scalars in AdS$_5$ is

$$m^2R^2=\frac{3}{4}k(3k-8)\ , \tag{13.23}$$

therefore the dimensions of the dual operators follow from

$$\Delta_\pm=2\pm\left|\frac{3}{2}k-2\right|\ . \tag{13.24}$$

For $k\geq 2$ there is only one possible quantization, and the dimension must be

$$\Delta_+=\frac{3}{2}k\ . \tag{13.25}$$

However for $k=1$ both quantizations are allowed: $\Delta_+=\frac{5}{2}$, $\Delta_-=\frac{3}{2}$. As we will see in a moment in FT, the one compatible with supersymmetry is the unusual one:

$$k=1\ :\qquad\qquad \Delta_-=\frac{3}{2}\ . \tag{13.26}$$

Thus we have here an example in which the alternative quantization is chosen: the boundary conditions fix the subleading mode, as opposed to the leading mode.

In the field theory, the single-trace chiral primaries are

$$\mathcal{O}=\mathrm{Tr}\Big(A_{i_1}B_{j_1}\dots A_{i_k}B_{j_k}\Big)\qquad\qquad k\geq 1\ .$$

From the superpotential $W=\mathrm{Tr}\left(A_1B_1A_2B_2-A_1B_2A_2B_1\right)$ one obtains the F-term relations

$$A_{[i}\,B_{|j|}\,A_{k]}=0\ ,\qquad\qquad B_{[j}\,A_{|i|}\,B_{l]}=0 \tag{13.27}$$

antisymmetrized in $[ik]$ and $[jl]$ respectively. It follows that the single-trace chiral primaries are fully symmetrized in $(i_1\dots i_k)$ and $(j_1\dots j_k)$ separately, and thus are in representation $\left(\frac{k}{2},\frac{k}{2}\right)$ of $SU(2)^2$. The R-symmetry of A,B is fixed by the superpotential:

$$R_A=\frac{1}{2} \tag{13.28}$$

and it cannot be renormalized. It follows that

$$R[\mathcal{O}] = k \;, \qquad\qquad \Delta[\mathcal{O}] = \frac{3}{2} k \;. \tag{13.29}$$

They precisely match the spectrum computed in supergravity.

There is an important difference between the S^5 and $T^{1,1}$ case. In the S^5 case, all modes in the KK reduction to AdS$_5$ sit in short (and thus protected) multiplets. In this sense, we are not learning something new about the spectrum. In the case of $T^{1,1}$, many of the KK modes on AdS$_5$ and in long *unprotected* multiplets. AdS/CFT makes the prediction that their dimensions remain finite in the large λ limit, and it gives a way to compute spectrum and correlators.

13.4 Dibaryons as wrapped D3-branes

No KK mode in AdS$_5$ is charged under $U(1)_B$, because the baryonic current comes from $C_{(4)}$ and the SUGRA modes are not charged under $C_{(4)}$.

In the field theory, what operators are charged under $U(1)_B$? For instance the "dybaryons"

$$\mathcal{B}_{i_1 \dots i_N} = (A_{i_1})^{\alpha_1}{}_{\beta_1} \, \dots \, (A_{i_N})^{\alpha_N}{}_{\beta_N} \, \epsilon_{\alpha_1 \dots \alpha_N} \, \epsilon^{\beta_1 \dots \beta_N} \;. \tag{13.30}$$

This object is, by construction, totally symmetric in $(i_1 \dots i_N)$, and thus it is in representation

$$\Big(\frac{N}{2}, 0 \Big)_{\frac{N}{2}, N} \qquad \text{of} \qquad SU(2)^2 \times U(1)_R \times U(1)_B \;.$$

In particular these are $N+1$ operators, and are chiral primaries.

There is of course a second dibaryon operator $\widetilde{\mathcal{B}}_{j_1 \dots j_N}$ constructed out of B_j, in representation

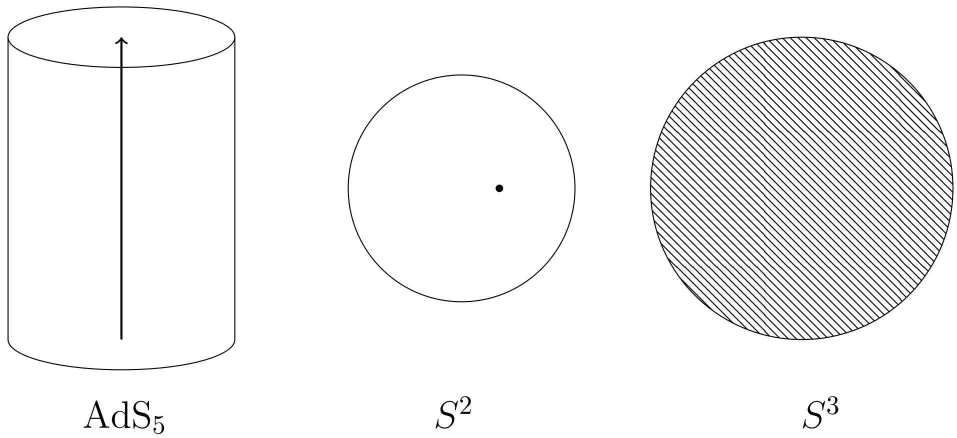
$$\Big(0, \frac{N}{2} \Big)_{\frac{N}{2}, -N} \;.$$

The dimension of these operators is

$$\Delta = \frac{3}{4} N \;, \tag{13.31}$$

in particular it is large as $N \to \infty$. These operators are not described by the KK reduction of 10d massless string modes. They are also not described by massive perturbative string modes, since for those the dimension scales as $\Delta \sim \lambda^{1/4}$. They are described by *solitonic* string states, in fact D-branes.

Since $T^{1,1} \cong S^2 \times S^3$, we can wrap a D3-brane on S^3 to get a particle in AdS$_5$.



A D3-brane is charged under $C_{(4)}$, thus the particle is charged under $U(1)_B$. It turns out that a D3-brane on a minimal-surface S^3 is also supersymmetric, thus the particle will be described by a chiral field. Such a chiral field, $\Phi_{\mathcal{B}}$, is then singled-out to be the bulk field dual to the dibaryon operator.

A minimal area S^3 in $T^{1,1}$ is given by

$$(\theta_1, \varphi_1) = \text{fixed} \qquad \forall\, \theta_2, \varphi_2, \psi \;.$$

Its volume is

$$\text{Vol}(S^3) = \frac{8}{9}\pi^2 R^3 \;. \tag{13.32}$$

Since the D3-brane has a tension, the mass of the particle is

$$m_{\text{D3}} = \text{Vol}(S^3)\,\frac{\sqrt{\pi}}{\kappa} \qquad \kappa = \sqrt{8\pi G_{\text{N}}} = 8\pi^{7/2} g_s \alpha'^2 \;. \tag{13.33}$$

For the precise coefficients see Polchinski's book or [Kle00]. Finally we find the dimension of the operator dual to $\Phi_{\mathcal{B}}$:

$$R^2 m_{\text{D3}}^2 = \frac{9}{16} N^2 \;, \qquad \Delta = 2 + \sqrt{2 + R^2 m^2} = \frac{3}{4} N + \mathcal{O}(1) \;. \tag{13.34}$$

We reproduce the dimension of $\mathcal{B}$, at leading order in N.

Classically, we have one such field $\Phi_{\mathcal{B}}$ for each value of (θ_1, φ_1). At the quantum level, we have a moduli space CP^1 that we should quantize, in the sense that we should find wavefunctions on this space. Reducing on S^3, we find a point particle on S^2 immersed into N units of magnetic flux (from the reduction of $F_{(5)}$). The quantization of this system, more precisely of the fermionic oscillations, leads to $N+1$ Landau levels: the vacuum is degenerate with $N+1$ states. Such states form an $(N+1)$-dimensional representation of the rotation group $SU(2)$, *i.e.* of spin $\frac{N}{2}$.

Thus we reproduce the fact that there are $N+1$ such operators, transforming in the spin $\frac{N}{2}$ representation of the first $SU(2)$.

13.5 Fractional branes and the Klebanov-Tseytlin solution

Another interesting object to introduce is M D5-branes wrapped on S^2 of $T^{1,1}$. This object is a domain wall in AdS_5, that separates two phases with a different 5d effective theory.

It is not obvious to understand what happens on the other side of the wall, and to make things simple we use some intuition. We take the baryonic particle coming from a D3-brane on S^3 and follow it as it crosses the wall. When a D3 and M D5's cross, by "brane-creation effect" [DFK97] M D1-strings are created between the two. As a result, the dibaryon operator has M free gauge index (attached to an external heavy quark). For instance

$$(A_{i_1})^{\alpha_1}_{\beta_1} \; \dots \; (A_{i_{N+M}})^{\alpha_{N+M}}_{\beta_{N+M}} \; \epsilon_{\alpha_1 \dots \alpha_N} \; \epsilon^{\beta_1 \dots \beta_{N+M}} \; .$$

This is possible if the gauge group has become

$$SU(N) \times SU(N+M) \; .$$

There are more refined arguments to draw this conclusion.

The wall is not BPS and is not really stable in AdS_5: its energy scales as r^4, thus the wall wants to "fall inside AdS" towards the horizon at $r=0$.

The stable configuration is without the wall (the wall has disappeared behind the horizon), but with a magnetic flux on S^3 of the RR 3-form $F_{(3)}$ which is the remnant of its passage.

Another way to think about the system is before taking the near-horizon limit: at the tip of the conifold we can place N D3-branes, as well as M D5-branes that wrap the vanishing S^2 cycle at the tip and thus "look like 3-branes" — these are called *fractional branes*.

Thus we are after a supergravity solution with

$$\frac{1}{4\pi\alpha'} \int_{S^3} F_{(3)} = M \; , \qquad\qquad \frac{1}{(4\pi\alpha')^2} \int_{T^{1,1}} \overset{?}{=} N \; . \tag{13.35}$$

We are studying a SUSY theory, and we are looking for SUSY ground states. Thus we can solve the BPS equations

$$\begin{cases} \delta\Psi_{\mu\alpha} = 0 & \text{gravitino} \\ \delta\lambda_\alpha = 0 & \text{dilatino} \; . \end{cases} \tag{13.36}$$

These equations imply

$$H_3 = g_s *_6 F_{(3)} \; , \tag{13.37}$$

therefore the solution should have a non-trivial NS 3-form flux H_3 as well. The Bianchi identity for $F_{(5)}$ is

$$dF_{(5)} = H_3 \wedge F_3 \neq 0 \qquad\qquad \text{because} \qquad F_{(5)} = dC_{(4)} + B_2 \wedge F_3 \; . \tag{13.38}$$

It follows that the number N defined above is *not* constant! It is a function of r.

These considerations lead to the following supergravity solution, constructed by Klebanov and Tseytlin [KT00]. The metric is a "warped product" of $\mathsf{R}^{3,1}$ and the conifold:

$$ds^2_{10} = h(r)^{-1/2} dx^2_{3,1} + h(r)^{1/2}\big(dr^2 + r^2 ds^2_{T^{1,1}}\big) \tag{13.39}$$

where we recall

$$ds^2_{T^{1,1}} = \frac{1}{6}\sum\nolimits_{i=1,2}\Big(d\theta_i^2 + \sin^2\theta_i\, d\varphi_i^2\Big) + \frac{1}{9}\Big(d\psi - \sum\nolimits_{i=1,2}\cos\theta_i\, d\varphi_i\Big)^2 \; . \tag{13.40}$$

To write the fluxes, we introduce two forms proportional to the volume forms on S^2 and S^3:[89]

$$\omega_2 = \frac{1}{2}\big(\sin\theta_1\, d\theta_1 \wedge d\varphi_1 - \sin\theta_2\, d\theta_2 \wedge d\varphi_2\big) \propto d\mathrm{vol}_{S^2} \; , \qquad \omega_3 = d\psi \wedge \omega_2 \propto d\mathrm{vol}_{S^3} \; . \tag{13.41}$$

They also satisfy $\omega_2 \wedge \omega_3 = 54\, d\mathrm{vol}_{T^{1,1}}$. Then

$$F_3 = \alpha' \frac{M}{2}\omega_3 \; , \qquad\qquad H_3 = g_s\alpha' \frac{3M}{2}\frac{dr}{r} \wedge \omega_2 \; . \tag{13.42}$$

The running 5-form flux is

$$F_{(5)} = (1+*)\, 27\pi\alpha'^2 N_{\text{eff}}(r)\, d\mathrm{vol}_{T^{1,1}} \; , \qquad N_{\text{eff}}(r) = N + \frac{3}{2\pi} g_s M^2 \log\frac{r}{r_0} \; . \tag{13.43}$$

Finally the warp factor is given by

$$h(r) = \frac{L^4}{r^4}\log\frac{r}{r_s} \qquad \text{for some } r_s \; , \qquad L^2 = \frac{9 g_s M \alpha'}{2\sqrt{2}} \; . \tag{13.44}$$

This supergravity solution encodes an incredibly rich physics. Let us explore some aspects. An excellent review on Seiberg duality and the physics of the KW theory is by Strassler [Str05].

[89] A representative for S^2 is $\theta_1 = \theta_2$, $\varphi_1 = -\varphi_2$. A representative for S^3 is $(\theta_2, \varphi_2) = $ fixed.

Exact β-function. In 4d supersymmetric $\mathcal{N}=1$ gauge theories there is an "exact" expression for the gauge β-function, if we use a holomorphic scheme $\mathcal{L}=\frac{1}{g^2}\operatorname{Tr} F_{\mu\nu}F^{\mu\nu}$. This is called the NSVZ beta function [NSVZ83, NSVZ86]:[90]

$$\beta_{\text{NSVZ}}\Big[\frac{8\pi^2}{g^2}\Big] = 3C_2(G) - \sum_{\text{chirals}} C(\mathfrak{R})\,(1-\gamma)\;. \tag{13.46}$$

If we neglect γ, this is the standard one-loop beta function. Instead

$$\gamma = \frac{\partial \log Z_\Phi}{\partial \log \mu} \qquad\qquad \Delta[\Phi] = 1 + \frac{1}{2}\gamma_\Phi \tag{13.47}$$

is the anomalous dimension of the chiral field Φ. This formula expresses the exact beta function, as a function of the unknown anomalous dimensions which receive contributions to all orders.

Applied to the KW theory with gauge group $SU(N+M)\times SU(N)$:

$$\begin{aligned}\frac{\partial}{\partial \log\mu}\frac{8\pi^2}{g_1^2} &= 3(N+M) - 2N(1-\gamma)\\ \frac{\partial}{\partial \log\mu}\frac{8\pi^2}{g_2^2} &= 3N - 2(N+M)(1-\gamma)\;.\end{aligned} \tag{13.48}$$

We need to determine γ. For $M=0$ the theory is conformal and $\Delta[A,B]=\frac{3}{4}$, then $\gamma=-\frac{1}{2}$. It follows that γ has an expansion in $\frac{M}{N}$. Since the theory is invariant under

$$M\to -M\;, \qquad\qquad N\to N+M \tag{13.49}$$

but this changes sign to the first order term, it follows that

$$\gamma = -\frac{1}{2} + \mathcal{O}\big(\tfrac{M}{N}\big)^2\;. \tag{13.50}$$

We compute

$$\tfrac{\partial}{\partial\log\mu}\big(\tfrac{8\pi^2}{g_1^2}+\tfrac{8\pi^2}{g_2^2}\big) = \mathcal{O}\big(M\cdot\tfrac{M}{N}\big)\;, \qquad\qquad \tfrac{\partial}{\partial\log\mu}\big(\tfrac{8\pi^2}{g_1^2}-\tfrac{8\pi^2}{g_2^2}\big) = 6M\Big(1+\mathcal{O}\big(\tfrac{M}{N}\big)^2\Big)\;. \tag{13.51}$$

[90] The standard one-loop beta function is

$$\beta(g) = -\frac{g^3}{16\pi^2}\bigg(\frac{11}{3}C_2(G) - \frac{2}{3}\sum_{\text{Weyl}} C(\mathfrak{R}_{\text{W}}) - \frac{1}{6}\sum_{\text{scalar}_{\mathbb{R}}} C(\mathfrak{R}_{\text{s}})\bigg)\;. \tag{13.45}$$

Here $C(\mathfrak{R})$ is the Dynkin index, or quadratic Casimir, of the representation $\mathfrak{R}$: $\operatorname{Tr} T^aT^b = C(\mathfrak{R})\,\delta^{ab}$. Instead $C_2(G)=C(\text{adj})$ is the dual Coxeter number.

We can compute the same runnings in gravity. The gauge couplings are identified through

$$\frac{8\pi^2}{g_1^2} + \frac{8\pi^2}{g_2^2} = \frac{2\pi}{g_s e^{\phi}} , \qquad \frac{8\pi^2}{g_1^2} - \frac{8\pi^2}{g_2^2} = \frac{2}{g_s e^{\phi}} \left(\frac{1}{2\pi\alpha'} \int_{S^2} B_2 - \pi \pmod{2\pi} \right) . \tag{13.52}$$

These formulae can be understood as follow. The sum (or average) of the inverse gauge couplings is the inverse gauge coupling on a D3-brane; this makes sense if we go on the Coulomb branch. The sum of the two expressions gives

$$\frac{8\pi^2}{g_1^2} = \frac{1}{2\pi\alpha' g_s e^{\phi}} \int_{S^2} B_2 .$$

This is the 4d inverse coupling that one obtains for a D5-brane on a 2-cycle, from the DBI action

$$\frac{1}{g_s} \int_{\mathbb{R}^{3,1}\times S^2} e^{-\phi} \sqrt{\det(g + B_2 + F_2)}$$

when the 2-cycle is vanishing.

In the SUGRA solution the dilaton is constant, $e^{\phi} = 1$, and this reproduces the vanishing of $\beta_1 + \beta_2$. To compute $\beta_1 - \beta_2$, we write

$$B_2 = g_s \alpha' \, \frac{3M}{2} \, \omega_2 \log r \tag{13.53}$$

as a potential for F_3, and identify the scale $\log\mu$ with $\log r$. This reproduces exactly

$$\beta_1 - \beta_2 = 6M \tag{13.54}$$

at leading order in N.

The chiral anomaly. For $M \neq 0$, $U(1)_R$ becomes anomalous.[91] Very roughly, the non-trivial β-function contributes to the trace anomaly $\langle T^{\mu}_{\mu} \rangle$, and by supersymmetry this is related to the chiral anomaly $\partial^{\mu} J^R_{\mu}$.

In field theory the chiral anomaly is one-loop exact and given by

$$\partial^{\mu} J^R_{\mu} = \frac{1}{16\pi^2} \mathcal{A}_{Rab} \, F^a_{\mu\nu} \widetilde{F}^{b\mu\nu} \qquad \qquad \widetilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \tag{13.55}$$

with the anomaly $\mathcal{A}$ following from triangle diagrams:

$$\mathcal{A}_{Rab} = \text{Tr}_{\text{fermions}} \, R\, T^a T^b . \tag{13.56}$$

[91] We stress that this is an R-gauge-gauge anomaly that spoils the symmetry, not a 't Hooft R-R-R anomaly as we computed in $\mathcal{N} = 4$ SYM.

The simple computation in the $SU(N+M)\times SU(N)$ KW theory gives

$$\partial^\mu J^R_\mu = \frac{M}{16\pi^2}\Big(F^a_{\mu\nu}\widetilde{F}^{a\mu\nu} - G^a_{\mu\nu}\widetilde{G}^{a\mu\nu}\Big) \;, \tag{13.57}$$

where F and G are the field strengths for the two groups, respectively.

From Noether's theorem, if we perform an R-symmetry rotation by $e^{i\epsilon}$ and then a "gauge" transformation $A^R_\mu \to A^R_\mu + \partial_\mu \epsilon$ for an external gauge field coupled to the R-symmetry, the action changes by

$$\delta S = \int d^4x\, J^\mu_R\, \delta A^R_\mu = -\int d^4x\, \epsilon\, \partial^\mu J^R_\mu \;. \tag{13.58}$$

Because of the anomaly, this is a shift of the theta angles in the theory, and therefore it is not a symmetry. The θ-angle terms are

$$S \;\supset\; \int d^4x \left(\frac{\theta_1}{32\pi^2} F^a_{\mu\nu}\widetilde{F}^{a\mu\nu} + \frac{\theta_2}{32\pi^2} G^a_{\mu\nu}\widetilde{G}^{a\mu\nu} \right) \;, \tag{13.59}$$

thus an R-symmetry rotation induces a shift of the θ-angles

$$\theta_1 \to \theta_1 + 2M\epsilon \;, \qquad\qquad \theta_2 \to \theta_2 - 2M\epsilon \tag{13.60}$$

and it is not a symmetry. However, since $\theta_i \cong \theta_i + 2\pi$, a residual discrete subgroup of $U(1)_R$ remains unbroken:

$$U(1)_R \quad \to \quad \mathsf{Z}_{2M} \;.$$

How does this anomaly appear in supergravity? Although F_3 is invariant under $U(1)_R$ rotations, its RR potential C_2 cannot be. We can choose, for instance,

$$F_3 = \alpha' \frac{M}{2} \omega_3 \qquad \Rightarrow \qquad C_2 = \alpha' \frac{M}{2}\, \psi\, \omega_2 \;. \tag{13.61}$$

Of course C_2 is not a gauge invariant (it is a gauge potential), however

$$\frac{1}{2\pi\alpha'} \int_{S^2} C_2 = M\, \psi \tag{13.62}$$

is a gauge-invariant (recall $\delta C_2 = d\lambda_1$), but it is not invariant under shifts of ψ. Such a gauge invariant is an axion, with period 2π, thus

$$\psi \;\to\; \psi + \frac{4\pi}{2M} \tag{13.63}$$

is a symmetry, and since $\psi \cong \psi + 4\pi$ we conclude that there is a residual Z_{2M} discrete symmetry.

In fact we can see the parallel with FT even better. The axion is the bulk mode dual to the operator sourced by the difference of the theta angles, while the other RR axion is dual to the operator sourced by the sum:[92]

$$\frac{1}{\pi\alpha'}\int_{S^2} C_2 = \theta_1 - \theta_2 \;, \qquad\qquad C_{(0)} \sim \theta_1 + \theta_2 \;. \tag{13.65}$$

We see that a shift $\delta\psi = 2\epsilon$ leaves $\theta_1 + \theta_2$ fixed, while shifting

$$\theta_1 - \theta_2 \;\to\; \theta_1 - \theta_2 + 4M\epsilon \;, \tag{13.66}$$

as in the FT calculation.

The picture we gave is in 10 dimensions, and the anomaly appears as a spontaneous breaking in the internal directions: the solution is not invariant. How does that appear in AdS_5? Global symmetries on the boundary are gauge symmetries in the bulk, and of course a gauge symmetry cannot be anomalous. Indeed, what happens is that, because of the spontaneous breaking, the gauge field get massive by Higgs mechanism in the bulk. Thus

$$\text{Anomalous symmetry on the boundary} \qquad \Leftrightarrow \qquad \text{Massive vector in the bulk} \;.$$

The computation can be found in [HKO02]. Since the vector is massive, the dual current operator gets an anomalous dimension $\Delta > 3$. AdS/CFT allows to compute such a dimension:

$$\Delta(J_\mu) = 2 + \sqrt{1 + m^2R^2} \;\simeq\; 3 + \frac{(g_sM)^2}{\pi g_s N} \;. \tag{13.67}$$

Cascading RG flow. The $SU(N+M)\times SU(N)$ theory is non-conformal. It has a very peculiar RG flow, that can be understood in terms of a "cascade" of Seiberg dualities.

We have computed the β-functions (both in FT and SUGRA) at leading order:

$$\frac{\partial}{\partial \log\mu}\frac{8\pi^2}{g_1^2} = 3M \;, \qquad\qquad \frac{\partial}{\partial \log\mu}\frac{8\pi^2}{g_2^2} = -3M \;. \tag{13.68}$$

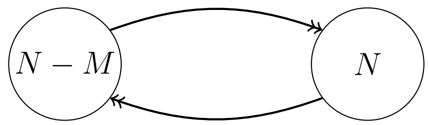
Thus the gauge group $SU(N+M)$ goes towards strong coupling in the IR, while $SU(N)$ goes towards weak coupling. At a scale

$$\Lambda = \mu\, e^{-\frac{1}{3M}\frac{8\pi^2}{g_1^2(\mu)}} \tag{13.69}$$

[92] Notice that these modes are constant in AdS_5, consistently with the fact that the dual operators, $F^a_{\mu\nu}\widetilde{F}^{a\mu\nu} \pm G^a_{\mu\nu}\widetilde{G}^{a\mu\nu}$ have dimension 4. The operator map follows from the WZ D-brane action

$$S_{\rm WZ} = \int_{{\rm D}p} e^{2\pi\alpha'(\mathcal{F}_2 + B_2)} \sum\nolimits_{\rm RR} C_{\rm RR} \;. \tag{13.64}$$

the group $SU(N+M)$ is strongly coupled. Since $N_c = N+M$ and $N_f = 2N$, we can use a Seiberg-dual description: $\widetilde{N}_c = N_f - N_c = N - M$. Working out the details (and integrating out massive fields) we obtain



and the conifold superpotential is reproduced. This is the same theory has before, but with

$$N \;\to\; N - M \;.$$

Now the role of the gauge groups gets exchanged: $SU(N)$ goes towards strong coupling and $SU(N-M)$ towards weak coupling, until we perform another duality on $SU(N) \to SU(N-2M)$, and we keep going. In fact this process continues indefinitely in the UV.[93]

The "running" of the gauge ranks appears in supergravity as the fact that

$$\int_{T^{1,1}} F_{(5)} \text{ is not constant },$$

and in particular

$$N_{\text{eff}}(r) = N + \frac{3}{2\pi} g_s M^2 \log \frac{r}{r_0} \;. \tag{13.70}$$

The precise coefficients, expressing the "RG distance" between one duality and the next, can be successfully compared with field theory.[94]

13.6 Chiral symmetry breaking, confinement, and the Klebanov-Strassler solution

It is clear that the cascading RG flow can continue indefinitely in the UV, but not in the IR. In supergravity, there is a naked singularity in the IR (small r) where either $N_{\text{eff}}(r)$ or $h(r)$ become negative. This signals that new physics is needed in the IR.

Luckily, it turns out that the resolution is possible within the supergravity approximation!

[93] This peculiar RG flow is a clever way that the theory has to remain strongly-coupled at all energies, even though the couplings run and so one would expect them to reach weak coupling at some point (either UV or IR). This would necessarily make the supergravity approximation break down.

[94] More precisely, the integer ranks should be compare with the Page charge $\int_{T^{1,1}} dC_{(4)}$, which is integer and it jumps discontinuously.

Let us first understand the physics (explained in depth in [Str05]). Suppose that towards the end of the cascade we reach

$$SU(2M) \times SU(M) \, .$$

Now $SU(2M)$ goes to strong coupling. However this time $N_f = N_c$, and there is no Seiberg duality in this case. Instead, the theory confines, mesons $\mathcal{M}$ and baryons $\mathcal{B}$, $\widetilde{\mathcal{B}}$ become the fundamental fields, but they are subjected to a quantum deformed constraint

$$\det \mathcal{M} - \mathcal{B}\widetilde{\mathcal{B}} = \Lambda^{2N_c} \, . \tag{13.71}$$

The constraint can be imposed by a Lagrange multiplier X in the superpotential. The mesons are $\mathcal{M}_{ji} = B_j A_i$. Thus the superpotential is

$$W = -\operatorname{Tr}\big(\mathcal{M}_{11}\mathcal{M}_{22} - \mathcal{M}_{12}\mathcal{M}_{21}\big) + X\bigg(\det \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{pmatrix} - \mathcal{B}\widetilde{\mathcal{B}} - \Lambda^{4M}\bigg) \, . \tag{13.72}$$

Because of the first term, the mesons are massive and can be integrated out. The baryons get a VEV, $\mathcal{B}\widetilde{\mathcal{B}} = -\Lambda^{4M}$, and leave a flat direction. Since the baryons are neutral under $SU(M)$, we are left with pure $SU(M)$ SYM. This theory has gaugino condensation, chiral symmetry breaking

$$\mathsf{Z}_{2M} \quad \to \quad \mathsf{Z}_2 \, ,$$

M inequivalent vacua and confinement.

Thus, the IR physics we were missing is *chiral symmetry breaking*. One can include such an ingredient by substituting the conifold by the *deformed conifold*:[95]

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = \varepsilon^2 \, . \tag{13.73}$$

The parameter ε breaks the R-symmetry, that rotates the variables z_a, to Z_2. The deformed conifold is still Calabi-Yau. It is not a cone, rather it is a smooth three-fold with a finite-size S^3 at the tip:

figure

The Klebanov-Strassler solution has metric

$$ds_{10}^2 = h(\tau)^{-1/2} dx_{3,1}^2 + h(\tau)^{1/2} ds_6^2(\text{def conifold}) \, , \tag{13.74}$$

3-form flux F_3 on S^3, H_3 along $dr \wedge d\text{vol}_{S^2}$, and it asymptotes the KT solution far from the tip. However it is everywhere smooth, and according to our discussion on confinement, it gives confinement of flux tubes with a discrete spectrum of glueballs.

[95] A better way to motivate the deformed conifold is to study the moduli space of $SU(M+1) \times SU(1)$.

14 More on response functions and AdS/CMT

14.1 Real-time correlators from AdS/CFT

Note: this section overlaps with Section 12.2.

We have learned how to compute generic correlators in Euclidean signature. To compute transport coefficients we need two-point functions in Lorentz signature. In principle, they are related by analytic continuation, but if we have a numerical result we don't really know how to do the continuation. We should then understand how to directly compute correlators in real time.

At the boundary there is not much difference. The EOM for a scalar in AdS_{d+1} gives asymptotic behavior

$$\Phi(z,t,x) = e^{-i\omega t + ikx}\Big(z^{d-\Delta}\phi_0\left(1+O(z)\right) + z^{\Delta}\phi_1\big(1+O(z)\big)\Big) . \tag{14.1}$$

We impose Dirichlet boundary conditions for one of the two modes, typically ϕ_0. This makes ϕ_0 be identified with the source and ϕ_1 with the VEV.

Things are quite different in the interior, *i.e.* at the horizon. For instance, in AdS the two exact solutions for time-like k^μ (*i.e.* on-shell, since $\omega^2 > \vec{k}^2$) are

$$z^{d/2}\,K_{\pm\nu}(iqz) , \qquad q = \sqrt{\omega^2 - \vec{k}^2} > 0 , \qquad \nu = \Delta - \frac{d}{2} .$$

They behave as

$$e^{\pm iqz} \qquad \text{for } z\to\infty .$$

They are both regular at the horizon. This is because there are many different real-time Green's functions we can construct in QFT.

To construct *retarded* Green's functions we have to choose *infalling* boundary conditions, *i.e.* energy should move towards larger z as time passes, thus falling inside the horizon. It can be motivated in three ways:

- Such boundary conditions correspond to a causal behavior.
- Such boundary conditions reproduce the correct analytic structure of $G_R(\omega)$, *i.e.* no poles in the upper half-plane.
- They can be derived from a holographic version of the Schwinger-Keldysh prescription.

Infalling boundary conditions break time reversal. In the example of AdS, we should choose the solution that behaves like $e^{-i\omega t+iqz}$ (for $\omega > 0$), that is

$$e^{-i\omega t+ikx} z^{d/2} K_{+\nu}(iqz) \, .$$

More generally, we should

Impose regular boundary conditions at future horizons.

A future horizon is a null surface beyond which events cannot causally propagate back to the boundary. On a future horizon, regularity requires that modes are infalling.

For a horizon at non-zero temperature located at $z = z_+$, the standard black hole metric looks like[96]

$$ds^2 = -f(z)\, dt^2 + \frac{dz^2}{f(z)} \qquad\qquad f(z) \simeq 4\pi T(z_+ - z) \tag{14.2}$$

and g_{tt} has a simple zero at the horizon. To analyze the modes around the horizon, we can use Kruskal coordinates

$$\rho \pm \tau = e^{\frac{1}{2}\log(z_+ - z) \pm 2\pi T t} \, , \tag{14.3}$$

which bring the metric to the simple form $ds^2 = (\pi T)^{-1}(-d\tau^2 + d\rho^2)$. The future horizon is at $\rho = \tau > 0$, the past horizon is at $\rho = -\tau > 0$. The solutions to the massless Klein-Gordon equation are $f_+(\rho + \tau)$ and $f_-(\rho - \tau)$. Imposing the time dependence $\Phi \sim e^{-i\omega t}$ we find

$$f_+ = (\rho + \tau)^{-i\omega/2\pi T} \, , \qquad\qquad f_- = (\rho - \tau)^{i\omega/2\pi T} \, . \tag{14.4}$$

The solution f_- is singular at the future horizon, while f_+ is regular. This singles out f_+, and therefore the infalling mode behaves as

$$\phi(z) \sim e^{-\frac{i\omega}{4\pi T}\log(z_+ - z)} \, . \tag{14.5}$$

For a zero-temperature horizon located at $z = z_+$, the component g_{tt} has a double zero and the metric looks like AdS_2:

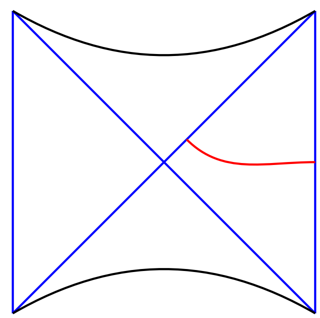
$$ds^2 \simeq -\frac{(z_+ - z)^2}{R^2}\, dt^2 + \frac{R^2}{(z_+ - z)^2}\, dz^2 \, . \tag{14.6}$$

This is mapped to our standard AdS metric by $z_+ - z = r = R^2/\zeta$. We already saw that infalling modes behaves as $e^{i\omega\zeta}$ (for zero momentum), thus

$$\phi(z) \sim e^{i\omega R^2/(z_+ - z)} \, . \tag{14.7}$$

[96] Go to Euclidean time and perform the coordinate change $\rho^2 = (z_+ - z)/\pi T$. The metric becomes approximately $ds^2 \simeq d\rho^2 + 4\pi^2 T^2 dt^2$, which is smooth R^2 if $t \cong t + T^{-1}$.

We fix the boundary conditions on the future horizon because a space-like surface terminating on a future horizon provides a good Cauchy surface from which to evolve initial data. A space-like surface ending on a past horizon does not: one would need to specify what is coming out of the "white hole".



Furthermore, the presence of a future horizon allows energy to be lost behind the horizon, which is the holographic manifestation of dissipation.

At this point, the two-point functions are computed in the standard way:

$$G_R^{\phi\phi}(\omega, k) = \frac{\delta\phi_1}{\delta\phi_{(0)}}\bigg|_{\omega,k} . \qquad (14.8)$$

14.2 A setup for condensed matter problems

We have studied in some detail the solutions to Einstein gravity with negative cosmological constant Λ. The simplest solution is

$$\mathrm{AdS}_{d+1} ,$$

representing a CFT in its conformal vacuum. A more interesting solution is

$$\text{Schwarzschild BH in } \mathrm{AdS}_{d+1} ,$$

representing the same CFT but in a thermal ensemble.

To do more, we need to add other fields. It is common in condensed matter systems to have a $U(1)$ symmetry.

The most common situation is that it is the electromagnetic $U(1)$. Of course that is a gauge symmetry, while AdS/CFT describes *global* symmetries on the boundary. However in many condensed matter systems the photon can be thought of as non-dynamical (in the low-energy effective theory).

1. The electromagnetic coupling is small.

2. Electromagnetic interactions are usually screened in a charged medium, thus the effective theory usually does not contain photons.

Thus, we consider bulk theories with a $U(1)$ gauge field, dual to CFTs with a $U(1)$ global symmetry.

This allows us to obtain *background* electric and magnetic fields on the boundary, as well as to add a chemical potential μ that induces a charge (and particle) density ρ.

For instance, consider the following model in AdS$_4$:

$$S_{(4)} = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}\,\Big(\mathcal{R}+\frac{6}{L^2}\Big) - \frac{1}{4g^2}\int d^4x\,\sqrt{-g}\,F_{MN}F^{MN}\ . \tag{14.9}$$

One obvious solution is AdS$_4$ with radius L. This is dual to the conformally-invariant vacuum of some $2+1$ dimensional CFT.

More interesting solutions are the dyonic black membranes, *i.e.* black holes with temperature, electric and magnetic charges, that span $\mathsf{R}^{2,1}$. The metric takes the familiar form

$$ds^2 = L^2\left[\frac{dz^2}{z^2 f(z)} + \frac{1}{z^2}\Big(-f(z)\,dt^2 + dx^2 + dy^2\Big)\right] \tag{14.10}$$

with

$$f(z) = 1 + (h^2+q^2)\alpha\frac{z^4}{z_h^4} - \big(1+(h^2+q^2)\alpha\big)\frac{z^3}{z_h^3}\ , \qquad \alpha = \frac{\kappa^2 z_h^2}{2g^2L^2}\ , \tag{14.11}$$

and there is also a non-trivial electro-magnetic field

$$A = \frac{h}{z_h}\,x\,dy - q\Big(1-\frac{z}{z_h}\Big)dt\ . \tag{14.12}$$

These are closed cousins of the Reissner-Nordström black hole. The boundary is at $z=0$ and the metric is asymptotically AdS$_4$, while there is a horizon at $z=z_h$. An integration constant in A is chosen in such a way that $A_t(z_h)=0$, hence $A_MA_Ng^{MN}<\infty$ is regular.

As we learned, regularity of the Euclidean solution at the horizon gives the temperature (both of the BH and of the CFT):

$$T = \frac{3-(h^2+q^2)\alpha}{4\pi z_h}\ . \tag{14.13}$$

The boundary value of the spatial components of the field strength give a magnetic field F_{xy}:

$$F_{xy}(z=0) = \frac{h}{z_h}\ . \tag{14.14}$$

Therefore the magnetic charge of the BH corresponds to a constant magnetic field in the boundary theory. To extract the charge density, we analyze the asymptotic behavior of the gauge field:

$$A_\mu = a_\mu + b_\mu z + \dots \tag{14.15}$$

The first constant is the source:

$$a_t = -q \equiv \mu \qquad\qquad \text{chemical potential } \mu \;. \tag{14.16}$$

The second constant is, up to a proportionality constant that follows from differentiating the action, the VEV:

$$b^t = g^2 \langle J^t \rangle \equiv g^2 \rho = -\frac{q}{z_h} = \frac{\mu}{z_h} \qquad\qquad \text{charge density } \rho \;. \tag{14.17}$$

Therefore the electric field of the BH corresponds to a chemical potential, which induces a charge density in the system.

14.3 The holographic superconductor

We have discussed scalar fields ϕ in the bulk. They can be dual to relevant operators, which can be used to trigger RG flows. If these fields are charged — meaning that the dual operators are charged under the global $U(1)$ symmetry on the boundary — they can become *order parameters* for broken symmetries.

This is very interesting because symmetry breaking is at the heart of CM physics (recall that the Higgs mechanism has been discovered by Anderson, a CM physicist, first).

We consider a simple model in which we add a charged scalar ϕ to the Einstein-Maxwell theory discussed before:

$$S_\phi = -\int d^4x\,\sqrt{-g}\,\Big(|D\phi|^2 - 2\frac{|\phi|^2}{L^2} \Big) \;, \qquad\qquad D = \partial - iA \;. \tag{14.18}$$

In general we could take a potential $V\big(|\phi|\big)$: we have taken the simplest choice, a mass term. The mass has been chosen arbitrarily (to do numerics we have to choose one), such that

$$m^2 L^2 = -2 \;, \qquad\qquad \Delta_+ = 2 \;, \qquad \Delta_- = 1 \;. \tag{14.19}$$

We are in the range of double quantization, therefore this model can describe an order parameter of dimension 2 or 1. We set

$$h = 0 \tag{14.20}$$

for simplicity: no magnetic field. But we do turn on temperature T and chemical potential μ.

The central observation is that the charge density acts as an effective z-dependent negative contribution to the mass of the scalar:

$$m_{\text{eff}}(z) = m^2 + g^{tt} A_t^2 = m^2 - \frac{q^2}{L^2} \frac{z^2}{f(z)} \big(1 - z/z_h\big)^2 \,. \tag{14.21}$$

When q is large enough, the mass becomes too negative, there is an instability and the scalar develops a non-trivial profile. However the effective mass vanishes at $z = z_h$ and $z = 0$, therefore the runaway direction is "stabilized by the curvature".

We can make the simplifying assumption

$$\kappa^2 \ll g^2 L^2 \,. \tag{14.22}$$

This is the weak gravity or *probe limit*: the gauge and scalar field do not have enough energy to curve spacetime, and we can study their dynamics on a fixed background. The metric simplifies to

$$f(z) = 1 - \frac{z^3}{z_h^3} \,. \tag{14.23}$$

One can write down the EOMs for ϕ and A_t, choosing a gauge where $\phi \in \mathsf{R}$:

$$z^2 \Big(\frac{f\,\phi'}{z^2}\Big)' = \Big(\frac{m^2}{z^2} - \frac{A_t^2}{f}\Big)\phi \,, \qquad\qquad A_t'' = \frac{2g^2}{z^2 f}\phi^2 A_t \,. \tag{14.24}$$

The boundary conditions at the boundary are

$$A_t = \mu - g^2 \rho\, z + \dots \,, \qquad\qquad \phi = a\, z + b\, z^2 + \dots \qquad\qquad \text{for } z \to 0 \,. \tag{14.25}$$

In the canonical ensemble we keep the density ρ fixed (and let μ be determined dynamically). If we choose $\mathcal{O}$ of dimension $\Delta = 2$, we insist that there is no source, $a = 0$. At the horizon we impose $\phi < \infty$ and $A_t = 0$ for regularity.

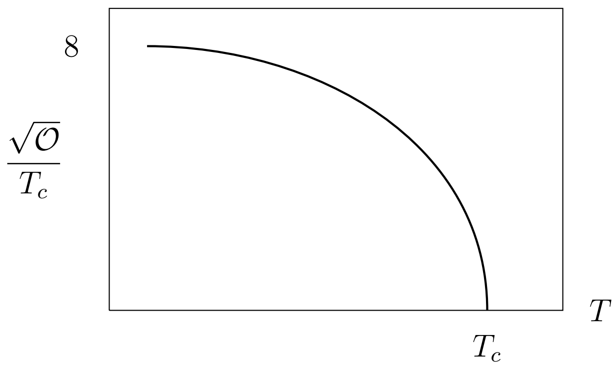
The equations have to be solve numerically. For T larger than a critical value T_c (proportional to $\rho^{1/2}$) the only solution is[97]

$$\phi = 0 \,, \qquad\qquad A_t = g^2 \rho (1 - z) \,. \tag{14.26}$$

At T_c, ϕ develops a normal mode: the linearized EOMs have a solution with no source. For $T < T_c$, there is a solution of the non-linear equations with non-trivial profile for ϕ (and it

[97]Here we have set $L = 1$ and $z_h = 1$. All physical quantities depend on ρ/T^2.

turns out that this solution has lower free energy):



The model has a second-order phase transition at T_c. The behavior of the condensate is

$$\langle \mathcal{O} \rangle \sim (T_c - T)^{1/2} \tag{14.27}$$

with the classical $\frac{1}{2}$ mean-field exponent of Landau-Ginzburg theory.

14.4 Conductivity

To compute the conductivity, we need to study the linearized equations of motion for a fluctuation of the gauge field in the background of the condensate. The equation is

$$\left(f\, A_x'\right)' - \frac{\omega^2}{f} A_x = \frac{2g^2}{z^2} \phi^2 A_x \ . \tag{14.28}$$

The boundary conditions near the boundary are

$$A_x = \frac{E_x}{i\omega} + \frac{g^2}{\alpha} J^x z + \dots \tag{14.29}$$

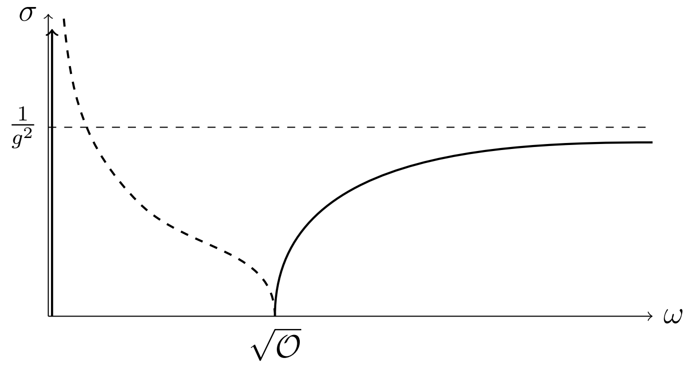
while at the horizon we impose infalling boundary conditions. Finally

$$\sigma_{xx}(\omega) = \frac{J^x}{E_x}\Big|_\omega \ . \tag{14.30}$$

Unfortunately the equations cannot be solved analytically, have to be solved numerically.

For $T > T_c$ the optical conductivity is constant and equal to $\sigma_{xx}(\omega) = 1/g^2$. The fact that it does not depend on ω is an artifact of the probe approximation.

For $T \ll T_c$ one finds, qualitatively:



For large frequencies, $\mathsf{Re}\,\sigma_{xx}$ approaches the normal-phase value, but at low frequencies there is a "gap". This is interpreted as a gap in the Fermi surface, and the fact that to conduce one needs to break a Cooper pair. Thus the gap is associated to twice the gap in the Fermi surface. At $\omega = 0$ there is a delta function, characteristic of a superconductor, which is derived from the $1/\omega$ behavior of the imaginary part and the Kramers-Kronig relations. This shape is qualitatively similar to the textbook one of BCS superconductors, however the gap is much larger than in the BCS case.

15 Entanglement entropy

A good review is hep-th/0905.0932 by Nishioka, Ryu, Takayanagi [NRT09]. Consider a quantum mechanical system with many degrees of freedom, such as a spin chain, a lattice model or a QFT.

In general, we could consider pure states $|\psi\rangle$ or mixed states described by density matrices

$$\rho = \sum_i |\psi_i\rangle\langle\psi_i| \qquad \text{with} \qquad \mathrm{Tr}\,\rho = 1 \,. \tag{15.1}$$

Notice that ρ is Hermitian and positive definite, and can be diagonalized in an orthonormal basis $|\widetilde{\psi}_j\rangle$:

$$\rho = \sum_j \rho_j |\widetilde{\psi}_j\rangle\langle\widetilde{\psi}_j| \qquad \text{with} \qquad 0 \le \rho_j \le 1 \,, \qquad \textstyle\sum_j \rho_j = 1 \,. \tag{15.2}$$

Here ρ_j are probabilities. Density matrices can also describe pure states, if

$$\rho = |\psi\rangle\langle\psi| \tag{15.3}$$

for some (normalized) state $|\psi\rangle$. Expectation values are computed by

$$\langle\mathcal{O}\rangle_\rho = \mathrm{Tr}\,\rho\,\mathcal{O} \,. \tag{15.4}$$

For pure states this reduces to the standard $\langle\mathcal{O}\rangle = \langle\psi|\mathcal{O}|\psi\rangle$.

A density matrix is pure if and only if it is a projector,

$$\rho^2 = \rho \,, \tag{15.5}$$

and because of the normalization condition this automatically implies that it projects to a one-dimensional subspace. Alternatively, we can compute the von Neumann entropy:

$$S = -\,\mathrm{Tr}\,\rho \log \rho \,. \tag{15.6}$$

This is ≥ 0, and zero if and only if ρ is a pure state. If $S > 0$, then e^S is a rough measure of the number of states involved in ρ.[98]

Put the system at zero temperature. Assuming no ground-state degeneracy, the system is in its ground state $|\Psi\rangle$, which is a pure state. The density matrix is

$$\rho_{\mathrm{tot}} = |\Psi\rangle\langle\Psi| \,. \tag{15.7}$$

[98] If ρ is uniformly distributed over n orthogonal states, namely $\rho = \frac{1}{n}\sum_{j=1}^n |\widetilde{\psi}_j\rangle\langle\widetilde{\psi}_j|$, then $S = \log n$.

The von Neumann entropy of this state, $S_{\text{tot}} = -\operatorname{Tr} \rho_{\text{tot}} \log \rho_{\text{tot}} = 0$, vanishes because ρ_{tot} is a pure state.

Next we divide the system into two subsystems, A and B, in such a way that the total Hilbert space factorizes into a tensor product:

$$\mathcal{H}_{\text{tot}} = \mathcal{H}_A \otimes \mathcal{H}_B \; . \tag{15.8}$$

For instance, we divide the sites of the spin chain or the lattice model into two groups (possibly connected), or we divide the space where a local QFT lives into two regions.[99] An observer that only has access to the subsystem A will feel as if the system is described by the *reduced density matrix*

$$\rho_A = \operatorname{Tr}_B \rho_{\text{tot}} \tag{15.9}$$

where the trace is over the subsystem B.

We define the entanglement entropy of the subsystem A as the von Neumann entropy of its reduced density matrix

$$S_A = -\operatorname{Tr}_A \rho_A \log \rho_A \; . \tag{15.10}$$

This quantity measures how much the systems A and B are entangled in the quantum state $|\Psi\rangle$ (or how much the state is "quantum").[100]

Example. Consider two particles of spin $\frac{1}{2}$: each has two states $|\uparrow\rangle$ and $|\downarrow\rangle$. Suppose they are in the state

$$|\Psi\rangle = |\uparrow\rangle_A \otimes |\downarrow\rangle_B \; .$$

Then

$$\rho_A = |\uparrow\rangle_A \langle\uparrow|_A \; , \qquad\qquad S_A = 0 \; . \tag{15.11}$$

In this state the two particles are not entangled: one is clearly $|\uparrow\rangle_A$ while the other one is clearly $|\downarrow\rangle_B$.

Suppose, instead, that they are in the state

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \Big(|\uparrow\rangle_A \otimes |\downarrow\rangle_B + |\downarrow\rangle_A \otimes |\uparrow\rangle_B \Big) \; .$$

This is the state if the two particles are created from the decay of a scalar particle. This time

$$\rho_A = \frac{1}{2} \Big(|\uparrow\rangle_A \langle\uparrow|_A + |\downarrow\rangle_A \langle\downarrow|_A \Big) \; , \qquad\qquad S_A = \log 2 \; . \tag{15.12}$$

[99] The case of a QFT is tricky because the Hilbert space does not really factorize [Wit18]. For this discussion we will assume it does.

[100] As explained below, this is a good measure of entanglement only if the state is pure in $\mathcal{H}_{\text{tot}}$.

In this state the two particles are entangled: we cannot determine the state of each one separately, rather, if one is $|\uparrow\rangle$ then the other one is $|\downarrow\rangle$ and vice versa. ■

We can define the entanglement entropy in arbitrary states, not just the ground state, and also in mixed states (described by general density matrices). For instance, we can define the entanglement entropy at finite temperature $T = \beta^{-1}$ by using for ρ_{tot} the thermal density matrix

$$\rho_{\text{thermal}} = \frac{1}{Z}\, e^{-\beta H} \qquad\qquad Z = \operatorname{Tr} e^{-\beta H} \tag{15.13}$$

where H is the Hamiltonian. When A is the total system, $S_A(\beta)$ is simply the thermal entropy. Indeed

$$\log\rho = -\beta H - \log Z \tag{15.14}$$

and so

$$-\operatorname{Tr}\rho\log\rho = \beta\operatorname{Tr}\rho H + \log Z \operatorname{Tr}\rho = \beta\langle H\rangle + \log Z = \beta\big(\langle H\rangle - \mathcal{F}\big) = S\ , \tag{15.15}$$

where $Z = e^{-\beta\mathcal{F}}$ and $\mathcal{F} = E - TS$ is the Helmholtz free energy.[101]

Two important properties of the entanglement entropy are:

- When the density matrix ρ_{tot} is pure, then

  $$S_A = S_B\ . \tag{15.16}$$

  Exercise. Prove it, for a finite-dimensional Hilbert space.

- *Strong subadditivity.* Given non-intersecting subsystems A, B and C (not necessarily covering the whole system):

  $$\begin{aligned} S_{A+B+C} + S_B &\leq S_{A+B} + S_{B+C} \\ S_A + S_C &\leq S_{A+B} + S_{B+C}\ . \end{aligned} \tag{15.17}$$

  These relations are quite non-trivial and we will not prove them.

[101] The Helmholtz free energy corresponds to the thermodynamic ensemble at constant temperature and volume. This is the correct ensemble for a QFT, in which we are supposed to work at fixed finite volume and then take the infinite volume limit at the end.

15.1 Entanglement entropy in QFT

Consider a local QFT on

$$\mathsf{R}_t \times N \,, \tag{15.18}$$

where N is a d-dimensional spatial manifold. We define the subregion A as a region $A \subset N$ at fixed time t_0. We call B its complement in N.

In a local QFT, let us suppose that the Hilbert space factorizes as a tensor product

$$\mathcal{H}_{\text{tot}} = \mathcal{H}_A \otimes \mathcal{H}_B \,. \tag{15.19}$$

This allows us to define S_A as before.

Such entanglement entropy is usually divergent in a continuum theory, and its definition requires a UV cutoff $\Lambda = \frac{1}{a}$ (a is a lattice spacing). Then the coefficient in front of the divergence is proportional to the area of the boundary ∂A:

$$S_A = \gamma \cdot \frac{\text{Area}(\partial A)}{a^{d-1}} + \text{subleading} \,. \tag{15.20}$$

This behavior is intuitively understood as the fact that the entanglement between A and B occurs at the boundary ∂A most strongly. The coefficient γ depends on the theory, but usually also on the renormalization scheme. This is clear: if you redefine the cutoff $a \to \lambda a$ then $\gamma \to \lambda^{d-1}\gamma$.

The behavior in 2D is different (then $d = 1$, and the boundary has dimension zero). For instance in a CFT, the entanglement entropy of an interval of lenght ℓ in R is [CC04]

$$S_A = \frac{c}{3} \log \frac{\ell}{a} \,, \tag{15.21}$$

where c is the central charge. This time the coefficient in front of the divergent term is scheme-independent, because if we redefine $a \to \lambda a$ we add a constant, but do not change the coefficient in front of $\log a$.

The form of entanglement entropy in CFTs in generic dimension is

$$S_A = p_1 \Big(\frac{\ell}{a}\Big)^{d-1} + p_3 \Big(\frac{\ell}{a}\Big)^{d-3} + \ldots + \begin{cases} p_{d-2}\Big(\frac{\ell}{a}\Big)^2 + \tilde{c}\,\log\frac{\ell}{a} & (d+1) \text{ even} \\ p_{d-1}\Big(\frac{\ell}{a}\Big) + p_d & (d+1) \text{ odd,} \end{cases} \tag{15.22}$$

where ℓ is a typical length scale of A. This can be understood because the various terms are controlled by local counterterms constructed with the metric and the extrinsic curvature of the boundary. Most terms are scheme dependent, as one infers by considering redefinitions $a \to \lambda a$ of the cutoff. However, $\tilde{c}$ and p_d are *scheme independent* and carry physical information.

For instance, in $(3+1)$D $\tilde{c}$ is proportional to the central charge a — the one involved in the a-theorem which is monotonic along unitary RG flows. Indeed one can use strong subadditivity to provide an alternative proof of the a-theorem [CTT17].

In $(2+1)$D one can use

$$p_2 \equiv F \tag{15.23}$$

to define a "central charge", which also has been proven to be monotonic along RG flows [CH12, CHMY15].

Replica trick. To evaluate the entanglement entropy in QFT we use the "replica trick". First we evaluate

$$\mathrm{Tr}_A\, \rho_A^n\ , \tag{15.24}$$

which is simpler because there is no log. This is called Rényi entropy. Then we analytically continue from $n \in \mathsf{N}$ to $n \in \mathsf{R}$, and then use

$$\frac{\partial}{\partial n} \mathrm{Tr}\, \rho^n = \frac{\partial}{\partial n} \mathrm{Tr}\, e^{n \log \rho} = \mathrm{Tr}\, \rho^n \log \rho \tag{15.25}$$

to write:

$$S_A = -\frac{\partial}{\partial n} \mathrm{Tr}_A\, \rho_A^n \Big|_{n=1} = -\frac{\partial}{\partial n} \log \mathrm{Tr}_A\, \rho_A^n \Big|_{n=1}\ . \tag{15.26}$$

The second equality is because $\mathrm{Tr}_A\, \rho_A = 1$.

To compute $\mathrm{Tr}_A\, \rho_A^n$ we use the path-integral formalism. Let us consider 2D for simplicity, and take A to be an interval at $t=0$ in Euclidean signature. The ground-state wavefunctional Ψ is found by integrating from $t = -\infty$:

$$\Psi\big(\phi_0(x)\big) = \int_{t=-\infty}^{\phi(t=0,x)=\phi_0(x)} \mathcal{D}\phi\, e^{-S[\phi]}\ . \tag{15.27}$$

The density matrix $\rho = |\Psi\rangle\langle\Psi|$ has matrix elements

$$[\rho]_{\phi_-,\phi_+} = \Psi(\phi_-)\, \overline{\Psi}(\phi_+)\ . \tag{15.28}$$

$\overline{\Psi}$ is obtained by integrating from $t=0$ to $t=+\infty$.

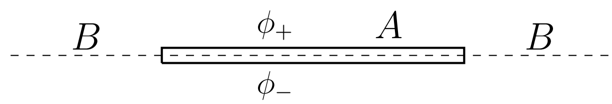
To obtain the reduced density matrix ρ_A we need to integrate over B, namely we set

$$\phi_-(x) = \phi_+(x) \equiv \phi_0 \qquad \text{for } x \in B \tag{15.29}$$

and integrate over ϕ_0 on B. We are left with discontinuous boundary conditions along A:

$$[\rho_A]_{\phi_-,\phi_+} = \frac{1}{Z_1} \int_{t=-\infty}^{t=+\infty} \mathcal{D}\phi \, e^{-S[\phi]} \prod_{x \in A} \delta\big(\phi(0_-,x) - \phi_-(x)\big) \, \delta\big(\phi(0_+,x) - \phi_+(x)\big) \, . \tag{15.30}$$

Here Z_1 is the vacuum partition function, necessary to guarantee that $\mathrm{Tr}_A \, \rho_A = 1$.



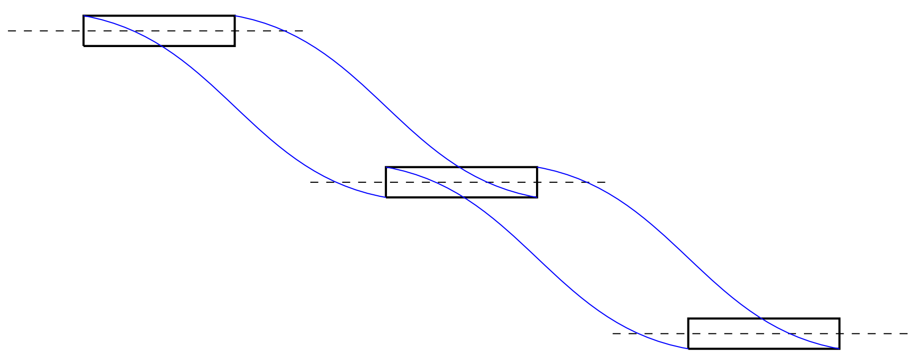
Now, ρ_A^n is given by n copies of ρ_A:

$$[\rho_A]_{\phi_{1,-}\phi_{1,+}} \cdots [\rho_A]_{\phi_{n,-}\phi_{n,+}} \, .$$

Multiplication is obtained by gluing the boundary conditions,

$$\phi_{j,+}(x) = \phi_{j+1,-}(x) \qquad \text{for } x \in A \, , \tag{15.31}$$

and integrating.



The trace is given by gluing and integrating the first and last boundary conditions.

Thus $\mathrm{Tr}_A \, \rho_A^n$ is the path-integral over an n-sheeted Riemann surface Σ_n, which is an n-fold covering of R^2 branched over ∂A:

$$\mathrm{Tr}_A \, \rho_A^n = \frac{1}{(Z_1)^n} \int_{\Sigma_n} \mathcal{D}\phi \, e^{-S[\phi]} \equiv \frac{Z_n}{(Z_1)^n} \, . \tag{15.32}$$

The construction in higher dimensions is the same.

15.2 Holographic entanglement entropy

Given a CFT with holographic dual, how do we compute the entanglement entropy of a region A?

Let us consider the Poincarè patch of AdS_{d+2}:

$$ds^2 = R^2 \frac{dz^2 + dx_{d,1}^2}{z^2} \ . \tag{15.33}$$

First we choose a cutoff $z \geq a$. The region A is at the boundary on R^d, and ∂A is a $(d-1)$-dimensional surface in R^d.

The prescription is that we should find a d-dimensional surface γ_A in AdS_{d+2}, at fixed time, that ends on ∂A, is homotopic to A, and has *minimal surface*. Then

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{(d+2)}} \ . \tag{15.34}$$

This is called the Ryu-Takayanagi formula. This prescription can be derived with a version of the replica trick in AdS_{d+2}.

Sketch of a derivation. We compute entanglement entropy as the analytically continued limit of the Renyi entropies. The latter is the partition function on an n-sheeted space $\mathcal{R}_n$, with deficit angle

$$\delta = 2\pi(1-n) \tag{15.35}$$

along the boundary ∂A. We should find a $(d+2)$-dimensional geometry $\mathcal{S}_n$ that solves Einstein equations and asymptotes to $\mathcal{R}_n$ at the boundary $z \to 0$. This is a technically difficult and unsolved problem, so we will use a trick.

We assume that $\mathcal{S}_n$ is an n-sheeted covering of AdS_{d+2}, with a deficit angle δ along a codimension-2 surface γ_A that asymptotes to the boundary ∂A. The Ricci scalar of the bulk spacetime has a δ-function along the surface:

$$R = 4\pi(1-n)\,\delta(\gamma_A) + R_0 \ , \tag{15.36}$$

where R_0 is the curvature of AdS_{d+2}. Then we plug this in the supergravity action

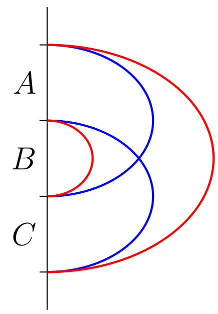
$$S_{\text{AdS}} = -\frac{1}{16\pi G_N} \int d^{d+2}x\, \sqrt{g}\,(R + \Lambda) + \dots \tag{15.37}$$

The missing terms give a contribution that cancels out in the ratio as we send $n \to 1$. Applying AdS/CFT:

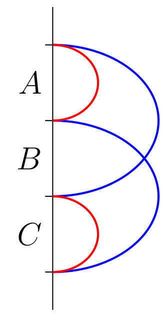
$$S_A = -\frac{\partial}{\partial n} \log \text{Tr}\, \rho_A^n \Big|_{n=1} = -\frac{\partial}{\partial n}\left[\frac{(1-n)\,\text{Area}(\gamma_A)}{4G_N}\right]_{n=1} = \frac{\text{Area}(\gamma_A)}{4G_N} \ . \tag{15.38}$$

Moreover, the action principle in gravity becomes the variational principle for the Area and selects the minimal-area surface γ_A.

Holographic proof of strong subadditivity. It is easy to prove with pictures:



$S_{A\cup B} + S_{B\cup C} \geq S_{A\cup B\cup C} + S_B$



$S_{A\cup B} + S_{B\cup C} \geq S_A + S_C$

In both pictures, the sum of the lengths (in AdS) of the blue lines is greater than that of the red line, because in both cases one can decompose the blue lines in such a way that they connect the same pair of points as the red lines, but the red lines are minimal length for those pairs, while the blue lines are not (they are minimal for different pairs).

Entanglement entropy of interval in 2D CFT. We use the relation between the radius of AdS_3 and the central charge c of the CFT_2:

$$c = \frac{3R}{2G_N^{(3)}} \,. \tag{15.39}$$

Take AdS_3 (its Poincarè patch) and an interval of length ℓ. We need the geodesic between the two points

$$\Big(-\frac{\ell}{2}, a\Big) \qquad \text{and} \qquad \Big(\frac{\ell}{2}, a\Big) \,. \tag{15.40}$$

This is given by the half-circle

$$(x, z) = \frac{\ell}{2}\big(\cos u\,, \sin u\big)\,, \qquad \epsilon \leq u \leq \pi - \epsilon\,, \qquad \epsilon = \frac{2a}{\ell} \tag{15.41}$$

for $a \to 0$. Then the length is easily computed ($ds^2 = R^2 du^2/\sin^2 u$ along the curve):

$$\text{Length}(\gamma_A) = 2R\int_\epsilon^{\pi/2} \frac{du}{\sin u} \simeq -2R\log\frac{\epsilon}{2} = 2R\log\frac{\ell}{a} \,. \tag{15.42}$$

The the entanglement entropy is

$$S_A = \frac{\text{Length}(\gamma_A)}{4G_N^{(3)}} = \frac{c}{3}\log\frac{\ell}{a}\ , \tag{15.43}$$

reproducing the field theory result.

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