SISSA
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BPS blowup surface defects and Hurwitz chiral ring expansions

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Abstract

In this thesis we study a class of BPS surface defects of $SU(2)$ supersymmetric gauge theories defined on a blown-up geometry. We show that in the Nekrasov-Shatashvili limit the partition function in presence of these defects is in general a $\mathcal{T}$-function satisfying some Painlevé Hirota bilinear equation. Using a topological version of operator/state correspondence we compute the expansion of the $\mathcal{T}$-function in an integer basis, given in terms of the moduli of the quantum Seiberg-Witten curve. In the four dimensional case we study the modular properties of these solutions and show that they do directly lead to BCOV holomorphic anomaly equations for the corresponding topological string partition function. The resulting $\mathcal{T}$-functions are holomorphic and modular and as such they provide a natural non-perturbative completion of topological strings partition functions. In the five-dimensional setting, we discuss a UV completion of a class of Argyres-Douglas (AD) theories in the Ω-background in terms of a renormalisation group flow from five dimensional $\mathcal{N} = 1$ superconformal field theories (SCFT) on S^1. This is obtained via analysing these theories in the light of (q-)Painlevé/gauge theory correspondence, which allows to compute the five dimensional BPS partition functions as an expansion in the circular Wilson loop vev with integer q-polynomials coefficients. We discuss in detail the phase diagram of the four dimensional limits, pinpointing the special AD loci. Explicit computations are reported for $\tilde{E}_1$ SCFT and its limit to $\mathrm{H}_0 = (A_1, A_2)$ AD theory.

Declaration

I hereby declare that, except where specific reference is made to the work of others, the contents of this thesis are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. The discussion is based on the following works:

- **Surface observables in gauge theories, modular Painlevé tau functions and non-perturbative topological strings**, with Giulio Bonelli, Pavlo Gavrylenko and Alessandro Tanzini, arXiv:2410.17868 [hep-th] ,

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- **On a 5D UV completion of Argyres-Douglas theories**, with Giulio Bonelli, Pavlo Gavrylenko and Alessandro Tanzini, arXiv:2508.05610 [hep-th] ,

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Contents

Introduction

Quantum Field theory (QFT) is a universal theoretical framework to describe physical phenomena. At the fundamental level, it allows to unify the principles of Quantum Mechanics and Special Relativity and has lead to the formulation of the Standard Model of particle physics which describes three of the four fundamental interactions of nature (except gravity) and has produced several experimental predictions which were tested to very high precision. It is also a powerful tool to study the behaviour of condensed matter systems e.g. critical phenomena, phases of matter and response functions.

Despite its enormous success, we have a systematic understanding of QFT only when interactions are weak such that a perturbative treatment is available. When QFT are strongly coupled we generally lose control of their quantum effects and the behaviour of the theory can be radically different, with a lot of new interesting phenomena appearing, such as confinement, mass gap, dynamical symmetry breaking and a rich and complicated structure of the spectrum. Furthermore, strongly coupled QFTs often present a rich set of dualities where the dynamics admits completely different descriptions that are physically equivalent. All these phenomena still lack full analytical understanding because the usual perturbative techniques are not available. Furthermore, one of the most important guiding principles which has emerged in the study of string theory and Quantum Gravity, the holographic principle, whose best known realization is the AdS/CFT correspondence, directly connects the dynamics of Quantum Gravity (QG) to the study of strongly coupled QFTs describing their holographic duals, making even more important a systematic and quantitative understanding of these theories. Finally, also string theory itself still lacks a non-perturbative formulation which is needed for a full understanding of QG.

One of the strategies to understand theories at the non-perturbative level is to look first at simplified models which capture some of the essential features of the dynamics of more realistic models but which are simple enough to allow for a full analytic non-perturbative treatment. The hope is that studying these models we can gain insights on the strongly coupled dynamics of more realistic ones, using them as a theoretical laboratory to test our principles and to find results which may be valid more in general.

A powerful tool to realize this program, which has led to a plethora of interesting results, is supersymmetry, a symmetry that exchanges bosonic and fermionic d.o.f.. Given some QFT we can make it supersymmetric adding some new d.o.f. promoting the fields to “supermultiplets” and then the bosonic and fermionic quantum corrections compensate each other and the dynamics is much more under control thanks to several constraints coming from the holomorphicity and integrability of the supersymmetric theories. In particular, because all the fundamental interactions in the Standard Model are described by Yang-Mills gauge theories, we are naturally lead to study supersymmetric gauge theories.

The "solvability" of some susy QFT depends on its number $\mathcal{N}$ of supersymmetries. In the case of 4d $\mathcal{N} = 2$ theories the dynamics is still sufficiently rich to allow for interesting phenomena but presents a rich mathematical structure which is deeply related to geometry and integrability which often allows to obtain exact results. The geometric structure of these supersymmetric QFTs naturally embed them in string theory and there is then a deep connections and interplay between the two.

Given a QFT, a basic question is to understand the structure of its vacua and its low-energy dynamics. In this respect, one of the most striking results in the context of supersymmetric QFTs is the Seiberg-Witten (SW) solution [1, 2] for $\mathcal{N} = 2$ gauge theories where the low-energy theory is completely fixed by the holomorphic prepotential $\mathcal{F}$ and can be exactly determined using the geometry of a complex curve, The SW curve. This curve can be interpreted as the spectral curve of some integrable system, whose dynamics is completely solvable, and the gauge theory low-energy dynamics is then directly related to the one of the integrable system. Finally, the SW curve can be interpreted physically by embedding the $\mathcal{N} = 2$ gauge theory in string theory, i.e., realizing it as the worldvolume theory of a system of branes [3] or as the compactification of string theory on a suitable local Calabi-Yau (CY) manifold through the so called "geometric engineering" [4]. The SW solution can also be extended to five-dimensional gauge theories compactified on a circle, the corresponding integrable system becomes relativistic and the relation with string theory becomes more direct because the SW curve is the "mirror curve" defining the type IIB CY geometry and is naturally dual to an M-theory compactification.

After the discovery of the SW result, there were several attempts to derive it more directly from the path integral of the supersymmetric gauge theory [5–9]. To do this one can use a powerful property of supersymmetric field theory which is localization. The basic idea of localization, which will be explained more in detail in the following, is that for certain protected observables the semiclassical approximation of the path integral is exact. In this way the supersymmetric path integral can be reduced to some finite-dimensional integral on the moduli space of instantons. These integrals are still difficult to handle because of the non-compactness of the moduli space of instantons but this problem can be cured improving the localization procedure to the so called equivariant localization which exploits the full symmetries of the theory and drastically simplifies the computations.

This was the key idea that led to the work of Nekrasov [10] where the exact non-perturbative partition function $Z_{Nek}(a, \epsilon_1, \epsilon_2)$ in presence of the so called Ω-background was computed from first principles using supersymmetric equivariant localization for the path integral of the microscopic theory. The Ω-background is a gravitational background that depends on two parameters (ϵ_1, ϵ_2) and acts as a regulator for the theory confining the dynamics on some finite volume $V = \frac{1}{\epsilon_1 \epsilon_2}$ and in this way one can compute exactly all the instanton non-perturbative corrections which effectively localize in the origin. In the limit in which we remove the Ω-background we can recover the SW prepotential as the free energy density

$$\mathcal{F}(a) = \lim_{\epsilon_1,\epsilon_2\to 0} -\frac{1}{V}\log Z_{Nek}(a,\epsilon_1,\epsilon_2) = \lim_{\epsilon_1,\epsilon_2\to 0} -\epsilon_1\epsilon_2 \log Z_{Nek}(a,\epsilon_1,\epsilon_2)\ . \tag{0.1}$$

It was soon realized that the Ω-background was not just some convenient regularization but encoded much more informations about the dynamics, well beyond the original SW result. Indeed, Nekrasov's result leads to a set of correspondences between several different theories

- 4d $\mathcal{N}=2$ Supersymmetric gauge theories;
- Integrable systems and isomonodromic deformations;
- Two-dimensional Conformal Field Theory;
- Topological string on local CY manifolds.

In this thesis we are going to use many sides of these correspondences, which are summarized in figure 1.

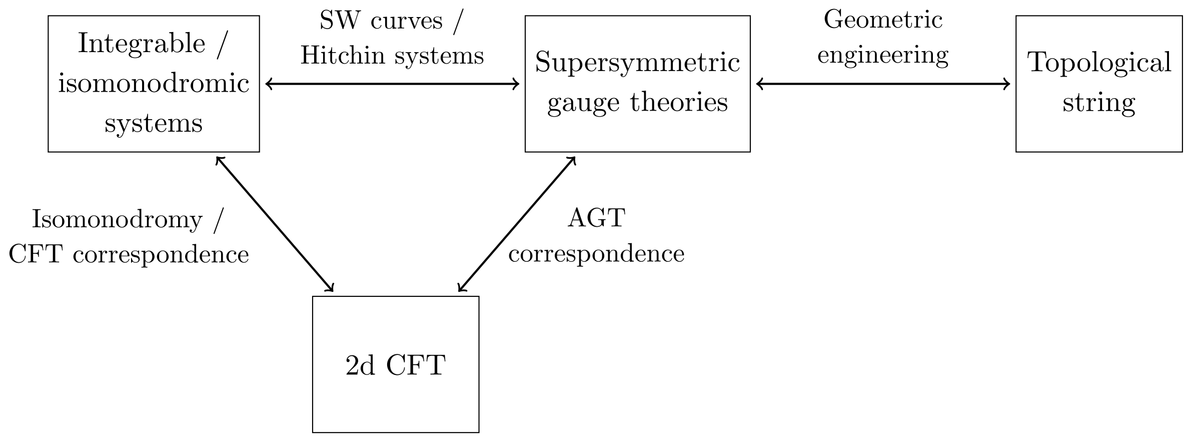

Figure 1.

In particular, in the Ω-background the connection between SW curves and integrable systems (based on isospectral deformations) is extended to the more general framework of (quantum) isomonodromic deformations. This is a generalization of integrable systems where the conservation of the hamiltonians of the system is replaced by the conservation of some monodromy properties of some linear problem which encodes the dynamics of the original system. These monodromy properties are the same of the so called *conformal blocks* of 2d CFTs which encode completely the structure of the correlation functions determined by the conformal symmetry.

Indeed, for a large class of $\mathcal{N}=2$ theories, the so called class S theories [11], it can be shown that the Nekrasov instanton partition function corresponds precisely to the conformal blocks of the 2d CFT leading to a direct connection between susy gauge theories and 2d CFT, the Alday-Gaiotto-Tachikawa (AGT) correspondence [12], from the name of the authors. The AGT correspondence is a manifestation of a more general principle, the BPS/CFT correspondence, which states that some protected sector of the supersymmetric gauge theory, the so-called BPS sector, is described by some 2d CFT [13]. This means that the dynamics of this sector has an enhanced set of symmetries, given by some infinite-dimensional algebra, which makes it completely solvable.

The connection with 2d CFT opened a new perspective on the relation between the dynamics of $\mathcal{N} = 2$ gauge theories and integrable systems after the discovery that the general solution of Painlevé equations, which are central in the theory of isomonodromic deformations, are given in terms of $SU(2)$ Nekrasov partition functions [14, 15]. This leads to the formulation of the so called "Painlevé-gauge correspondence" [16] where the dynamics of the $SU(2)$ $\mathcal{N} = 2$ theory is associated to the one of some isomonodromic system, governed by Painlevé equations. In particular, the Nekrasov partition function of the gauge theory in the self-dual background $\epsilon_1 = -\epsilon_2 = \epsilon$ is related to the so called $\mathcal{T}$-function, which generates the Painlevé hamiltonian $\zeta \sim \partial_t \log \mathcal{T}$ and encodes the full information about the solutions, through the "Kyiv formula". This gives the $\mathcal{T}$-function in the form of a discrete Fourier-like transform, called *Zak transform*

$$\mathcal{T} \propto \sum_n e^{n\rho} Z_{Nek}(a + n\epsilon, \epsilon, -\epsilon) \ . \tag{0.2}$$

As a consequence, the Nekrasov partition function, in terms of $\mathcal{T}$, can be defined as a solution of a differential equation instead of the result of computing some localized path integral. This was a crucial step to study the gauge theory in the strongly coupled regime going beyond the standard localization techniques which are valid only in the regime where the theory admits a weakly-coupled lagrangian description. Indeed, from the point of view of Painlevé equations different choices of coupling just correspond to different choices of the Painlevé time and then a weak/strong coupling expansion can be related to a short/long time expansion. This is especially important for non-lagrangian theories such as Argyres-Douglas theories, where Painlevé equations become a powerful tool to study the theory without relying on a standard lagrangian approach.

Since then there have been several further studies concerning late time expansion of the Painlevé equations [17] and its relation to strongly coupled phases of gauge theory [16], extension to more general isomonodromic deformation problems [18–24] and to q-difference equations [14, 25–29]. Moreover, the relation to Fredholm determinants [30–32] allowed to establish a link to the proposal of [33] for the non-perturbative completion of topological strings in terms of spectral theory [26, 34–37]. Other proposals for a non-perturbative formulation of topological strings in terms of isomonodromic $\mathcal{T}$-functions appeared in [38–40].

It is natural to ask if the Painlevé integrable structure can be directly derived from the properties of the gauge theory itself. An intriguing aspect of the Painlevé equations that govern the gauge theory dynamics is the fact that this structure is directly related to the so called blowup equations [41]. Blowup equations are relations describing how the gauge theory on a manifold M behaves under the replacement of a regular (or singular) point of M with the local geometry of a two-sphere E. This procedure is familiar in the case of 2d

CFT where one can glue some handle to the worldsheet Σ. Cutting a small disk contaning the handle this defines some state on the boundary of the disk and one can replace this state with some vertex operator via the state/operator correspondence.

The same happens for the BPS sector of the 4d theory. The state $|\Psi(d)\rangle$ living on the boundary of a ball which contains some defect living on the exceptional sphere E, labelled by some coupling d, is equivalent to a BPS local operator which in general will be given as an expansion in the chiral ring basis $\mathcal{O}_i$ of the BPS sector of the theory

$$\begin{array}{ccc} \text{Blowup state (defect)} & & \text{Sum of local BPS operators (chiral ring)} \\ |\Psi(d)\rangle & = & \sum_j B_j(d)\mathcal{O}_j \end{array}$$

where $B_j(d)$ are the coefficients in the chiral ring basis and depend on the defect coupling d. Furthermore, one can equivalently compute the blowup state "microscopically" decomposing the blowup partition function with the defect insertion $I(d)$ in the two patches of the exceptional sphere E. This gives the blowup partition function as a convolution of the two patches where we sum over all the magnetic fluxes on E, schematically

$$\begin{aligned} &\langle I(d)\rangle_{\hat{X}} = \hat{Z}(d) \sim \sum_n D_d(Z^{(1)}, Z^{(2)}) \ , \\ &Z^{(1)} = Z(a+n\epsilon_1, \epsilon_1, \epsilon_2-\epsilon_1) \ , \quad Z^{(2)} = Z(a+n\epsilon_2, \epsilon_1-\epsilon_2, \epsilon_2) \ , \end{aligned} \tag{0.3}$$

where D_d is some differential (or difference in the 5d case) bilinear operator the shifts in the partition functions $Z^{(1)}, Z^{(2)}$ correspond to the equivariant action on the affine coordinates $\xi_1 = z_2/z_1,\ \xi_2 = z_1/z_2$ of E. As we will review more in detail in subsection 3.6, one can derive the Hirota equation for the Painlevé $\mathcal{T}$-function starting exactly from the blowup equations for the corresponding Nekrasov partition function [42, 43]. The main idea is that the Painlevé equations of the $\mathcal{T}$-function can be written as Hirota bilinear equation for a suitable differential (or difference in the discrete q-Painlevé case) bilinear operator D

$$D(\mathcal{T}, \mathcal{T}) = 0 \ . \tag{0.4}$$

Via the Kyiv formula (0.2), this operator is nothing but the Zak transformed version of some relation involving the convolution of two copies of the Nekrasov partition function, schematically

$$\sum_n D(Z^{(1)}, Z^{(2)}) = 0 \ . \tag{0.5}$$

This is obtained by taking a vanishing combination of blowup equations

$$\alpha_{d_1}\hat{Z}_{d_1} + \cdots + \alpha_{d_m}\hat{Z}_{d_m} = \sum_j (\alpha_{d_1}B_j(d_1) + \cdots + \alpha_{d_m}B_j(d_m))\, \langle \mathcal{O}_j\rangle = 0 \ , \tag{0.6}$$

where α_l are differential operators in the coupling Λ of the gauge theory, and rewriting $\hat{Z}_d$ in terms of the convolution (0.3).

The aim of this thesis is to study more in detail the relation between the Painlevé equations and the OPE coefficients $B_j(d)$ that express the blowup defect in terms of the local BPS chiral ring basis $\mathcal{O}_j$.

The original results of this thesis are the following. Generalizing some earlier observations [44–46], we will show that in the Nekrasov-Shatashvili limit $\epsilon_2 \to 0$, not only the coefficients $B_j(d)$ determine the Painlevé equations of the $\mathcal{T}$-function but that they define *themselves* $\mathcal{T}$-functions of the Painlevé equations, and therefore they encode also the information about the gauge theory partition function. Schematically we have

$$\left.\frac{\hat{Z}(\epsilon, \epsilon_2, d)}{Z(\epsilon, \epsilon_2)}\right|_{\epsilon_2 \to 0} \sim \sum_j B_j(d) \langle O_j \rangle \sim \mathcal{T} \propto \sum_n e^{n\rho} Z(a + n\epsilon, \epsilon, -\epsilon) \ , \tag{0.7}$$

where in the last step we take the $\epsilon_2 \to 0$ limit of the shifted partition functions $Z^{(1)}, Z^{(2)}$ defined in (0.3) which reduce respectively to the self-dual Ω-background $\epsilon_1 = -\epsilon_2 = \epsilon$ and NS Ω-background $\epsilon_2 \to 0, \epsilon_1 = \epsilon$ partition functions.

This implies the surface defects living on the blowup carry the full informations about the integrability properties of the theory. We will show that this gives a new realization of the Painlevé $\mathcal{T}$-function as an expansion in the chiral ring basis vevs $\langle \mathcal{O}_j \rangle$ where the surface defect parameter d plays the role of time for the Painlevé equation.

This alternative realization of the $\mathcal{T}$-function presents a number of remarkable properties

- It is a convergent, or even finite, expansion, *analytic* in all the gauge theory moduli. For this reason it is valid in any point of the moduli space.
- The OPE coefficients are *Hurwitz integral*, that is, they are polynomials with integer coefficients. This naturally relate them to topological invariants and BPS counting.
- The resulting $\mathcal{T}$-function has manifest *modular* properties.

Relying on these properties, we will find two different applications of the chiral ring expansions of the $\mathcal{T}$-function in the simple setup of $SU(2)$ supersymmetric gauge theories both for 4d theories and for 5d theories compactified on a circle S^1_β of length β. The main ideas of our analysis are based on the identifications reported in the figure 2

Painlevé equations		**Gauge theory on the blowup**		**Topological string on CY X**
Painlevé $\mathcal{T}$-function	=	Blowup partition function with surface defect	=	Non-perturbative partition function $\mathcal{Z}_X$ = holomorphic + modular

Figure 2.

The first one is related to the so called holomorphic anomaly equations of the topological string, that encode the dependence of the string amplitudes $\mathcal{F}_g$ on the anti-holomorphic moduli. The CY geometries X which engineer the four-dimensional gauge theories correspond to elliptic curves[1] [47, 48]. For these geometries we will show that the holomorphic anomaly equations directly follow from the simple statement that the Painlevé $\mathcal{T}$-function has the structure of a "grand-canonical" partition function given by the Kyiv formula and is an analytic function of the chiral ring generator $\mathrm{Tr}\,\phi^2$ which then carry all the modular properties of the amplitudes $\mathcal{F}_g$. Schematically

$$\partial_{E_2}\mathcal{T} = 0 \Leftrightarrow \begin{array}{c}\text{BCOV holomorphic}\\ \text{anomaly equation.}\end{array}, \tag{0.8}$$

this also allows to construct a full non-perturbative completion of the topological string partition function which is both holomorphic and modular and as such it is background independent, i.e. independent on the point of expansion in the CY moduli space. The non-perturbative corrections appear as trans-series like corrections $e^{n\rho}$ inside the Kyiv formula, where ρ is an appropriate period of the SW curve[2].

The second result will be based on the fact that the chiral ring basis gives a *global* description of the partition function Z, therefore it can be easily studied in the strongly coupled regime and holds also for non-lagrangian theories such as Argyres-Douglas theories. We will compute explicitly the chiral ring expansion for the rank 1 AD theories. This carries the full information about the Nekrasov self-dual partition function. Once we lift the analysis to the 5d setting we will use this fact to study a novel completion of AD theories [28, 49] in terms of 5d gauge theories following explicitly the RG flow 5d→4d at the level of the chiral ring expansion, we will analyze in detail the flow from the $\tilde{E}_1$ SCFT (UV completion of 5d SYM with Chern-Simons level $k = 1$) and the AD H_0 SCFT .

The structure of this thesis is the following. In the first part I we will review some background material that we will use in the thesis. In the second part II we start with the original results of this thesis. We will formulate the four-dimensional gauge theory on the blowup and show that the generating function of the blowup defect, that we call *blowup factor*, in the NS limit is a Painlevé $\mathcal{T}$-function which can be expanded in an integer chiral ring basis. We will study in detail the modular properties of the blowup factor and derive from this the BCOV holomorphic anomaly equations for the topological string. We will then compute explicitly the chiral ring expansion of the blowup factor in terms of recurrence relations coming from the Painlevé equations. In the third part III, we will generalize this construction to the five-dimensional gauge theory compactified on a circle. The corresponding Painlevé equations become discrete and the chiral ring expansion is expressed in terms of Wilson loops wrapped along the 5d circle. We will compute then explicitly this expansion for the pure 5d SYM theory with Chern-Simons level $k = 0, 1$ and we will show that in the $k = 1$ case we can take a 4d limit to the AD theory H_0 giving a 5d UV completion of this non-lagrangian theory as the perturbation of a finite-coupling point in the 5d gauge theory moduli space. Finally, in the appendix IV we recap our conventions on Weierstrass elliptic functions, we collect some details on the derivation

[1]These can be obtained from the CY-threefold geometries that engineer 5d gauge theories taking the so called geometric engineering limit [4].

[2]This is done in the so called 4d geometric engineering limit of the topological string. The full topological string theory corresponds to the 5d setting.

of the holomorphic anomaly equations and we display the coefficients of the chiral ring Hurwitz expansions for the theories we studied.

As a final comment, we remark that, for convenience of the analysis, the notations in the different parts may be slightly different. The notations and conventions adopted are self-consistent and explained in each part.

I Background

1 Seiberg-Witten theory

We start our analysis reviewing the Seiberg-Witten solution for the exact low-energy theory of 4d $SU(2)\,\mathcal{N}=2$ SYM pure theory. We will then analyze the case of $SU(2)\,N_f=1$ to illustrate the basic example of an Argyres-Douglas (AD) theory which is a strongly coupled SCFT which arise as a singularity in the moduli space of the gauge theory.

1.1 $\mathcal{N}=2$ supersymmetry

In order to define $\mathcal{N}=2$ susy gauge theories and to set some notation we first review briefly the 4d supersymmetry algebra and its representations which are called "supermultiplets". The Poincaré algebra $\mathfrak{iso}(1,3)$ which generates the isometries of Minkowski spacetime is

$$
\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= -i\eta_{\mu\rho}M_{\nu\sigma} + i\eta_{\nu\rho}M_{\mu\sigma} - i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\mu\sigma}M_{\nu\rho} \ , && (1.1)\\
[M_{\mu\nu}, P_\rho] &= -i\eta_{\mu\rho}P_\nu + i\eta_{\nu\rho}P_\mu \ , && (1.2)\\
[P_\mu, P_\nu] &= 0 \ , && (1.3)
\end{aligned}
$$

where $\eta_{\mu\nu} = (+,-,-,-)$ is the Minkowski metric, P_μ are the generators of spacetime translations and $M_{\mu\nu}$ are the generators of the Lorentz group given by rotations and Lorentz boosts. The super-Poincaré algebra is an extension of the Poincaré algebra where we add $\mathcal{N}$ pairs of Grassmann-odd generators, the *supercharges* $Q^I_\alpha, \bar{Q}^J_{\dot\alpha}$ with $I, J = 1, \dots, \mathcal{N}$ and where the indices $\alpha, \dot\alpha$ transform as left-handed and right-handed Weyl spinors respectively. This transformation property together with the graded version of Jacobi identity imposes the following commutation relations on the supercharges

$$
\begin{aligned}
\{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\} &= 2\sigma^\mu_{\alpha\dot\beta}P_\mu\delta^{IJ} \ , && (1.4)\\
\{Q^I_\alpha, Q^J_\beta\} &= \epsilon_{\alpha\beta}Z^{IJ} \ , && (1.5)\\
\{\bar{Q}^I_{\dot\alpha}, \bar{Q}^J_{\dot\beta}\} &= \epsilon_{\dot\alpha\dot\beta}(Z^{IJ})^* \ , && (1.6)\\
[M_{\mu\nu}, Q^I_\alpha] &= i(\sigma_{\mu\nu})_\alpha{}^\beta Q^I_\beta \ , && (1.7)\\
[M_{\mu\nu}, \bar{Q}^I_{\dot\alpha}] &= i(\bar\sigma_{\mu\nu})_{\dot\alpha}{}^{\dot\beta}\bar{Q}^I_{\dot\beta} \ , && (1.8)\\
[P_\mu, Q^I_\alpha] &= 0 \ , && (1.9)\\
[P_\mu, \bar{Q}^I_{\dot\alpha}] &= 0 \ , && (1.10)
\end{aligned}
$$

The susy representations are given by direct sums of Poincaré representations. For $\mathcal{N}=1$ supersymmetry with spin ≤ 1 we have essentially two representations:

- Chiral (or Wess-Zumino) multiplet: (ϕ, ψ_α) ,
- Vector multiplet: (λ_α, A_μ) ,

where $\psi_\alpha, \lambda_\alpha$ are the fermionic "superpartners" of the bosonic fields ϕ, A_μ. The $\mathcal{N}=2$ supersymmetry representation are direct sums of $\mathcal{N}=1$ representations and we will be interested essentially in the short or BPS representations which are of two types

- $\mathcal{N}=2$ vector supermultiplet: $(\phi, \psi_\alpha) \oplus (\lambda_\alpha, A_\mu)$ (chiral multiplet plus $\mathcal{N}=1$ vector multiplet) ,
- Hypermultiplet: $(\phi_1, \psi_{1,\alpha}) \oplus (\phi_2, \psi_{2,\alpha})$ (sum of two chiral multiplets) .

Finally, all generators of the supersymmetry algebra commute with the mass and the internal symmetry generators therefore all fields in the same multiplet have different spins but the same mass and the same internal quantum numbers. In particular, all the fields in the vector multiplet are in the adjoint representation.

If $\mathcal{N}>1$ then we have some non-trivial automorphisms acting on the algebra of the supercharges which preserve their commutation relations which is called "R-symmetry". For $\mathcal{N}=2$ susy we have only one non-trivial central charge $Z^{IJ}=\epsilon^{IJ}Z$ and this is related to the mass of the BPS states by

$$M_{BPS}=\sqrt{2}|Z| \ , \tag{1.11}$$

and the $\mathcal{R}$-symmetry is given by $USp(2)_\mathcal{R} \simeq SU(2)_\mathcal{R}$ which rotates the two fermions in the $\mathcal{N}=2$ supermultiplets.

1.2 Low-energy theory on the Coulomb branch

The space of vacua of a $\mathcal{N}=2$ theory forms in general a moduli space which is protected by supersymmetry so is not lifted by quantum corrections. The moduli space of vacua splits in different branches which are chacterized by the different Higgsing patterns driven by the scalars in the vector multiplets and Hypermultiplets. We are interested in the case where the adjoint scalar $\phi=\phi^a T^a$ in the vector multiplet acquires a vev due to the presence of the potential

$$V=\mathrm{Tr}\big[\bar{\phi},\phi\big]^2 \ . \tag{1.12}$$

The minima are given by the Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$ of the gauge algebra $\mathfrak{g}$ which is the maximal abelian subalgebra of $\mathfrak{g}$ and its dimension $\dim\mathfrak{h}=r$ is called the rank of the gauge group[3]. The gauge group G is then broken to the abelian group $U(1)^r$ and for this reason we call this space of vacua the "Coulomb branch". For $SU(N)$ gauge theories, up

[3]We assume $\mathfrak{g}$ to be semisimple.

to gauge transformations, the vacua are[4]

$$\langle\phi\rangle = \mathrm{diag}(a_1,\dots,a_N)\ ,\quad \sum_{j=1}^{N} a_j = 0\ . \tag{1.13}$$

In a generic point of the moduli space, at the scale $\langle\phi\rangle$ the coupling stops running because we integrate out the massive W bosons and in the infrared we have an abelian $\mathcal{N}=2$ effective susy gauge theory which is given by a supersymmetric non-linear sigma model of r abelian gauge bosons and r neutral scalars $\phi_i(x)$. Then by $\mathcal{N}=2$ supersymmetry the lagrangian is completely fixed by a single holomorphic (multivalued) function, the prepotential[5] $\mathcal{F}_0(a)$, which is integrated over the chiral $\mathcal{N}=2$ superspace

$$\mathcal{L}_{eff} = \int \mathrm{d}\theta^4 \mathcal{F}_0(\Phi) = \mathrm{Im}\left[\frac{\tau_{ij}(\phi)}{16\pi}\left(F^i_{\mu\nu}F^{j,\mu\nu} + \partial_\mu\bar{\phi}^i\partial^\mu\phi^j\right)\right] + \mathrm{Re}\left[\frac{\tau_{ij}(\phi)}{16\pi}\left(F^i_{\mu\nu}\tilde{F}^{j,\mu\nu}\right)\right] + \text{fermions}\ , \tag{1.14}$$

where $\tilde{F}^i_{\mu\nu}$ is the Hodge dual and the complex matrix $\tau_{ij}(a)$ is the generalized gauge coupling which is related to the holomorphic prepotential $\mathcal{F}_0(a)$ by

$$\tau_{ij}(a) = \frac{\partial^2\mathcal{F}_0}{\partial a^i\partial a^j}(a)\ . \tag{1.15}$$

and the Kähler metric of the moduli space is given exactly by $\mathrm{Im}[\tau_{ij}]$. In some special point of the moduli space, due to strong coupling effects, some extra d.o.f. can become massless and the metric $\mathrm{Im}[\tau_{ij}]$ becomes singular because the extra light fields must be included in the effective description.

It is important to observe that by unitarity $\mathrm{Im}[\tau_{ij}]$ should be positive definite, $\mathrm{Im}[\tau_{ij}] > 0$. This means that a non-trivial prepotential $\mathcal{F}_0(a)$ is not a globally defined holomorphic function on the moduli space, because $\mathrm{Im}[\tau_{ij}]$ is harmonic so cannot be bounded everywhere, and a_i cannot be a global coordinate. Indeed, the low-energy description is actually not unique. Defining the dual field

$$a^i_D = \frac{\partial\mathcal{F}_0}{\partial a^i}\ , \tag{1.16}$$

under an electromagnetic duality[6] $Sp(2r,\mathbb{Z})$

$$\begin{pmatrix} a_D \\ a\end{pmatrix} \to \begin{pmatrix} A & B \\ C & D\end{pmatrix}\begin{pmatrix} a_D \\ a\end{pmatrix}\ ,\quad \begin{pmatrix} A & B \\ C & D\end{pmatrix} \in Sp(2r,\mathbb{Z})\ , \tag{1.17}$$

the lagrangian (1.14) keeps the same form with the gauge coupling τ_{ij} that transforms projectively

$$\tau_{ij} \to [(A\tau+B)(C\tau+D)^{-1}]_{ij}\ . \tag{1.18}$$

[4] There is still some gauge redundancy coming from the Weyl reflections so we have to quotient the space of minima a_j by the Weyl group.

[5] The reason for the subscript 0 is that, as we will review later, the *prepotential* can be identified with the genus zero free energy of the topological string theory associated to the SW geometry.

[6] At the level of the full $\mathcal{N}=2$ superfield Φ(which contains all the fields of the $\mathcal{N}=2$ vector supermultiplet this reproduces the standard electromagnetic duality for the abelian gauge bosons. The fact that the matrix is integer valued comes from the Dirac-Zwanziger quantization condition.

Therefore, the electromagnetic duality is not a symmetry but gives a physically equivalent description of the infrared theory which generally is more suitable to describe some different region of the moduli space. As a consequence, if we follow some loop γ in the moduli space of the theory once we go back to the initial point we obtain the same theory only up to a duality transformation $M_\gamma \in Sp(2r, \mathbb{Z})$. This is precisely the case when the loop encircles some singularity of the moduli space and the informations about the monodromy matrices M_γ encode the geometry of the moduli space of vacua and together with some asymptotic conditions at infinity are sufficient to completely reconstruct the exact prepotential $\mathcal{F}_0$.

The BPS spectrum of the theory contains electrically charged W-bosons, coming from the Higgsing, and solitonic objects such as magnetic monopoles or dyons. If (n_m, n_e) are the charge vectors of a BPS state, its mass is given by

$$M = \sqrt{2}|Z| = \sqrt{2}|a \cdot n_e + a_D \cdot n_e| \ . \tag{1.19}$$

In particular a corresponds to the mass of the W-bosons and a_D to the mass of the monopoles.

In the following we will focus on the case of $SU(2)$ theories and we discuss the original derivation by Seiberg and Witten of the exact prepotential for $\mathcal{N} = 2$ $SU(2)$ SYM theory.

1.3 The Seiberg-Witten solution

In this section we will review the original Seiberg-Witten (SW) solution for $SU(2)$ and then explain how it can be extended to a framework valid for more general $\mathcal{N} = 2$ theories. The lagrangian of pure $SU(2)$ gauge theory is

$$S = \int \mathrm{d}^4x \, \mathrm{Im}\, \mathrm{Tr}\, \frac{\tau}{16\pi} \left[\int \mathrm{d}^2\theta W_\alpha W^\alpha + \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \ \bar{\Psi} e^{2gV} \Psi \right] \ , \tag{1.20}$$

where (ψ_1, A_μ) and $\Psi = (\phi, \psi_2)$ are the vector and chiral $\mathcal{N} = 1$ superfields respectively and

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} \ , \tag{1.21}$$

is the bare complexified gauge coupling. In $\mathcal{N} = 2$ language this action is given by the microscopic prepotential $\mathcal{F}_{UV}$

$$S = \frac{1}{16\pi} \int \mathrm{d}^4x \, \mathrm{Im} \int \mathrm{d}^2\theta \mathrm{d}^2\tilde{\theta} \mathcal{F}_{UV}(\Phi) \ , \quad \mathcal{F}_{UV}(\Phi) = \frac{1}{2} \mathrm{Tr}\, \tau \Phi^2 \ . \tag{1.22}$$

Following the general discussion above, the low-energy theory is given by a $\mathcal{N} = 2$ $U(1)$ SYM theory.

We start from a semiclassical analysis. Up to gauge transformations the classical vacua are

$$\langle \phi \rangle = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \ . \tag{1.23}$$

There is a residual gauge invariance corresponding to the Weyl reflection $a \to -a$ therefore to label the vacua we can use the gauge-invariant Casimir

$$u(a) = \frac{1}{2} \left\langle \mathrm{Tr}\, \phi^2 \right\rangle = a^2 \ , \tag{1.24}$$

which we will see that this is the correct order parameter also in the full quantum regime. At $a = 0$ there is some singularity where the full non-abelian gauge symmetry $SU(2)$ is restored.

Due to quantum corrections, the coupling $\tau(\mu)$ runs with the scale μ until the Higgs scale a is reached. The perturbative β function and the corresponding effective coupling at scale a is

$$\beta_g = -\frac{g^3}{4\pi^2} \quad \Rightarrow \quad \tau(a) = \frac{i}{\pi}\log\left(\frac{a^2}{\Lambda^2}\right) \ . \tag{1.25}$$

where Λ is the dynamically generated strong coupling scale $1/g^2(\Lambda) = 0$. After this scale the gauge group is broken to $U(1)$ and the gauge coupling is freezed to the value $\tau(a)$. From If $a >> \Lambda$ the coupling essentially does not run and we are in the semiclassical regime where we can trust perturbation theory and the previous semiclassical analysis. If $a \sim \Lambda$ instead the theory becomes strongly coupled and the quantum corrections can, and will, modify completely the classical moduli space.

Generically, we will have that at the full quantum level the moduli space has some singularities at infinity and at some points $u_1, \ldots, u_k$, where some extra light d.o.f. appear. The exact prepotential contains also non-perturbative quantum corrections, which are given by instanton contributions encoded in some constants $\mathcal{F}_0^{(k)}$, and has the form[7]

$$\mathcal{F}_0(a) = \frac{i}{2\pi}a^2\log\left(\frac{a^2}{\Lambda^2}\right) + a^2\sum_{k=1}^{+\infty}\mathcal{F}_0^{(k)}\left(\frac{\Lambda}{a}\right)^{4k} \ , \tag{1.26}$$

where the powers of the strong coupling scale Λ are fixed by the anomaly of the $U(1)$ R-charge [50].

The key observation is now that, thanks to the holomorphicity, $\mathcal{F}_0$ is completely fixed by its behaviour around the singularities of the moduli space. Fixing some point of the moduli space u_0 we consider some loop $\delta u \to \delta u\, e^{2\pi i}$ at this point which encircles some singularity u_j producing some monodromy matrix M_j. Then the monodromies must form a representation of the fundamental group of the moduli space therefore we have the condition

$$M_\infty = M_1 \cdots M_k \ . \tag{1.27}$$

The monodromy at infinity is obtained doing a loop $u \to e^{2\pi i}u$ for $a \to \infty$ where the semiclassical approximation $a \simeq \sqrt{u}$ holds

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} -a_D + 2a \\ -a \end{pmatrix} \quad \Rightarrow \quad M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \ . \tag{1.28}$$

We have now to find the singularities at strong coupling. Classically we have a singularity at $a = 0$ where the full $SU(2)$ gauge symmetry is restored and extra massless gauge bosons

[7] The prepotential is defined up to a quadratic part which depend on the choice of the scheme and can be reabsorbed in a redefinition of Λ.

appear. However, at the full quantum level this is excluded[8] and the only possibility is that some collective excitations, such as monopole or dyons, become massless in these points [1]. The Seiberg-Witten assumption, which has been verified a posteriori in several ways, was that there are exactly two singularities $\pm u_0$ which must be related by the symmetry $u \to -u$ of the anomalous $U(1)_{\mathcal{R}}$ symmetry $\mathbb{Z}_8/\mathbb{Z}_4 \simeq \mathbb{Z}_2$. A BPS state (n_m, n_e) which becomes massless in the singularity u_* must be invariant under the monodromy M_* around it, i.e. it is a left eigenvector of M_*, which implies

$$M_* = \begin{pmatrix} 1 + 2n_e n_m & 2n_e^2 \\ 2n_m^2 & 12n_e n_m \end{pmatrix} . \tag{1.29}$$

Using this fact the monodromies in the two singularities $\pm u_0$ can be determined from the equation

$$M_\infty = M_{u_0} M_{-u_0} , \tag{1.30}$$

and the solution is

$$M_{u_0} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} , \quad M_{-u_0} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} . \tag{1.31}$$

The corresponding BPS states which becomes massless at $\pm u_0$ are a monopole[9] $(1, 0)$ and a dyon $(1, -1)$.

At this point we found that $a(u), a_D(u)$ are some functions which get some monodromies when they encircle the singularities.

This has a nice geometrical interpretation. One can prove that the group generated by the monodromy matrices $M_\infty, M_{u_0}, M_{-u_0}$ is the $SL(2, \mathbb{Z})$ subgroup $\Gamma[2]$ of the matrices congruent to the identity modulo 2. The u-plane $\mathcal{M} = \mathbb{C}\backslash\{u_0, -u_0, \infty\}$ is the three-punched sphere and can be "uniformized" as the quotient $\mathcal{M} \simeq H/\Gamma[2]$, where $H = \{\mathrm{Im}\,\tau > 0\}$ is the upper half-plane that parametrizes the complex structures of the torus. The domain $\mathcal{M} \simeq H/\Gamma[2]$ is a cover of the standard fundamental domain of the modular group $SL(2, \mathbb{Z})$ and parametrizes the family of elliptic curves Σ_u

$$y^2 = (x - u)(x^2 - \Lambda^4) . \tag{1.32}$$

The curve Σ_u is a double cover of the complex plane with branch points[10] at $\pm\Lambda^2, \infty, u$ and topologically corresponds to a torus. This can be realized by gluing two copies of $\mathbb{C}$ with branch cuts from $-\Lambda^2$ to Λ^2 and from u to ∞. We can choose then a canonical basis

[8] The argument goes as follows. The first observation is that the monodromy representation cannot be abelian otherwise it will commute with the sign flip $a \to -a$ coming from M_∞ and a^2 will be a good global coordinate. As previously said this is not possible because otherwise the metric $\mathrm{Im}\,\tau(a)$ will be globally defined. The representation must be then non-abelian which means that we must have at least two extra singularities in the u-plane. Assuming only two extra singularities $\pm u_0$ this means that $u_0 \neq 0$ so we cannot have massless gauge bosons otherwise this will be in contrast with superconformal invariance in the IR limit.

[9] Conversely, assuming that a monopole becomes massless in the singularity one can study the dual magnetic weakly coupled theory around the singularity and determine the monodromy which was the original SW approach.

[10] The precise value of $u_0 = \Lambda^2$ is obtained matching with the perturbative result for $u \to \infty$.

of homology cycles A, B which has intersection number $A \cdot B = 1, A \cdot A = B \cdot B = 0$ and can be taken, e.g., with A going around the cut from $-\Lambda^2$ to Λ^2 and B going across the two cuts passing from one sheet to the other. The effective coupling $\tau(u) = \tau(a(u))$ can then be identified with the complex structure of the torus Σ_u and it is given by the ratio of two period integrals

$$\tau(u) = \frac{\partial a_D}{\partial a} = \frac{\omega_2(u)}{\omega_1(u)} \quad \Rightarrow \quad \omega_1(u) = \frac{\partial a}{\partial u}(u) \ , \quad \omega_2(u) = \frac{\partial a_D}{\partial u}(u) \ , \tag{1.33}$$

where ω_1, ω_2 are the period integrals of the unique, up to normalization, holomorphic differential $\Omega(u)$ of the elliptic curve Σ_u

$$\omega_1(u) = \int_A \Omega(u) \ , \quad \omega_2(u) = \int_B \Omega(u) \ , \quad \Omega(u) = \frac{\mathrm{d}x}{y} \ . \tag{1.34}$$

As a consequence of the Riemann bilinear relations, this guarantees the positivity condition $\mathrm{Im}\,\tau(u) > 0$ required by unitarity

$$\int_A \bar{\Omega}(u) \int_B \Omega(u) = \omega_1^* \omega_2 = |\omega_1(u)|^2 \,\mathrm{Im}\,\tau(u) > 0 \quad \Rightarrow \quad \mathrm{Im}\,\tau(u) > 0 \ . \tag{1.35}$$

The quantities a, a_D correspond then to the periods of a meromorphic differential λ_{SW} which generates Ω and is defined up to the addition of an exact form (which does not change the periods)

$$\Omega = \frac{\partial \lambda_{SW}}{\partial u} \quad \Rightarrow \quad \lambda_{SW} = \frac{2y\,\mathrm{d}x}{x^2 - \Lambda^4} = \frac{2\sqrt{x-u}}{\sqrt{x^2-\Lambda^4}}\mathrm{d}x \ , \tag{1.36}$$

and we can compute the periods explicitly in terms of Hypergeometric functions

$$a(u) = \int_A \lambda_{SW}(u) = \int_{-\Lambda^2}^{\Lambda^2} \frac{2\sqrt{x-u}}{\sqrt{x^2-\Lambda^4}}\mathrm{d}x = \sqrt{2(u+\Lambda^2)} \cdot {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; \frac{2\Lambda^2}{u+\Lambda^2}\right) \ , \tag{1.37}$$

$$a_D(u) = \int_B \lambda_{SW}(u) = \int_{\Lambda^2}^{u} \frac{2\sqrt{x-u}}{\sqrt{x^2-\Lambda^4}}\mathrm{d}x = \frac{i}{\pi}\sqrt{2(u-\Lambda^2)} \cdot {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; \frac{2\Lambda^2}{u-\Lambda^2}\right) \ . \tag{1.38}$$

We observe that the cycles A, B, and the corresponding periods a, a_D, are defined up to $SL(2,\mathbb{Z})$ transformations which correspond to the electromagnetic duality of the low-energy theory.

The singularities of the curve correspond to points where some cycle of the torus shrinks which happens if and only if some of the branch points $\pm\Lambda^2, u, \infty$ collide. For $u \to \infty$ we are in the semiclassical regime and the A cycle shrinks[11] giving the singularity around ∞. In this point we have[12]

$$a \simeq \sqrt{u} \ , \quad a_D = \frac{i}{\pi}\sqrt{u}\log\left(\frac{u}{\Lambda^2}\right) + \frac{i}{\pi}\sqrt{u} \ . \tag{1.39}$$

[11] Notice that the A-cycle around $(-\Lambda^2, \Lambda^2)$ is homologically equivalent to the one around (u, ∞).

[12] The periods do not vanish here because the differential λ_{SW} has a pole at ∞ which give a finite contribution coming from the residue.

and we recover the perturbative result. At $u = \Lambda^2$ we have that the B cycle shrinks and the periods are

$$a = \Lambda^2 \ , \quad a_D = 0 \ . \tag{1.40}$$

therefore we have a massless monopole $(1, 0)$ which appears in this point, as expected. Finally, at $u = -\Lambda^2$ the cycle $B - A$ shrinks and we have

$$a = \Lambda^2 \ , \quad a_D = \Lambda^2 \ \Rightarrow a_D - a = 0 \ , \tag{1.41}$$

which corresponds to a dyon $(1, -1)$ becoming massless. This are precisely the monopole and dyon points $u = \pm\Lambda^2$ where we have $a_D = 0$ or $a_D - a = 0$ respectively. Finally, we can invert the expression of a in terms of u and compute $a_D(a)$ to extract the prepotential $\mathcal{F}(a)$ and in this way we are able to fix all the instanton corrections. We can also invert in terms of a_D to get the magnetic description valid around the singularities $\pm\Lambda^2$.

The previous result can be generalized in several directions. For a gauge group G the low-energy theory is a $U(1)^r$ theory, r being the rank of G, and we have r pairs of periods $a_i, a_{D,i}$. The SW curve is again some algebraic curve of the form

$$y^2 = P(x, \vec{u}, \Lambda) \ . \tag{1.42}$$

which in this case has genus $g = r$ and has r moduli $\vec{u} = u_1, \ldots, u_r \in \mathbb{C}^r/W_G$ where W_G is the Weyl group of G. This defines a family of Riemann surfaces $\Sigma_{\vec{u}}$ obtained as some cover where we glue different copies of $\mathbb{C}$ along branch cuts. We have a singularity when some of the branch points collide which define some singular loci in the moduli space $\vec{u} \in \mathcal{M}$.

Another possible generalization is to add some Hypermultiplets to the theory. This is particularly interesting because in this way one can realize superconformal theories which admits some S-duality [11]. In presence of matter the central charge is

$$Z = a \cdot n_e + a_D \cdot n_e + \frac{1}{\sqrt{2}} \sum_{i=1}^{N_f} m_i q_i \tag{1.43}$$

where m_i are the bare masses of the Hypermultiplets and q_i their charges. In this case the SW differential has some extra poles whose residues give the mass of the matter fields.

The most systematic way to understand the SW construction is to study the integrable system associated to the SW curve which we will review later. The theories with matter or higher-rank group present new phenomena because they have a more general structure of the Coulomb branch singularities. In the following we will review some important class of these singularities which are the so called Argyres-Douglas theories.

1.4 Argyres-Douglas theories

The monopole or dyon singularity which arises pinching a cycle of the SW curve is the most simple singularity that can appear. As we saw it admits a dual weakly coupled description in the appropriate duality frame. However, this is not the general situation. Indeed, we can have more complicated type of singularities, which do not admit any duality frame where the theory is weakly coupled. A large class of examples of this kind are the Argyres-Douglas (AD) theories where mutually non-local degrees of freedom, such as monopoles and W-bosons, become massless simultaneously. This implies that the theory does not admit a local lagrangian description and it is a strongly coupled interacting Superconformal Field Theory (SCFT)[13]. In the following we will review the simplest example of AD theory the $\mathrm{H}_0 = (A_1, A_2)$ theory which was the one originally found by Argyres and Douglas and appears as a singularity of the moduli space of $SU(3)$ SYM [51] or $SU(2)$ $N_f = 1$ [52]. We will focus on this second, rank 1, realization because is more useful for our analysis. The Seiberg-Witten curve of the $N_f = 1$ theory is

$$y^2 = P(x) = x^2(x-u) + 2m\Lambda_1^3 x - \Lambda_1^6 \ , \tag{1.44}$$

where m is the mass of the hypermultiplet and Λ_1 is the strong coupling scale of the theory. We observe that we can recover the SW curve of the pure gauge theory through holomorphic decoupling of the massive dof, sending $m \to \infty$ and keeping fixed $\Lambda^4 = m\Lambda_1^3$

We want now to study the structure of the singularities of the curve to identify the strongly coupled points. The case in which only one cycle shrinks corresponds to the case in which only two singularities collide $x_1 = x_2 = x_*$. This means that at the singularity the curve has the polynomial has the form

$$P(x) = (x - x_*)^2 (x - k) \ , \tag{1.45}$$

and the can always pass to a dual description in terms of the light d.o.f. where the theory is weakly coupled. A more singular case is when all the three finite[14] roots of $P(x)$ coincide and the polynomial has the form of a cubic cusp

$$P(x) = (x - x_*)^3 \ . \tag{1.46}$$

Geometrically, this means that in this point two different intersecting cycles of the torus shrink at the same time and the corresponding light BPS states, having non-vanishing Dirac pairing, are mutually non-local. To find the point (u_*, m_*) in the moduli space for which we get a cusp singularity, we equate the coefficients of (1.45) with the ones of the curve (1.44) and we obtain the H_0 SCFT AD point

$$x_* = \omega\Lambda_1^2 \ , \quad u_* = 3\omega\Lambda_1^2 \ , \quad m_* = \frac{3}{2}\omega^2\Lambda_1 \ , \quad \omega^3 = 1 \ . \tag{1.47}$$

We can study the scaling dimensions of the parameters of this theory isolating the physics of the light fields through holomorphic decoupling of the massive dof. Physically, this means that we "zoom in" around the SCFT point and capture only the IR physics of its relevant deformations. For rank 1 theories this is given by the Weierstrass parametrization[15]

$$y^2 = x^3 - g_2(u, \Lambda, m_i)x - g_3(u, \Lambda, m_i) \ , \tag{1.48}$$

[13] From the discussion of the previous subsection we cannot have a point of gauge symmetry enhancement in the IR and indeed what happens in this points is that we have an interacting Coulomb phase which necessarily requires the presence of mutually non-local dof.

[14] Except the point at ∞.

[15] The Weierstrass parametrization is valid for any elliptic curve so it applies to arbitrary rank 1 theories. What happens, however, is that in the SCFT case we have $g_2 = g_3 = 0$ at the SCFT point.

where $g_2 = g_3 = 0$ at the SCFT point. For the above theory this is obtained from the following scaling[16]

$$u \to u_* + 2\Lambda c\varepsilon^2 + \varepsilon^3 u \ , \quad m \to m_* + \varepsilon^2 c \ , \quad x \to x_* + \varepsilon x \ , \quad y \to \varepsilon^{3/2} y \ . \tag{1.49}$$

In the limit $\varepsilon \to 0$ we obtain the following curve

$$y^2 = x^3 - cx - u \tag{1.50}$$

which corresponds to the SW curve of the H_0 theory. The physical dimensions of the couplings can be fixed observing that in order for the periods a, a_D to have dimension of a mass the SW differential λ_{SW} should also have the same dimension and we have

$$\lambda_{SW} \sim u \frac{\mathrm{d}x}{y}. \tag{1.51}$$

Together with the curve (1.50) this fixes the scaling dimensions of the variables to be

$$[x] = \frac{2}{5} \ , \quad [y] = \frac{3}{5} \ , \quad [u] = \frac{6}{5} \ , \quad [c] = \frac{4}{5} \ . \tag{1.52}$$

We observe that the scaling dimensions are fractional and very different from the ones of the free theory which signals the strongly coupled nature of the AD theory.

One can proceed in a similar way in the case of $N_f = 2$ and $N_f = 3$ theory and find the corresponding AD points which are denoted as $\mathrm{H}_1 = (A_1, A_3)$ and $\mathrm{H}_2 = (A_1, D_4)$ respectively.

The lack of lagrangian description make the AD theories difficult to handle with the standard methods e.g. coming from supersymmetric localization but nevertheless they can be still studied by relying on their integrability properties and their superconformal symmetry as we will see in the next sections.

2 Topological twist and Nekrasov partition function

In this section we will review the exact computation of the partition function of the $\mathcal{N} = 2$ susy gauge theory on $\mathbb{R}^4$ from the supersymmetric localization of its path integral. As usual in QFT, this quantity is ill-defined at infinite volume therefore one needs an infrared regulator to make the computation. There is a very natural IR regulator one can introduce, the so called Ω-background, which preserves all the supersymmetries of the theory and allows to compute all the non-perturbative instanton corrections explicitly.

Furthermore, the resulting partition function has an interesting combinatorial structure which is deeply related to the integrability of the protected BPS sector of the theory and as we will see in the next section defines some new special functions which obey differential

[16] It is instructive to see what happens for a generic value of the scaling. In this case we obtain just a trivial (genus 0) curve and the corresponding theory is just a free theory. This happens because we make infinitely massive all the W-bosons and hypermultiplets and we end up with a purely abelian free phase. The AD theory corresponds to a special tuning of the coupling where the mutually non-local dof remain light in the decoupling limit.

equations related to isomonodromic problems and conformal blocks of 2d CFT. This point of view will be very useful to actually define the partition function in the strongly coupled regime where the supersymmetric localization is not available anymore but we still can define and solve the differential equation.

2.1 Localization

The partition function of a (euclidean) gauge theory is defined schematically by the path integral

$$Z = \int_{b.c.} \mathcal{D}\Phi\, e^{-S[\Phi]} \ , \tag{2.1}$$

where Φ is the field content of the theory and $S[\Phi]$ is the action and we have to assign some boundary conditions at the boundary ∂M of the spacetime M, e.g. at spacetime infinity, to the fields Φ which select the vacuum of the QFT.

In supersymmetric gauge theories sometimes it is possible to compute exactly the partition function Z, even with certain operator insertions, through the powerful technique of localization. The basic idea of localization is that the saddle-point approximation of the supersymmetric path integral becomes *exact* and then the path integral reduces to a finite dimensional integral. Physically, this means that we are computing some BPS protected quantity, i.e. preserved by some supersymmetry, therefore its quantum corrections are suppressed.

More precisely, the BPS sector of a susy QFT has the structure of a cohomological QFT. Let Q be a fermionic[17] symmmetry of the action S

$$QS = 0 \ , \quad Q^2 = \delta_B \ , \tag{2.2}$$

where δ_B is a bosonic symmetry of the theory. From this, it follows immediately that the expectation value of any Q-exact observable vanishes

$$\langle Q\mathcal{O}[\Phi]\rangle = \int \mathcal{D}\Phi\, e^{-S[\Phi]} Q\mathcal{O}[\Phi] = 0 \ , \tag{2.3}$$

where in the last step we integrated by parts[18] and we assumed that the measure is Q-invariant, i.e. Q is non-anomalous. It is then natural to consider the subset of observables defined by the Q-cohomology

$$H_Q^\bullet = \frac{\ker Q}{\mathrm{im} Q} \ , \tag{2.4}$$

i.e. observables invariant under the bosonic symmetry $\delta_B \mathcal{O} = 0$ that are Q-closed $Q\mathcal{O} = 0$ up to Q-exact terms $\mathcal{O} \sim \mathcal{O} + Q\chi$. From (2.3) we have that the expectation value of a Q-closed observable depends only on its Q-cohomology class

$$\langle \mathcal{O} + Q\chi \rangle = \langle \mathcal{O} \rangle \ . \tag{2.5}$$

[17] Otherwise Q^2 does not satisfy the Leibniz rule and cannot be a symmetry.

[18] In principle there can be some boundary terms, here we ignore this possibility.

Furthermore, if we deform the action with a Q-exact term $tQV[\Phi]$ the expectation value of Q-closed observables is independent on the deformation[19]

$$\partial_t \langle \mathcal{O} \rangle_t = \int \mathcal{D}\Phi \, e^{-S[\Phi]-tQV[\Phi]} \, O[\Phi] QV = 0 \; , \tag{2.6}$$

where again we integrated by parts and we used $Q\mathcal{O} = 0$. Because $\langle \mathcal{O} \rangle_t$ is t independent, we have $\langle \mathcal{O} \rangle = \langle \mathcal{O} \rangle_t$ and we can take the limit $t \to +\infty$ of the rhs. In this limit all the quantum fluctuations are suppressed and the saddle-point approximation becomes exact

$$\langle \mathcal{O} \rangle = \int_{BPS} \mathcal{D}\Phi \, e^{-S[\Phi_{cl}]} \frac{1}{\mathrm{SDet}' \frac{\delta^2 QV}{\delta \Phi^2} \big|_{\Phi=\Phi_{cl}}} \; , \tag{2.7}$$

which is the localization formula. The BPS configurations are the critical points of $QV[\Phi]$ and we have to integrate over their zero modes, and SDet$'$ is the super-determinant, or 1-loop determinant, which is the ratio between bosonic and fermionic determinants of the fluctuations, with the zero modes removed.

2.2 Witten's topological twist of SYM

We want now to apply the above localization procedure to the $\mathcal{N} = 2$ gauge theory, for simplicity we will focus on the $SU(2)$ SYM theory and, for later purpose, it will be convenient to do this for an arbitrary manifold M which can be curved. A systematic procedure to construct the charge Q for any four-manifold (M, g), where g is the metric on M, is the so called topological twist which is due to Witten [53].

In general if we put a supersymmetric theory on an arbitrary euclidean curved manifold M of dimension d all its supersymmetries are broken because the supersymmetry parameters ξ must be covariantly constant, that is they have to satisfy the equation $\nabla_\mu \xi = 0$ which in general may have no solution. This problem can be cured if the theory has some $\mathcal{R}$-symmetry. In that case we can modify its spin connection ω_μ coupling the theory with a background gauge field $A_\mu^{\mathcal{R}}$ for the $\mathcal{R}$-symmetry and the covariant derivative becomes

$$\nabla_\mu \to D_\mu = \partial_\mu + \omega_\mu + A_\mu^{\mathcal{R}} \; , \tag{2.8}$$

so if we choose $A_\mu^{\mathcal{R}} = -\omega_\mu$ the spinors ξ become effectively scalars for a new "twisted" rotation group that is obtained as the diagonal spin subbundle of the $\mathcal{R}$-symmetry and spacetime rotations and the equation $\nabla_\mu \xi = 0$ trivializes. In particular, in even dimension the spin bundle always decomposes as a direct sum of a left and right part and we can choose to twist just one of the two. We observe that the restriction to the diagonal subbundle changes the spin of the fields in the theory because the old rotation transformation is compensated by some $\mathcal{R}$-symmetry transformation.

[19] This is just the generalization of the usual BRST quantization of gauge theories. In that case $Q = Q_{BRST}$ and the Q-cohomology identifies gauge-invariant observables. The localization argument becomes then simply the statement that the gauge fixed action $S_{tot} = S + S_{g.f.}$ is independent on the gauge fixing term $S_{g.f.} = Q_{BRST} F$ and the BPS configurations where the path integral localizes correspond to a gauge slice $F = 0$. The difference here is that the Q-cohomology one uses to compute the BPS observables is finite-dimensional and therefore much more tractable.

We consider now the $\mathcal{N} = 2$ theory. The spacetime symmetries of the theory are the spacetime rotations $SO(4) \simeq SU(2)_L \times SU(2)_R$, the $\mathcal{R}$-symmetry $SU(2)_\mathcal{R}$ and the $\mathcal{R}$-charge $U(1)_\mathcal{R}$. Following the previous discussion we introduce a background gauge field $A_\mu^\mathcal{R}$ for the $\mathcal{R}$-symmetry $SU(2)_\mathcal{R}$ which cancel the spin connection ω_μ^R associated to the rotation subgroup $SU(2)_R$ and define some new "twisted" spacetime rotations $SO(4)'$ taking the diagonal subgroup of $SU(2)_R \times SU(2)_\mathcal{R}$

$$SO(4)' = SU(2)_L \times SU(2)'_R \ , \quad SU(2)'_R = \mathrm{diag}(SU(2)_R \times SU(2)_\mathcal{R}) \ . \tag{2.9}$$

With respect to the old spacetime symmetries $SU(2)_L \times SU(2)_R \times SU(2)_\mathcal{R} \times U(1)_\mathcal{R}$ we have the following supercharges and fields

$$Q_\alpha^I \sim (2,1,2)_{1/2} \ , \quad \bar{Q}_{\dot\alpha}^J \sim (1,2,2)_{-1/2} \ , \tag{2.10}$$
$$\phi \sim (1,1,1)_2 \ , \qquad \bar\phi \sim (1,1,1)_{-2} \ , \qquad A_\mu \sim (2,2,1)_0 \ , \tag{2.11}$$
$$\psi_\alpha^I \sim (2,1,2)_1 \ , \qquad \bar\psi_{\dot\alpha}^I \sim (1,2,2)_{-1} \ , \tag{2.12}$$

In the twisted theory the spacetime symmetries become $SU(2)_L \times SU(2)'_R \times U(1)_\mathcal{R}$ and the new supercharges and fields are

$$Q_\alpha^I \quad \to \quad G_\mu \sim (2,2)_{1/2} \ , \tag{2.13}$$
$$\bar{Q}_{\dot\alpha}^J \quad \to \quad Q \sim (1,1)_{-1/2} \ , \quad Q_{\mu\nu}^+ \sim (0,2)_{-1/2} \tag{2.14}$$
$$\phi \sim (1,1)_2 \ , \quad \bar\phi \sim (1,1)_{-2} \ , \quad A_\mu \sim (2,2)_0 \ , \tag{2.15}$$
$$\psi_\alpha^I \quad \to \quad \psi_\mu \sim (2,1,2)_1 \ , \tag{2.16}$$
$$\bar\psi_{\dot\alpha}^I \quad \to \quad \eta \sim (1,1)_{-1} \ , \quad \chi_{\mu\nu}^+ \sim (0,2)_{-1} \ , \tag{2.17}$$

and the supersymmetry algebra gives the relation

$$\{Q, G_\mu\} = P_\mu \ , \tag{2.18}$$

We observe that in this new basis the fields can be written as differential forms $\omega_p = \omega_{\mu_1,\ldots,\mu_p} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}$ with values in the Lie algebra of the gauge group $SU(2)$. The supersymmetric transformations of the fields with respect to Q are then

$$\begin{aligned} &QA = \psi \ , \quad Q\psi = D\phi \ , \quad Q\phi = 0 \ , \\ &Q\chi^+ = H^+ \ , \quad QH^+ = [\phi, \chi^+] \ , \end{aligned} \tag{2.19}$$

where H^+ is some Lagrange multiplier needed to close the supersymmetry algebra off-shell, and it turns out that the stress-energy tensor is Q-exact

$$T_{\mu\nu} = \{Q, G_{\mu\nu}\} \ , \tag{2.20}$$

therefore the expectation values of observables in the Q-cohomology are independent of the metric

$$\begin{aligned} &\frac{1}{\sqrt{g(x)}} \frac{\delta}{\delta g_{\mu\nu}(x)} \langle O_1(x_1) \ldots O_n(x_n) \rangle_g = \\ &= \langle T_{\mu\nu}(x) O_1(x_1) \ldots O_n(x_n) \rangle_g = \langle Q G_{\mu\nu}(x) O_1(x_1) \ldots O_n(x_n) \rangle_g = 0 \ . \end{aligned} \tag{2.21}$$

This means that the BPS sector defined by Q is actually a topological QFT (TQFT) of the so called Witten, or cohomological, type and the expectation values of Q-observables correspond to topological invariants of the four-manifold M. Indeed, this was the original motivation of Witten to introduce the topological twisting procedure.

It interesting to observe that for a Q-closed local observable $O(x)$ we have

$$\partial_\mu O(x) = P_\mu O(x) = \{Q, G_\mu\} O(x) = Q O_{1,\mu}(x) \ , \quad O_{1,\mu}(x) \equiv G_\mu O(x) \ . \tag{2.22}$$

This procedure can be iterated to define some p-form observable $O_p(x) = O_{p,\mu_1,\dots,\mu_p}(x) \mathrm{d}x^{\mu_1} \wedge \dots \wedge \mathrm{d}x^{\mu_p}$ through the so called descent equation

$$dO_p = Q O_{p+1} \ , \tag{2.23}$$

and then we can automatically define some Q-closed observable pairing O_p with a homology cycle $\gamma_p \in H_p(M)$

$$Q \int_{\gamma_p} O_p = \int_{\gamma_p} dO_{p-1} = \int_{\partial\gamma_p} O_{p-1} = 0 \ . \tag{2.24}$$

Applying this procedure to the gauge invariant local observable

$$I_0(x) = \frac{1}{2} \operatorname{Tr} \phi^2(x) \ , \tag{2.25}$$

we can construct the following topological observables

$$I_1(\gamma_1) = \int_{\gamma_1} \operatorname{Tr} \phi\psi \ , \tag{2.26}$$

$$I_2(\gamma_2) = \int_{\gamma_2} \operatorname{Tr} F\phi + \frac{1}{2} \psi \wedge \psi \ , \tag{2.27}$$

$$I_3(\gamma_3) = \int_{\gamma_3} \operatorname{Tr} F\psi \ , \tag{2.28}$$

$$I_4(M) = \frac{1}{2} \int_M \operatorname{Tr} F \wedge F \ , \tag{2.29}$$

where $\gamma_k \in H_k(M)$ and by construction $Q I_k(\gamma_k) = 0$. Therefore the correlation functions of $I_k(\gamma_k)$ correspond to topological invariants.

These correlators can be explicitly computed through localization. The action of the topological twisted theory is Q-exact $S = QV$ with [53]

$$V = \int \mathrm{d}^4x \sqrt{g} \, \frac{1}{4} \operatorname{Tr} F^+_{\mu\nu} \chi^{\mu\nu} + \frac{1}{2} \operatorname{Tr} \psi_\mu D^\mu \phi - \frac{1}{4} \operatorname{Tr} \eta [\phi, \phi^*] \ . \tag{2.30}$$

where $F^+ = F - \tilde{F}$. This means that the deformation parameter we use to localize is the gauge coupling itself therefore the computation is independent on the value of the coupling can be done in the semiclassical limit $g \to 0$. Sending $g \to 0$ the path integral localizes on the BPS configurations which are given by

$$\delta S = 0 \quad \Leftrightarrow \quad \tilde{F} = -F \ , \tag{2.31}$$

that correspond to an instanton configuration. We recall that for a $SU(2)$ gauge theory in $\mathbb{R}^4$ instantons/anti-instantons are the minima of the Euclidean YM action

$$S[A] = \frac{1}{g^2}\int \mathrm{d}x^4\, \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} \ , \tag{2.32}$$

which are given by the solutions of the anti self-dual/self-dual YM equations

$$\tilde{F} = \mp F \ , \tag{2.33}$$

and saturate the BPS bound

$$S[A] \geq \frac{8\pi^2|k|}{g^2} \ , \tag{2.34}$$

where k is the instanton number given by the second Chern-class

$$\frac{1}{8\pi^2}\int \mathrm{d}^4x\, \mathrm{Tr}\, F\wedge F = k \ . \tag{2.35}$$

For fixed k instantons form a moduli space $\mathcal{M}_k$ of dimension $8k$ where the 8 moduli parametrize the spacetime position (4 moduli), the orientation (3 moduli) associated to the global $SU(2)$ gauge rotations, and the size (1 modulus) associated to the classical scaling invariance of the theory. More generally, on a compact manifold (M, g) the instanton moduli space has dimension[20]

$$\dim \mathcal{M}_k = 8k - \frac{3}{2}(\chi(M) + \sigma(M)) \ . \tag{2.36}$$

where $\chi(M)$ is the Euler characteristic of M and $\sigma(M)$ is the signature of the intersection form[21]. In the supersymmetric theory, we also have some instanton solutions, but by supersymmetry we now have a fermionic zero mode for each bosonic modulus of $\mathcal{M}_k$ so the number of fermionic zero modes should be equal to the dimension of $\mathcal{M}_k$.

Once we localize a Q-closed observable $O[A, \psi, \phi]$ on the moduli space it becomes a function $O_m(\alpha, \psi) = O(\alpha)_{j_1,\ldots,j_m}\psi^{j_1}\cdots\psi^{j_m}$ where α^j are the instantons' moduli, $j = 1, \ldots, \dim\mathcal{M}_k$, which parametrize $\mathcal{M}_k$ and ψ^j are the corresponding fermionic zero modes. The expectation value of O can be non-vanishing only if the number of zero modes which enter in the localized observable $O_m(\alpha, \psi)$ is equal to the dimension of the instanton moduli space $m = \dim\mathcal{M}_k$ and in that case $O_m(\alpha, \psi)$ defines a top form $O_m = O_m(\alpha)_{j_1,\ldots,j_m}\mathrm{d}\alpha^{j_1}\cdots\mathrm{d}\alpha^{j_m}$ on $\mathcal{M}_k$ and we have

$$\langle O\rangle = \int_{\mathcal{M}_k} O_m \ . \tag{2.37}$$

[20]More precisely, this is the dimension of the unframed moduli space, where we do not have gauge transformations at infinity. For the moduli spaces of framed instantons the formula is different because one has to include the framing parameters. For example, as we will see in the next analysis, for ADHM moduli space of $SU(2)$ framed instantons on S^4 is $\dim\mathcal{M}_k = 8k$.

[21]This is defined as follows. The intersection form is a non-degenerate bilinear form on $H_2(M)$, from Poincaré duality, and we can define its positive and negative eigenspaces, which correspond to self-dual anti self-dual forms respectively, which are of dimension b_2^+ and b_2^- respectively. The signature is then $\sigma(M) = b_2^+ - b_2^-$. We also have $\chi(M) = 2 - 2b_1 + b_2$ with $b_2 = b_2^+ + b_2^- = \dim H_2(M)$.

More generally, if we have the product of observables $O_1[A,\psi,\phi]\cdots O_n[A,\psi,\phi]$ each of them localizes to some m_j-form $O_{m_j}(\alpha,\psi)$ on $\mathcal{M}_k$ and the product is mapped to the wedge product[22] $O_m(\alpha,\psi)\wedge\cdots\wedge O_m(\alpha,\psi)$. Their correlation function is non-trivial if $\sum_j m_j = \dim\mathcal{M}_k$ and is given by the integral on the instanton moduli space

$$\langle O_1\cdots O_n\rangle_M = \int_{\mathcal{M}_k} O_{m_1}\wedge\cdots\wedge O_{m_n}\ . \tag{2.38}$$

If we apply this procedure to the observables $I_k(\gamma_k)$ the it turns out that $\psi_\mu(x)$ is mapped to the fermion zero modes ψ^j, the scalar field to a 2-form $\phi(x)\to\phi_2(a)\sim f_{ij}\psi^i\psi^j$ and the gauge field $A_\mu(x)$ to the corresponding instanton configurations $A_\mu^{inst}(x;\alpha)$. This implies that the observables $I_p(\gamma_p)$ defined by homology cycles $\gamma_p\in H_p(M)$ localize to $4-p$-differential forms of $\mathcal{M}_k$ of degree $4-k$ which gives the famous Donaldson μ-map

$$\mu : H_p(M)\to H^{4-p}(\mathcal{M}_k)\ , \tag{2.39}$$

which maps homology cycles of the four-manifold M in the moduli space of instantons $\mathcal{M}_k$. The correlators of the observables $I_p(\gamma_p)$ are then exactly the Donaldson invariants of the four-manifold and geometrically they correspond to intersection numbers in the instanton moduli space $\mathcal{M}_k$. In particular, given some homology classes $P=pP_0\in H_0(M)$, $\Sigma=s\Sigma_0\in H_2(M)$ the most interesting invariants are given by polynomials

$$D(P,\Sigma) = \sum_{2l+4m} d_{l,m}p^{4l}s^{2m}\in \mathrm{Symm}(H_0(M)\oplus H_2(M))\ , \tag{2.40}$$

$$d_{l,m} = \left\langle I_0(P_0)^l I_2(\Sigma_0)^m\right\rangle_M = \int_{\mathcal{M}_k}\mu(P_0)^l\wedge\mu(\Sigma_0)^m\ . \tag{2.41}$$

which can be organized in the Donaldson-Witten partition function

$$Z(P,\Sigma) = \left\langle e^{I_0(P)+I_2(\Sigma)}\right\rangle_M\ . \tag{2.42}$$

We observe that if M is non-compact the integrals on the instanton moduli space are ill-defined because there is an IR divergence due to the fact that instantons can go to ∞. A natural way to make sense of this case is to consider an equivariant version of the theory that we review in the following. We will see that the equivariant version of $Z(p,S)$ is given by the Nekrasov partition function and allows to recover the full SW prepotential $\mathcal{F}_0$ from the equivariant localization on the instanton moduli space.

2.3 Equivariant localization

The topological twist of Witten captures the topological informations of the four-manifold M where the susy QFT lives encoded in the Donaldson invariants. It is natural

[22]In principle it is not guaranteed that the product of the observables $O_j[A,\psi,\phi]$ is mapped into the product of the localized ones $O_{m_j}(\alpha,\psi)$ but in this case this is true due to the absence of contact terms produced by the Wick contractions between different fields. We will see that this is not the case when the same expectation values are computed in terms of the abelian effective low-energy theory given by the SW prepotential $\mathcal{F}_0$ where some non-trivial contact term appear.

to ask if we can use a similar technique to extract dynamical informations about the theory, in particular we want to derive the SW prepotential $\mathcal{F}_0(a)$ directly from the path integral of the theory.

A natural way to do this was introduced by Nekrasov in [10]. The idea is just to compute the partition function of the theory on $\mathbb{R}^4$ with respect to a *new* supercharge which $Q_{\epsilon_1,\epsilon_2}$ which is obtained deforming the original topological charge Q.

As we already observed, for a non-compact four-manifold such as $\mathbb{R}^4$ the integrals on the instanton moduli space $\mathcal{M}_k$ are actually ill-defined because there is some IR divergence coming from the fact that instantons can escape to ∞. This can be cured in a natural way passing to *equivariant* cohomology of the four-manifold M associated to the action of some vector field v. We will consider the case of toric actions. Let then v_i be the vector fields that generate some $U(1)^m$ action on some $2n$-dimensional manifold M. We define the equivariant differential

$$\mathrm{d}_\xi = \mathrm{d} + \xi^i \iota_{v_i} \ , \tag{2.43}$$

where ι_{v_i} is the contraction with respect to the vector field v_i. This has the property

$$\mathrm{d}_\xi^2 = (\mathrm{d} + \xi^i \iota_{v_i})^2 = \xi^i(\mathrm{d}\iota_{v_i} + \iota_{v_i}\mathrm{d}) = \xi^i \mathcal{L}_{v_i} \ , \tag{2.44}$$

where $\mathcal{L}_{v_i}$ is the Lie derivative with respect to the vector field v_i. Therefore, d_ξ^2 vanishes on the equivariant (multi)forms

$$\Omega \in H^\bullet(M) \ , \quad \mathcal{L}_{v_i}\Omega = 0 \ , \tag{2.45}$$

and we can define the equivariant cohomology $H_\xi^\bullet(M)$ as equivariantly closed forms $d_\xi\Omega = 0$ modulo equivariantly exact forms $\Omega_\xi = d_\xi\eta$. In particular, if we have a symplectic manifold (M,ω) and the action of the vector fields v_i is generated by a moment map

$$\mu : \mathbb{R}^m = \mathrm{Lie}(\mathbb{T}^m) \to C^\infty(M) \ , \quad \xi^i \iota_{v_i}\omega = -\mathrm{d}\mu(\xi) \ , \tag{2.46}$$

then we can define the equivariant symplectic form

$$\omega_\xi = \omega + \mu(\xi) \ , \quad \mathrm{d}_\xi\omega_\xi = (\mathrm{d} + \xi^i\iota_{v_i})(\omega + \mu(\xi)) = \xi^i\iota_{v_i}\omega + \mathrm{d}\mu(\xi) = 0 \ , \tag{2.47}$$

and the equivariant volume is given by the Duistermaat-Heckmann (DH) formula

$$\mathrm{Vol}(M)_\xi = \int_M \exp(\omega_\xi) = \int_M \frac{\omega^n}{n!}\, e^{\mu(\xi)} = \sum_{P|v_i(P)=0} \frac{e^{\mu[\xi](P)}}{\prod_{i=1}^n w_i[\xi](P)} \tag{2.48}$$

where $vol(\omega) = \omega^n/n!$ is the symplectic volume form and $w_i[\xi]$ are the weights of the representation of $\mathbb{T}^m$ on the tangent space $T_P M$ at the fixed point P. We observe that the localization formula (2.7) is just the infinite-dimensional analogue of the DH formula where the path integral measure plays the role of the symplectic volume form.

We want to apply this procedure to the instanton moduli space $\mathcal{M}_{k,N}$ of the $U(N)$ theory. We choose the toric action given by the torus $\mathbb{T}^2 \times \mathbb{T}^N$. The first factor $\mathbb{T}^2 \subset SO(4)'$ corresponds to the Cartan torus of the spacetime rotations acting on $\mathbb{C}^2$

$$(z_1, z_2) \to (z_1 e^{i\epsilon_1}, z_2 e^{i\epsilon_2}) \ , \quad (\epsilon_1, \epsilon_2) \in \mathrm{Lie}(\mathbb{T}^2) \ . \tag{2.49}$$

which is generated by the vector field $v_{\epsilon_1,\epsilon_2} = \epsilon_1 v_1 + \epsilon_2 v_2$ with

$$v_j = i(z_j\partial_j - \bar{z}_j\bar{\partial}_j) \ , \tag{2.50}$$

where $\partial_j = \partial/\partial z_j$. The second factor correspond to the action of the Cartan torus of the gauge group $a \in \mathbb{T}^N \subset U(N)$, that act on the boundary condition $g_\infty(x)$, which is called a "framing", at ∞ as $g_\infty \to g_\infty t$.

The equivariant cohomology of this action is realized in the susy gauge theory defining the equivariant supercharge

$$Q_{\epsilon_1,\epsilon_2} = Q + v^\mu_{\epsilon_1,\epsilon_2} G_\mu \ , \quad Q^2_{\epsilon_1,\epsilon_2} = \mathcal{L}_{v_{\epsilon_1,\epsilon_2}} + \mathcal{G}_\phi \tag{2.51}$$

where $\mathcal{G}_\phi$ are infinitesimal gauge transformations of parameter ϕ. This gives the following supersymmetry transformations

$$\begin{aligned} &Q_{\epsilon_1,\epsilon_2} A = \psi \ , \quad Q_{\epsilon_1,\epsilon_2}\psi = \iota_{v_{\epsilon_1,\epsilon_2}} F + D\phi \ , \quad Q_{\epsilon_1,\epsilon_2}\phi = 0 \ , \\ &Q_{\epsilon_1,\epsilon_2}\chi^+ = H^+ \ , \quad Q_{\epsilon_1,\epsilon_2} H^+ = \mathcal{L}^A_{v_{\epsilon_1,\epsilon_2}}\chi^+ + [\phi,\chi^+] \ , \end{aligned} \tag{2.52}$$

and we want to study the observables defined by the cohomology of $Q_{\epsilon_1,\epsilon_2}$. Similarly to the Q-cohomology, we can construct equivariant $Q_{\epsilon_1,\epsilon_2}$-closed observables through descent equations, in particular we are interested in the equivariant lift of the observables $I_k(\gamma_k)$. We observe that the observable $I_p(\gamma_p)$ defined on a homology cycle γ_p can be defined equivalently in terms of $4-p$ cohomology classes

$$I(\Omega_{4-p}) = I(\gamma_k) = \int \Omega_{4-p} \wedge O_p \tag{2.53}$$

where Ω_{4-p} is the Poincaré dual of γ_p. The equivariant lift of these observables can be obtained in a compact way introducing the generalized Bianchi identity for the curvature of the universal bundle [54]

$$\mathbf{DF} = (-Q + D + \iota_v)(F + \psi + i\phi) = 0 \ . \tag{2.54}$$

For any ad-invariant polynomial $\mathcal{P}(\mathbf{F})$ of the Lie algebra of the gauge group we have then

$$Q\mathcal{P}(\mathbf{F}) = (d + \iota_v)\mathcal{P}(\mathbf{F}) \ . \tag{2.55}$$

Using this fact we can easily construct $Q_{\epsilon_1,\epsilon_2}$-closed observables as follows. Consider an equivariant cohomology class $\Omega \in H^\bullet_{\epsilon_1,\epsilon_2}(X)$ then from (10.7) the observable

$$O(\Omega, \mathcal{P}) = \int_X \Omega \wedge \mathcal{P}(\mathbf{F}) \ , \tag{2.56}$$

is $Q_{\epsilon_1,\epsilon_2}$-closed.

In particular, a natural equivariant cohomology class is the equivariant symplectic form $\omega_{\epsilon_1,\epsilon_2} = \omega + H$ where H is the moment map and ω is the standard symplectic form of $\mathbb{C}^2$

$$H = \frac{\epsilon_1}{2}|z_1|^2 + \frac{\epsilon_2}{2}|z_2|^2 \ , \quad \omega = \frac{i}{2}\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \frac{i}{2}\mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_2 \ , \quad (d + \iota_v)(\omega + H) = 0 \ . \tag{2.57}$$

which defines the equivariant lift of the surface observable $I_2(\gamma_2)$

$$O^{(4d)}(\omega_{\epsilon_1,\epsilon_2}, \mathbf{F}^2) = -\frac{1}{4\pi^2}\int_M (\omega+H)\wedge \mathrm{Tr}\,\mathbf{F}^2 = -\frac{1}{4\pi^2}\int_M \omega\wedge 2\,\mathrm{Tr}\left(\phi F + \frac{1}{2}\psi\wedge\psi\right) + H\,\mathrm{Tr}\,F\wedge F\ . \tag{2.58}$$

We now consider the following expectation value

$$Z(a,\Lambda,\epsilon_1,\epsilon_2) \equiv \langle 1\rangle_{\mathbb{C}^2} = \left\langle \exp O^{(4d)}(\omega_{\epsilon_1,\epsilon_2}, \mathbf{F}^2)\right\rangle_{\mathbb{C}^2}\ . \tag{2.59}$$

One can show [10] that the above observable is in the same class of the identity 1 so it corresponds to the partition function of the gauge theory and is called the *Nekrasov partition function*.

The key property of this observable is that one can see that in the limit ϵ_1, ϵ_2 the leading contribution to the free-energy is given precisely by the Seiberg-Witten prepotential

$$\log Z(a,\Lambda,\epsilon_1,\epsilon_2) = \frac{1}{\epsilon_1\epsilon_2}\mathcal{F}_0(a,\Lambda) + O(1) \quad \Rightarrow \quad \mathcal{F}_0(a,\Lambda) = \lim_{\epsilon_1,\epsilon_2\to 0}\epsilon_1\epsilon_2 \log Z(a,\Lambda,\epsilon_1,\epsilon_2) \tag{2.60}$$

This was rigorously proved in [41] using the so called blow-up equations that we review in the following.

In the next subsection we will see that one can explicitly compute $Z(a,\Lambda,\epsilon_1,\epsilon_2)$ from localization and in this way we recover the Seiberg-Witten result from first principles.

2.4 Nekrasov partition function

At this point one can apply the localization procedure. We can do this in two steps. In the first step we just localize on the moduli space of instantons and the partition functions reduces to a sum over instanton sectors weighted by the equivariant volumes of the moduli spaces $\mathcal{M}_{k,N}$

$$Z(a,\Lambda,\epsilon_1,\epsilon_2) = \sum_{k=0}^{+\infty}\Lambda^k \int_{\mathcal{M}_{k,N}} e^{\omega+\mu(a,\epsilon_1,\epsilon_2)}\ . \tag{2.61}$$

where $\mu(a,\epsilon_1,\epsilon_2)$ is the moment map associated to the toric action. The computation can then be done applying the DH localization formula (2.48) on $\mathcal{M}_{k,N}$. We review now this computation following [8, 10, 55].

An efficient way to describe the instanton moduli space $\mathcal{M}_{k,N}$ is in terms of the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction. Let V and W be complex vector spaces of dimension k and N respectively and define the space of ADHM data

$$\mathbb{X} = \mathrm{Hom}(V,V)\oplus\mathrm{Hom}(V,V)\oplus\mathrm{Hom}(W,V)\oplus\mathrm{Hom}(V,W)\ni(B_1,B_2,I,J)\ . \tag{2.62}$$

This space has an hyperkahler structure given by the three complex structures

$$I(B_1,B_2,I,J) = (iB_1, iB_2, iI, iJ)\ , \quad J(B_1,B_2,I,J) = (B_2^\dagger, -B_1^\dagger, J^\dagger, -I^\dagger)\ , \quad K = IJ\ . \tag{2.63}$$

Then we can consider the following moment maps

$$\mu_R = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J\ , \tag{2.64}$$

$$\mu_C = [B_1, B_2] + IJ\ . \tag{2.65}$$

and one can prove that the instanton moduli space can be realized as the hyperkahler quotient[23]

$$\mathcal{M}_{k,N} = \mathbb{X}///U(k) = \mu_j^{-1}(0)/U(k) \ , \tag{2.66}$$

which has dimension $\dim \mathcal{M}_{k,N} = 4kN$. The action of the torus $\mathfrak{t} \in (\epsilon_1, \epsilon_2, a_1, \ldots, a_N) \in \mathbb{T}^2 \times \mathbb{T}^N$ on the ADHM data is

$$\mathfrak{t} \cdot B_1 = e^{\epsilon_1} B_1 \qquad \mathfrak{t} \cdot B_2 = e^{\epsilon_2} B_2 \ , \tag{2.67}$$

$$\mathfrak{t} \cdot I = I e^{a}, \qquad \mathfrak{t} \cdot J = e^{-a} e^{\epsilon_1+\epsilon_2} J \ , \tag{2.68}$$

where $a = \mathrm{diag}(a_1, \ldots, a_N)$. The fixed points correspond to study the conditions under which the ADHM data are fixed up to $U(k)$ gauge transformations

$$e^{\epsilon_j} B_j = g(\mathfrak{t}) B_j g(\mathfrak{t})^{-1}, \qquad j = 1, 2, \tag{2.69}$$

$$I e^{a} = g(\mathfrak{t}) I \ , \tag{2.70}$$

$$e^{-a} e^{\epsilon_1+\epsilon_2} J = J g(\mathfrak{t})^{-1} \ , \tag{2.71}$$

for some $g(\mathfrak{t}) \in U(k)$. The solutions to these equations can be classified in terms of N Young diagrams $(Y_1, \ldots, Y_N)$. Given a partition of an integer $k = k_1 + \cdots + k_r$ of non-increasing integers $y_1 \geq y_2 \geq ... \geq y_N > 0$ a Young diagram Y is obtained as a collection of left-justified boxes organized in columns and we choose the convention where starting from the left the i-th column contains y_i boxes. It is useful to define also the transposed diagram Y^T with column of lenght y_j^T given by the lenght of the j-th row of Y. Let $l(Y) = y_1^T$ denote the number of columns of Y and $l(Y^T) = y_1$ the number of rows. For a box $s = (i, j)$ and a Young diagram Y we define the arm $a_Y(s)$ and the length $l_Y(s)$ of s with respect to Y as

$$a_Y(i,j) = y_i - j \ , \quad l_Y(i,j) = y_j^T - i \ , \tag{2.72}$$

and we set $y_i = 0$ if $i > l(Y)$ and $y_j^T = 0$ if $j > l(Y^T)$ and notice that in general s may not lie in Y. The figure 3 illustrates an example of Young diagram.

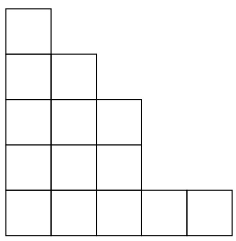

Figure 3. Young diagram for the partition $\{5, 4, 3, 1, 1\}$.

[23]This has a nice physical interpretation in string theory. The $U(N)$ gauge theory is realized as the worldvolume theory of a stack of N $D3$ branes and k instantons correspond to a collection of k $D(-1)$branes which lie on the $D3$ branes. The ADHM data are then simply the configurations of the fields of the 0d $U(k)$ gauge theory living on the stack of k D(-1)branes. The moduli space $\mathcal{M}_{k,N}$ is given by the susy vacua of this theory, the conditions $\mu_R = 0$ and $\mu_C = 0$ are just the D-term and F-term equations for the vacua and we need to quotient by the $U(k)$ gauge transformations of the theory.

Finally, we need to compute the 1-loop determinant associated to each configuration. The 1-loop determinant is obtained computing the weights of the linearized action (2.67) of $\mathbb{T}^2 \times \mathbb{T}^r$ on the tangent space $T_{\vec{Y}}\mathcal{M}_{k,N}$ of a fixed point labelled by Young diagrams $\vec{Y}$. The final result is

$$Z_{\text{vec}}(\vec{a}, \vec{Y}) = \frac{1}{\prod_{\alpha,\beta=1}^{N} \prod_{s \in Y_\alpha} E_{\alpha\beta}(a, \vec{Y}) \times \prod_{s \in Y_\beta} (\epsilon_1 + \epsilon_2 - E_{\beta\alpha}(\vec{a}, \vec{Y}))} , \tag{2.73}$$

and if we can also include the contribution of an Hypermultiplet in the fundamental or adjoint representation

$$Z_{\text{fund}}(\vec{a}, \vec{Y}, m) = \prod_{\alpha=1}^{N} \prod_{s \in Y_\alpha} (\phi(a_i, s) - m + \epsilon_1 + \epsilon_2) , \tag{2.74}$$

$$Z_{\text{adj}}(\vec{a}, \vec{Y}) = \prod_{\alpha,\beta=1}^{N} \prod_{s \in Y_\alpha} (E_{\alpha\beta}(a, \vec{Y}) - m) \times \prod_{s \in Y_\beta} (\epsilon_1 + \epsilon_2 - E_{\beta\alpha}(\vec{a}, \vec{Y}) - m) , \tag{2.75}$$

In the previous formulas $a_Y(s), l_Y(s)$ are the arm and length associated to the box s and

$$E_{\alpha\beta}(a, \vec{Y}) = -l_{Y_\beta}(s)\epsilon_1 + (a_{Y_\alpha}(s) + 1)\epsilon_2 + a_\beta - a_\alpha , \tag{2.76}$$

$$\phi(a, s) = a + \epsilon_1(i - 1) + \epsilon_2(j - 1) . \tag{2.77}$$

Summing over all instantons contributions we get the Nekrasov instanton partition function

$$Z^{inst}(\vec{a}, \Lambda, \vec{m}, \epsilon_1, \epsilon_2) = \sum_{\vec{Y}} \Lambda^{|\vec{Y}|} Z_{\text{vec}}(\vec{a}, \vec{Y}) \prod_{i=1}^{N_f} Z_{\text{fund}}(\vec{a}, \vec{Y}, m_i) \prod_{l=1}^{N_A} Z_{\text{adj}}(\vec{a}, \vec{Y}, m_l) , \tag{2.78}$$

To obtain the full partition function we need to consider also the perturbative contribution coming from the classical and 1-loop parts

$$Z^{\text{vec}}_{\text{pert}}(\vec{a}, \epsilon_1, \epsilon_2) = \prod_{\alpha \neq \beta}^{N} \frac{1}{\Gamma_2(a_\alpha - a_\beta \mid \epsilon_1, \epsilon_2)} , \tag{2.79}$$

$$Z^{\text{fund}}_{\text{pert}}(\vec{a}, m, \epsilon_1, \epsilon_2) = \prod_{\alpha=1}^{N} \Gamma_2(a_\alpha + m \mid \epsilon_1, \epsilon_2) , \tag{2.80}$$

$$Z^{\text{adj}}_{\text{pert}}(\vec{a}, m, \epsilon_1, \epsilon_2) = \prod_{\alpha \neq \beta}^{N} \Gamma_2(a_\alpha - a_\beta + m \mid \epsilon_1, \epsilon_2) , \tag{2.81}$$

$$Z_{\text{cl}} = \exp\left(-\frac{2\pi i \tau_0}{2\epsilon_1\epsilon_2} \sum_{\alpha=1}^{N} a_\alpha^2 \right) . \tag{2.82}$$

where $\Gamma_2(x \mid \epsilon_1, \epsilon_2)$ is the Barnes double gamma function and τ_0 is the bare UV coupling. The full Nekrasov partition function is

$$Z(\vec{a}, \Lambda, \vec{m}, \epsilon_1, \epsilon_2) = Z_{\text{cl}} Z_{1-\text{loop}} Z_{\text{inst}} . \tag{2.83}$$

As a final comment, we observe that the Nekrasov partition function possess a rich combinatorial structure which contains much more informations than the original SW prepotential. Indeed, as we will review in the following sections, this structure leads to several non-trivial correspondences between susy gauge theories, isomonodromic deformations, 2d CFT and topological strings.

2.5 Five-dimensional theories

The Nekrasov construction can be lifted to the five-dimensional setting, considering a 5d $\mathcal{N}=1$ (8 real supercharges) gauge theory compactified on a circle S^1_β of length β. This is actually the natural setup for the Nekrasov partition function.

In five dimensions the gauge coupling $1/g_5^2$ has the dimension of a mass and the theory is non-renormalizable. However, as showed by Seiberg [56], this is too naive and some susy gauge theories admit a UV completion to some 5d non-trivial strongly coupled SCFTs. On $\mathbb{R}^5$ the theory is exact at 1-loop and we don't have instanton contributions. However, once we compactify on a circle, instantons contributions appear and are given by particles extended along the circle, charged under the $U(1)$ current

$$J_I = \star(F \wedge F) \ , \tag{2.84}$$

and of mass

$$m_I = \frac{1}{g_5^2} \ , \tag{2.85}$$

so their contribution to the classical action is $e^{-S_I} = e^{-\beta m_I}$ which is suppressed once we decompactify $\beta \to 0$.

In five dimensions, the Ω-background can be interpreted as a 5d gravitational background in which we "twist" the 4d directions $\mathbb{R}^4$ along the circle S^1_β with a holonomy given by a Cartan element $(\epsilon_1, \epsilon_2) \in \mathbb{T}^2$ (see figure 4), that is a $\mathbb{C}^2$ bundle over S^1_β

$$(z_1, z_2, 0) \sim (z_1 e^{i\beta\epsilon_1}, z_2 e^{i\beta\epsilon_2}, \beta) \ . \tag{2.86}$$

The four-dimensional Ω-background is then the geometry that we obtain compactifying the circle S^1_β sending $\beta \to 0$.

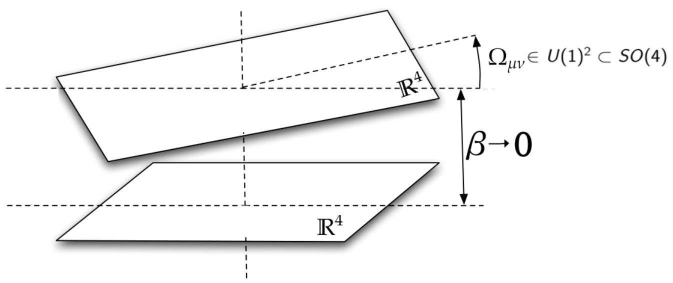

Figure 4. Geometry of the five-dimensional Ω-background and its dimensional reduction.

Once we localize the Nekrasov partition function becomes a sum of supersymmetric indices [57]

$$Z(a, z, \epsilon_1, \epsilon_2, \beta) = \sum_{k=0}^{+\infty} z^k \operatorname{Tr}_{\mathcal{H}_{k,N}} (-1)^F e^{i\beta(\epsilon_1 J_1 + \epsilon_2 J_2 + \sum_{\alpha=1}^{N} a_\alpha C_\alpha + \sum_{j=1}^{N_f} m_j F_j)} . \tag{2.87}$$

where

$$z = e^{-\frac{8\pi^2 \beta}{g_5^2}} \tag{2.88}$$

is the 5d instanton counting scale, $\mathcal{H}_{k,N}$ is the Hilbert space of the supersymmetric quantum mechanics on the instanton moduli space $\mathcal{M}_{k,N}$, J_1, J_2 are the operators associated to the generators of $\mathbb{T}^2$, C_α are the operators associated to the gauge transformations $\mathbb{T}^r$, F_j are the operators associated to flavour symmetries and F is the fermion number.

The indices can be computed explicitly localizing on the ground states of the susy QM, corresponding to the fixed point of the equivariant action. The contribution of a fixed point for vector multiplets and hypermultiplets are [58]

$$Z_{\text{vec}}(\vec{a}, \vec{Y}, \beta) = \prod_{\alpha,\beta=1}^{N} \frac{1}{N_{Y_\alpha, Y_\beta}(u_\alpha/u_\beta; q_1, q_2)} , \tag{2.89}$$

$$Z_{\text{fund}}(\vec{a}, \vec{Y}, \beta) = \prod_{i=1}^{N_f} \prod_{\alpha=1}^{N} N_{Y_\alpha, \varnothing}(Q_i u_\alpha) , \tag{2.90}$$

with

$$N_{Y_1,Y_2}(u, q_1, q_2) = \prod_{s \in Y_1} \left(1 - u q_1^{l_{Y_1}(s)} q_2^{-a_{Y_2}(s)-1}\right) \prod_{s \in Y_2} \left(1 - u q_1^{-l_{Y_2}(s)-1} q_2^{a_{Y_1}(s)}\right) . \tag{2.91}$$

We can also include in the theory a Chern-Simons term with level $k = 0, \ldots, N$ which gives the contribution

$$Z_{CS}(a, \vec{Y}, \beta) = \prod_{\alpha=1}^{N} T_{Y_\alpha}(u_\alpha, q_1, q_2) , \quad T_Y(u, q_1, q_2) = \prod_{(i,j) \in Y} u q_1^{i-1} q_2^{j-1} . \tag{2.92}$$

The instanton partition function is then

$$Z_{\text{inst}}(a, z, \epsilon_1, \epsilon_2, \beta) = \sum_{\vec{Y}} z^{|\vec{Y}|} Z_{\text{vec}}(\vec{a}, \vec{Y}, \beta) Z_{CS}(\vec{a}, \vec{Y}, \beta) Z_{\text{fund}}(\vec{a}, \vec{Y}, \beta) . \tag{2.93}$$

The perturbative contribution is given by

$$Z_{\text{cl}}(a, z, \epsilon_1, \epsilon_2, \beta) = \exp\left(-\log z \sum_{\alpha=1}^{N} \frac{(\log u_\alpha)^2}{2 \log q_1 \log q_2} - k \sum_{\alpha=1}^{N} \frac{(\log u_\alpha)^3}{6 \log q_1 \log q_2}\right) \tag{2.94}$$

$$Z_{1-\text{loop}}(a, z, \epsilon_1, \epsilon_2, \beta) = \frac{\prod_{1 \le \alpha \neq \beta \le N} (u_\alpha/u_\beta; q_1, q_2)_\infty}{\prod_{i=1}^{N_f} \prod_{\alpha=1}^{N} (Q_i u_\alpha; q_1, q_2)_\infty} , \tag{2.95}$$

where we introduced the double q-Pochhmammer symbol

$$(u;q_1,q_2)=\prod_{i,j=0}^{\infty}(1-uq_1^iq_2^j)\ . \tag{2.96}$$

Finally, the full 5d Nekrasov partition function is

$$Z(a,z,\epsilon_1,\epsilon_2,\beta)=Z_{\text{cl}}Z_{1-\text{loop}}Z_{\text{inst}}\ . \tag{2.97}$$

3 Integrability and Painlevé-gauge correspondence

In this section we will discuss the integrability properties of the supersymmetric gauge theory. We will see that the SW curve can be interpreted naturally as the so called spectral curve of some classical integrable system which encodes the low-energy gauge theory dynamics. For a special class of theories, the class S theories, obtained from the compactification of a 6d theory, the appearance of this integrable system and the corresponding SW spectral curve are completely encoded in the geometry of the compactification. We will then see that the Ω-background turns this classical integrable system in some isomonodromic system that, for $SU(2)$ theories, corresponds to the Painlevé equations. This gives the so called Painlevé-gauge correspondence which allows to study the Nekrasov partition function using the dynamics of Painlevé equations which are defined in the full moduli space of the theory, including the strongly coupled regime.

3.1 Seiberg-Witten curve and integrable systems

The appearance of the SW curve in the exact solution for the prepotential $\mathcal{F}_0$ of the $\mathcal{N}=2$ low-energy theory has a natural interpretation in the theory of integrable systems.

The familiar notion of integrable system from hamiltonian mechanics is the Liouville-Arnold integrability. A system with hamiltonian $H(q,p)$ in $\mathbb{R}^{2n}$ is integrable in the Liouville sense if there exist n Poisson-commuting conserved[24] quantities f_j

$$\{H,f_i\}=0\ ,\quad \{f_i,f_j\}=0\ ,\quad \forall i,j\ . \tag{3.1}$$

where $\{\ ,\ \}$ are the Poisson bracket associated to the standard symplectic form

$$\omega=\mathrm{d}x^i\wedge\mathrm{d}p_i\ . \tag{3.2}$$

The common level sets $\mathbb{T}(f)=\{f_j(q,p)=f_j\}$ are n-dimensional invariant manifolds of the phase space which define a foliation of it. For bounded motions, they are diffeomorphic to tori $\mathbb{T}(f)\simeq\mathbb{T}^n$ and on them one can always pass to angle-action coordinates (θ,I), with the I_i's functions of the f_j's, where the flow reduces to a trivial linear motion

$$\theta_i(t)=\theta_i+\omega_i(I)t\ ,\quad \dot{I}_j=0\ , \tag{3.3}$$

[24]This is always possible locally. To have a non-trivial statement, the quantities f_j should be functionally independent (the vector fields ∇f_j are linearly independent) and globally defined.

on the invariant torus $\mathbb{T}(I)$ which can labelled by the n conserved action variables I_j. The action variables I_j are then defined by the period integrals

$$I_j = \int_{\gamma_j} \lambda \ , \tag{3.4}$$

where λ is the symplectic potential

$$\lambda = -p_i \mathrm{d}x^i \ , \quad d\lambda = \omega \ . \tag{3.5}$$

and γ_i is a homology basis of $\mathbb{T}(I)$. Therefore, geometrically the phase space is a fibration of tori $\mathbb{T}(I)$ parametrized by the action variables I.

A more general and powerful definition of integrability comes from the concept of Lax Pair. We say a system is integrable if it can be written in the form of a Lax equation

$$\frac{\mathrm{d}}{\mathrm{d}t} L \equiv \dot{L} = [M, L] \ , \tag{3.6}$$

for some operators $M = M(q,p,t), L = L(q,p,t)$. In the finite dimensional case $(q,p) \in \mathbb{R}^{2n}$ these operators are just matrices of dimension n. In these representation the existence of n conserved quantities is manifest because the spectrum of the Lax matrix $L(t)$ is preserved by the dynamics, i.e. the flow of the Lax equation is isospectral[25], in particular all the traces $\operatorname{Tr} L^k$ are conserved. In general the Lax Pair depends also on some extra spectral parameter z and the characteristic equation defines then the so called spectral curve Σ_f

$$\Sigma_f : \det(L(z) - \lambda) = 0 \ . \tag{3.7}$$

Thanks to isospectrality the whole curve (3.7) is conserved by the time evolution and therefore its coefficients correspond to the conserved quantities f_j which are the moduli of the curve.

In our analysis we are interested in complex integrable systems which means that we work in a complexified phase space $(q,p) \in \mathbb{C}^{2n}$ foliated by complex[26] Liouville tori $\mathbb{T}(f)$ of dimension n. If we consider then an integrable system with a Lax Pair $L(z)$ we have $z \in \mathbb{C}$ and the spectral curve defines a complex algebraic curve Σ_f of genus g. One can show that the complex Liouville torus $\mathbb{T}(f)$ corresponds to the Jacobian variety of Σ_f

$$\mathbb{T}(f) \simeq J(\Sigma_f) = \mathbb{C}^g / L(\Sigma_f) \ , \tag{3.8}$$

where $L(\Sigma_f)$ is the lattice in $\mathbb{C}^g$ generated by the $2g$ period vectors[27] π_j associated to a basis of holomorphic differentials $\omega_1, \ldots, \omega_g \in H^1(\Sigma)$ computed on a homology basis $\gamma_1, \ldots, \gamma_{2g} \in H_1(\Sigma)$

$$L(\Sigma_f) = \left\{ \sum_{j=1}^{2g} n_j \pi_j \middle| \, n_j \in \mathbb{Z} \right\} \ , \quad \pi_j = \left(\int_{\gamma_j} \omega_1 \ , \ldots , \ \int_{\gamma_j} \omega_g \right) \ . \tag{3.9}$$

[25] This can be easily seen observing that the solution of the Lax equation is given by a similarity transformation $L(t) = U(t)L(0)U(t)^{-1}$ with $M(t) = \dot{U}(t)U(t)^{-1}$

[26] That is real tori of dimension $2n$ which are endowed with a complex structure.

[27] In a canonical basis $A_i \cdot A_j = B_i \cdot B_j = 0$, $A_i \cdot B_j = \delta_{ij}$, the period $(\pi_{A_i})_j = \delta_{ij}$, $(\pi_{B_i})_j = \tau_{ij}$ where τ_{ij} is the Riemann period matrix and by the Riemann bilinear relation it follows that the vectors π_j are linearly independent over $\mathbb{R}$.

The relation with SW theory is now clear. The SW curve can be interpreted as the spectral curve of an integrable system and the Coulomb moduli u_j are the conserved hamiltonians of this system. Finally, the periods $a_i, a_{D,i}$ are exactly the action variables given by the periods of the meromorphic differential λ_{SW}.

For a large class of $\mathcal{N} = 2$ theories there is a systematic way to construct this integrable system and the corresponding SW spectral curve. In the following we review this construction.

3.2 Class S construction

A large class of $\mathcal{N} = 2$ theories can be realized with the so called class S construction[28] [11]. We start from a $(2,0)$ SCFT with gauge algebra $\mathfrak{g}$ in six dimensions[29], which is the maximal dimension where a superconformal algebra can exist. The field content of this theory is given by tensor supermultiplets $(B_{\mu\nu}, \Phi^I, \lambda)$ where $B_{\mu\nu}$ is a two-form, Φ^I are the five real scalars associated to the R-symmetry $SO(5)$ and λ are fermions.

The second step is to compactify this 6d $(2,0)$ SCFT on a Riemann surface $\mathcal{C}$ the so called "Gaiotto" or "UV" curve[30].

As we discussed in section 2 the problem is that generically if we compactify the 6d theory on a surface $\mathcal{C}$ we may not preserve enough supersymmetry to obtain a $\mathcal{N} = 2$ susy theory in 4d. This problem can be cured by doing first a "partial topological twist". More precisely, the bosonic part of the spacetime symmetry of the 6d theory is $SO(2,6) \times SO(5)_{\mathcal{R}}$ i.e. the 6d conformal group and the $\mathcal{R}$-symmetry which rotates the five scalars. Once we compactify on $\mathcal{C}$ we have the subgroup

$$SO(1,3) \times SO(2)_{\mathcal{C}} \times SO(3)_{\mathcal{R}} \times SO(2)_{\mathcal{R}} \; . \tag{3.10}$$

The partial topological twist then correspond to consider the diagonal subgroup $U(1)_{\mathcal{R}} \equiv SO'(2)_{\mathcal{R}} \subset SO(2)_{\mathcal{C}} \times SO(2)_{\mathcal{R}}$. After the twist, the fields Φ^3, Φ^4, Φ^5 become scalars on $\mathcal{C}$ and are rotated by the $SO(3)_{\mathcal{R}} \simeq SU(2)_R$ group and the fields Φ_1, Φ_2 become the components of a vector field on $\mathcal{C}$. Using complex coordinates z on $\mathcal{C}$ it is convenient to define the complex fields

$$\Phi_z = \Phi^1 + i\Phi^2 \; , \quad \Phi_{\bar{z}} = \Phi^1 - i\Phi^2 \; , \tag{3.11}$$

which define $(1,0)$ and $(0,1)$ forms $\Phi_z \mathrm{d}z, \Phi_{\bar{z}} \mathrm{d}\bar{z}$ on $\mathcal{C}$ called the Hitchin (Higgs) fields. In terms of the Hitchin field Φ_z the SW curve is given by the N-sheeted cover[31] of $\mathcal{C}$

$$\langle \det(x - \Phi_z) \rangle = 0 \; , \tag{3.12}$$

[28]The letter S stands for "six" because of the six-dimensional origin of this theories.

[29]These theories have no lagrangian description but can be realized from M-theory as the worldvolume theory on a stack of $M5$ parallel branes, for $A_{N-1} = \mathfrak{su}(N)$ theories, or more generally by studying the theory on an ADE singularity.

[30]To understand this construction is useful first to recall how the low-energy SW theory can be realized geometrically from M-theory. In that case the idea is to consider the worldvolume theory of a single $M5$ brane wrapped on $\Sigma_u \times \mathbb{R}^{1,3}$ where Σ_u is the SW curve associated to the vacuum u and the mass of the BPS states corresponds to the tension of $M2$ branes that end on Σ_u [3]. This corresponds exactly to compactify the original 6d (2,0) $\mathfrak{su}(N)$ SCFT on the SW curve Σ_u.

[31]This has a nice physical meaning in M-theory. The N $M5$ branes have 4 directions defining the bulk $\mathbb{R}^4$ where the gauge theory lives and two dimensions parametrized by the UV curve $z \in \mathcal{C}$. The brane

and the SW differential is just the canonical differential

$$\lambda_{SW} = x\mathrm{d}z \ . \tag{3.13}$$

Finally, to complete the construction we can add some punctures on $\mathcal{C}$. This can be interpreted as codimension 2 defects of the $(2,0)$ $6d$ SCFT and are specified assigning some singular behaviour of the Hitchin field Φ_z in that point. For a defect inserted in $z = z_j$ the behaviour of Φ_z around it is

$$\Phi_z(z) = \sum_{l=1}^{r_j+1} \frac{A_{j,l}}{(z-z_j)^l} + \ldots \tag{3.14}$$

the number r_j is called the Poincaré rank and can be integer or half-integer. If $r_j = 0$ the puncture is called regular, otherwise it is called irregular. The gauge invariant data of the residues $A_{j,l}$ correspond to masses[32] and the order of the poles which appear in the puncture determine the flavour group of the theory.

In general, the couplings of the gauge theory are related to the moduli of the UV curve $\mathcal{C}$. In our analysis we will be interested in the asymptotically free or conformal $SU(2)$ theories. If we choose $\mathcal{C}$ as the sphere with four regular punctures, then by conformal symmetry we can fix three of them at $0, 1, \infty$ and we are left with a dimensionless modulus t which parametrize the position of the fourth puncture. In the gauge theory each puncture give rise to a massive hypermultiplet of mass m_i the t correspond to the gauge coupling of the theory. This gives the class S realization of the $SU(2)$ $N_f = 4$ theory[33] The other $SU(2)$ theories are obtained by colliding some of the regular punctures to form more singular behaviours. In this way one can realize all asymptotically free lagrangian theories $N_f < 4$ and the AD theories H_0,H_1,H_2.

3.3 AGT correspondence

The six-dimensional origin of the $\mathcal{N} = 2$ class S theories has some interesting consequences on the BPS sector of the theory. Indeed, the BPS quantities descend from the ones of the $(2,0)$ 6d SCFT on $\mathbb{R}^4 \times \mathcal{C}$ and because they are protected they can be studied also on the 2d theory reduced to the Gaiotto curve $\mathcal{C}$ itself. This suggest that we can associate to some 4d BPS observable some quantity in the 2d theory living on $\mathcal{C}$. This is the content

fluctuations along the 5 transverse directions are parametrized by the fields Φ^I. In particular, the fields $\Phi_z, \Phi_{\bar{z}}$ the two directions transverse to $\mathcal{C}$ which together with $\mathcal{C}$ give the cotangent bundle $T^*\mathcal{C}$. Let (z, x) be coordinates on $T^*\mathcal{C}$, then for a UV configuration of the branes given by the Hitchin field Φ_z the transverse positions x of the branes in the low-energy theory are given by the eigenvalues of the Hitchin field that are precisely given by the equation (3.12).

[32] More precisely, the gauge invariant data correspond to the conjugacy classes under $\mathfrak{g}$. If the matrices $A_{j,l}$ can be put in Jordan form, their eigenvalues correspond to the mass parameters. If instead they are nilpotent then there are no mass parameters introduced by the defect.

[33] From this construction all the operations on the gauge theory can be realized modifying the topology of $\mathcal{C}$. In particular, there is a manifest $SO(8)$ symmetry obtained permuting the four punctures and an S-duality corresponding to the different pants decompositions of the sphere with four punctures.

of the so called AGT correspondence. Its simplest realization is the one corresponding to the $N_f = 4$ $SU(2)$ gauge theory on the squashed sphere S^4_b

$$b(x_1^2 + x_2^2) + \frac{1}{b}(x_3^2 + x_4)^2 + x_5^2 = R^2 \ . \tag{3.15}$$

The supersymmetric partition function on S^4_b can be computed using localization and the result can be expressed in terms of the instanton Nekrasov partition function

$$Z_{S^4}(q, b, R) = \int \mathrm{d}a Z_{\mathrm{Cl}}(a, q, \bar{q}) Z_{1-\mathrm{loop}}(a, \epsilon_1, \epsilon_2) Z_{\mathrm{inst}}(a, q, \epsilon_1, \epsilon_2) Z_{\mathrm{inst}}(a, \bar{q}, \epsilon_1, \epsilon_2) \ , \tag{3.16}$$

where $q = e^{i\pi\tau_{UV}}$ is the UV gauge coupling and the ϵ_j parameters are given by

$$\epsilon_1 = bR \ , \quad \epsilon_2 = \frac{R}{b} \ . \tag{3.17}$$

The AGT correspondence states that

$$Z_{S^4_b} = \langle V_{p_1}(z_1) \dots V_{p_4}(z_4) \rangle_{CFT} \tag{3.18}$$

where the rhs correspond to the four-point function of the Liouville CFT vertex operators with momenta given by the masses m_j

$$V_p(z) =: e^{p\phi(z)} : \ , \quad p = mR + \frac{Q}{2} \ , \tag{3.19}$$

and central charge

$$c = 1 + 6Q^2 \ , \quad Q = b + \frac{1}{b} \ . \tag{3.20}$$

Once we decompose the CFT four-point function in terms of conformal blocks we obtain schematically the following identifications

- $Z_{\mathrm{inst}} \quad \leftrightarrow \quad$ Conformal blocks ,
- $Z_{1-\mathrm{loop}} \quad \leftrightarrow \quad$ Three-point function .

The correspondence can be generalized to arbitrary class S theories and to more general observables. The AGT correspondence is particularly interesting for non-lagrangian theories, such as Argyres-Douglas theories, where we don't have a way to compute the Nekrasov partition function from localization because we lack a $\mathcal{N} = 2$ lagrangian description of the theory, but we can still study the corresponding CFT conformal blocks. In the following we will see that the identification with conformal blocks gives a direct interpretation of the Nekrasov partition function in terms of the so called $\mathcal{T}$-function which encodes the monodromy properties of the conformal blocks around the punctures of $\mathcal{C}$.

3.4 Isomonodromic deformations and Painlevé equations

As we discussed previously the SW low-energy theory is associated to an integrable system, which for class S theories correspond to the Hitchin system defined on the UV curve $\mathcal{C}$. The class of integrable systems can be enlarged to a more general, time-dependent, dynamics which is a naturally obtained studying the associated linear system defined by its Lax pair. Indeed, we observe that the Lax equations

$$\dot{L} = [M, L] \; , \tag{3.21}$$

arise as the compatibility conditions of the following linear system

$$\dot{\psi}(z,t) = M\psi(z,t) \; , \quad L\psi(z,t) = 0 \; . \tag{3.22}$$

From the previous construction the Lax pair $L(z)$ for a theory of class S is exactly the Hitchin field $\Phi(z)$ which has the structure

$$L(z) = \Phi(z) = \sum_{j=0}^{n} \sum_{l=0}^{r_j+1} \frac{A_{j,l}}{(z-z_j)^l} \; , \tag{3.23}$$

where r_j is the Poincaré rank of the puncture. A natural way to deform this integrable system is to study the linear ODE defined by the Lax Matrix

$$\partial_z \psi = L(z)\psi \; , \tag{3.24}$$

because of the poles in $L(z)$, the solutions of the above system are not single valued but they must be analytically continued around each singularity. This gives some monodromy θ_i associated to these points. A natural question is then the so called Riemann-Hilbert problem, which ask if one can reconstruct the ODE given by $L(z)$ starting from the monodromy data θ_i. The solution of this problem is not unique, and is given in general by a family $L(z, t_1, \dots, t_n)$ of matrices, the *isomonodromic deformations* of the system where the parameters $t_1, \dots, t_n$ are the isomonodromic times. We will focus on the case in which we have just one isomonodromic time t. The condition of isomonodromy can be expressed in terms of some linear ODE for the vector ψ and we have the linear system

$$\partial_z \psi = L(z)\psi \; , \quad \partial_t \psi = M\psi \; , \tag{3.25}$$

whose compatibility condition is

$$\partial_z \partial_t \psi = \partial_t \partial_z \psi \quad \rightarrow \quad \partial_t L = \partial_z M + [M, L] \; . \tag{3.26}$$

The resulting equations are non-autonomous, i.e. time dependent, and give a system of non-linear PDE's called the Schlesinger equations.

In the case of $L(z) \in SL(2, \mathbb{C})$ and $\mathcal{C}$ corresponding to the sphere with four punctures this system reduces to an ODE, the sixth Painlevé equation PVI. Then, colliding some of the punctures, to obtain more singular, irregular, behaviours one can obtain all the other

Painlevé equations. These equations were first discovered by Painlevé in an attempt to classify all possible system of second order

$$\ddot{x} = F(x, \dot{x}, t) \ . \tag{3.27}$$

However, this problem is difficult to handle. The reason for this is that in general we can have equations whose solutions present branch cuts or essential singularities which depend on the initial conditions but do not appear in the equation itself. Painlevé and his collaborators found that the correct condition was then to study equations where these phenomena are not present i.e. where the only singularities that depend on initial conditions, the so called movable singularities, are poles. This is now known as the Painlevé property. With this requirement they found 50 types of different equations most of which can be solved by quadrature or in terms of known special functions. There are then six non-linear equations, the Painlevé equations $PI, \dots, PVI$, whose solutions are not reducible to any known special function and define *new* special functions, the Painlevé transcendents.

Once we pass to the isomonodromic deformations the hamiltonians of the integrable system depend on the isomonodromic times t_i and they are generated by a single function, the isomonodromic $\mathcal{T}$-function

$$H_i = \frac{\partial}{\partial t_i} \log \mathcal{T} \ . \tag{3.28}$$

The $\mathcal{T}$-function is a central object for the theory of the isomonodromic system. It contains the full information about the solutions of the system and in terms of it one can see that all Painlevé equations can be written as Hirota bilinear equations of the form

$$D(\mathcal{T}, \mathcal{T}) = 0 \ . \tag{3.29}$$

Where D is a suitable differential operator. As we will see in the following the $\mathcal{T}$-function is directly related to the Nekrasov partition function of the gauge theory.

3.5 Painlevé-gauge correspondence

At this point we can construct a full dictionary between the gauge theory and the isomonodromic system. In the case of $SU(2)$ gauge theory this is the so called Painlevé-gauge correspondence. We first notice that the integrable system associated to the gauge theory is precisely the one defined by the Lax matrix given by the Hitchin field $\Phi_z(z) = L(z)$. This can be directly if we further compactify the theory on a circle and in this way we obtain a 3d theory. Because the vacuum sector is BPS protected the order of the compactification is irrelevant and we can equivalently study the vacua of the $5d$ SYM theory obtained compactifying on the circle which lives in the space $\mathbb{R}^{1,2} \times \mathcal{C}$. A direct computation shows then that the BPS equations for the vacua of the theory are given by the so called Hitchin equations which define an integrable system whose Lax matrix is $\Phi_z(z)$ and whose spectral curve is exactly (3.12). The relation between the punctures of the UV curve $\mathcal{C}$ which realize the $SU(2)$ gauge theory and the corresponding Painlevé equation is reported in figure 5,

We can now state the dictionary. From the previous analysis we have that the moduli of $\mathcal{C}$ correspond both to the isomonodromic times and to the gauge couplings and the masses

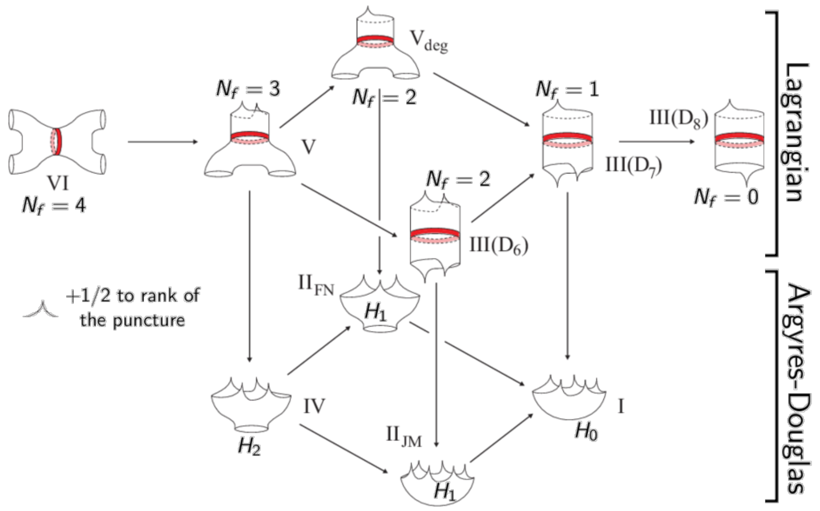

Figure 5. Painlevé-gauge correspondence. The number of "spikes" corresponds to 1/2 the Poincaré rank of the puncture.

are given by residues on the punctures and therefore corresponding to monodromy data for the isomonodromic problem. The hamiltonians of the integrable system correspond to the Coulomb branch parameters. As we saw the isospectral limit, which make the system autonomous, give the original integrable system associated to SW theory and this suggests that the Ω-background parameters ϵ_j play the role of deautonomization parameters. This dictionary is reported in table 3.5.

Gauge theory	**Painlevé equation**
Hitchin field $\Phi_z(z)$	Painlevé Lax matrix $L(z)$
SW curve	spectral curve
gauge coupling Λ	Painlevé time t
masses m_j	monodromy parameters θ_j
coulomb branch parameter u	Painlevé hamiltonian ζ
SW limit $\epsilon \to 0$	autonomous limit

We can also find a direct relation between the gauge theory partition function $Z(a, \Lambda, \epsilon_1, \epsilon_2)$ in the self-dual case $\epsilon_2 = -\epsilon_1 = \epsilon$. This can be better understood via the AGT correspondence, studying the CFT conformal blocks for $c = 1$ which corresponds to study the Nekrasov partition function in the self-dual limit

$$\epsilon_1 = -\epsilon_2 = \epsilon \quad \leftrightarrow \quad b = \sqrt{\frac{\epsilon_1}{\epsilon_2}} = i \ , \quad c = 1 \ . \tag{3.30}$$

Indeed, the conformal blocks have monodromy properties which give a natural solution of the isomonodromy problem. This leads to the so called Kyiv formula of Gamayun-Iorgov-Lisovvy [14, 59]

$$\mathcal{T} \propto Z_D(a, \rho, \Lambda, \epsilon) \equiv \sum_n e^{n\rho} Z(a + n\epsilon, \Lambda, \epsilon, -\epsilon) \ . \tag{3.31}$$

This is the key relation we are going to use in our analysis because it allows to map the partition function of the gauge theory in the Painlevé $\mathcal{T}$-function. This gives a way to *define* the partition function even in absence of any lagrangian description if we are able to compute the $\mathcal{T}$-function.

The Nekrasov-like formulas can indeed be obtained studying the solutions around some singularity of the Painlevé equations [16]. This give access to the regions of the theory where we have some duality frame where the theory is weakly coupled such as the monopole point in SYM. For AD theories this corresponds to some expansion far from the SCFT point. As we will see in the following analysis, there are however other solutions one can study which do not depend on a specific choice of frame because they are related to studying the theory around some movable singularity, which depends on the initial conditions and does not appear in the equation. This implies that these solutions can be constructed in any point of the moduli space of the gauge theory including the strongly coupled points and for this reason they are suitable to study the AD points.

3.6 Painlevé equations and Blowup equations

Painlevé equations can be directly derived from the properties of the gauge theory Nekrasov partition function $Z(a, \Lambda, \epsilon_1, \epsilon_2)$ using the so called blow-up equations. We will sketch here the idea of the proof [60] and for simplicity we do this for the pure theory.

Consider the gauge theory on the blowup $\hat{X}$ of the orbifold $\mathbb{C}^2/\mathbb{Z}_2$. Geometrically, as we will review more in detail in II, this corresponds to replace the origin of $\mathbb{C}^2/\mathbb{Z}_2$ with a sphere E, the exceptional divisor. On this divisor we can then insert d copies of the surface observable considering

$$\hat{Z}(a, \Lambda, \epsilon_1, \epsilon_2, d) = \left\langle O^{(4d)}(\omega_{\epsilon_1,\epsilon_2}, \mathbf{F}^2)^d \right\rangle_{\hat{X}} \ . \tag{3.32}$$

where $\omega_{\epsilon_1,\epsilon_2}$ is the equivariant lift of the Poincaré dual of E. Using the relation between Nekrasov partition functions and 2d conformal blocks given by the AGT correspondence, one can then prove [42, 46] from the representation theory of 2d CFTs that we have the following $\mathbb{C}^2/\mathbb{Z}_2$ blowup equations[34]

$$\hat{Z}(a, \Lambda, \epsilon_1, \epsilon_2, d) = B_d(\Lambda, \epsilon_1\epsilon_2\Lambda\partial_\Lambda, \epsilon_1, \epsilon_2) Z(a, \Lambda, \epsilon_1, \epsilon_2) \ , \tag{3.33}$$

where the operator $\epsilon_1\epsilon_2\Lambda\partial_\Lambda$ corresponds to the insertion of the local operator $\mathrm{Tr}\,\phi^2$. One can then analyze the blowup partition function decomposing the blowup geometry in the two patches of the exceptional sphere E and in this way we obtain a relation of the form

$$\hat{Z}_d(a, \Lambda, \epsilon_1, \epsilon_2) = \sum_n D^{(d)}_{\log\Lambda}(Z(a + n\epsilon_1, \Lambda, 2\epsilon_1, \epsilon_2 - \epsilon_1) Z(a + n\epsilon_2, \Lambda, \epsilon_1 - \epsilon_2, 2\epsilon_2)) \ , \tag{3.34}$$

where the bilinear operator $D^{(d)}_{\log\Lambda}$ is the Hirota derivative defined by

$$f(x+h)g(x-h) = \sum_{k=0}^{\infty} D^{(k)}_x(f,g)\frac{h^k}{k!} \ \Rightarrow \ D^{(k)}_x(f,g) = \frac{\mathrm{d}^k}{\mathrm{d}h^k} f(x+h)g(x-h)\bigg|_{h=0} \ . \tag{3.35}$$

[34]These are slightly different from the Nakajima-Yoshioka blowup equations [41] for $\hat{\mathbb{C}}^2$ that we will use in the following. Here the advantage of using the equations for the blowup of $\mathbb{C}^2/\mathbb{Z}_2$ is that we can obtain bilinear relations for the self-dual partition function, i.e. $c = 1$ conformal blocks.

At this point we can write a combination of blowup equations (3.33) which is vanishing

$$B_4 - 2\Lambda\partial_\Lambda B_2 + (1+Q^2)B_2 + 4\Lambda^4 B_0 = 0 \ , \tag{3.36}$$

with $Q = b+b^{-1}$. Substituting (3.34) and taking the limit self-dual limit $\epsilon_1 = -\epsilon_2 = \epsilon/2 \leftrightarrow Q = 0$ we obtain some bilinear operator inside the convolution of two partition functions

$$\sum_n D^{\mathrm{III}_3}(Z(a+n\epsilon,\Lambda,\epsilon,-\epsilon)Z(a-n\epsilon,\Lambda,\epsilon,-\epsilon)) = 0 \ , \tag{3.37}$$

where we defined

$$D^{\mathrm{III}_3} = D^{(4)}_{\log\Lambda} - 2\Lambda\partial_\Lambda D^{(2)}_{\log\Lambda} + D^{(2)}_{\log\Lambda} + 4\Lambda^4 D^{(0)}_{\log\Lambda} \ . \tag{3.38}$$

Finally, if we define the $\mathcal{T}$-function by the Kyiv formula

$$\mathcal{T} = \sum_n e^{n\rho} Z(a+n\epsilon,\Lambda,\epsilon,-\epsilon) \ , \tag{3.39}$$

from (3.38) we obtain the PIII$_3$ Hirota equation for the Painlevé $\mathcal{T}$-function

$$D^{\mathrm{III}_3}(\mathcal{T},\mathcal{T}) = 0 \ , \tag{3.40}$$

where we observe that, in agreement with Painlevé-gauge correspondence, the coupling Λ corresponds to the time of the Painlevé equation. In a similar way one can derive the other Painlevé equations in Hirota form for the $\mathcal{T}$-function starting from the blowup equations of the corresponding gauge theory.

In the following we will study the Hirota equations for the Painlevé $\mathcal{T}$-function and we will show that the coefficients B_d, the "blowup factors", themselves can be seen as $\mathcal{T}$-functions which satisfy these Hirota equations.

4 Topological strings and geometric engineering

As done for the case of supersymmetric gauge, theory, is it possible to study also a topological twisted version of string theory, which captures some of the BPS protected quantities of the physical theory. This can be thought as a toy model of string theory, which arises as a topological version of bosonic string and lives in a target CY manifold X. The resulting theory is very rich but still enough simple to be solved and had lead to non-trivial mathematical results, such as Mirror symmetry. For some special class of CY manifolds, the partition function of the topological string theory has a direct physical meaning in the gauge theory because it gives the prepotential terms of the field theory obtained compactifying string theory on these manifolds. These corrections are the same as the Ω-background corrections and we obtain then a non-trivial connection between the susy gauge theory and the topological string.

4.1 Topological string theory

To define the topological string we start from a 2d non-linear $\mathcal{N} = (2,2)$ supersymmetric sigma model with target a Calabi-Yau manifold X

$$S = \int d^2z d^4\theta \, K(\Phi, \bar{\Phi}) \ , \tag{4.1}$$

where Φ^i are chiral superfields and $K(\Phi, \bar{\Phi})$ is the Kähler potential which gives the metric of the target Calabi-Yau manifold X $g_{i\bar{j}}(\Phi, \bar{\Phi}) = \partial_i \partial_{\bar{j}} K(\Phi, \bar{\Phi})$. The most interesting version of the theory corresponds to the case when X is a CY threefold, i.e. X has complex dimension 3.

The $\mathcal{N} = (2,2)$ algebra has four supercharges $Q_\pm, \bar{Q}_\pm$ and two $U(1)$ R-symmetries. We can obtain a topological theory applying the twisting procedure with respect to one of the two $U(1)$ $\mathcal{R}$-symmetries. The two possibilities correspond to the A-twist and B-twist and the corresponding topological scalar charges are[35]

$$Q_A = \bar{Q}_+ + Q_- \ , \quad Q_B = \bar{Q}_+ + \bar{Q}_- \ . \tag{4.2}$$

The topological observables are given by the Q-cohomology (A or B) and define the chiral ring. Given a basis of the chiral ring ϕ_i and its conjugate $\bar{\phi}_i$ we can define some non-local observables by the descent procedure, in particular some surface observables $\int d^2z \, \phi_i^{(2)}$ and $\int d^2z \, \bar{\phi}_i^{(2)}$. It is then interesting to deform the action by some complex parameters $t^i, \bar{t}^j$

$$S \to S + t^i \int d^2z \, \phi_i^{(2)} + \bar{t}^i \int d^2z \, \bar{\phi}_i^{(2)} \ , \tag{4.3}$$

which correspond to marginal deformations of the original $\mathcal{N} = (2,2)$ SCFT if the chiral fields ϕ_i have conformal dimensions $h_i = (1/2, 1/2)$. In terms of the geometry of the target CY manifold X these correspond to move along its moduli space and their nature depends on the twisting procedure

- A-model: $t^i, \bar{t}^i$ parametrize the Kähler structures, locally $\simeq H^{1,1}(X)$,
- B-model: $t^i, \bar{t}^i$ parametrize the complex strucures, locally $\simeq H^{1,2}(X)$.

The two-twisting are physically equivalent and are related by the so called Mirror symmetry [47] which states that the A-model on some CY three-fold is equivalent to the B-model on some different, mirror, CY three-fold. An important observation is that the observable $\int d^2z \, \bar{\phi}_i^{(2)}$ is Q-exact

$$\int d^2z \, \bar{\phi}_i^{(2)} = Q(\dots) \ , \tag{4.4}$$

so it should decouple from the physical amplitudes. In the following we will see that actually this fails because of the so called holomorphic anomaly.

[35]There are other possible choices of $Q_{A,B}$ but they are related to these by automorphism of the (2,2) algebra.

For both twistings both cases the stress-energy tensor is Q-exact

$$T_{zz}(z) = QG_{zz}(z) \ , \quad T_{\bar{z}\bar{z}}(\bar{z}) = QG_{\bar{z}\bar{z}}(\bar{z}) \ , \tag{4.5}$$

and therefore the theory is independent of the metric on the worldsheet Σ.

The topological string theory is then obtained coupling this twisted $\mathcal{N} = (2,2)$ model to gravity. For a worldsheet Σ_g of genus g we define the partition function of the topological string theory coupled to gravity as the integral over the moduli space $\mathfrak{M}_g$ of Riemann surfaces of genus g

$$\mathcal{F}_g(t,\bar{t}) = \int_{\mathfrak{M}_g} \left\langle \prod_{i=1}^{3g-3} G_{zz}(\mu_i) \prod_{i=1}^{3g-3} G_{\bar{z}\bar{z}}(\bar{\mu}_{\bar{i}}) \right\rangle_{\Sigma_g} \ , \tag{4.6}$$

where $\mu_i \in H^{1,1}(\Sigma_g)$ are the Beltrami differentials which give the measure on $\mathfrak{M}_g$. We can consider also the all genus partition function $\mathcal{Z}(t,\bar{t},g_s)$ summing over all genus contributions $\mathcal{F}_g(t)$ and exponentiating

$$\mathcal{Z}(t,\bar{t},g_s) = \exp\left(\sum_{g=0}^{\infty} \mathcal{F}_g(t,\bar{t}) g_s^{2g-2}\right) \ , \tag{4.7}$$

where g_s is the string coupling. The partition function $\mathcal{Z}(t,\bar{t},g_s)$ contains then the full information about the topological string theory.

4.2 Holomorphic anomaly equations

Once we compute the amplitudes $\mathcal{F}_g(t)$ we can also study correlation functions of the theory taking (covariant[36]) derivatives with respect to the moduli t^i

$$D_{i_1} \cdots D_{i_n} \mathcal{F}_g(t,\bar{t}) = \mathcal{C}^g_{i_1,\dots,i_n} = \int_{\mathfrak{M}_g} \left\langle \prod_{i=1}^{3g-3} G_{zz}(\mu_i) \prod_{i=1}^{3g-3} G_{zz}(\mu_i) \int \mathrm{d}^2 z \phi^{(2)}_{i_1} \cdots \int \mathrm{d}^2 z \phi^{(2)}_{i_n} \right\rangle_{\Sigma_g} \ . \tag{4.8}$$

on the other side, the derivatives with respect to $\bar{t}^{\bar{i}}$ in principle should be trivial because it gives a Q-exact contribution and therefore we expect the amplitudes to be holomorphic

$$\bar{\partial}_{\bar{i}} \mathcal{F}_g(t,\bar{t}) = \int_{\mathfrak{M}_g} \left\langle \prod_{i=1}^{3g-3} G_{zz}(\mu_i) \prod_{i=1}^{3g-3} G_{zz}(\mu_i) Q(\dots) \right\rangle_{\Sigma_g} \quad \overset{?}{\Longrightarrow} \quad \bar{\partial}_{\bar{i}} \mathcal{F}_g(t,\bar{t}) = 0 \ . \tag{4.9}$$

However, this is not correct because the BRST symmetry of the theory is *anomalous* and gives some non-trivial boundary terms. This gives the so called holomorphic anomaly equation of Bershasky-Cecotti-Ooguri-Vafa (BCOV)

$$\bar{\partial}_{\bar{i}} \mathcal{F}_g = \frac{1}{2} \bar{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} (D_j D_k \mathcal{F}_{g-1} + \sum_{g'=1}^{g-1} D_j \mathcal{F}_{g'} D_k \mathcal{F}_{g-g'}) \ . \tag{4.10}$$

[36]The reason for this covariantization is some contact term introduced by the descendant operators $\phi^{(2)}_i$. The space of marginal deformations define a manifold with the so called Zamolodchikov metric and doing the computation some contact term proportional to the connection of this metric arise and make all the derivatives covariant.

The two contributions come from the two possible singularities in the moduli space of Riemann surfaces $\mathfrak{M}_g$. The first term correspond to pinch a cycle of Σ_g to lower the genus by 1 and replaces it with two punctures, i.e. insertions of the vertex operators $\int \mathrm{d}^2 z \phi_i^{(2)}$. The other contribution comes from pinching a cycle to split the Riemann surface in two surfaces of lower genus and replacing the point where they are glued by a vertex operator.

The holomorphic anomaly equations (4.10) give a powerful method to solve the topological string computing the amplitudes $\mathcal{F}_g$ recursively. This can be done up to some holomorphic function of t, which is not determined by the equations. This is the so called "holomorphic ambiguity" which is fixed using some extra conditions, and in some cases, as in the example of toric CY can be obtained sistematically with the so called topological vertex.

These equations can be extended also to the correlation functions and summing over all genus. To do this it is convenient to define the generating function

$$W(g_s, t, \bar{t}, x) = \sum_{g=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{n!} g_s^{2g-2} \mathcal{C}^g_{i_1,\dots,i_n} x^{i_1} \cdots x^{i_n} + \left(\frac{\chi}{24} - 1 \right) \log g_s \ . \tag{4.11}$$

and we observe that for $x = 0$ we recover the topological string partition function

$$W(g_s; t, \bar{t}, x) \Big|_{x=0} = \mathcal{Z}(g_s, t, \bar{t}) \ . \tag{4.12}$$

The holomorphic anomaly equation for W is then the so called BCOV master equation [61]

$$\frac{\partial}{\partial \bar{t}^{\bar{i}}} e^W = \left[\frac{g_s^2}{2} \bar{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} \frac{\partial^2}{\partial x^j \partial x^k} + G_{\bar{i}j} x^j \left(g_s \frac{\partial}{\partial g_s} + x^k \frac{\partial}{\partial x^k} \right) \right] e^W \ . \tag{4.13}$$

and for $x = 0$ we get a linear equation for the partition function

$$\frac{\partial}{\partial \bar{t}^{\bar{i}}} \mathcal{Z} = \frac{1}{2} g_s^2 \bar{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} D_j D_k \mathcal{Z} \ . \tag{4.14}$$

An interesting physical interpretation of this last equation was given by Witten in [62].

The idea is that the dependence on $\bar{t}^{\bar{i}}$ select a special point on the moduli space of the CY, that is the point where $\bar{t}^{\bar{i}} = (t^i)^*$ i.e. where the theory is unitary. If no BRST anomaly arises this will be not the case. The consequence of this is that the theory depends on the choice of a base-point in the moduli space, parametrized by $\bar{t}^{\bar{i}}$, and therefore is not independent on the background where we study the topological string.

However, as Witten argues, although perturbatively, i.e. order by order in the genus expansion in g_s, the topological strig theory is not background independent it can actually be interpreted as a background independent theory *non-perturbatively*. The idea is to treat $\mathcal{Z}$ as a vector in an abstract Hilbert space where the equation (4.14) is interpreted as the condition for parallel transport along the moduli space and in this sense $\mathcal{Z}$ is $\bar{t}$ independent.

4.3 Geometric engineering of gauge theories

There is an alternative realization of the 4d $\mathcal{N} = 2$ supersymmetric gauge theories as compactification of string theories on some CY threefold, the so called "geometric engineering" [4]. The idea is to start with some surface $\mathcal{C}$ which is fibered by ADE singularities.

An ADE singularity is a singular space of the form $\mathbb{C}^2/\Gamma$ where $\Gamma \subset SU(2)$ is a discrete subgroup of $SU(2)$. To make this space non-singular we blow-up the origin and we get a collection of exceptional 2-cycles E_j whose intersections are given by a Cartan matrix A_{ij} of the Lie algebra ADE family. This determines the gauge algebra $\mathfrak{g}$ of the theory, the number of exceptional 2-cycles being the rank of $\mathfrak{g}$. The CY manifold is then obtained fibering the ADE singularity over a Riemann surface, which specifies the field content of the theory. In particular, $\Gamma = \mathbb{Z}_N$ generates the algebra A_N.

This construction give rise to a so called local toric CY, which are non-compact CY manifolds that locally can be described as the total space of some vector bundle over a compact base.

One simple example is the case of local $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ which engineers the pure $SU(2)$ theory. This theory corresponds to the ADE singularity $\mathbb{C}^2/\mathbb{Z}_2$ fibered over the Riemann sphere $\mathbb{P}^1$. The volume Q_b of the base $\mathbb{P}^1$ corresponds to the gauge coupling of the theory and the volume Q_f of the $\mathbb{P}^1$ in the fiber corresponds to the Coulomb branch modulus.

If we consider M-theory on the background $\mathbb{R}^{1,3} \times X \times S^1_\beta$ and we decouple gravity, the resulting theory is a five-dimensional $\mathcal{N} = 1$ susy gauge theory compactified on a circle of radius β. The SW curve $P(x, y)$ of this 5d theory is the so called mirror curve which defines a CY manifold

$$uv = P(x, y) \ , \tag{4.15}$$

which is the mirror of the CY geometry X.

To obtain the four-dimensional theory we have then to do a dimensional reduction with a suitable scaling of the parametrs. As an example, for $\mathbb{P}^1 \times \mathbb{P}^1$ we have the following scaling of the parameters

$$Q_b = \beta^4 \Lambda^4 \ , \quad Q_f = e^{\beta a} \ , \tag{4.16}$$

with Λ the 4d coupling scale, a the 4d scalar vev. An important observation is that in M-theory the BPS states correspond to $M2$ branes wrapping holomorphic cycles. Once we compactify this can be equivalently seen as BPS states of the supersymmetric gauge theory living on $\mathbb{R}^{1,3} \times S^1_\beta$ or as BPS configuration of the topological string theory, which captures the protected sector of the string theory, on the CY manifold X, and this equivalence remains true also in the geometric engineering limit $\beta \to 0$.

Indeed, we have the equality

$$Z_{SD}(a, \Lambda, \vec{m}, \epsilon) = \lim_{\bar{a} \to \infty} \mathcal{Z}_X(a, \bar{a}, \Lambda, \vec{m}, , g_s) \ . \tag{4.17}$$

where Z_{SD} is the Nekrasov partition function in the self-dual limit $\epsilon_1 = -\epsilon_2 = \epsilon$, $\vec{m}$ are mass parameters, we identify $g_s = \epsilon$ and we have to decouple[37] the antiholomorphic part $\bar{a} \to \infty$. which gives a direct connection between the gauge theory and the topological string. In particular the SW prepotential is given precisely by the genus 0 amplitude $\mathcal{F}_0(a, \Lambda, \vec{m})$.

[37]This does not really cancel the anomaly. What happens is that the partition function becomes then non-modular, see 7 for more details. This can be seen as a different choice of polarization in the CY moduli space [63].

Furthermore, this suggests a more refined version of topological string theory where we also consider the insertion of some operators in order to match the Nekrasov partition function for arbitrary values of the Ω-background parameters ϵ_1, ϵ_2. The refined expansion becomes

$$\log Z(a, \vec{m}, \Lambda, \epsilon_1, \epsilon_2) \sim \sum_{g=0}^{+\infty} \sum_{k \in \{\frac{1}{2}\} \cup \mathbb{Z}_{\geq 0}} (-\epsilon_1 \epsilon_2)^{g-1} (\epsilon_1 + \epsilon_2)^{2k} \mathcal{F}_{g,k}(a, \vec{m}, \Lambda) \; . \tag{4.18}$$

where the second index in $\mathcal{F}_{g,k}(a, \vec{m}, \Lambda)$ it corresponds to the refinement and is related to the Nekrasov-Shatashvili contributions at $\epsilon_2 \to 0$.

The relation (4.17) implies that all the integrable structure of the gauge theory encoded in the Painlevé equations can be applied also to study the partition function of the topological string on the CY manifold X that engineers the gauge theory.

Using this relation with Painlevé equations we will see in 7 that we can *derive* the BCOV equation just from Painlevé equations themself. In this sense the Painlevé equations give a non-perturbative completion of the topological string. Furthermore, the Kiev formula of the Painlevé $\mathcal{T}$-function gives a natural "grand-canonical" version which is explicitly background-independent, in the sense that it does not depend on the moduli $\bar{t}^{\bar{i}}$, making manifest the restoration of background independence of $\mathcal{Z}$ at the non-perturbative level argued by Witten.

II Four-dimensional gauge theory on the blowup

In this part we start the original content of this thesis, that is based on the paper [64]. We consider the generating function of surface observables on the blow-up of the plane, called *blow-up factor* (5.5), which, in the NS limit, is essentially the Painlevé $\mathcal{T}$-function itself. We show that this produces a new expansion of the latter, in terms of *gauge invariant* quantities describing the infrared (quantum) effective Seiberg-Witten geometry. This new expansion displays a number of remarkable features:

- it is a convergent expansion around one of its zeros, being thus *analytic*
- it is a *Hurwitz integral* series[38]
- the resulting $\mathcal{T}$-function has manifest *modular* properties

The third property allows to establish a clear and direct link to the holomorphic anomaly equations governing the topological string amplitudes which geometrically engineer the corresponding gauge theory. In particular, we obtain Painlevé $\mathcal{T}$-functions which are holomorphic and modular in their arguments, and are such that when expanded around a saddle point in the Ω-background and in the source s of the surface observable they reproduce the standard genus expansion of observables in topological strings, satisfying the corresponding holomorphic anomaly equations. Furthermore, the above properties do not rely on the genus expansion of topological strings being exact in the string coupling. We thus propose these $\mathcal{T}$-functions as realising a non-perturbative[39] (therefore, background independent) formulation of topological string theory for these geometries. Schematically we have

$$\partial_{E_2}\mathcal{T} = 0 \Leftrightarrow \begin{array}{c}\text{BCOV holomorphic}\\ \text{anomaly equation.}\end{array}, \tag{4.19}$$

where the latter directly appears in the master form [61]. The physical principle at the basis of our derivation is operator/state correspondence. Indeed, this correspondence implies

[38]The integrality of the coefficients is presently only a conjectural property that we experimentally verified to very high orders in the expansion (see Appendix E). For earlier observations on this theme, see [65, 66].

[39]We observe that in 4d the Nekrasov partition function is itself a non-perturbative completion of the topological string partition function because it is resummed in ϵ_1, ϵ_2. However, it is expected that in the full non-perturbative formulation the theory must be also background independent (i.e. holomorphic and modular).

that the surface observable insertion can be expressed through a dependence on the chiral ring generator. As a consequence of this, the modular properties of the equivariant gauge theory on the blow-up are be dictated by its dependence on this generator only. See (7.6) and section 7 for the specific discussion of this.

The fact that the expansion is in terms of gauge invariant quantities allows to easily study it around any point of the Coulomb moduli space. We display expansions around the Argyres-Douglas points of $SU(2)$ gauge theory with massive hypers.

In the Seiberg-Witten (SW) limit the expansion we find has a universal structure. Indeed, in this case the $\mathcal{T}$-function is proportional to the Weierstrass σ-function (see [44])

$$\mathcal{T}(s) \to e^{-\frac{1}{2}Ts^2}\sigma(s, g_2, g_3) \ , \tag{4.20}$$

and is completely fixed by two data, the elliptic invariants g_2, g_3 of the Weierstrass parametrization of the SW curve

$$y^2 = 4x^3 - g_2 x - g_3 \ , \tag{4.21}$$

and the contact term T which arises because the algebra of surface observables gets changed under the renormalization group flow [44]. This happens because at the intersection of two surfaces Σ_1 and Σ_2 there can be singularities which are given by a local observable $\tilde{T}$

$$I(\Sigma_1)I(\Sigma_2) \to I_{IR}(\Sigma_1)I_{IR}(\Sigma_2) + \tilde{T}\Sigma_1 \cdot \Sigma_2 \ , \tag{4.22}$$

where $\Sigma_1 \cdot \Sigma_2$ is the intersection number and $\tilde{T}$ is related to T by a suitable shift as in (6.14). As a consequence we have that the generating function of the surface observable $I(\Sigma)$ has extra contributions given by Wick contractions of $\tilde{T}(u)$

$$e^{I(\Sigma)} \to e^{I_{IR}(\Sigma)+\tilde{T}(u)\Sigma^2} \ . \tag{4.23}$$

The remarkable feature of the result (4.20) is that the details of the theory are completely encoded in the invariants g_2, g_3 and in T, so that different theories differ just by the specific form of the parametrization of the geometrical data g_2, g_3, T in terms of the physical parameters of theory. The universal structure of the SW result implies the remarkable features of the $\mathcal{T}$-functions. Indeed, these are realized in a universal common way because of the classical power expansion of the Weierstrass σ-function itself in terms of integer polynomials of the elliptic invariants g_2 and g_3. Painlevé $\mathcal{T}$-functions preserve the above properties, but in a more general basis of polynomials. We indeed find that the Painlevé $\mathcal{T}$-function can be universally written as a quantum Weierstrass σ-function, that is, it can be obtained from this latter by acting with a suitable differential operator whose explicit form depends on the corresponding Painlevé equation, see (7.58).

The content of this part is the following. In Sect.5 we review the geometry of topology changing operators in topological quantum field theory, provide a brief reminder of blowup equations in supersymmetric gauge theory in four dimensions and focus on the NS limit of the blowup factors. In Sect.6 we discuss the chiral ring expansion of the NS blowup factor by explaining its autonomous (i.e. Seiberg-Witten) limit with respect to the Weierstrass σ-function and the blowup equations. We also explain the expansion of the blowup factor according to state/operator correspondence. In Sect.7 we discuss the modular properties of the Painlevé $\mathcal{T}$-function, which lead to the proposal that they provide a non-perturbative description of topological string. We propose concrete expansion formulas as universal ϵ-deformations of the Weierstrass σ-function in the form of quantum versions of the latter and show how the holomorphic anomaly equations follow from the modular properties of the blowup factor implied by state/operator correspondence. In Sect.8 we explicitly give the Hurwitz expansions of the Painlevé $\mathcal{T}$-functions around their generic zeros. We specifically discuss the cases of PVI, PIV, PIII_2, PI and PIII_3, which according to Painlevé/gauge correspondence describe respectively $SU(2)$ supersymmetric gauge theories with $N_f = 4$ fundamental hypers, H_2 AD-point, $N_f = 1$, H_0 AD theory and pure SYM. In Sect. 9 we discuss some open problems and possible generalizations of our results. The appendix is reported in part IV. In App.A we recap our conventions on Weierstrass elliptic functions, in App.B we collect some details of the computation of modular invariants appearing in the expansion of PVI and in App.C we report some technical points relevant for section 7. In the extra App.E (located after the bibliography for convenience) we collect numerical tables of integer polynomial coefficients corresponding to the Hurwitz expansions of section 8.

5 Topology changing operations in TQFT

5.1 Handle gluing operators in two dimensions

Topology changing operators are naturally studied in connection to operator/state correspondence. Let us discuss this with an example in two dimensional topological field theory (TFT2). A well-known topology changing operator is the handle-gluing operator on the Riemann surface Σ_g where the TFT2 is formulated. This is given by the operator changing the Riemann surface by the addition of a handle. This operation can be realised in two equivalent ways related by operator/state correspondence.

One is the creation of an handle on Σ_g by replacing it with $\Sigma_g \# T^2$, the connected sum of Σ_g and the two torus T^2. This is obtained by excising a disk on Σ_g and on T^2 and gluing the two components along the boundary circles. From this view point the string partition function at genus $g+1$ is expressed as the pairing of the two component TFT2 wave functions along the boundaries

$$\mathcal{Z}_{g+1}(\Sigma_g \# T^2) = < T^2 | \Sigma_g > .$$

Elongating the cylinder containing the gluing circle to conformal infinity defines the TFT2 handle operator $\mathcal{H}$ via operator/state correspondence

$$< T^2 | \Sigma_g > = < \mathcal{H} >_{\Sigma_g} .$$

This depends on three complex parameters, namely the modulus of the elliptic curve, the center of the excised disk in Σ_g and the length/twist parameter of the gluing. These are the three extra Teichmuller parameters needed to correctly account for the complex structure of the genus $g+1$ Riemann surface $\Sigma_g \# T^2$.

The second way is the creation of a handle on the string world sheet Σ_g by excising two non overlapping disks on the Riemann surface and by gluing their boundary circles generating the new Riemann surface $\mathcal{H}\Sigma_g$. The corresponding TFT2 amplitude depends on the three extra Teichmuller parameters needed to correctly account for the complex structure of the genus $g+1$ Riemann surface $\mathcal{H}\Sigma_g$, namely the position of the centers of the two excised disks in Σ_g and the length/twist parameter of their gluing. The state/operator correspondence relates then the resulting TFT2 partition function to the normalized trace of two point functions

$$\mathcal{Z}_{g+1}(\mathcal{H}\Sigma_g) = C^{AB} < \mathcal{O}_A \mathcal{O}_B >_{\Sigma_g} \quad .$$

These two ways are indeed equivalent by operator/state correspondence and define the handle gluing operator in terms of local ones. Indeed, the circle in the first construction can be deformed from the base of the handle to a pair of circles linking the handle itself, but with opposite orientations. While elongating at conformal infinity the first was defining the handle operator $\mathcal{H}$ as a topology changing operator, elongating to conformal infinity the pair of resulting circles expresses the handle operator in terms of local ones. Indeed one obtains the "handle gluing formula"

$$\mathcal{H} = C^{AB} \mathcal{O}_A \mathcal{O}_B$$

and one gets

$$\mathcal{Z}_{g+1}(\mathcal{H}\Sigma_g) = \mathcal{Z}_{g+1}(\Sigma_g \# T^2)$$

as depicted in Fig.6. The above construction was stated in more general terms in [67].

5.2 Blow-up operator in four dimensions

The blow up of a smooth point of a four dimensional Riemannian manifold X is obtained by removing a four-dimensional ball around that point and gluing back the component at infinity of $\overline{\mathbb{CP}}^2$. This is the connected sum $\hat{X} = X \# \overline{\mathbb{CP}}^2$. The local description of $\mathbb{CP}^2$ around its irreducible divisor is the total space of $\mathcal{O}(-1)_{\mathbb{CP}^1}$, which as also a toric manifold and can be described as the GIT quotient $\left(\mathbb{C}^3 \setminus 0\right)/\mathbb{C}^*$ with weights $(1, 1, -1)$ on the three coordinates. Equivalently, it can be described by the Kähler quotient of $\mathbb{C}^3$ w.r.t. a $U(1)$ action whose moment map is set to $|z_1|^2 + |z_2|^2 - |z_0|^2 = \zeta > 0$. This space is the blow up of the complex plane, usually denoted as $\hat{\mathbb{C}}^2$. The geometry of $\hat{\mathbb{C}}^2$ can be analysed in terms of its Kähler potential. This is obtained by extremising the $\mathbb{C}^*$ invariant potential

$$U[v, z_0, z_1, z_2] = e^{-2v}(|z_1|^2 + |z_2|^2) + e^{2v}|z_0|^2 + 2v\zeta \ ,$$

with respect to v. At the stable critical point

$$v_* = \frac{1}{2}\ln\left[((-\zeta + \sqrt{\zeta^2 + 4|z_0|^2(|z_1|^2 + |z_2|^2)})/(2|z_0|^2))\right] \ ,$$

and setting $K[z_0, z_1, z_2] = U[v_*, z_0, z_1, z_2]$. One has

$$K[z_0, z_1, z_2] = \sqrt{\zeta^2 + 4|z_0|^2(|z_1|^2 + |z_2|^2)} + \zeta \ln\left[(-\zeta + \sqrt{\zeta^2 + 4|z_0|^2(|z_1|^2 + |z_2|^2)})/(2|z_0|^2)\right] \ ,$$

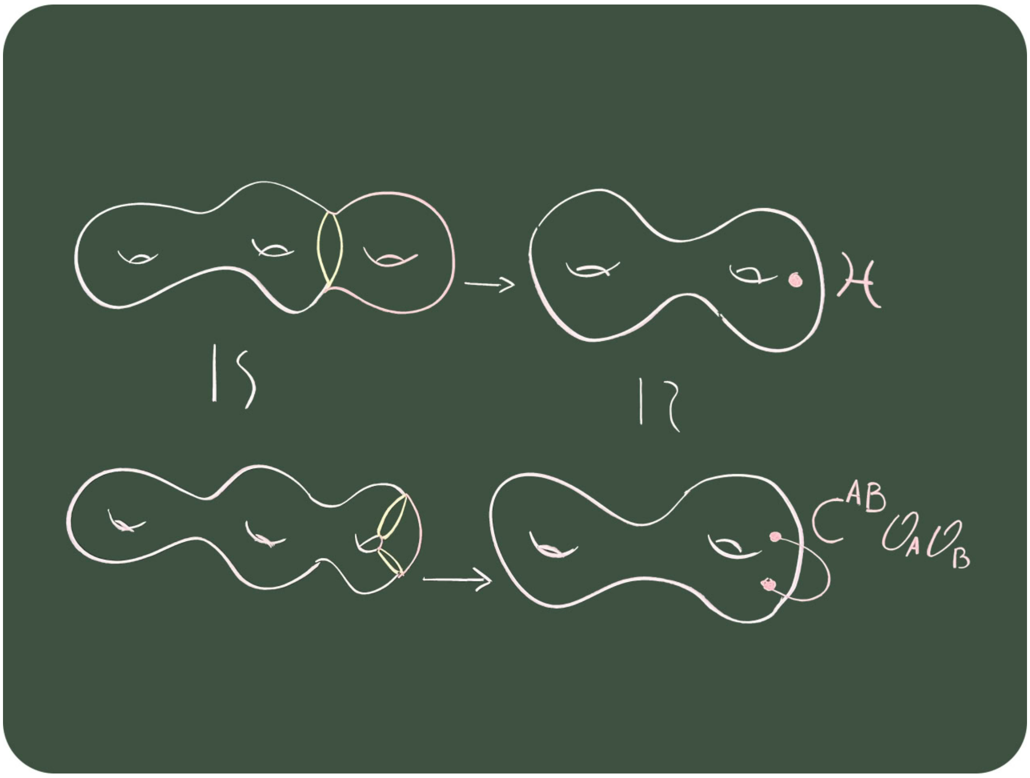

Figure 6. Handle gluing operator and state/operator correspondence.

In the blow down limit $\zeta \sim 0$ one can reduce to the region $z_0 \sim \sqrt{|z_1|^2 + |z_2|^2}$ and obtain

$$K \sim 2(|z_1|^2 + |z_2|^2) - \frac{\zeta^2}{4(|z_1|^2 + |z_2|^2)} + O(\zeta^4) \ ,$$

which is the flat Kähler potential on $\mathbb{C}^2$ with corrections due to the infinitesimal blow up. In the blown up limit $\zeta \sim +\infty$ one can instead rescale to $z_0 \sim 1$ and expand

$$K \sim \zeta(1 - \ln(\zeta)) + \zeta \ \ln\bigl(|z_1|^2 + |z_2|^2\bigr) + \zeta^{-1}(|z_1|^2 + |z_2|^2) + O(\zeta^{-3}) \ ,$$

which is the Kahler potential of the Burns metric (up to a constant).

The topology of the blow up $\hat{X}$ differs by that of X by the addition of a non contractible two-sphere E, which is called exceptional divisor. The BPS partition partition function of the $\mathcal{N} = 2$ gauge theory can be formulated after the topological twist. As the stress-energy tensor of the twisted theory is exact under the scalar supersymmetry defining the BPS partition function, the latter does not depend on the Kähler parameter ζ. Therefore the partition functions on $\hat{X}$ and X coincide[40]. Moreover, on $\hat{X}$ one can study BPS surface observables located on the exceptional divisor E. Since the theory is topological, the multiple insertions of the BPS surface observable in the theory on $\hat{X}$ can be traded for the insertions of the corresponding local operators on X [44]. This realizes the topological

[40]This should be carefully normalized and holds in the appropriate topological sector. See later for a more precise statement.

version of state/operator correspondence. Indeed, the limit $\zeta \to 0$ corresponds to elongating the location of the surface observable on the exceptional divisor along an infinite cylinder and its insertion is therefore equivalent to that of a local operator located at the point at infinity. See figure 7. The equivalence of the two pictures implies the blow up equations of Fintushel and Stern [68].

When X is a toric manifold and one considers a toric blowup, the set of toric divisors of $\hat{X}$ gets correspondingly augmented by the element corresponding to the exceptional divisor[41]. $\mathcal{N}=2$ gauge theories on X and on $\hat{X}$ can be formulated equivariantly under the toric isometry. The exceptional divisor lifts to an equivariant one and therefore defines an extra set of equivariant surface observables for the gauge theory on $\hat{X}$. Indeed state/operator correspondence lifts to the equivariant set up and the corresponding blow-up equations were studied by Nakajima and Yoshioka on $\hat{\mathbb{C}}^2$ in [69].

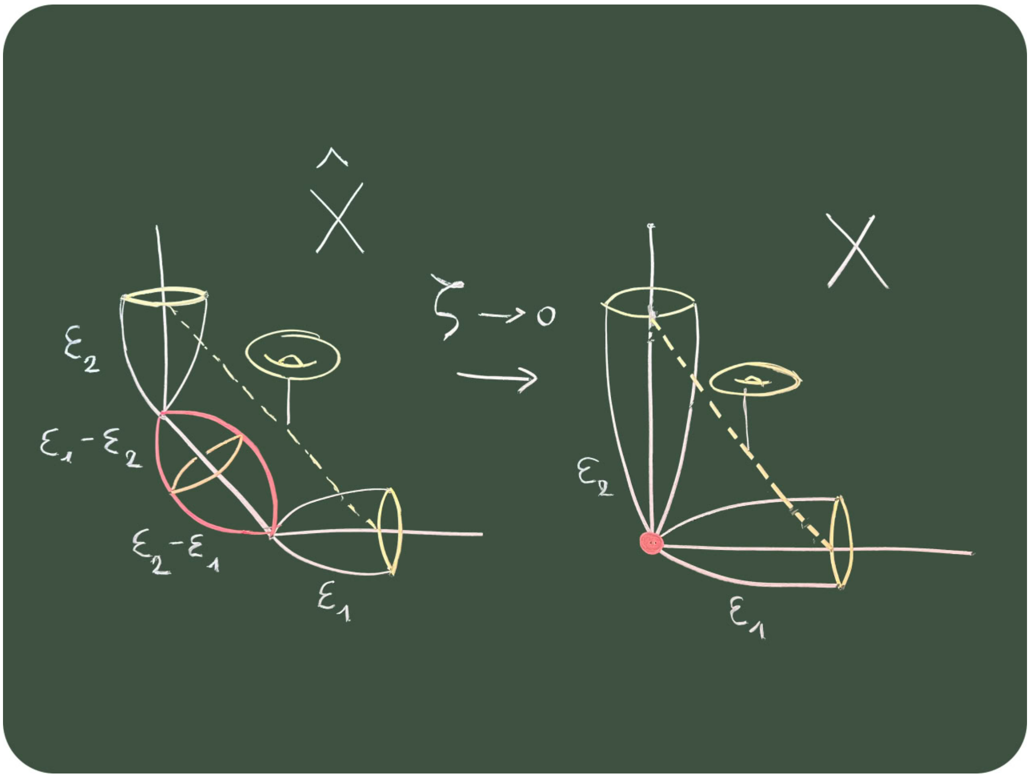

Figure 7. Toric diagram of the blow down of the exceptional divisor E (in red) and state/operator correspondence.

For the generating function of the $SU(2)$ surface observable $I(E)$

$$\hat{Z}(a, \vec{m}, \Lambda, \epsilon_1, \epsilon_2, s) = \left\langle e^{sI(E)} \right\rangle_{\hat{\mathbb{C}}^2} , \tag{5.1}$$

[41] See [47] for details.

they read[42]

$$\hat{Z}(a,\vec{m},\Lambda,\epsilon_1,\epsilon_2,s)=\sum_{n\in\mathbb{Z}+\frac{1}{2}} Z(a+n\epsilon_1,\vec{m},\Lambda_{\epsilon_1 s},\epsilon_1,\epsilon_2-\epsilon_1)Z(a+n\epsilon_2,\vec{m},\Lambda_{\epsilon_2 s},\epsilon_1-\epsilon_2,\epsilon_2)\ , \tag{5.2}$$

where $\hat{Z}|_{s=0}=0$ and we defined the shifted coupling

$$\Lambda_{\epsilon_i s}=\Lambda\exp(\epsilon_i s)\ . \tag{5.3}$$

In (5.2) a is the $SU(2)$ Cartan parameter, $\vec{m}=\{m_i\}$ are the masses of the hypermultiplets, Λ is the gauge coupling, and ϵ_1,ϵ_2 are the Ω-background parameters. The structure of the formula (5.2) is simple: the two factors correspond to the contributions of the two patches of the exceptional divisor E, and are convoluted through the sum over the magnetic fluxes supported on E. The effect of the exponentiated surface observable is to shift the couplings as in (5.3). Each patch corresponds to an affine chart of the blowup geometry $\hat{\mathbb{C}}^2$ described by two coordinates $(z_l,\ \xi_l)$, $l=1,2$. The coordinates $\xi_l=z_j/z_l$ with $j\neq l$ is an affine coordinate on $\mathbb{C}P^1$ and the corresponding equivariant action is

$$\xi_l\to e^{i(\epsilon_j-\epsilon_l)}\xi_l\ , \tag{5.4}$$

which gives the shifts in ϵ_j which appear in the rhs of (5.2).

The non-equivariant limit of the above blowup equations was studied in detail in [69], showing that they indeed reduce to the equation of Fintushel and Stern. The blowup factor in the latter is given by the Weierstrass σ-function. This is naturally arising in the Seiberg-Witten description of the low-energy effective field theory in the Coulomb phase and was indeed used in [69, 70] to provide a proof of Nekrasov's conjecture about the recontruction of the Seiberg-Witten prepotential from supersymmetric localisation.

5.3 The blowup factor in the NS limit

In the following we will study the NS limit of the blowup factor in the Ω-background parameters, namely $(\epsilon_1,\epsilon_2)\to(\epsilon,0)$ and show that it is related to Painlevé $\mathcal{T}$-function expanded around its zero. This will allow us to study the new expansions of Painlevé $\mathcal{T}$-functions which correspond to study the gauge theory around weakly coupled, monopole and Argyres-Douglas points, and their integrality and modularity properties.

The $SU(2)$ theory can be naturally associated to an integrable system, the Hitchin system, if we further compactify on a circle. In this way we obtain a $3d$ $\mathcal{N}=4$ sigma model whose target space $\mathcal{M}$ is the moduli space of the Hitchin system [71]. It turns out that this Hitchin system can be related to a Painlevé Lax pair and in this way we get a direct relation between gauge theory and Painlevé equations, the so called "Painlevé-gauge theory correspondence" [16]. This will serve us as a guide to identify the relevant Painlevé equation for a given gauge theory.

[42] Here we consider the gauge theory on the blowup with first Chern class $c_1=1$. This fixes the odd lattice in (5.2). The other sector $c_1=0$ produces the non-vanishing blowup equations [69].

Let us define the blowup factor

$$\mathcal{B}(a,\vec{m},\Lambda,\epsilon_1,\epsilon_2,s) = \frac{\hat{Z}(a,\vec{m},\Lambda,\epsilon_1,\epsilon_2,s)}{Z(a,\vec{m},\Lambda,\epsilon_1,\epsilon_2)} , \tag{5.5}$$

as the normalised generating functional of the correlation functions of the surface observable $I(E)$. The partition function Z of gauge theory in the Ω-background can be expanded as a refined topological string theory

$$\log Z(a,\vec{m},\Lambda,\epsilon_1,\epsilon_2) \sim \sum_{g=0}^{+\infty} \sum_{k\in\{\frac{1}{2}\}\cup\mathbb{Z}_{\geq 0}} (-\epsilon_1\epsilon_2)^{g-1}(\epsilon_1+\epsilon_2)^{2k}\mathcal{F}_{g,k}(a,\vec{m},\Lambda) . \tag{5.6}$$

The tree level term $\mathcal{F}_0 \equiv \mathcal{F}_{0,0}$ corresponds to the Seiberg-Witten prepotential of the gauge theory. The only term with half-integer n is given explicitly by[43]

$$\mathcal{F}_{0,\frac{1}{2}}(a,\vec{m},\Lambda) = -i\pi a . \tag{5.7}$$

Substituting (5.6) in (5.5) and taking the NS limit $\epsilon_2 \to 0$ with fixed $\epsilon_1 \equiv \epsilon$ we obtain

$$\begin{aligned} \mathcal{B}_{NS}(a,\vec{m},\Lambda,\epsilon,s) &\equiv \lim_{\epsilon_2\to 0}\mathcal{B}(a,\vec{m},\Lambda,\epsilon,\epsilon_2,s) = \\ &= e^{\alpha-\frac{\mathbf{u}s}{\epsilon}} \sum_{n\in\frac{1}{2}+\mathbb{Z}} e^{-\frac{n\rho}{\epsilon}} Z_{SD}(a+n\epsilon,\vec{m},\Lambda_{\epsilon s},\epsilon) = \\ &= e^{\alpha-\frac{\mathbf{u}s}{\epsilon}} Z_D(a,\vec{m},\rho,\Lambda_{\epsilon s},\epsilon) , \end{aligned} \tag{5.8}$$

where Z_{SD} is the Nekrasov partition function in the self-dual background $\epsilon_1 = -\epsilon_2 = \epsilon$ and we have defined

$$\rho = \epsilon\frac{\partial}{\partial a}W(a,\Lambda,\epsilon) , \quad \alpha = \frac{\partial}{\partial\epsilon}W(a,\Lambda,\epsilon) , \quad \mathbf{u} = \epsilon\Lambda\frac{\partial}{\partial\Lambda}W(a,\Lambda,\epsilon) , \tag{5.9}$$

and W is the twisted superpotential

$$W(a,\Lambda,\epsilon) = \sum_{k\in\{\frac{1}{2}\}\cup\mathbb{Z}_{\geq 0}} \mathcal{F}_{0,k}(a,\Lambda)\epsilon^{2k-1} . \tag{5.10}$$

As we discussed previously, the identification between the dual partition function in (5.8) and the Painlevé $\mathcal{T}$-function was found in [14] - known as *Kyiv formula*. Their $\mathcal{T}$-function is expanded around a generic initial datum for the Painlevé equation. The blowup formulae we are considering here instead lead to an expansion around a *zero* of the $\mathcal{T}$-function. This imposes a reparametrization of the time variable, that is of the gauge coupling, as the one determined by the insertion of the surface observable (5.3), s being the Painlevé time.

From formula (5.8) we see that in the limit $\epsilon_2 \to 0$ the blowup factor reduces simply to the dual partition function, up to an exponential prefactor $\exp(\alpha - \mathbf{u}s/\epsilon)$. The factor $\exp(\alpha)$ fixes the normalization of the Painlevé $\mathcal{T}$-function. We finally obtain

$$\mathcal{T} = e^{\frac{\mathbf{u}s}{\epsilon}}\mathcal{B}_{NS} . \tag{5.11}$$

[43] This is true for $N_f \leq 3$. In the case of $N_f = 4$ we have an extra contribution coming from the instanton corrections which is independent on a.

We observe that in this way the knowledge of the self-dual Ω-background partition function, i.e.of the Painlevé $\mathcal{T}$-function, is equivalent to the knowledge of the blowup factor in the NS limit. This is precisely the effect of the blowup relations (5.2) where the extra shifts in the ϵ_j variables that enter into the partition function of the patches Z allow to relate partition functions in different Ω backgrounds.

6 Chiral ring expansion of the blowup factor

6.1 Autonomous limit and Weierstrass σ-function

In this subsection we preliminarly discuss the limit $\epsilon \to 0$ of the the NS blowup factor $\mathcal{B}_{NS}$. This is the SW limit, already discussed in [69], and corresponds to the autonomous limit of the Painlevé dynamics. It turns out that in this limit the Painlevé $\mathcal{T}$-function universally reduces to the Weierstrass σ-function and correspondingly the blow up factor reduces to the one of Fintushel and Stern, that is the σ-function itself - up to a Gaussian prefactor - computed in the modular invariants parametrizing the SW curve of the specific gauge theory.

Let us underline that here one does not have to assume the theory to be Lagrangian and therefore the analysis holds also at the Argyres-Douglas points in the moduli space.

In the following, for simplicity, we omit the explicit dependence on the masses $\vec{m}$ $\mathcal{F}_0(a,\Lambda) \equiv \mathcal{F}_0(a,\vec{m},\Lambda)$, as it plays no role in the derivation.

We compute now the SW limit $\epsilon \to 0$. We start from the expression of the NS blowup factor in (5.8)

$$\mathcal{B}_{NS}(a,\Lambda,\epsilon,s) = e^{\alpha - \frac{\mathbf{u}s}{\epsilon}} \sum_{n\in\mathbb{Z}+\frac{1}{2}} e^{-\frac{n\rho}{\epsilon}} Z_{SD}(a+n\epsilon,\Lambda_{\epsilon s},\epsilon)\ . \tag{6.1}$$

with $\rho = \rho(a,\Lambda,\epsilon)$ given by (5.9). Using the genus expansion (5.6) we obtain

$$\mathcal{B}_{NS}(a,\Lambda,\epsilon,s) = e^{\alpha - \frac{\mathbf{u}s}{\epsilon}} \sum_{n\in\mathbb{Z}+\frac{1}{2}} \exp\left(-\frac{n}{\epsilon}\rho + \frac{1}{\epsilon^2}\mathcal{F}_0(a+n\epsilon,\Lambda_{\epsilon s}) + \mathcal{F}_{1,0}(a+n\epsilon,\Lambda_{\epsilon s}) + O(\epsilon)\right)\ . \tag{6.2}$$

Expanding in ϵ the a-dependence we get

$$\mathcal{B}_{NS}(a,\Lambda,\epsilon,s) = \tag{6.3}$$

$$= e^{\alpha - \frac{\mathbf{u}s}{\epsilon}} \sum_{n\in\mathbb{Z}+\frac{1}{2}} \exp\left[\frac{1}{\epsilon^2}\mathcal{F}_0(a,\Lambda_{\epsilon s}) - \frac{n}{\epsilon}\left(\rho - \frac{\partial\mathcal{F}_0}{\partial a}(a,\Lambda_{\epsilon s})\right) + \frac{1}{2}n^2\frac{\partial^2\mathcal{F}_0}{\partial a^2}(a,\Lambda_{\epsilon s}) + \mathcal{F}_{1,0}(a,\Lambda_{\epsilon s}) + O(\epsilon)\right]\ . \tag{6.4}$$

Expanding further in the instanton counting scale $\Lambda_{\epsilon s} = \Lambda + \epsilon\Lambda s + O(\epsilon^2)$ we obtain

$$
\begin{aligned}
&\mathcal{B}_{NS}(a,\Lambda,\epsilon,s) = \\
&= \exp\left[\alpha + \frac{1}{\epsilon^2}\mathcal{F}_0(a,\Lambda) + \mathcal{F}_{1,0}(a,\Lambda)\right] \times && (6.5)\\
&\times \exp\left[\frac{1}{\epsilon}\left(u\left(a,\Lambda\right) - \mathbf{u}\right)s + \frac{1}{2}u\left(a,\Lambda\right)s^2 + \frac{1}{2}\Lambda^2\frac{\partial^2\mathcal{F}_0}{\partial\Lambda^2}\left(a,\Lambda\right)s^2\right] \times \\
&\times \sum_{n\in\mathbb{Z}+\frac{1}{2}} \exp\left[-\frac{n}{\epsilon}\left(\rho - \frac{\partial\mathcal{F}_0}{\partial a}\left(a,\Lambda\right)\right) + n\frac{\partial u}{\partial a}\left(a,\Lambda\right)s + \frac{1}{2}n^2\frac{\partial^2\mathcal{F}_0}{\partial a^2}\left(a,\Lambda\right) + O(\epsilon)\right] , && (6.6)
\end{aligned}
$$

where we defined $u(a,\Lambda) = \Lambda\partial_\Lambda\mathcal{F}_0(a,\Lambda)$. Finally, we expand in ϵ all the NS quantities

$$
\rho = \frac{\partial\mathcal{F}_0}{\partial a} + \epsilon\partial_a\mathcal{F}_{0,\frac{1}{2}} + O(\epsilon^2)\ ,\ \alpha = -\frac{1}{\epsilon^2}\mathcal{F}_0 + \mathcal{F}_{0,1} + O(\epsilon)\ ,\ \mathbf{u} = u + \epsilon\Lambda\partial_\Lambda\mathcal{F}_{0,\frac{1}{2}} + O(\epsilon^2)\ , \quad (6.7)
$$

and using the following relations

$$
\begin{aligned}
&\tau \equiv \tau(a,\Lambda) = \frac{1}{2\pi i}\frac{\partial^2\mathcal{F}_0}{\partial a^2}\left(a,\Lambda\right)\ , && (6.8)\\
&\frac{1}{2\omega_1} = \frac{1}{2\pi i}\frac{\partial u}{\partial a}\left(a,\Lambda\right) = \frac{1}{2\pi i}\Lambda\frac{\partial^2\mathcal{F}_0}{\partial\Lambda\partial a}\left(a,\Lambda\right)\ , && (6.9)\\
&\partial_a\mathcal{F}_{0,\frac{1}{2}}\left(a,\Lambda\right) = -i\pi\ ,\quad \Lambda\partial_\Lambda\mathcal{F}_{0,\frac{1}{2}} = 0\ ,\quad \exp(\mathcal{F}_{1,0} + \mathcal{F}_{0,1}) = \frac{2\omega_1\Lambda}{\theta_1'(0)}, && (6.10)
\end{aligned}
$$

where τ is the complexified gauge coupling of the IR theory and $\omega_1 \equiv \omega_1(a,\Lambda)$ is the lattice half-period associated to a of the Seiberg-Witten curve, we obtain

$$
\begin{aligned}
\mathcal{B}_{NS}(a,\Lambda,\epsilon,s) =& \frac{2\omega_1\Lambda}{\theta_1'(0)}\exp\left[\frac{1}{2}us^2 + \frac{1}{2}\Lambda^2\frac{\partial^2\mathcal{F}_0}{\partial\Lambda^2}\left(a,\Lambda\right)s^2\right] \times \\
&\times \sum_{n\in\mathbb{Z}+\frac{1}{2}}(-1)^n\exp\left[2in\left(\frac{\pi s}{2\omega_1}\right) + \pi i\tau n^2 + O(\epsilon)\right] . \qquad (6.11)
\end{aligned}
$$

Using the relations (A.16) and (A.17) we finally get

$$
\mathcal{B}_{SW}(a,\Lambda,s) = \lim_{\epsilon\to 0}\mathcal{B}_{NS}(a,\Lambda,\epsilon,s) = \Lambda e^{-\frac{1}{2}Ts^2}\sigma(s;\omega_1,\omega_2)\ . \qquad (6.12)
$$

Here we have defined $\omega_2 = \omega_1\tau$ and the contact term

$$
T = -\Lambda\frac{\partial u}{\partial\Lambda} + \frac{\pi^2}{12}\frac{E_2(\tau)}{\omega_1^2}\ , \qquad (6.13)
$$

which can be computed from SW theory and depends on the specific theory, see appendix B. Notice that in [44] the contact term is given by

$$
\tilde{T} = -\Lambda\partial_\Lambda u = T - \frac{\pi^2}{12}\frac{E_2(\tau)}{\omega_1^2}\ . \qquad (6.14)
$$

We isolated the contribution coming from $E_2(\tau)$ to express the result in terms of the Weierstrass σ function and T is then modular invariant. The previous result shows that the blowup factor is "universal" in the sense that, up to contact terms, it always reduces to the Weierstrass σ-function in the SW limit. The information about the gauge theory is all encoded in the parametrization of the half-periods $\omega_j(a, \vec{m}, \Lambda), j = 1, 2$, or equivalently of the elliptic invariants $g_2(u, \vec{m}, \Lambda), g_3(u, \vec{m}, \Lambda)$ of the SW curve, where we have reintroduced the dependence on the masses $\vec{m}$ of the hypermultiplets.

6.2 Blowup equations and expansion in the chiral ring

In the previous section we verified that in the autonomous limit the $\mathcal{T}$-function given by the Kyiv formula reduces to the Weierstrass σ-function up to Gaussian prefactors. It is interesting to observe that $\sigma(s, g_2, g_3)$ admits the following Hurwitz integral expansion in terms of s, see for example [72]

$$\sigma(s; g_2, g_3) = \sum_{m,n=0}^{\infty} a_{mn} \left(\frac{g_2}{2}\right)^m (2g_3)^n \frac{s^{4m+6n+1}}{(4m+6n+1)!} , \tag{6.15}$$

where a_{mn} are integer coefficients (see appendix A).

Since σ gives the SW limit of $\mathcal{B}_{NS}$, it is natural to ask if a similar expansion holds also in the presence of Ω-background.

This is indeed the case: in this section we propose that the chiral ring has the natural basis to be exploited, and in the next sections we provide evidence of the existence of a Hurwitz integral expansion for the NS blow-up factor $\mathcal{B}_{NS}$. An expansion similar to (6.15) can be derived from the NY blowup equations. To this end we rewrite the partition function on the blowup as

$$\hat{Z}(a, \Lambda, \vec{m}, \epsilon_1, \epsilon_2, s) = D_{NY}(s) Z(a, \Lambda, \vec{m}, \epsilon_1, \epsilon_2) \tag{6.16}$$

which allows to compute it by acting on a single copy of the Nekrasov partition function on $\mathbb{C}^2$ with a suitable linear differential operator $D_{NY}(s)$. The structure of $D_{NY}(s)$ has been derived from the AGT correspondence using a CFT analysis in [46][44].

$$D_{NY}(s) = \sum_{n=0}^{\infty} D_n(-\epsilon_1\epsilon_2\Lambda\frac{\partial}{\partial\Lambda}, \Lambda, \vec{m}, \epsilon_1, \epsilon_2)\frac{s^{n+1}}{(n+1)!} , \tag{6.17}$$

and the operator vanishes for $s = 0$. The c_n are normal-ordered polynomials in which all the derivatives $\Lambda\partial_\Lambda$ are moved to the right. In the NS limit $\epsilon_2 \to 0, \epsilon_1 = \epsilon$ we observe that using (5.6) and (5.9) we get

$$\lim_{\epsilon_2\to 0} \frac{1}{Z} \left(-\epsilon\epsilon_2\Lambda\frac{\partial}{\partial\Lambda}\right)^k Z = \mathbf{u}^k , \tag{6.18}$$

correspondingly, the blowup factor in the NS limit has the following structure

$$\mathcal{B}_{NS}(s) = \lim_{\epsilon_2\to 0} \frac{1}{Z} D_{NY}(s) Z = b_0 \sum_{n=0}^{\infty} c_n(\mathbf{u}, \Lambda, \vec{m}, \epsilon)\frac{s^{n+1}}{(n+1)!} , \tag{6.19}$$

[44] It would be interesting to derive it also from a pure gauge theory treatment, for example by analysing an equivariant version of the u-plane integral. The expression (6.16) is valid for $SU(2)$ gauge theories. By operator/state correspondence, an higher rank extension of this should exist. A possible mathematical setup for this generalization could be the wall crossing approach to the blow-up of [73].

where $b_0 = \langle I(E) \rangle$ is the one-point function of the surface observable and $c_0 = 1$. Such limit of the blowup equations was studied before in cases with surface defect in [45, 74], and also in cases without surface defect, which we are studying here, in [75–79].

The physical interpretation of the above expansion is given by the operator/state correspondence. The blowup factor is an observable in the topologically twisted gauge theory. The blowup is a local change of the topology of X, therefore in the limit of very high distances, that is in the low-energy theory, it must be possible to reproduce the effect of the blowup $\hat{X}$ as a local operator of the theory on X. As the local observables of the equivariant chiral ring are generated by $\mathbf{u}$, we get correspondingly a series in this variable resulting from the OPE of the local equivariant observables. The coefficients of this OPE are precisely the polynomials c_n. Let us underline that the above analysis is valid independently whether the theory is Lagrangian or not.

From the previous analysis we obtain the following dictionary

Gauge theory	**Painlevé equation**
blowup factor $\mathcal{B}_{NS}$	$\mathcal{T}$-function
gauge coupling Λ	position of the zero of $\mathcal{T}$
surface observable source s	Painlevé time
chiral ring expansion of $\mathcal{B}_{NS}$	expansion around a zero of $\mathcal{T}$
SW theory	autonomous limit

In section 8 we will apply this map to compute the coefficients of the expansion (6.19) from Painlevé equations.

Before ending this section we observe that in the SW limit $\epsilon \to 0$, the coefficients c_n are given precisely by the expansion (6.15) of the Weierstrass σ-function. Therefore they are modular polynomials with integer coefficients in the elliptic invariants g_2, g_3 of the SW curve. Modularity is a consequence of the electromagnetic duality of the low-energy theory A thorough analysis of the modular properties of the Painlevé $\mathcal{T}$-function and its relation with the holomorphic anomaly equations will be performed in the next sections.

7 Modular properties of the $\mathcal{T}$-function

The Nekrasov partition function $Z_{SD}(a, \Lambda, \epsilon)$ in the self-dual Ω-background corresponds to the holomorphic limit $\bar{a} \to \infty$ of the partition function of topological strings $\mathcal{Z}_X(a, \bar{a}, g_s, \Lambda)$ in the local Calabi-Yau (CY) X which geometrically engineers the corresponding gauge theory[45]. Namely, denoting by $a, \bar{a}$ the moduli of X and by $g_s = \epsilon$, we have

$$Z_{SD}(a, \Lambda, \epsilon) = \lim_{\bar{a} \to \infty} \mathcal{Z}_X(a, \bar{a}, \Lambda, g_s) \ . \tag{7.1}$$

The $\bar{a}$ dependence of $\mathcal{Z}_X$ is controlled by the BCOV holomorphic anomaly equation [61]. This takes a particularly simple form for $SU(2)$ gauge theories where all the non-holomorphic

[45]These geometries correspond to elliptic curves. For a given 4d gauge theory this elliptic curve is the Jacobian variety of the SW curve which is its mirror.

dependence enters only through the non-holomorphic extension of the second Eisenstein series [48]

$$\hat{E}_2(\tau,\bar{\tau}) = E_2(\tau) - \frac{3}{\pi \operatorname{Im}\tau} \ , \tag{7.2}$$

where τ is the IR gauge coupling. This is modular thanks to the non-holomorphic contribution coming from $\operatorname{Im}\tau$. In the holomorphic limit $\operatorname{Im}\tau \to \infty$ $\hat{E}_2(\tau,\bar{\tau})$ reduces to $E_2(\tau)$ which is indeed holomorphic but not modular. Therefore, the holomorphic limit of $\mathcal{Z}_X$ is not modular, the lack of modularity being encoded in its E_2 dependence.

Let us recall the relation (5.11) between the NS blowup factor $\mathcal{B}_{NS}$ and the Painlevé $\mathcal{T}$-function

$$\mathcal{B}_{NS}(a,\Lambda,\epsilon,s) = e^{-\frac{\mathbf{u}s}{\epsilon}}\mathcal{T}(a,\Lambda,\epsilon,s) \ , \tag{7.3}$$

where

$$\mathcal{T}(a,\Lambda,\epsilon,s) = e^{\frac{\partial W}{\partial\epsilon}} \sum_{n\in\mathbb{Z}+\frac{1}{2}} e^{-\frac{n\rho}{\epsilon}} Z_{SD}(a+n\epsilon,\Lambda_{\epsilon s},\epsilon) \ , \tag{7.4}$$

W is the twisted superpotential (5.10), and $\rho = \epsilon\partial_a W$ is the dual quantum period.

In the previous section we showed that, thanks to the topological operator/state correspondence, the NS blowup factor $\mathcal{B}_{NS}$ can be interpreted as a local observable of the equivariant chiral ring, and therefore is generated by $\mathbf{u}$. This is a direct consequence of the blowup equations in presence of the surface observable $I(E)$ (6.16) on the exceptional divisor E. Since $\mathcal{B}_{NS} \leftrightarrow \mathcal{T}$ is a function of $\mathbf{u}$ only, all the E_2 dependence is carried by $\mathbf{u}$ and there is no further explicit E_2 dependence. Therefore the E_2 derivative of the $\mathcal{T}$ function is

$$D_{E_2}\mathcal{T} = \frac{\partial\mathcal{T}}{\partial\mathbf{u}} D_{E_2}\mathbf{u} = \partial_a\mathcal{T}\frac{D_{E_2}\mathbf{u}}{\partial_a\mathbf{u}} \ , \tag{7.5}$$

that is

$$D_{E_2}\mathcal{T}\partial_a\mathbf{u} - \partial_a\mathcal{T}D_{E_2}\mathbf{u} = 0 \ . \tag{7.6}$$

We will show that the condition (7.6) implies the holomorphic anomaly equations both for the SD and NS partition functions. The derivation holds for general $SU(2)$ gauge theories, but we omit the explicit dependence on the masses because it is not relevant for the analysis.

The reason why we can obtain both the equations, starting from the single condition (7.6), is due to the insertion of the surface observable $I(E)$ which shifts the coupling $\Lambda \to \Lambda_{\epsilon s}$ of the SD contribution in (7.4). This effectively decouples the scales of the SD and NS contributions and allows to study separately their E_2 dependence.

For this purpose it is convenient to write the $\mathcal{T}$-function as a series in the Weierstrass σ-function and its derivatives. To do this we define

$$\mathcal{F}^{x,s}_{\text{st}}(a,\Lambda,\epsilon) = \mathcal{F}(a+\epsilon x,\Lambda_{\epsilon s},\epsilon) - \frac{1}{\epsilon^2}\mathcal{F}_0(a,\Lambda) - \frac{x}{\epsilon}\partial_a\mathcal{F}_0(a,\Lambda) - \frac{x^2}{2}\partial_a^2\mathcal{F}_0(a,\Lambda) \ , \tag{7.7}$$

where the subscript "st" stands for "stable" and

$$\mathcal{F}(a+\epsilon x,\Lambda_{\epsilon s},\epsilon) = \log Z_{SD}(a+\epsilon x,\Lambda_{\epsilon s},\epsilon) \ . \tag{7.8}$$

The prepotential $\mathcal{F}_{\rm st}^{x,s}$ can be interpreted as the BCOV generating functional for the correlation functions of the chiral ring of X. The correlation functions are obtained taking derivatives with respect to x, s and setting $x = 0, s = 0$ at the end. Therefore $\mathcal{F}_{\rm st}^{x,s}$ encodes the full physical information about the perturbative topological string theory.

7.1 Algebra of operators on the modular ring

In order to study E_2-dependence of the tau function we first study the action of ∂_τ derivative on the space of modular functions with respect to some subgroup Γ of $SL(2,\mathbb{Z})$ (e.g. the duality group of $SU(2)$ Seiberg-Witten theory). It is easy to see [48] that if f_k is a modular form of weight k, then

$$D_\tau f_k = \frac{\partial_\tau}{2\pi i} f_k - \frac{k}{12} E_2 f_k \ , \tag{7.9}$$

is again modular. We also have the relation

$$\frac{\partial_\tau}{2\pi i} E_2 = \frac{1}{12} E_2^2 - \frac{1}{12} E_4 \ . \tag{7.10}$$

We now introduce the weight operator $\widehat{d}$

$$\widehat{d} f_k = k f_k \ , \qquad \widehat{d} E_2 = 2 E_2 \ , \qquad \widehat{d} \tau = 0 \ , \qquad \widehat{d}(fg) = \widehat{d}(f) g + f \widehat{d}(g) \ , \tag{7.11}$$

and the derivation with respect to E_2

$$D_{E_2} F(E_2, \tau, g_1, \ldots g_n) = \left. \frac{\partial F(E_2, \tau, g_1, \ldots g_n)}{\partial E_2} \right|_{\tau, g_1, \ldots, g_n} \ , \tag{7.12}$$

where g_i are the generators of the ring of holomorphic modular forms with respect to Γ. One can check by explicit computation that

$$[D_{E_2}, \frac{\partial_\tau}{2\pi i}] = \frac{1}{12} \widehat{d} \ . \tag{7.13}$$

We emphasize that D_{E_2} and $\hat{d}$ are defined only on the ring generated by τ, E_2, and modular forms, but not on the space of arbitrary functions of τ. We also introduce explicit τ derivative ∂'_τ by

$$\partial'_\tau F(E_2, \tau, g_1, \ldots g_n) = \left. \frac{\partial F(E_2, \tau, g_1, \ldots g_n)}{\partial \tau} \right|_{E_2, g_1, \ldots, g_n} \ . \tag{7.14}$$

As

$$\tau = \frac{1}{2\pi i} \frac{\partial^2}{\partial a^2} \mathcal{F}_0(a, \Lambda, \vec{m}) \ , \tag{7.15}$$

we can express the a, Λ, and m_i derivatives by

$$\partial_a = \frac{\partial^3 \mathcal{F}_0}{\partial a^3} \frac{\partial_\tau}{2\pi i} \ , \quad \Lambda \partial_\Lambda = \Lambda \frac{\partial^3 \mathcal{F}_0}{\partial \Lambda \partial a^2} \frac{\partial_\tau}{2\pi i} + \Lambda \partial'_\Lambda \ , \quad \partial_{m_i} = \frac{\partial^3 \mathcal{F}_0}{\partial m_i \partial a^2} \frac{\partial_\tau}{2\pi i} + \partial'_{m_i} \ . \tag{7.16}$$

In general, ∂' means the derivative with all other variables kept fixed. We notice that $\partial_a^3\mathcal{F}_0$ is modular of weight -3, since

$$\partial_{a_D}\frac{-1}{\tau} = \frac{1}{\tau^2}\frac{\partial a}{\partial a_D}\partial_a\tau = \frac{1}{\tau^3}\partial_a\tau \ . \tag{7.17}$$

Analogously, since $u = \Lambda\partial_\Lambda\mathcal{F}_0$ is modular invariant, by using (7.16) one gets that $\pi i\omega_1^{-1} = \Lambda\partial_\Lambda\partial_a\mathcal{F}_0$ is modular of weight -1. The canonically normalized holomorphic differential is given by

$$d\omega = \partial_a dS^{SW} = \Lambda\partial_\Lambda\partial_a\mathcal{F}_0\frac{\partial}{\partial u}dS^{SW} \ , \tag{7.18}$$

where dS^{SW} is the SW differential. By using the above and (6.14) one gets

$$D_{E_2}\Lambda\partial_\Lambda\partial_a^2\mathcal{F}_0 = -\frac{1}{12}\Lambda\partial_\Lambda\partial_a\mathcal{F}_0\cdot\partial_a^3\mathcal{F}_0 \ . \tag{7.19}$$

Using this one can check that the above operators satisfy the following algebra

$$\begin{gathered}
[\partial_a, \Lambda\partial_\Lambda] = 0 \ , \\
[D_{E_2}, \partial_a] = \frac{1}{12}\partial_a^3\mathcal{F}_0(a,\Lambda)\widehat{d} \ , \\
[D_{E_2}, \Lambda\partial_\Lambda] = \frac{1}{12}\Lambda\partial_\Lambda\partial_a^2\mathcal{F}_0(a,\Lambda)\widehat{d} - \frac{1}{12}\Lambda\partial_\Lambda\partial_a\mathcal{F}_0(a,\Lambda)\partial_a \ , \\
[\widehat{d}, \Lambda\partial_\Lambda] = -\Lambda\partial_\Lambda\partial_a^2\mathcal{F}_0(a,\Lambda)\frac{\partial'_\tau}{\pi i} \ , \qquad [\widehat{d}, \partial_a] = -\partial_a - \partial_a^3\mathcal{F}_0(a,\Lambda)\frac{\partial'_\tau}{\pi i} \ , \qquad [\widehat{d}, D_{E_2}] = -2D_{E_2} \ .
\end{gathered} \tag{7.20}$$

In order to be able to compute the action of this algebra on the $\mathcal{F}_{g,n}$'s we display the modular properties of the first terms of the prepotential

$$\begin{gathered}
D_{E_2}\Lambda\partial_\Lambda\mathcal{F}_0(a,\Lambda) = 0 \ , \quad D_{E_2}\partial_a^3\mathcal{F}_0(a,\Lambda) = 0 \ , \\
\widehat{d}\Lambda\partial_\Lambda\mathcal{F}_0 = 0 \ , \quad \widehat{d}\partial_a^3\mathcal{F}_0 = -3\partial_a^3\mathcal{F}_0 \ , \quad D_{E_2}\mathcal{F}_{1,0}(a,\Lambda) = 0 \ , \quad D_{E_2}\mathcal{F}_{0,1}(a,\Lambda) = 0 \ ,
\end{gathered} \tag{7.21}$$

while the modular weights of the $\mathcal{F}_{g,n}$ are fixed by

$$\widehat{d}\mathcal{F}_{g,n}(a,\Lambda) = -\frac{1}{2}\delta_{g,1}\delta_{n,0} \ , \ g+n \geq 1 \ . \tag{7.22}$$

Now we study the genus expansion of the prepotential in more detail. For $g+n > 1$ we can write [48]

$$\mathcal{F}_{g,n} = (\Delta(\tau))^{2-2g-2n}\sum_{\substack{l+k\leq 3g+3n-3 \\ k\leq 3g+2n-3}} P_{l,k}(\Lambda,\vec{m})u^l X^k \ , \tag{7.23}$$

where $\Delta(\tau) = g_2^3 - 27g_3^2$ is the discriminant of the SW curve, $P_{l,k}$ are polynomials of the appropriate scaling dimensions, and

$$X = \frac{\pi^2}{64}\frac{E_2(\tau)}{\Lambda^2\omega_1^2} \ . \tag{7.24}$$

For a gauge theory $\mathtt{T}$ we will express our solution (7.23) in the following ring $R_{\mathtt{T}}$ of polynomials

$$R_{\mathtt{T}} = \mathbb{C}[X, \{SW\}_{\mathtt{T}}, \Delta_{\mathtt{T}}^{-1}, \omega_1^{-1}][\omega_1, \tau] \ , \tag{7.25}$$

where $\{SW\}_{\mathrm{T}}$ is the set of coefficients of the SW curve and the generator ω_1^{-1} guarantees that R_{T} is closed under the action of derivatives. $\mathcal{F}_{g,n}$'s with $g+n>1$ belong to the first part of (7.25), the only terms that involve the extension by ω_1, τ are $\mathcal{F}_0$ and $\mathcal{F}_{0,\frac{1}{2}}$. The terms $\mathcal{F}_{1,0}$ and $\mathcal{F}_{0,1}$ do not lie in the ring, but $e^{12\mathcal{F}_{1,0}}$ and $e^{24\mathcal{F}_{0,1}}$ do (see the example below).

For the pure theory $N_f=0$ this ring can be described by[46]

$$R_{N_f=0}=\mathbb{C}[X,U,\Lambda,\Delta^{-1},(\theta_2\theta_3)^{-1}][\theta_2\theta_3,\tau]\ , \tag{7.26}$$

with

$$U=-\frac{1}{2}\left(\frac{\theta_4^2}{\theta_2^2}+\frac{\theta_2^2}{\theta_4^2}\right)=\frac{u}{8\Lambda^2}\ ,\quad X=\frac{E_2}{\theta_2^2\theta_3^2}\ ,\quad \Delta=\Lambda^{12}(U^2-1)\ ,\quad \omega_1=\frac{\pi}{8\Lambda}\theta_2\theta_3\ . \tag{7.27}$$

Explicit formulas for the first $\mathcal{F}_{g,n}$'s in terms of these functions for the pure $SU(2)$ gauge theory are:

$$\mathcal{F}_0 \ =\frac{4}{3}\Lambda^2(5U-2X)-\frac{4}{9}\pi i\tau\theta_2^2\theta_3^2\Lambda^2(X-U)^2\ , \tag{7.28}$$

$$\mathcal{F}_{0,\frac{1}{2}} \ =-i\pi a=-\frac{2}{3}\pi(U-X)\theta_2\theta_3\Lambda\ , \tag{7.29}$$

$$\mathcal{F}_{1,0} \ =-\frac{1}{12}\log\bigl(U^2-1\bigr)-\frac{1}{2}\log\left(\theta_2\theta_3\right)+\frac{1}{6}\log\Lambda\ , \tag{7.30}$$

$$\mathcal{F}_{0,1} \ =-\frac{1}{24}\log\bigl(U^2-1\bigr)-\frac{1}{6}\log\Lambda\ , \tag{7.31}$$

while, for the elements of the ring (7.25) we have

$$\mathcal{F}_{0,2} \ =\frac{U(75+4U^2+5UX)}{17280(U^2-1)^2\Lambda^2}\ , \tag{7.32}$$

$$\cdots\quad\cdots$$

$$\mathcal{F}_{g,n} \ =\bigl(\Lambda(U^2-1)\bigr)^{2-2g-2n}\sum_{\substack{m+k\leq 3g+3n-3\\ k\leq 3g+2n-3\\ 2\nmid g+n+k+m}} c_{k,m}X^kU^m\ , \tag{7.33}$$

and according to (7.23) all other $\mathcal{F}_{g,n}$'s are rational functions of U and X multiplied by the appropriate power of Λ. The explicit action of the derivatives in a and Λ (7.16) on the ring $R_{N_f=0}$ is given by

$$\partial_a \ =\frac{\partial^3\mathcal{F}_0}{\partial a^3}\frac{\partial_\tau}{2\pi i}=\frac{2i}{\Lambda\theta_2^3\theta_3^3(U^2-1)}\frac{\partial_\tau}{2\pi i}\ , \tag{7.34}$$

$$\Lambda\partial_\Lambda \ =\Lambda\frac{\partial^3\mathcal{F}_0}{\partial\Lambda\partial a^2}\frac{\partial_\tau}{2\pi i}+\Lambda\partial'_\Lambda=\frac{4(X-U)}{3(U^2-1)\theta_2^2\theta_3^2}\frac{\partial_\tau}{2\pi i}+\Lambda\partial'_\Lambda\ , \tag{7.35}$$

where ∂_Λ is the derivative in Λ with fixed a, while ∂'_Λ is the derivative in Λ with fixed θ_2, θ_3, τ, U, X. The modular weight operator is given explicitly by

$$\widehat{d}=\frac{1}{2}\left(\theta_3\partial'_{\theta_3}+\theta_2\partial'_{\theta_2}\right)\ , \tag{7.36}$$

[46] See (A.18) for definiteness.

and the derivative with respect to E_2 has the following expression

$$D_{E_2} = \frac{1}{\theta_2^2\theta_3^2}\partial'_X \ . \tag{7.37}$$

The explicit expression of the ∂_τ operator acting on $R_{N_f=0}$ (7.26) is given by

$$\begin{aligned}\frac{1}{\theta_2^2\theta_3^2}\frac{\partial_\tau}{2\pi i} &= \frac{1}{12}\left(3-4U^2+2UX-X^2\right)\partial'_X + \frac{1}{2}(U^2-1)\partial'_U \\ &+ \frac{1}{24}(X-U)(\theta_2\partial'_{\theta_2}+\theta_3\partial'_{\theta_3}) + \frac{1}{8}\sqrt{U^2-1}(\theta_3\partial'_{\theta_3}-\theta_2\partial'_{\theta_2}) + \frac{1}{\theta_2^2\theta_3^2}\frac{\partial'_\tau}{2\pi i} \ .\end{aligned} \tag{7.38}$$

One can check that the operators ∂_a, $\Lambda\partial_\Lambda$, D_{E_2}, and $\widehat{d}$ acting as derivations on $R_{N_f=0}$ actually satisfy the algebra (7.20). Let us also display the explicit expression of the first few derivatives of the prepotential:

$$\partial_a\mathcal{F}_0 = a_D = \frac{8i\Lambda}{\theta_2\theta_3} + \frac{4}{3}\pi\tau\Lambda\theta_2\theta_3(U-X) \ , \tag{7.39}$$

$$\partial_a^2\mathcal{F}_0 = 2\pi i\tau \ , \tag{7.40}$$

$$\partial_a^3\mathcal{F}_0 = \frac{2i}{\Lambda\theta_2^3\theta_3^3(U^2-1)} \ , \tag{7.41}$$

$$\Lambda\partial_\Lambda\mathcal{F}_0 = 8U\Lambda^2 \ , \tag{7.42}$$

$$\Lambda\partial_\Lambda\partial_a\mathcal{F}_0 = \frac{8i\Lambda}{\theta_2\theta_3} \ , \tag{7.43}$$

$$(\Lambda\partial_\Lambda)^2\mathcal{F}_0 = \frac{16}{3}(2U+X)\Lambda^2 \ . \tag{7.44}$$

All the other derivatives are in the ring $R_{N_f=0}$.

7.2 $\mathcal{T}$-function as a quantum Weierstrass σ-function

Using (7.7) the $\mathcal{T}$-function (7.4) can be written as

$$\mathcal{T}(a,\Lambda,\epsilon,s) = e^{\frac{\partial W_{\rm st}}{\partial\epsilon}}\sum_{n\in\mathbb{Z}+\frac{1}{2}} e^{i\pi\tau n^2} e^{\mathcal{F}^{n,s}_{\rm st}(a,\Lambda_{\epsilon s},\epsilon)-\frac{\rho_{\rm st}n}{\epsilon}} \ , \tag{7.45}$$

where we defined

$$W_{\rm st} = W - \frac{1}{\epsilon}\mathcal{F}_0 \ , \quad \rho_{\rm st} = \rho - a_D \ . \tag{7.46}$$

We can now rewrite (7.45) in terms of the σ-function using (A.16), (A.17)

$$\sum_{n\in\mathbb{Z}+\frac{1}{2}} e^{i\pi\tau n^2+inx} = \theta_1\left(\frac{x}{2};\tau\right) = -\pi\eta(\tau)^3\sigma\left(\frac{x}{\pi};1,\tau\right)e^{-\frac{1}{24}E_2x^2} \ , \tag{7.47}$$

$$\sum_{n\in\mathbb{Z}+\frac{1}{2}} e^{i\pi\tau n^2}f(n) = f(-i\partial_x)\theta_1\left(\frac{x}{2};\tau\right)\Big|_{x=0} = \theta_1\left(-\frac{i}{2}\partial_x;\tau\right)f(x)\Big|_{x=0} \ , \tag{7.48}$$

then we get

$$\mathcal{T} = e^{\partial_\epsilon W_{\rm st}}\eta(\tau)^3\left(\sigma(ip)e^{\frac{E_2}{24}p^2}, e^{\mathcal{F}^{x,s}_{\rm st}(a,\Lambda,\epsilon)-\frac{\rho_{\rm st}x}{\epsilon}}\right) , \tag{7.49}$$

where $\eta(\tau)$ is the Dedekind eta function, $\sigma(ip) \equiv \sigma(ip; \pi, \pi\tau)$ is the Weierstrass σ-function, and we defined the pairing

$$(f(p), g(x)) = f(\partial_x) g(x)|_{x=0} \ . \tag{7.50}$$

The formula (7.49) can be used to compute the ϵ-expansion of the $\mathcal{T}$-function and present it as a quantum Weierstrass σ-function. In order to do this we separate all functions in the exponentials into singular, constant, and higher order corrections in ϵ as

$$\partial_\epsilon W_{\rm st} + \mathcal{F}_{\rm st}^{x,s} - \frac{1}{\epsilon}\rho_{\rm st} x = \frac{s}{\epsilon}\Lambda\partial_\Lambda \mathcal{F}_0 + xs\Lambda\partial_\Lambda\partial_a\mathcal{F}_0 + \frac{s^2}{2}\left(\Lambda\partial_\Lambda\right)^2\mathcal{F}_0 + \mathcal{F}_{0,1} + \mathcal{F}_{1,0} + \widetilde{\mathcal{F}}_{\rm st}(x,s) \ , \tag{7.51}$$

where

$$\begin{aligned}\widetilde{\mathcal{F}}_{\rm st}(x,s) = & \sum_{n=2}^{\infty}\epsilon^{2n-2}(2n-1)\mathcal{F}_{0,n} - \sum_{n=1}^{\infty}\epsilon^{2n-1} x\partial_a\mathcal{F}_{0,n} \\ & + \sum_{k+l\geq 3}^{\infty}\frac{\epsilon^{k+l-2}s^k x^l}{k!\ l!}\left(\Lambda\partial_\Lambda\right)^k\partial_a^l\mathcal{F}_0 + \sum_{k,l,g}\frac{\epsilon^{k+l+2g-2}s^k x^l}{k!\ l!}\left(\Lambda\partial_\Lambda\right)^k\partial_a^l\mathcal{F}_{g,0} = \\ & = \sum_{n=1}^{\infty}\epsilon^n p_{n+2}(i\pi x/\omega_1, s) \ , \end{aligned} \tag{7.52}$$

where $p_{n+2}(x,s)$ are some polynomials of degree at most $n+2$ in x, s.

$$\mathcal{T} = \eta(\tau)^3 e^{\frac{s}{\epsilon}\Lambda\partial_\Lambda\partial_a\mathcal{F}_0 + \mathcal{F}_{0,1} + \mathcal{F}_{1,0} + \frac{s^2}{2}(\Lambda\partial_\Lambda)^2\mathcal{F}_0}\left(\sigma(ip)e^{\frac{E_2}{24}p^2}, e^{\pi i\frac{xs}{\omega_1}}e^{\widetilde{\mathcal{F}}_{\rm st}(x,s)}\right) \ . \tag{7.53}$$

Now we introduce

$$\sigma(-\pi s/\omega_1) = -\frac{\pi}{\omega_1}\widetilde{\sigma}(s) \ , \quad \mathcal{T}_0 = -\frac{\pi}{\omega_1}\eta(\tau)^3 e^{\frac{s}{\epsilon}\Lambda\partial_\Lambda\partial_a\mathcal{F}_0 + \mathcal{F}_{0,1} + \mathcal{F}_{1,0} + \frac{s^2}{2}(\Lambda\partial_\Lambda)^2\mathcal{F}_0} \ , \tag{7.54}$$

and we move the term $e^{\pi i\frac{xs}{\omega_1}}$ to the left

$$\mathcal{T} = \mathcal{T}_0 \cdot \left(\widetilde{\sigma}(s - ip\omega_1/\pi)e^{-\frac{\pi^2 E_2}{24\omega_1^2}(s - ip\omega_1/\pi)^2}, e^{\widetilde{\mathcal{F}}_{\rm st}(x,s)}\right) \ . \tag{7.55}$$

Rescaling the x and p variables simultaneously

$$\mathcal{T} = \mathcal{T}_0 \cdot \left(\widetilde{\sigma}(s+p)e^{-\frac{\pi^2 E_2}{24\omega_1^2}(s+p)^2}, e^{\widetilde{\mathcal{F}}_{\rm st}(-ix\omega_1/\pi, s)}\right) \ , \tag{7.56}$$

exchanging the two arguments of the pairing, and substituting the expansion (7.52), we get

$$\mathcal{T} = \mathcal{T}_0 \cdot \left(e^{\sum\limits_{n=1}^{\infty}\epsilon^n p_{n+2}(p,s)}, \widetilde{\sigma}(s+x)e^{-\frac{\pi^2 E_2}{24\omega_1^2}(s+x)^2}\right) \ . \tag{7.57}$$

This expression can be rewritten equivalently as

$$\mathcal{T} = \mathcal{T}_0 \ : e^{\sum\limits_{n=1}^{\infty}\epsilon^n p_{n+2}(\partial_s, s)} : e^{-\frac{\pi^2 E_2}{24\omega_1^2}s^2}\widetilde{\sigma}(s) \ , \tag{7.58}$$

where the normal ordering : : moves all the ∂_s to the right. In this way we obtain a representation of the $\mathcal{T}$-function as a differential operator acting on the σ-function

$$\mathcal{T} = -\frac{\pi}{\omega_1}\eta(\tau)^3 e^{\frac{s}{\epsilon}\Lambda\partial_\Lambda\partial_a\mathcal{F}_0 - T\frac{s^2}{2}+\mathcal{F}_{0,1}+\mathcal{F}_{1,0}} \exp\left(\sum_{n=1}^{\infty}\sum_{k,l=0}^{n+2}\epsilon^n\Delta^{-n}c_{n,kl}s^k\partial_s^l\right)\widetilde{\sigma}(s)\ , \tag{7.59}$$

where Δ is the discriminant of the elliptic curve. For the $N_f = 0$ case it is

$$\Delta = 64\Lambda^8(u^2-\Lambda^4)\ . \tag{7.60}$$

The explicit form of the first few matrices of coefficients for the $N_f = 0$ case is[47]

$$c_1^{N_f=0} = \begin{pmatrix} 0 & 0 & 0 & -\frac{8\Lambda^8}{3} \\ \frac{16}{3}\Lambda^8\left(u^2-\Lambda^4\right) & 0 & \frac{8\Lambda^8 u}{3} & 0 \\ 0 & \frac{8}{9}\Lambda^8\left(8u^2-9\Lambda^4\right) & 0 & 0 \\ \frac{8}{81}\Lambda^8 u\left(28u^2-27\Lambda^4\right) & 0 & 0 & 0 \end{pmatrix}, \tag{7.61}$$

$$c_2^{N_f=0} = \begin{pmatrix} \frac{256}{9}\Lambda^{16}u\left(2\Lambda^4+u^2\right) & 0 & -\frac{128}{3}\Lambda^{16}\left(\Lambda^4-3u^2\right) & 0 & \frac{128\Lambda^{16}u}{3} \\ 0 & \frac{128}{9}\Lambda^{16}u\left(3u^2-7\Lambda^4\right) & 0 & \frac{64}{9}\Lambda^{16}\left(7u^2-15\Lambda^4\right) & 0 \\ -\frac{256}{27}\Lambda^{16}u^2\left(3u^2-4\Lambda^4\right) & 0 & -\frac{128}{9}\Lambda^{16}u\left(u^2-3\Lambda^4\right) & 0 & 0 \\ 0 & -\frac{64}{81}\Lambda^{16}\left(27\Lambda^8+44u^4-63\Lambda^4u^2\right) & 0 & 0 & 0 \\ -\frac{128}{243}\Lambda^{16}u\left(27\Lambda^8+20u^4-48\Lambda^4u^2\right) & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7.62}$$

We can compute such expansions for other theories by using the ansatz (7.59) for the corresponding Painlevé equation and finding the coefficients of the expansion. For example, such expansion for the case of H_0 Argyres Douglas theory (corresponding to the Painlevé I case), is

$$\mathcal{T} = \exp\left(\sum_{n=1}^{\infty}\sum_{k,l=0}^{n+2}\epsilon^n\Delta^{-n}c^{H_0}_{n,kl}s^k\partial_s^l\right)\sigma(s;g_2,g_3)\ , \tag{7.63}$$

where

$$\Delta = g_2^3 - 27g_3^2\ , \tag{7.64}$$

and

$$c_1^{H_0} = \begin{pmatrix} 0 & 0 & 0 & -g_2 \\ 0 & 0 & \frac{9g_3}{2} & 0 \\ 0 & -\frac{g_2^2}{4} & 0 & 0 \\ \frac{g_2g_3}{8} & 0 & 0 & 0 \end{pmatrix}, \tag{7.65}$$

$$c_2^{H_0} = \begin{pmatrix} \frac{81g_2^2g_3}{2} & 0 & \frac{21g_2^3}{4}+\frac{405g_3^2}{4} & 0 & 27g_2g_3 \\ 0 & -27g_2^2g_3 & 0 & -\frac{13g_2^3}{4}-\frac{297g_3^2}{4} & 0 \\ \frac{5g_2^4}{16}+\frac{189g_2g_3^2}{16} & 0 & \frac{27g_2^2g_3}{2} & 0 & 0 \\ 0 & -\frac{3g_2^4}{16}-\frac{135g_2g_3^2}{16} & 0 & 0 & 0 \\ \frac{g_2^3g_3}{8}+\frac{27g_3^3}{16} & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{7.66}$$

[47] In these formulas s actually denotes $s' = \frac{s}{4}$. Otherwise we have huge powers of 2.

$$
c_3^{H_0} =
\begin{pmatrix}
0 & -\frac{467g_2^5}{80} - \frac{34047g_2^2g_3^2}{80} & 0 & -\frac{4149g_2^3g_3}{8} - \frac{27945g_3^3}{8} & 0 & -\frac{507g_2^4}{40} - \frac{32967g_2g_3^2}{40} \\
\frac{3531g_2^4g_3}{160} + \frac{44631g_2g_3^3}{160} & 0 & \frac{451g_2^5}{16} + \frac{34479g_2^2g_3^2}{16} & 0 & \frac{4095g_2^3g_3}{16} + \frac{29403g_3^3}{16} & 0 \\
0 & -\frac{3525g_2^4g_3}{32} - \frac{44793g_2g_3^3}{32} & 0 & -\frac{141g_2^5}{16} - \frac{11745g_2^2g_3^2}{16} & 0 & 0 \\
\frac{85g_2^6}{192} + \frac{3951g_2^3g_3^2}{64} + \frac{3159g_3^4}{16} & 0 & \frac{1137g_2^4g_3}{32} + \frac{15957g_2g_3^3}{32} & 0 & 0 & 0 \\
0 & -\frac{27g_2^6}{128} - \frac{3879g_2^3g_3^2}{128} - \frac{243g_3^4}{2} & 0 & 0 & 0 & 0 \\
\frac{207g_2^5g_3}{1280} + \frac{9963g_2^2g_3^3}{1280} & 0 & 0 & 0 & 0 & 0
\end{pmatrix} . \tag{7.67}
$$

7.3 Derivation of the holomorphic anomaly equations

We can now proceed with the derivation of the holomorphic anomaly equations. In view of the condition (7.6) we start computing the derivatives of the $\mathcal{T}$-function. Taking the E_2 derivative we get

$$
\begin{aligned}
D_{E_2}\mathcal{T} &= e^{\partial_\epsilon W_{\mathrm{st}}}\eta(\tau)^3 \left(\sigma(ip)e^{\frac{E_2}{24}p^2}, \left(D_{E_2} + D_{E_2}\partial_\epsilon W_{\mathrm{st}} + \frac{\partial_x^2}{24}\right) e^{\mathcal{F}_{\mathrm{st}}^{x,s} - \frac{\rho_{\mathrm{st}}x}{\epsilon}}\right) = \qquad (7.68)\\
&= e^{\partial_\epsilon W_{\mathrm{st}}}\eta(\tau)^3 \left(\sigma(ip)e^{\frac{E_2}{24}p^2}, e^{-\frac{\rho_{\mathrm{st}}x}{\epsilon}} \left(D_{E_2} - \frac{x}{\epsilon}D_{E_2}\rho_{\mathrm{st}} + D_{E_2}\partial_\epsilon W_{\mathrm{st}} + \frac{\partial_x^2}{24} - \frac{\rho_{\mathrm{st}}}{12\epsilon}\partial_x + \frac{(\rho_{\mathrm{st}})^2}{24\epsilon^2}\right) e^{\mathcal{F}_{\mathrm{st}}^{x,s}}\right) ,
\end{aligned}
$$

while the a derivative reads

$$
\partial_a\mathcal{T} = e^{\partial_\epsilon W_{\mathrm{st}}}\eta(\tau)^3 \left(\sigma(ip)e^{\frac{E_2}{24}p^2}, \left(\partial_a + \partial_a\partial_\epsilon W_{\mathrm{st}} + i\pi\frac{\partial\tau}{\partial a}x^2\right) e^{\mathcal{F}_{\mathrm{st}}^{x,s}(a,\Lambda,\epsilon) - \frac{\rho_{\mathrm{st}}x}{\epsilon}}\right) . \tag{7.69}
$$

Here we just used the relations

$$
\partial_a\left(\eta(\tau)^3\sigma(s)e^{-\frac{1}{24}E_2s^2}\right) = -i\pi\frac{\partial\tau}{\partial a}\frac{\partial^2}{\partial s^2}\left(\eta(\tau)^3\sigma(s)e^{-\frac{1}{24}E_2s^2}\right) , \tag{7.70}
$$

$$
(f''(ip), g(x)) = (f(ip), -x^2g(x)) . \tag{7.71}
$$

The previous expression can be further simplified using the following equation

$$
\partial_a e^{\mathcal{F}_{\mathrm{st}}^{x,s}(a,\Lambda,\epsilon)} = \left(\frac{1}{\epsilon}\partial_x - i\pi\frac{\partial\tau}{\partial a}x^2\right) e^{\mathcal{F}_{\mathrm{st}}^{x,s}(a,\Lambda,\epsilon)} , \tag{7.72}
$$

which gives the relation between a and x derivatives and corresponds to the equation which imposes $\mathcal{F}_{\mathrm{st}}^{x,s}$ to be the generating function of the correlation functions of the chiral ring[48]. Substituting in (7.69) we obtain

$$
\partial_a\mathcal{T} = e^{\partial_\epsilon W_{\mathrm{st}}}\eta(\tau)^3 \left(\sigma(ip)e^{\frac{E_2}{24}p^2}, e^{-\frac{\rho_{\mathrm{st}}x}{\epsilon}} \left(\frac{1}{\epsilon}\partial_x + \partial_a\partial_\epsilon W_{\mathrm{st}} - \frac{x}{\epsilon}\partial_a\rho_{\mathrm{st}}\right) e^{\mathcal{F}_{\mathrm{st}}^{x,s}(a,\Lambda,\epsilon)}\right) . \tag{7.73}
$$

Inserting (7.68), (7.73) in (7.6) we obtain the following equation

$$
\begin{aligned}
&\left(D_{E_2} - \frac{x}{\epsilon}D_{E_2}\rho_{\mathrm{st}} + D_{E_2}\partial_\epsilon W_{\mathrm{st}} + \frac{\partial_x^2}{24} - \frac{\rho_{\mathrm{st}}}{12\epsilon}\partial_x + \frac{(\rho_{\mathrm{st}})^2}{24\epsilon^2}\right) e^{\mathcal{F}_{\mathrm{st}}^{x,s}(a,\Lambda,\epsilon)} \qquad (7.74)\\
&- \frac{D_{E_2}\mathbf{u}}{\partial_a\mathbf{u}}\left(\frac{1}{\epsilon}\partial_x + \partial_a\partial_\epsilon W_{\mathrm{st}} - \frac{x}{\epsilon}\partial_a\rho_{\mathrm{st}}\right) e^{\mathcal{F}_{\mathrm{st}}^{x,s}(a,\Lambda,\epsilon)} = 0 ,
\end{aligned}
$$

[48] See [61] appendix B.

where, thanks to the dependence on the source s, we removed the pairing[49]. Dividing (7.74) by $\exp(\mathcal{F}_{\rm st}^{x,s}(a,\Lambda,\epsilon))$ and rearranging some terms we can rewrite it as

$$\begin{aligned}&\left(D_{E_2}\mathcal{F}_{\rm st}^{x,s}(a,\Lambda,\epsilon)+\frac{1}{24}\partial_x^2\mathcal{F}_{\rm st}^{x,s}(a,\Lambda,\epsilon)+\frac{1}{24}\left(\partial_x\mathcal{F}_{\rm st}^{x,s}(a,\Lambda,\epsilon)\right)^2\right)\\&-\left(\frac{\rho_{\rm st}}{12}+\frac{D_{E_2}\mathbf{u}}{\partial_a\mathbf{u}}\right)\frac{1}{\epsilon}\partial_x\mathcal{F}_{\rm st}^{x,s}(a,\Lambda,\epsilon)\\&-\frac{x}{\epsilon}\left(D_{E_2}\rho_{\rm st}-\frac{D_{E_2}\mathbf{u}}{\partial_a\mathbf{u}}\partial_a\rho_{\rm st}\right)+\left(D_{E_2}\partial_\epsilon W_{\rm st}+\frac{(\rho_{\rm st})^2}{24\epsilon^2}-\frac{D_{E_2}\mathbf{u}}{\partial_a\mathbf{u}}\partial_a\partial_\epsilon W_{\rm st}\right)=0\;.\end{aligned}\tag{7.75}$$

Finally, by conjugating the first line of the above equation with the shift operators arising from the algebra of the vector fields (7.20) and expressing the ratio $D_{E_2}\mathbf{u}/\partial_a\mathbf{u}$ in terms of $W_{\rm st},\rho_{\rm st}$, we can rewrite the above equation as (see appendix C.1 and C.3 for details)

$$\begin{aligned}&\left(D_{E_2}\widehat{\mathcal{F}}(y,\Lambda',\epsilon)+\frac{\epsilon^2}{24}\partial_y^2\widehat{\mathcal{F}}(y,\Lambda',\epsilon)+\frac{\epsilon^2}{24}\left(\partial_y\widehat{\mathcal{F}}(y,\Lambda',\epsilon)\right)^2\right)\\&-\left(\left(\frac{y-a}{\epsilon}\right)\partial_a-\partial_\epsilon-\frac{\epsilon\partial_y\mathcal{F}(y,\Lambda',\epsilon)-(y-a)\partial_a\rho/\epsilon+\epsilon\partial_a\partial_\epsilon W}{\partial_a\mathbf{u}}\Lambda\partial_\Lambda\right)\times\\&\times\left(D_{E_2}W_{\rm st}+\frac{\epsilon}{24}(\partial_aW_{\rm st})^2\right)=0\;,\end{aligned}\tag{7.76}$$

where we defined the shifted modulus $y=\epsilon x+a$, the scale $\Lambda'=\Lambda_{\epsilon s}=\Lambda\exp(\epsilon s)$ and[50]

$$\widehat{\mathcal{F}}(a,\Lambda,\epsilon)=\mathcal{F}(a,\Lambda,\epsilon)-\frac{1}{\epsilon^2}\mathcal{F}_0(a,\Lambda)\;.\tag{7.77}$$

By using the separation of scales one can show (see appendix C.3 for details) that (7.76) is equivalent to the holomorphic anomaly equations obeyed by the SD and NS free energies

$$D_{E_2}W_{\rm st}+\frac{\epsilon}{24}(\partial_aW_{\rm st})^2=0\;,\tag{7.78}$$

$$D_{E_2}\widehat{\mathcal{F}}(y,\Lambda',\epsilon)+\frac{\epsilon^2}{24}\partial_y^2\widehat{\mathcal{F}}(y,\Lambda',\epsilon)+\frac{\epsilon^2}{24}\left(\partial_y\widehat{\mathcal{F}}(y,\Lambda',\epsilon)\right)^2=0\;.\tag{7.79}$$

We emphasize here that the formula (7.4) defining the $\mathcal{T}$-function can be obtained using the purely mathematical arguments [18, 76, 78, 79], so that Z_{SD} is given by explicit combinatorial expression [18], and W can be identified with the classical action on the trajectory [15]. Therefore, our derivation of the holomorphic anomaly equations (7.78), (7.79) is actually independent from any physical arguments and uses only the fact that the $\mathcal{T}$-functions of the Painlevé equation can be parameterized by the position of its zero Λ and the value of the Hamiltonian $\mathbf{u}$ at this point, and the fact that the formula (7.4) exists.

[49]The reason for this is that we can move the linear term $xs\Lambda\partial_\Lambda\partial_a\mathcal{F}_0$ coming from the ϵ expansion of $\mathcal{F}_{\rm st}^{x,s}$, to the lhs of the pairing. This corresponds to the substitution $p\to p+\Lambda\partial_\Lambda\partial_a\mathcal{F}_0 s$,. Expanding in p we have that the pairing must vanish for arbitrary linear combinations of $\sigma(s)$ and its derivatives, which are linearly independent functions of s.

[50]The notation $\widehat{\mathcal{F}}(a,\Lambda,\epsilon)$ denotes here the stable part of the SD free energy. This is to distinguish it from the BCOV generating function $\mathcal{F}_{st}^{x,s}$ and is not related in any way to the blowup.

7.4 Painlevé $\mathcal{T}$-function as a non-perturbative completion of topological strings partition function

As reviewed at the beginning of Sect. 7, the perturbative topological strings partition function $\mathcal{Z}_X$ displays a non-holomorphic dependence on the moduli. This can be interpreted as a background dependence of this formulation [62]. On the other hand, the partition function Z_{SD} is obviously holomorphic but it is not modular and as such frame dependent, thus suffering the same problem of background dependence. In this subsection we show that the very structure of the Painlevé $\mathcal{T}$-function contains non-perturbative corrections in the topological string coupling $\epsilon = g_s$ that are suited to cancel the modular anomaly. In particular, we show that there exists a suitable expansion point (7.82) for $\mathcal{T}$ which provides a natural non-perturbative completition of topological strings, whose modularity is explicitly implied by the BCOV holomorphic anomaly equation.

Let us recall eq.(7.4)

$$\mathcal{T}(a,\Lambda,\epsilon,s) = e^{\partial_\epsilon W(a,\Lambda,\epsilon)} \sum_{n\in\mathbb{Z}+\frac{1}{2}} e^{-\frac{n}{\epsilon}\rho(a,\Lambda,\epsilon)} Z_{SD}(a+n\epsilon,\Lambda_{\epsilon s},\epsilon) \ . \tag{7.80}$$

As we saw, the E_2 dependence of (7.80) enters only through the $\mathbf{u}$ modulus. Let us now show that this E_2 dependence can be reabsorbed by fixing the surface observable source $s = s_0$ such that

$$\rho(a,\Lambda,\epsilon) = \epsilon\partial_a W(a,\Lambda,\epsilon) = a_D(a,\Lambda_{SD}) = a_D(a,\Lambda e^{\epsilon s_0(a,\Lambda,\epsilon)}) \ , \tag{7.81}$$

which relates the dual quantum period ρ computed at the NS scale Λ to the SW period a_D at the SD scale[51]

$$\Lambda_{SD} = \Lambda e^{\epsilon s_0(a,\Lambda,\epsilon)} \ . \tag{7.82}$$

Thus, by expanding the $\mathcal{T}$-function (7.80) around s_0, we get

$$\mathcal{T}(a,\Lambda,\epsilon,s'+s_0) = e^{\partial_\epsilon W(a,\Lambda,\epsilon)} \sum_{n\in\mathbb{Z}+\frac{1}{2}} e^{-\frac{n}{\epsilon}a_D(a,\Lambda_{SD})} Z_{SD}(a+n\epsilon,\Lambda_{SD}e^{\epsilon s'},\epsilon) \ . \tag{7.83}$$

Up to a change of the overall normalization of the above $\mathcal{T}$-function, we then define

$$\mathcal{T}_{SD}(a,\Lambda_{SD},\epsilon,s') = e^{-\frac{1}{\epsilon^2}\mathcal{F}_0(a,\Lambda_{SD})} \sum_{n\in\mathbb{Z}+\frac{1}{2}} e^{-\frac{n}{\epsilon}a_D(a,\Lambda_{SD})} Z_{SD}(a+n\epsilon,\Lambda_{SD}e^{\epsilon s'},\epsilon) \ . \tag{7.84}$$

We will now show that this $\mathcal{T}$-function is E_2 independent and as such it is holomorphic and modular, being thus globally defined on the moduli space of X. It then provides a non-perturbative extension of the topological string partition function.

The dependence on τ is now measured at the appropriate scale Λ_{SD}, $\tau = \tau(a,\Lambda_{SD})$. Correspondingly also the algebra of derivations is referred to this scale. Applying the same manipulation as in (7.45)-(7.68) we can write

$$\mathcal{T}_{SD}(a,\Lambda_{SD},\epsilon,s') = \eta(\tau)^3 \left(\sigma(ip)e^{\frac{E_2}{24}p^2}, e^{\mathcal{F}_{\rm st}^{x,s'}(a,\Lambda_{SD},\epsilon)}\right) , \tag{7.85}$$

[51]We observe that in the WKB theory this seems to correspond to a procedure where we just take the classical period but on a "quantum corrected SW curve". The full NS quantum corrections are then directly encoded in this classical effective curve. We thank Fabrizio Del Monte for pointing this out.

Computing the E_2 derivative of $\mathcal{T}_{SD}$ and insisting on modular invariance we get

$$0 = D_{E_2}\mathcal{T}_{SD} = \eta(\tau)^3 \left(\sigma(ip)e^{\frac{E_2}{24}p^2}, \left(D_{E_2} + \frac{\partial_x^2}{24}\right) e^{\mathcal{F}_{\rm st}^{x,s'}(a,\Lambda_{SD},\epsilon)}\right) . \tag{7.86}$$

As we evaluate this quantity at arbitrary s', this implies

$$\left(D_{E_2} + \frac{\partial_x^2}{24}\right) e^{\mathcal{F}_{\rm st}^{x,s'}(a,\Lambda_{SD},\epsilon)} = 0 \ , \tag{7.87}$$

which is nothing but the holomorphic anomaly equation[52] for the generating function. In particular, setting to zero the sources $x, s' = 0$ we obtain also the usual holomorphic anomaly equation

$$\left(D_{E_2} + \frac{\epsilon^2}{24}\partial_a^2\right) e^{\widehat{\mathcal{F}}(a,\Lambda_{SD},\epsilon)} = 0 \ , \tag{7.88}$$

while, computing derivatives with respect to x of (7.87) and then setting $x, s = 0$ one obtains also the holomorphic anomaly equations for the correlation functions of the chiral ring.

Let us remark that (7.87) also appear in [80], were a reference point for the expansion in the SW moduli is taken to be the massless point. The relation between (7.87) and (7.88) can be explained in terms of shift operators based on the algebra (7.20). Namely, it is the non-trivial relation

$$\left(D_{E_2} + \frac{\partial_x^2}{24}\right) e^{\mathcal{F}_{0,\rm st}^{x,s}(a,\Lambda,\epsilon)} e^{\epsilon x\partial_a + \epsilon s\Lambda\partial_\Lambda} e^{\widehat{\mathcal{F}}(a,\Lambda,\epsilon)} = e^{\mathcal{F}_{0,\rm st}^{x,s}(a,\Lambda,\epsilon)} e^{\epsilon x\partial_a + \epsilon s\Lambda\partial_\Lambda} \left(D_{E_2} + \frac{\epsilon^2}{24}\partial_a^2\right) e^{\widehat{\mathcal{F}}(a,\Lambda,\epsilon)} \tag{7.89}$$

proved in the appendix C. We also notice that (7.87) is the usual heat equation, while (7.88) is not, because the derivative operators involved do not commute, see the second line of (7.20).

So we proved that E_2-independence of the $\mathcal{T}$-function implies the holomorphic anomaly equations. Clearly also the converse is true, the holomorphic anomaly equations (7.87) imply that $\mathcal{T}_{SD}$ is E_2 independent. Thus, the invariant $\mathcal{T}$-function $\mathcal{T}_{SD}$ is a natural candidate for a non-perturbative partition function of the corresponding topological string

$$\mathcal{T}_{SD} = \mathcal{Z}_X^{NP} \ . \tag{7.90}$$

8 Hurwitz expansions of Painlevé $\mathcal{T}$-functions

In [65] and [66] Hone, Ragnisco and Zullo (HRZ) found numerical evidence that the Painlevé $\mathcal{T}$-functions of PI, PII and PIV enjoy Hurwitz integrality when expanded around one of their zeros. Indeed, this can be seen as a natural deformation of the Hurwitz expansion (A.20) of the Weierstrass σ-function, while the recurrence relations for the corresponding coefficients of the deformed expansions generalizes the one of the coefficients of σ.

[52]In [61] in the equation for the generating function an extra term appears due to the contributions of two insertions of colliding chiral fields. However, this term explicitly depends on the Kähler potential and decouples in the holomorphic limit.

As we already observed in section (6), from the gauge theory point of view the expansion of the $\mathcal{T}$-function around a zero corresponds to the expansion of the blowup factor in terms of the source s of the surface observable $I(E)$ in the NS limit. It is then possible to determine all the coefficients of the blowup formula recursively from the Painlevé equations.

In this section we explicitly compute these recursion relations, generalizing the results of HRZ. As previously said, the relation (5.11) between the Painlevé $\mathcal{T}$-function and the blowup factor $\mathcal{B}_{NS}$, being based on operator/state correspondence in the topological theory, holds in general, also in the Argyres-Douglas cases[53]. We reproduce the Hurwitz integral expansions for the AD theories H_0 and H_2, which correspond to PI and PIV equations respectively, in a form which is more suitable to a gauge theory analysis and find new expansions for the Lagrangian theories with $N_f = 0, 1, 4$ corresponding to PIII$_3$, PIII$_2$, PVI respectively. In the case of PIII$_2$ we will show explicitly how one can recover the PI Hurwitz integral expansion from the one of PIII$_2$.

Before starting the detailed analysis we explain here the general strategy. The $\mathcal{T}$-function of Painlevé equations obeys an Hirota bilinear differential equation of the form

$$D(\mathcal{T}, \mathcal{T}) = 0 \ , \tag{8.1}$$

where $D(f, g)$ is generically a quartic order bilinear differential operator in the Painlevé time t (whose specific form depends on the specific equation) which is a combination of ordinary derivatives and Hirota derivatives $D_x^{(k)}(f, g)$ defined as

$$f(x+h)g(x-h) = \sum_{k=0}^{\infty} D_x^{(k)}(f,g)\frac{h^k}{k!} \ \Rightarrow \ D_x^{(k)}(f,g) = \frac{\mathrm{d}^k}{\mathrm{d}h^k} f(x+h)g(x-h)\bigg|_{h=0} \ . \tag{8.2}$$

Following the gauge theory interpretation, we consider then a zero t_0 of the $\mathcal{T}$-function and change the time variable $t = t(\epsilon, s)$ in such a way that $s = 0$ corresponds to the position of the zero t_0, and in the limit $\epsilon \to 0$ the equation becomes an autonomous equation in s. From the gauge theory point of view t_0 corresponds to the value of the gauge coupling Λ and we interpret the map $t = t(\epsilon, s)$ as the shifted[54] gauge coupling $\Lambda_{\epsilon s}$ due to the insertion of the surface observable $I(E)$.

From (5.11) we observe that $\mathcal{T}$ has a singular prefactor in the limit $\epsilon \to 0$. Therefore, to get a sensible limit, it is convenient to rewrite the equation in terms of $\mathcal{B}_{NS}$ which is regular in the limit $\epsilon \to 0$

$$\mathcal{B}_{NS}(\epsilon, s) = e^{-\frac{\mathbf{u}s}{\epsilon}} \mathcal{T}(\epsilon, s) \ . \tag{8.3}$$

In all cases, in the autonomous limit $\epsilon \to 0$ the Hirota equation reduces to

$$D_s^{(4)}\left(\mathcal{B}_{SW}, \mathcal{B}_{SW}\right) + 12 T D_s^{(2)}\left(\mathcal{B}_{SW}, \mathcal{B}_{SW}\right) + (12T^2 - g_2)\mathcal{B}_{SW}^2 = 0 \ , \tag{8.4}$$

[53] A consistency check is to show that the solution of the AD case corresponds exactly to the limit of the Lagrangian theory solution around the AD point. We will show explicitly this for the limit $(N_f = 1) \to H_0$.

[54] The precise form of the shift depends on the theory. In the Lagrangian theories the shift is multiplicative $t = t_0 \exp(\epsilon s)$. In the AD theories the shift is additive $t = t_0 + \epsilon^k s$, with $[s] = -k$.

where g_2, T depend on the parameters of the theory. In this limit the solution $\mathcal{B}_{NS}(\epsilon, s)$ reduces, up to a Gaussian prefactor which gives the contact term T, to the Weierstrass σ-function as

$$\mathcal{B}_{SW}(s) = \lim_{\epsilon \to 0} \mathcal{B}_{NS}(\epsilon, s) = \Lambda e^{-\frac{1}{2}Ts^2} \sigma(s; g_2, g_3) , \tag{8.5}$$

in agreement with the gauge theory result in the SW limit (6.12). The parameter g_3 can be determined from the sigma-form of the Painlevé equation, which is the equation for the Painlevé hamiltonian $\eta = \partial_s \log \mathcal{B}_{NS}$. In the autonomous limit $\epsilon \to 0$ the hamiltonian η always reduces to the form $\eta_{SW} = \zeta_W(s; g_2, g_3) - sT$ where $\zeta_W(s; g_2, g_3)$ is the Weierstrass zeta function (see appendix A) and the equation has the form[55]

$$\ddot{\eta}_{SW}^2 + 4\dot{\eta}_{SW}^3 + 12T\dot{\eta}_{SW}^2 + (12T^2 - g_2)\dot{\eta}_{SW} + 4T^3 - g_2 T + g_3 = 0 , \quad \eta_{SW} = \frac{\partial}{\partial s} \log \mathcal{B}_{SW} , \tag{8.6}$$

from which we can easily read the value of the parameter g_3 once T and g_2 are known. The parameters g_2, g_3 are exactly the elliptic invariants given by the Weierstrass parametrization of the SW curve.

In the Painlevé equation in Hirota bilinear form (8.1) we then substitute the power series expression

$$\mathcal{B}_{NS}(\epsilon, s) = b_0 \sum_{n=0}^{\infty} c_n \frac{s^{n+1}}{(n+1)!} , \tag{8.7}$$

which means that $\mathcal{B}_{NS}(\epsilon, s)$ is expanded (in s) around its zero t_0. This is motivated in the gauge theory by the possibility of expressing $\mathcal{B}_{NS}$ as a sum of local operators given by a series in s as defined by the differential operator D_{NY} in the NY blowup equations in presence of the surface observable $I(E)$ (6.19).

To construct the recursion relation it is useful to write the action of the Hirota derivative (8.2) in the power basis s^n. One has

$$\frac{1}{n!m!} D_s^{(k)}(s^n, s^m) = P_{nm}^k s^{n+m-k} , \tag{8.8}$$

where the coefficients P_{nm}^k are given by[56]

$$P_{nm}^k = \frac{k!}{n!m!} \sum_{l=0}^{k} (-1)^l \binom{n}{l} \binom{m}{k-l} , \quad P_{mn}^k = (-1)^k P_{nm}^k , \tag{8.9}$$

$$P_{nm}^0 = \frac{1}{n!m!} , P_{nm}^k = 0 \text{ if } n+m<k . \tag{8.10}$$

Therefore, using for $\mathcal{B}_{NS}$[57] the ansatz (8.7), we can write

$$D_s^{(k)}(\mathcal{B}_{NS}, \mathcal{B}_{NS}) = \sum_{n,m=0}^{+\infty} P_{n+1,m+1}^k c_n c_m = s^{4-k} \sum_{n=0}^{+\infty} s^{n-2} F_n^k , \quad F_n^k = \sum_{l=0}^{n} P_{l+1,n-l+1}^k c_l c_{n-l} . \tag{8.11}$$

[55]This equation follows simply from the equation of the SW curve in Weierstrass parametrization written in terms of η.

[56]We use the convention that $\binom{n}{k} = 0$ if $n < k$.

[57]In the following we set $b_0 = 1$. It can be restored simply rescaling $\mathcal{B}_{NS} \to b_0 \mathcal{B}_{NS}$ thanks to bilinearity of the Hirota equation.

The quantities F_n^k are the basic building blocks and the recursive relation can be written as linear combinations of the F_n^k's. To obtain more compact expressions we follow the convention $F_n^k = 0$ if $n < 0$. For all Painlevé equations the bilinear form (8.1) contains a quartic Hirota derivative $D_s^{(4)}(\mathcal{B}_{NS}, \mathcal{B}_{NS})$ and this is the term of maximal degree. We can write

$$F_n^4 = {F'}_n^4 + 2c_n P_{n+1,1}^4 \ , \quad {F'}_n^4 = \sum_{l=1}^{n-1} P_{l+1,n-l+1}^4 c_l c_{n-l} \ , \quad P_{n+1,1}^4 = \frac{n(n^2-1)(n-6)}{(n+1)!} \ , \tag{8.12}$$

where we isolated the factor containing the coefficient c_n form the others and used $c_0 = 1$. The operators s and ∂_s act as shift operators in the basis[58] s^{n-2}

$$s\sum_{n=0}^{+\infty} s^{n-2}\psi_n = \sum_{n=0}^{+\infty} s^{n-2}\psi_{n-1} \equiv \sum_{n=0}^{+\infty} s^{n-2}\hat{S}\psi_n \ , \quad \hat{S}\psi_n = \psi_{n-1} \ , \tag{8.13}$$

$$\frac{\partial}{\partial s}\sum_{n=0}^{+\infty} s^{n-2}\psi_n = \sum_{n=0}^{+\infty} s^{n-2}(n-1)\psi_{n+1} \equiv \sum_{n=0}^{+\infty} s^{n-2}\hat{S}^\dagger\psi_n \ , \quad \hat{S}^\dagger\psi_n = (n-1)\psi_{n+1} \ . \tag{8.14}$$

Therefore we obtain the following mapping

$$D_s^{(k)}(\mathcal{B}_{NS}, \mathcal{B}_{NS}) \to \hat{S}^{4-k} F_n^k \ , \quad s \to \hat{S}, \quad \frac{\partial}{\partial s} \to \hat{S}^\dagger \ . \tag{8.15}$$

Using this representation we obtain a recursion relation of the form

$$\frac{2n(n^2-1)(n-6)}{(n+1)!} c_n = -{F'}_n^4 + \sum_{k<4}\sum_{j<n} q_{jk} F_j^k \ , \tag{8.16}$$

where the coefficients q_{jk} depends on t_0, ϵ and the parameters of the specific Painlevè equation at hand and will be analyzed in full detail in the following subsections. It turns out that for $n = 0, 1, 6$ both sides of (8.16) vanish identically and the corresponding coefficients c_0, c_1, c_6 are undetermined. These coefficients correspond to *resonances* that give a parametric family of solutions and are related to the assignment of initial conditions for the equation. These free coefficients can be fixed from the gauge theory. The resonance at c_0 corresponds to the freedom of changing the normalization of the $\mathcal{T}$-function. With our choice of normalization it corresponds to the first coefficient of the blowup factor $\mathcal{B}_{NS}(s) = b_0 s + \ldots$ therefore we have always $c_0 = 1$. The coefficient c_1 corresponds to the s^2 term and it is related to the pole in the prefactor $\exp(\mathbf{u}s/\epsilon)$. We have $c_1 = 0$ because in the blowup factor $\mathcal{B}_{NS}$ this pole is removed. Finally, we consider the coefficient c_6.

The origin of this resonance is that the Hirota equation for $\mathcal{B}_{NS}$ is obtained by one derivative of the corresponding sigma-form equation for the Painlevé Hamiltonian η. This produces a further integration constant. This constant, which is related to the elliptic invariant g_3, determines the value of the coefficient c_6 which must be then fixed imposing that the solution $\mathcal{B}_{NS}$ satisfies the original sigma-form equation.

[58]The reason for the shift by -2 is that the leading order in the recursion relation is given by $D_s^{(4)}(\mathcal{B}_{NS}, \mathcal{B}_{NS})$.

It is clear from the recursion relation (8.16) that c_n are polynomials in the parameters of the theory and for dimensional reasons they are homogeneous polynomials. From the limit (8.5) we see that the coefficients c_n are deformations in ϵ of the Weierstrass σ-function coefficients c_n^σ.

The non-trivial result, which we checked numerically[59], but for which we don't have a proof, is that the polynomials c_n turn out to have integer coefficients. This property was already noticed in [65, 66] for the cases of PI, PII and PIV and we checked that it remains true also in the other cases.

In the following we will apply the analysis outlined above to Painlevé equations in Hirota form to derive the explicit structure of the recursion relation and of the coefficients c_n of the (8.7) expansion showing that it is an integral Hurwitz series in the ring of polynomials with integer coefficients of the corresponding invariants and we will comment on their specific structure.

Finally, sometimes it will be convenient to organize the expansion in terms of the IR parameters g_2, g_3, T of the SW curve also when $\epsilon \neq 0$. In this case, however, they will be evaluated in the NS modulus $\mathbf{u}$ and we will denote them by $\mathbf{g_2}, \mathbf{g_3}, \mathbf{T}$. We remark that in this section the normalization for the u modulus is different from the previous ones and is fixed by the explicit parametrization of the SW curve we use.

8.1 PVI alias $N_f = 4$

We start the analysis from $SU(2)$ gauge theory with $N_f = 4$ which corresponds to the PVI equation. The theory depends on the four masses m_i of the hypermultiplets and one dimensionless UV coupling $q = \exp(i\pi\tau_0)$. In the massless case $m_i = 0$ the UV coupling has no perturbative corrections, because the perturbative beta function vanishes, but is renormalized non-perturbatively by instanton corrections[60] as

$$q = \exp(i\pi\tau_0) = \frac{1}{16}\frac{\theta_2^4(\tau)}{\theta_3^4(\tau)} , \tag{8.17}$$

where τ is the IR effective coupling. The SW curve for this theory is

$$y^2 = x(x-u)(x-qu)-x^2(1-q)^2e_2-4x(1-q)q(2(1+q)p_4+(1-q)e_4)+16(1-q)q^2(up_4-(1-q)e_6), \tag{8.18}$$

the parameters e_k, p_4 are symmetric polynomials in the m_j's

$$e_2 = m_1^2 + m_2^2 + m_3^2 + m_4^2 , \quad e_4 = m_1^2m_2^2 + m_1^2m_3^2 + m_1^2m_4^2 + m_2^2m_3^2 + m_2^2m_4^2 + m_3^2m_4^2 , \tag{8.19}$$

$$e_6 = m_1^2m_2^2m_3^2 + m_1^2m_2^2m_4^2 + m_1^2m_3^2m_4^2 + m_2^2m_3^2m_4^2 , \quad p_4 = m_1m_2m_3m_4 , \tag{8.20}$$

and they make manifest the $SO(8)$ flavour symmetry of the theory. The PVI equation is the most general Painlevé equation, all the others can be obtained as coalescence limits of PVI scaling the parameters in a suitable way. In the class S construction of the theory in

[59]We checked this to very high order, $n \sim 100$.

[60]See e.g. [81].

terms of the Gaiotto curve this operation corresponds to colliding some punctures to form irregular ones. This is the geometrical realization of the renormalization group flow for the corresponding gauge theories. Later we will work in detail how to take the coalescence limit to obtain the simplest AD theory H_0 from the $N_f = 1$ Lagrangian theory.

We start now the analysis of the PVI equation. The Hirota bilinear equation is [60]

$$
\begin{aligned}
&(t-1)^3 D^{(4)}_{\log t}(\mathcal{T},\mathcal{T}) + 2(t-1)^2(1+t)\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right) D^{(2)}_{\log t}(\mathcal{T},\mathcal{T}) \\
&+ (1-t)(e_2' t^2 - t^2 + t - 1) D^{(2)}_{\log t}(\mathcal{T},\mathcal{T}) + t(t-1)\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 \mathcal{T}^2 \\
&- e_2' t^2 \left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)\mathcal{T}^2 + t(te_4' + 2(t-2)p_4')\mathcal{T}^2 = 0 \ ,
\end{aligned} \tag{8.21}
$$

while the Hamiltonian $\zeta(t) = t(t-1)\partial_t \log \tau(t)$ satisfies the sigma-form equation

$$
\begin{aligned}
&t^2(1-t)^2\ddot{\zeta}^2 - 4t(1-t)\dot{\zeta}^3 - (e_2' t^2 + 4(2t-1)\zeta)\dot{\zeta}^2 - (4p_4' - (e_4' + 2p_4')t - 2e_2' t\zeta - 4\zeta^2)\dot{\zeta} \\
&- (e_4' + 2p_4')\zeta - e_2'\zeta^2 - e_6' = 0 \ ,
\end{aligned} \tag{8.22}
$$

where e_k', p_4' are the symmetric polynomials in the normalized mass parameters $\mu_i = m_i/\epsilon$. To get the autonomous limit we use the mass parameters m_j

$$
\mu_j = \frac{m_j}{\epsilon} \ \Rightarrow \quad e_k' = \frac{e_k}{\epsilon^k} \ , \quad p_4' = \frac{p_4}{\epsilon^4} \ . \tag{8.23}
$$

For the Painlevé time we use the following parametrization

$$
t = q e^{(q-1)\epsilon s} \ , \tag{8.24}
$$

where we notice that in the formula an extra factor $(q-1)$ appears. From the point of view of the Painlevé VI equation this factor is required to cancel poles at $q = 1$ which is a non-movable singularity of PVI. In the gauge theory the map (8.24) gives the shift of the coupling q due to the insertion of the surface observable $I(E)$. This suggests that in the conformal case the surface observable contribution contains a non-trivial renormalization factor $(q-1)$, which is non-perturbative, similarly to the coupling renormalization (8.17). The expression for the $\mathcal{T}$-function (5.11) and for the hamiltonian ζ becomes

$$
\mathcal{T}(s) = e^{\frac{\mathbf{u}(q-1)s}{\epsilon}} \mathcal{B}_{NS}(s) = e^{\frac{\mathbf{u}_r s}{\epsilon}} \mathcal{B}_{NS}(s) \ , \quad \zeta(s) = \epsilon^{-1} r(t)\eta(s) + r(t)\epsilon^{-2}\mathbf{u}_r, \tag{8.25}
$$

where we defined $\mathbf{u}_r = \mathbf{u}(q-1)$ and the ratio

$$
r(t) = \frac{t-1}{q-1} \ , \quad r(t) \underset{\epsilon\to 0}{\to} 1 \ . \tag{8.26}
$$

Using (8.23), (8.24) the Hirota equation reads

$$
\begin{aligned}
&r(t)^3 D^{(4)}_s(\mathcal{B}_{NS},\mathcal{B}_{NS}) + 2\epsilon r(t)^2(t+1)\frac{\partial}{\partial s} D^{(2)}_s(\mathcal{B}_{NS},\mathcal{B}_{NS}) \\
&+ r(t)(4\mathbf{u}_r r(t)(1+t) - e_2 t^2 + \epsilon^2(t^2 - t + 1)) D^{(2)}_s(\mathcal{B}_{NS},\mathcal{B}_{NS}) \\
&+ \epsilon^2 r(t) t \frac{\partial^2}{\partial s^2}\mathcal{B}_{NS}^2 + \epsilon t\,(4r(t)\mathbf{u}_r - e_2 t)\frac{\partial}{\partial s}\mathcal{B}_{NS}^2 \\
&+ \left(4r(t)t\mathbf{u}_r^2 - 2t^2\mathbf{u}_r e_2 + t(q-1)(e_4 t + 2(t-2)p_4)\right)\mathcal{B}_{NS}^2 = 0 \ ,
\end{aligned} \tag{8.27}
$$

and the sigma-form equation becomes

$$
\begin{aligned}
&r(t)^2\left(r(t)\ddot{\eta}+\epsilon(t+1)\dot{\eta}\right)^2+4r(t)^4\dot{\eta}^3+r(t)^2\left(4r(t)(1+t)(\mathbf{u}_r+\epsilon\eta)-e_2t^2\right)\dot{\eta}^2\\
&+(4r(t)^2t(\mathbf{u}_r+\epsilon\eta)^2-2r(t)t^2e_2(\mathbf{u}_r+\epsilon\eta)+t(t-1)(2p_4(t-2)+e_4t))\dot{\eta}\\
&-t^2\left(e_6(q-1)^2+(2p_4-e_4)(\mathbf{u}_r+\epsilon\eta)(q-1)+e_2(\mathbf{u}_r+\epsilon\eta)^2\right)=0\ . \qquad (8.28)
\end{aligned}
$$

In the limit $\epsilon\to 0$ we have $\mathbf{u}\to u$ and the equations (8.27), (8.28) reduce to the autonomous form (8.4), (8.6) respectively, with the following values of the parameters

$$
\begin{aligned}
T_r^{PVI}&=\frac{u_r}{3}(1+q)-\frac{1}{12}e_2q^2\ ,\\
g_{2,r}^{PVI}&=-q(q-1)(2p_4(q-2)+e_4q)+\frac{1}{12}e_2^2q^4-\frac{2}{3}e_2q^2(q-2)u_r+\frac{4}{3}(1-q+q^2)u_r^2\ ,\\
g_{3,r}^{PVI}&=-q^2(q-1)^2e_6+\frac{1}{12}e_2(q-1)q^3(2p_4(q-2)+e_4q)-\frac{1}{216}e_2^3q^6\\
&\quad+\frac{1}{18}q\left((q-2)q(-6e_4(q-1)+e_2^2q^2)-12p_4(2-4q+q^2+q^3)\right)u_r\\
&\quad-\frac{1}{9}e_2q^2(5-5q+2q^2)u_r^2+\frac{4}{27}(2-3q-3q^2+2q^3)u_r^3\ .
\end{aligned}
$$

In the massless case $m_i=0$ they reduce to

$$
T_r^{PVI}=\frac{u_r}{3}(1+q)\ ,\quad g_{2,r}^{PVI}=\frac{4}{3}(1-q+q^2)u_r^2\ ,\quad g_{3,r}^{PVI}=\frac{4}{27}(2-3q-3q^2+2q^3)u_r^3\ . \qquad (8.29)
$$

The variables are not directly the physical parameters of the theory, because of the renormalization of the shifted coupling (8.24), which corresponds to the replacement $s\to(q-1)s$ in the SW result. This means that in the SW limit we have

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\mathcal{B}_{NS}^{PVI}(\epsilon,s)\propto e^{-\frac{1}{2}T^{PVI}(q-1)^2s^2}\sigma(s(q-1);g_2,g_3)= \qquad (8.30)\\
&=(q-1)e^{-\frac{1}{2}T^{PVI}(q-1)^2s^2}\sigma(s;(q-1)^4g_2,(q-1)^6g_3)=(q-1)e^{-\frac{1}{2}T_r^{PVI}s^2}\sigma(s;g_{2,r},g_{3,r})\ ,
\end{aligned}
$$

where we used the homogeneity of σ. Therefore, the physical variables are

$$
T^{PVI}=\frac{1}{(q-1)^2}T_r^{PVI}=-\frac{u}{3}\frac{1+q}{1-q}-\frac{1}{12}e_2'\left(\frac{q}{1-q}\right)^2\ ,\quad g_2^{PVI}=\frac{1}{(q-1)^4}g_{2,r}^{PVI}\ ,\quad g_3^{PVI}=\frac{1}{(q-1)^6}g_{3,r}^{PVI}\ . \qquad (8.31)
$$

in particular, in the massless limit we have (substituting back $u_r=(q-1)u$)

$$
T^{PVI}=-\frac{u}{3}\frac{1+q}{1-q}\ ,\quad g_2^{PVI}=\frac{4}{3}\frac{(1-q+q^2)}{(q-1)^2}u^2\ ,\quad g_3^{PVI}=\frac{4}{27}\frac{(2-3q-3q^2+2q^3)}{(q-1)^3}u^3\ , \qquad (8.32)
$$

which is consistent with the SW result[61].

As previously discussed, the recurrence relation for the coefficients of the Hurwitz expansion can be obtained expressing all the operators in the power basis s^n. Applying

---

[61] See appendix B.

the map (8.15) to (8.27) we obtain the following recurrence relation

$$\begin{aligned}\frac{2n(n^2-1)(n-6)}{(n+1)!}c_n =& -F'^4_n + (r(\hat t)^3-1)F^4_n + 2\epsilon r(\hat t)^2(\hat t+1)(n-1)F^2_{n-1}\\ &+r(\hat t)(4\mathbf{u}_r r(\hat t)(1+\hat t) - e_2\hat t^2 + \epsilon^2(\hat t^2-\hat t+1))F^2_{n-2}\\ &+\epsilon^2 r(\hat t)\hat t(n-1)nF^0_{n-2} + \epsilon\hat t\left(4r(\hat t)\mathbf{u}_r - e_2\hat t\right)(n-1)F^0_{n-3}\\ &+\left(4r(\hat t)\hat t\mathbf{u}_r^2 - 2\hat t^2\mathbf{u}_r e_2 + \hat t(q-1)(e_4+2(\hat t-2)p_4)\right)F^0_{n-4}\ ,\end{aligned} \tag{8.33}$$

where we defined the operator $\hat t = q\exp\Big(\epsilon(q-1)\hat S\Big)$. Solving the recurrence relation we obtain that the coefficients c_n have the following structure.

$$c_n^{PVI} = \sum_{2j+k+2i_1+4i_2+6i_3+h+l=n} a_{jkli_1i_2i_3h}\mathbf{u}^j q^k \left(\frac{e_2}{4}\right)^{i_1}\left(\frac{e_4}{2}\right)^{i_2} e_6{}^{i_3} p_4{}^h\left(\frac{\epsilon}{2}\right)^l\ , \tag{8.34}$$

i.e. they are homogeneous polynomials of all the parameters of the theory $\mathbf{u}, q, \vec m$, where the coefficients $a_{jkli_1i_2i_3h}$ are computed recursively using (8.33) and we verified that they are integers to very high order in n. The values of the first c_n^{PVI}'s are written in the appendix E.1.

It is interesting to notice that this expansion actually reorganizes in a non-trivial way in the SW limit where the coefficients c_n^{PVI} reduce just to the Weierstrass σ-function coefficients c_n^σ, after we removed the contact term Gaussian prefactor. This implies that in the SW limit the coefficients c_n are completely fixed by the invariants g_2, g_3 of the SW curve and by the contact term T.

8.2 PIV alias Argyres-Douglas H_2

As a first example of AD theory we consider H_2 which is obtained from the $N_f=3$ theory (PV equation) taking the coalescence limit, i.e. namely running the renormalization group flow from $N_f=3$ to the AD H_2 singularity. The resulting theory has two relevant massive deformations $\tilde m$ and m_-. The operator u has dimension $[u]=3/2$ and the corresponding coupling Λ has dimension $[\Lambda]=1/2$. The SW curve is

$$y^2 = x^2 + 2\Lambda x + (2\tilde m+\Lambda^2) + \frac{u+2\Lambda m_-}{x} + \frac{m_-^2}{x^2}\ . \tag{8.35}$$

The correspondent Painlevé equation is PIV. Its Hirota bilinear form reads[62]

$$D_t^{(4)}(\mathcal{T},\mathcal{T}) - (t^2-8(\theta_\infty+3\theta_0))D_t^{(2)}(\mathcal{T},\mathcal{T}) + t\frac{\partial}{\partial t}\mathcal{T}^2 + 32\theta_0(\theta_0+\theta_\infty)\mathcal{T}^2\ , \tag{8.36}$$

where θ_0, θ_∞ are the monodromy parameters. The sigma-form is

$$\ddot\zeta^2 - (t\dot\zeta-\zeta)^2 + 4\dot\zeta(\dot\zeta+4\theta_0)(\dot\zeta+2\theta_0+2\theta_\infty) = 0\ , \tag{8.37}$$

[62]The equation we use is a reparametrization of the one in [66]. Our parametrization is convenient to take the autonomous limit of PIV and recover the SW theory for H_2.

and the autonomous limit is obtained with the following change of variables

$$t = t_0 + \epsilon^{\frac{1}{2}} s \ , \quad \theta_0 = \frac{m_-}{\epsilon} \ , \quad \theta_\infty = -\frac{\tilde{m}}{\epsilon} \ , \quad \Lambda = \epsilon^{\frac{1}{2}} t_0 \ , \tag{8.38}$$

$$\mathcal{T}(s) = e^{\frac{\mathbf{u}s}{\epsilon}} \mathcal{B}_{NS}(s) \ , \quad \zeta = \epsilon^{-\frac{1}{2}} \eta + \epsilon^{-\frac{3}{2}} \mathbf{u} \ . \tag{8.39}$$

In these variables the equation (8.36) reads

$$\begin{aligned} &D_s^{(4)} (\mathcal{B}_{NS}, \mathcal{B}_{NS}) + (12T + 2\Lambda\epsilon s + \epsilon^2 s^2) D_s^{(2)} (\mathcal{B}_{NS}, \mathcal{B}_{NS}) \\ &+ \epsilon(\Lambda + \epsilon s) \frac{\partial}{\partial s} \mathcal{B}_{NS}^2 + 2\mathbf{u}\epsilon s \mathcal{B}_{NS}^2 + (12T^2 - \mathbf{g_2}) {\mathcal{B}_{NS}}^2 = 0 \ , \end{aligned} \tag{8.40}$$

the sigma-form becomes

$$\ddot{\eta}^2 - ((\Lambda + \epsilon s)\dot{\eta} - \epsilon\eta - \mathbf{u})^2 + 4\dot{\eta}(\dot{\eta} + 4m_-)(\dot{\eta} + 2m_- - 2\tilde{m}) = 0 \ , \tag{8.41}$$

and the values of the parameters of the autonomous equation (8.4) are

$$T^{PIV} = -\frac{\Lambda^2}{12} + 2m_- - \frac{2}{3}\tilde{m} \ , \quad g_2^{PIV} = 16m_-^2 - 2u\Lambda - 4m_-\Lambda^2 + \frac{1}{12}(8\tilde{m} + \Lambda^2)^2 \ , \tag{8.42}$$

$$g_3^{PIV} = -u^2 - 2T^{PIV} u\Lambda + 16m_-^2 T^{PIV} - 4(T^{PIV})^3 - 4m_- T^{PIV} \Lambda^2 + \frac{1}{12} T^{PIV} (8\tilde{m} + \Lambda^2)^2 \ . \tag{8.43}$$

Let us observe that all the dependence on the masses is reabsorbed in the parameters T, g_2, g_3. The recursion relation for the Hurwitz expansion (8.7) is

$$\begin{aligned} &\frac{2n(n^2-1)(n-6)}{(n+1)!} c_n = \\ &= -F'^4_n - 12TF^2_{n-2} + \Lambda\epsilon(2F^2_{n-3} - (n-1)F^0_{n-3}) + \epsilon^2(F^2_{n-4} - (n-2)F^0_{n-4}) \\ &\quad + (\mathbf{g_2} - 12T^2)F^0_{n-4} - 2\mathbf{u}\epsilon F^0_{n-5} \ , \end{aligned}$$

and the coefficients of the Hurwitz expansion have the following structure

$$c_n^{PIV} = \sum_{4j+6k+2p+q+3r+2l=n} a_{jkpqrl} \left(\frac{\mathbf{g_2}}{2}\right)^j (2\mathbf{g_3})^k T^p \mathbf{u}^r \Lambda^q \epsilon^l \ . \tag{8.44}$$

As in the previous cases, the coefficients a_{jkpql} are integers. Values of these coefficients are written in the appendix E.2. In the autonomous limit we recover again the Weierstrass σ-function coefficients c_n^σ (up to contact term).

A similar analysis can be done for the AD theory H_1, decoupling one of the mass parameters. The corresponding Painlevé equation is PII and the structure of the expansion remains the same and we have only to change the parametrizations of g_2, g_3, T.

8.3 PIII$_2$ alias $N_f = 1$

As an example of how we can get the AD theory from the coalescence limit of Painlevé equations, we consider now $N_f = 1$ theory which is the simplest $SU(2)$ Lagrangian theory with matter. The corresponding Painlevé equation is PIII$_2$.

The theory has an AD point where extra massless mutually non-local d.o.f. appear, and for this reason the theory at this point corresponds to an *interacting* Coulomb phase. This is the AD theory H_0, and the corresponding Painlevé equation is PI. In this subsection we will show how we can obtain it from a limit of PIII$_2$ equation. In the following subsection we will study then in detail the H_0 theory using the PI equation.

We start now the analysis of PIII$_2$. The SW curve for $N_f = 1$ is

$$y^2 = (x^2 - u)^2 - 4\Lambda^3(x + m) \ , \tag{8.45}$$

where m is the mass of the hypermultiplet. The Hirota bilinear equation for PIII$_2$ is [60]

$$D^{(4)}_{\log t}(\mathcal{T}, \mathcal{T}) - 2t\frac{\partial}{\partial t}D^{(2)}_{\log t}(\mathcal{T}, \mathcal{T}) + D^{(2)}_{\log t}(\mathcal{T}, \mathcal{T}) + 4t\mathcal{T}^2 = 0 \ , \tag{8.46}$$

and the sigma form equation for the hamiltonian $\zeta = t\partial_t \log \mathcal{T}$ is

$$(t\ddot{\zeta})^2 - 4\dot{\zeta}^2(\zeta - t\dot{\zeta}) + 4\dot{\zeta} = \frac{1}{\theta^2} \ . \tag{8.47}$$

We observe that the mass parameter $\theta = m/\epsilon$ does not appear directly in the Hirota form, but it appears in the parametrization of the elliptic invariants g_2, g_3 and in the sigma-form equation.

We now proceed with the derivation of the Hurwitz expansion. As usual, to obtain the autonomous limit we consider a time t_0 and we change variable as follows

$$t = t_0 e^{\epsilon s} \ , \quad \theta = \frac{m}{\epsilon} \ , \quad m\Lambda^3 = \epsilon^4 t_0 \ , \quad \mathcal{T}(s) = e^{\frac{\mathbf{u}s}{\epsilon}}\mathcal{B}_{NS}(s) \ , \quad \zeta(s) = \epsilon^{-1}\eta(s) + \epsilon^{-2}\mathbf{u} \ . \tag{8.48}$$

The Hirota equation becomes

$$D^{(4)}_s(\mathcal{B}_{NS}, \mathcal{B}_{NS}) - 2\epsilon\frac{\partial}{\partial s}D^{(2)}_s(\mathcal{B}_{NS}, \mathcal{B}_{NS}) + (\epsilon^2 - 4\mathbf{u})D^{(2)}_s(\mathcal{B}_{NS}, \mathcal{B}_{NS}) + 4m\Lambda^3 e^{\epsilon s}{\mathcal{B}_{NS}}^2 = 0 \ , \tag{8.49}$$

the sigma-form in autonomous variables is

$$(\ddot{\eta} - \epsilon\dot{\eta})^2 + 4\dot{\eta}^3 - 4\epsilon\dot{\eta}^2\eta - 4\mathbf{u}\dot{\eta}^2 + 4m\Lambda^3 e^{\epsilon s}\dot{\eta} = \Lambda^6 e^{2\epsilon s} \ , \tag{8.50}$$

and taking the limit $\epsilon \to 0$ we can find the contact term T and the elliptic invariants g_2, g_3

$$T^{PIII_2} = -\frac{u}{3} \ , \quad g_2^{PIII_2} = \frac{4}{3}u^2 - 4m\Lambda^3 \ , \quad g_3^{PIII_2} = -\frac{8}{27}u^3 + \frac{4}{3}um\Lambda^3 - \Lambda^6 \ . \tag{8.51}$$

Inserting the ansatz (8.7) in (8.49) we obtain the recursion for the coefficients c_n

$$\frac{2n(n^2-1)(n-6)}{(n+1)!}c_n = -{F'}^4_n + 2\epsilon(n-1)F^2_{n-1} - (\epsilon^2 - 4\mathbf{u})F^2_{n-2} - 4m\Lambda^3 e^{\epsilon\hat{S}}F^0_{n-4} \ , \tag{8.52}$$

where $\hat{S}$ is the shift operator[63] defined in (8.13). Therefore the polynomials c_n have the following[64] structure

$$c_n^{PIII_2} = \sum_{4j+6k+2p+l=n} a_{jkpl}\left(\frac{\mathbf{g_2}}{2}\right)^j (2\mathbf{g_3})^k\,\mathbf{T}^p\left(\frac{\epsilon}{2}\right)^l \ , \tag{8.53}$$

[63]Notice that $F^k_n = 0$ for $n < 0$ therefore for fixed n the action of $e^{\epsilon\hat{S}}$ is truncated.

[64]We observe that using the expression for g_2 the dependence on m, Λ can be eliminated. Therefore, the expansion depends only on the parameters g_2, g_3, T and ϵ.

and again numerically the coefficients a_{jklp} turn out to be integers, see appendix E.3.

As anticipated, an interesting feature of the Hurwitz expansion of the blowup factor $\mathcal{B}_{NS}$ is that it is defined in terms of the IR parameters $\mathbf{u}, \Lambda$ and therefore we can easily study the theory around strongly coupled points. In particular, we are interested in the AD superconformal point of $N_f = 1$ which corresponds to PI equation.

The PI equation can be obtained from PIII$_2$ as follows. First we remove the Gaussian contribution of the contact term[65]

$$\mathcal{B}_{NS}^{PIII_2}(s) = e^{\frac{\mathbf{u}}{6}s^2}\mathcal{B}_{NS}^{PI}(m^{\frac{4}{5}}s) \ , \tag{8.54}$$

and then we change variables[66]

$$s = m^{-\frac{4}{5}}\tilde{s} \ , \quad \mathbf{u} = \frac{3}{\sqrt{2}}m^2 - \frac{1}{2\sqrt{2}}m^{\frac{6}{5}}g_2^{PI} - \frac{1}{4}m^{\frac{4}{5}}\mathbf{g_3}^{PI} \ , \quad \Lambda^3 = \frac{3}{2}m^3 - \frac{3}{4}m^{\frac{11}{5}}g_2^{PI} \ , \tag{8.55}$$

which implies

$$g_2^{PIII_2} = m^{\frac{16}{5}}(g_2^{PI} + O(m^{-\frac{2}{5}})) \ , \quad g_3^{PIII_2} = m^{\frac{24}{5}}(g_3^{PI} + O(m^{-\frac{2}{5}})) \ . \tag{8.56}$$

After the change of variables the equation (8.49) becomes (renaming $\tilde{s} \to s$ for simplicity)

$$D_s^{(4)}(\mathcal{B}_{NS}, \mathcal{B}_{NS}) + 2\epsilon s\mathcal{B}_{NS}^2 + g_2^{PI}(2 - 3e^{\frac{\epsilon s}{m^{4/5}}})\mathcal{B}_{NS}^2 + (6(e^{\frac{\epsilon s}{m^{4/5}}} - 1)m^{4/5} - 4\epsilon s)\mathcal{B}_{NS}^2 + O(m^{-\frac{2}{5}}) = 0 \ , \tag{8.57}$$

and in the limit $m \to \infty$ we obtain the PI Hirota equation

$$D_s^{(4)}(\mathcal{B}_{NS}, \mathcal{B}_{NS}) + 2\epsilon s\mathcal{B}_{NS}^2 - g_2^{PI}\mathcal{B}_{NS}^2 = 0 \ . \tag{8.58}$$

We observe that the parameters of the AD theory have fractional dimensions $[g_2^{PI}] = 4/5, [g_3^{PI}] = 6/5$ and that all the contact term contributions are subleading and decouple in the limit $m \to \infty$. This means that in the AD point we don't have contributions to the contact term coming from the extra massless degrees of freedom and the contact term of H_0 theory vanishes, $T^{PI} = 0$.

8.4 PI alias Argyres-Douglas H_0

In the previous subsection we obtained the AD theory from holomorphic decoupling of the $N_f = 1$ theory and we showed that the corresponding Painlevé equation is PI. In this section we will study in detail the Hurwitz expansion for this theory. This expansion was the one originally studied by HRZ in [65] and has more special properties with respect to the Lagrangian one because its SW curve is

$$y^2 = 4x^3 - cx + 2u \ , \tag{8.59}$$

[65]This corresponds precisely to the contribution of the contact term coming from the states that are integrated out.

[66]As we will see later, in the H_0 theory we have $g_3^{PI} = -2u, g_2^{PI} = c$, where c is the coupling of the u observable. Therefore only g_3 is affected by the NS ϵ corrections.

therefore the parameters of the theory are exactly the elliptic invariants $g_2 = c, g_3 = -2u$. Furthermore, as we have shown before, the contact term vanishes, $T = 0$. This implies that the structure of the Hurwitz expansion is particularly simple.

The parameter u is the Coulomb branch parameter and has dimension $[u] = 6/5$, and c is the gauge coupling which corresponds to the source associated to the operator u for the AD theory and has dimension $[c] = 4/5$. The dimension of the source s of the surface observable can be fixed from the fact that $[us/\epsilon] = 0$, which implies $[s] = -1/5$.

We start now the analysis. The PI Hirota bilinear equation is [65]

$$D_t^{(4)}\left(\mathcal{T},\mathcal{T}\right) + 2t\mathcal{T}^2 = 0 \ , \tag{8.60}$$

and its sigma form for the hamiltonian $\zeta(t) = \partial_t \log \mathcal{T}$ is

$$\ddot{\zeta}(t)^2 + 4\dot{\zeta}(t)^3 - 2(\zeta - t\dot{\zeta}) = 0 \ . \tag{8.61}$$

We now consider a zero t_0 of $\mathcal{T}$ and changing variables[67]

$$t = \epsilon^{1/5} s + t_0 \ , \quad c = -2\epsilon^{4/5} t_0 \ , \quad \mathcal{T}(s) = e^{\frac{\mathbf{u}s}{\epsilon}}\mathcal{B}_{NS} \ , \quad \zeta(s) = \epsilon^{-\frac{1}{5}}\eta(s) + \epsilon^{-\frac{6}{5}}\mathbf{u} \ , \tag{8.62}$$

we get

$$D_s^{(4)}\left(\mathcal{B}_{NS},\mathcal{B}_{NS}\right) + 2\epsilon s\mathcal{B}_{NS}^2 - c\mathcal{B}_{NS}^2 = 0 \ , \tag{8.63}$$

which agrees with the result obtained from the coalescence limit (8.58). The sigma-form equation becomes

$$\ddot{\eta}^2 + 4\dot{\eta}^3 - 2\epsilon(\eta - s\dot{\eta}) - c\dot{\eta} - 2\mathbf{u} = 0 \ . \tag{8.64}$$

As usual, in the limit $\epsilon \to 0$, the equations become autonomous and reduces to the form (8.4), (8.6) with the following parameters

$$g_2^{PI} = c \ , \quad g_3^{PI} = -2u \ , \quad T^{PI} = 0 \ , \tag{8.65}$$

which agrees with the SW result (8.59). In this case, as already observed, the contact term vanishes, $T^{PI} = 0$.

We observe that the equation for $\mathcal{B}_{NS}$ is independent on $\mathbf{u}$. This is due to the fact that the PI equation is invariant under the "gauge transformations" of $\mathcal{T}$

$$\mathcal{T} \to \mathcal{T}' = e^{a+bs}\mathcal{T} \ , \tag{8.66}$$

as $D_s^{(k)}$ is covariant with respect to the gauge transformations

$$D_s^{(k)}\left(\mathcal{T}',\mathcal{T}'\right) = e^{2a+2bs} D_s^{(k)}\left(\mathcal{T},\mathcal{T}\right) \ , \tag{8.67}$$

which can be easily seen from the very definition of the Hirota derivative (8.2).

[67] As already observed for PIV, for AD theories the effect of the surface observable is an *additive* shift. From the analysis of PIII_2 this can be understood as a consequence of the decoupling limit (8.55) in which the shifted coupling $\Lambda \exp(\epsilon s)$ is truncated due to the scaling of s.

Substituting the general ansatz (8.7) we obtain the following recursion relation for the coefficients c_n

$$\frac{2n(n^2-1)(n-6)}{(n+1)!}c_n = -{F'}^4_n + g_2 F^0_{n-4} - 2\epsilon F^0_{n-5} \ . \tag{8.68}$$

The coefficients are homogeneous polynomials in the modular parameters

$$c_n^{PI} = \sum_{4j+6k+5l=n} a_{jkl} \left(\frac{g_2}{2}\right)^j (2\mathbf{g_3})^k \epsilon^l \ , \tag{8.69}$$

and numerically it can be verified that the coefficients a_{mnl} are integers, see appendix E.4. As usual, in the autonomous limit $\epsilon \to 0$ the coefficients c_n^{PI} reduces to the Weierstrass σ-function coefficients.

We observe that this expansion is exactly centered around the AD superconformal point which corresponds to the cubic singularity $g_2 = 0, g_3 = 0$. We notice that the structure of the Hurwitz expansion for AD theory H_0 is particular simple due to the absence of the contact term.

8.5 PIII$_3$ alias $N_f = 0$

As a last example we consider the $SU(2)$ pure gauge theory. The SW curve of this theory is

$$y^2 = (x^2 - u)^2 - 4\Lambda^4 \ . \tag{8.70}$$

The pure gauge theory corresponds to Painlevé PIII$_3$ and the Hirota bilinear equation for PIII$_3$ is the same as PIII$_2$

$$D^{(4)}_{\log t}(\mathcal{T},\mathcal{T}) - 2t\frac{\partial}{\partial t}D^{(2)}_{\log t}(\mathcal{T},\mathcal{T}) + D^{(2)}_{\log t}(\mathcal{T},\mathcal{T}) + 4t\mathcal{T}^2 = 0 \ . \tag{8.71}$$

The hamiltonian is $\zeta = t\partial_t \log \mathcal{T}$, and the corresponding sigma-form of the equation is

$$\left(t\ddot{\zeta}\right)^2 - 4\dot{\zeta}^2(\zeta - t\dot{\zeta}) + 4\dot{\zeta} = 0 \ , \tag{8.72}$$

obtained from the sigma form (8.47) of PIII$_2$ in the limit $\theta \to \infty$ which is the coalescence limit PIII$_2 \to$ PIII$_3$. From the gauge theory point of view, this limit corresponds exactly to the holomorphic decoupling $N_f = 1 \to N_f = 0$ obtained integrating out the hypermultiplet

$$\theta = \frac{m}{\epsilon} \ , \quad m \to \infty \ , \quad m\Lambda^3_{N_f=1} = \Lambda^4_{N_f=0} \ . \tag{8.73}$$

To obtain the autonomous limit, similarly to PIII$_2$, we change variables as follows

$$t = t_0 e^{\epsilon s} \ , \quad \Lambda^4 = \epsilon^4 t_0 \ , \quad \mathcal{T}(s) = e^{\frac{\mathbf{u}s}{\epsilon}}\mathcal{B}_{NS}(s) \ , \quad \zeta(s) = \epsilon^{-1}\eta(s) + \epsilon^{-2}\mathbf{u} \ , \tag{8.74}$$

and the equation (8.71) becomes

$$D^{(4)}_s(\mathcal{B}_{NS},\mathcal{B}_{NS}) - 2\epsilon\frac{\partial}{\partial s}D^{(2)}_s(\mathcal{B}_{NS},\mathcal{B}_{NS}) + (\epsilon^2 - 4\mathbf{u})D^{(2)}_s(\mathcal{B}_{NS},\mathcal{B}_{NS}) + 4\Lambda^4 e^{\epsilon s}{\mathcal{B}_{NS}}^2 = 0 \tag{8.75}$$

with $\Lambda^4 = \epsilon^4 t_0$. The sigma-form equation reads

$$(\ddot{\eta} - \epsilon\dot{\eta})^2 + 4\dot{\eta}^3 - 4\epsilon\dot{\eta}^2\eta - 4\mathbf{u}\dot{\eta}^2 + 4\Lambda^4 e^{\epsilon s}\dot{\eta} = 0 \ . \tag{8.76}$$

Again, in the limit $\epsilon \to 0$ the equations reduce to (8.4),(8.6) with

$$g_2^{PIII_3} = \frac{4}{3}u^2 - 4\Lambda^4 \ , \quad g_3^{PIII_3} = -\frac{8}{27}u^3 + \frac{4}{3}u\Lambda^4 \ , \quad T^{PIII_3} = -\frac{u}{3} \ . \tag{8.77}$$

The parameters g_2, g_3 are exactly the elliptic invariants of the SW curve (8.70).

Substituting the ansatz (8.7) we obtain the following recursion

$$\frac{2n(n^2-1)(n-6)}{(n+1)!}c_n = -F'^4_n + 2\epsilon(n-1)F^2_{n-1} - (\epsilon^2 - 4\mathbf{u})F^2_{n-2} - 4\Lambda^4 e^{\epsilon\hat{S}}F^0_{n-4} \ . \tag{8.78}$$

The coefficients c_n are homogeneous polynomials of the parameters of the equation

$$c_n^{PIII_3} = \sum_{4j+6k+2p+l=n} a_{jkpl}\left(\frac{\mathbf{g_2}}{2}\right)^j (2\mathbf{g_3})^k\, \mathbf{T}^p \left(\frac{\epsilon}{2}\right)^l \ , \tag{8.79}$$

and a_{jkpl} numerically are integer coefficients. Finally, in the autonomous limit the contact term T contribution reduces to a Gaussian prefactor and can be removed so that the coefficients reduce to the Weierstrass σ-function coefficients. We observe that we can easily study this expansion around the monopole or dyon point simply setting $u = \pm 2\Lambda^2$. At this point the expansion should simplify because the Weierstrass σ reduces to a trigonometric function.

9 Conclusions and future perspectives

In the previous analysis we discussed a new realization of the $\mathcal{T}$-function as a *blowup factor*, i.e. the generating function of the surface observable $I(E)$ inserted in the exceptional sphere of the blowup $\hat{\mathbb{C}}^2$ and with source parameter s. Using a topological version of operator/state correspondence we derived an analytic chiral ring expansion of this blowup defect in terms of the generator of the $SU(2)$ chiral ring $\mathrm{Tr}\,\phi^2$ and we showed that in the NS limit $\epsilon_2 \to 0$ this is precisely the Painlevé $\mathcal{T}$-function which is expanded around a zero, corresponding to the value $s = 0$. A non-trivial result is that this expansion turned out to be a Hurwitz integral expansion, that is, the coefficients are given by integer polynomials.

Because this expansion is expressed in terms of the global NS modulus $\mathbf{u} = \left\langle \mathrm{Tr}\,\phi^2 \right\rangle_{NS}$, it can be easily studied around any point of the moduli space, including the non-lagrangian AD points. One can also easily take the limit from lagrangian theories to AD theories, e.g. $N_f = 1 \to \mathrm{H}_0$.

The manifest modular properties of the chiral ring expansion led to the derivation of the BCOV holomorphic anomaly equations from Painlevé equations themselves, using just the structure of the Kyiv formula

$$\mathcal{T} \propto \sum_n e^{-\frac{n}{\epsilon}\rho_{NS}} Z_{SD}(a, \Lambda e^{\epsilon s}, \epsilon) \ , \tag{9.1}$$

for the Painlevé $\mathcal{T}$-function and the analyticity in the chiral ring variable $\mathbf{u}$. The very same structure allows also to construct a non-perturbative completion of the topological string partition function $\mathcal{Z}_X$

$$\mathcal{Z}_X^{NP} = e^{-\frac{1}{\epsilon^2}\mathcal{F}_0(a,\Lambda_{SD})} \sum_{n\in\mathbb{Z}+\frac{1}{2}} e^{-\frac{n}{\epsilon}a_D(a,\Lambda_{SD})} Z_{SD}(a+n\epsilon, \Lambda_{SD}e^{\epsilon s'}, \epsilon) \ , \tag{9.2}$$

where we expand around some new point $s = s' + s_0$ in order to include some non-perturbative contribution of the form $e^{-\frac{na_D}{\epsilon}}$ where a_D is the SW dual period. These corrections systematically cancel all the non-modular $E_2(\tau)$ dependence and make the $\mathcal{T}$-function $\mathcal{Z}_X^{NP}$ a background-independent non-perturbative completion of $\mathcal{Z}_X$.

We conclude this part discussing some open problems and further directions in which our results can be generalized.

The $\mathcal{T}$-function we propose as non-perturbative completion of topological strings share some features with earlier proposals, such as [82, 83] and the trans-series solution of holomorphic anomaly equations discussed in [84, 85]. It would be important to analyse in detail the relation among these various approaches. As we already noticed, Painlevé $\mathcal{T}$-functions can be regarded as a quantum version of the classical Weierstrass σ-function. The resulting expansion in the ϵ Ω-background parameter can be possibly compared with the results of [83, 86–90]. We also notice that this expansion should interestingly simplify at the singular strongly coupled points where the Weierstrass functions reduce to trigonometric ones. We further observe in comparing with TS/ST correspondence one should lift the $\rho = 0$ condition used in [34] and formulate a more general setup. Finally, we underline that the variables $(\mathbf{u}, \Lambda)$ we use in our expansions are the exactly the base of the fibration of the Joyce structure used in [40]. It would be very interesting to further compare the two approaches.

The blowup factor plays a key rôle in mathematics to study Donaldson invariants of four manifolds [68]. These are integral-valued by their very definition, as the singular part of the instanton moduli space does not contribute to their evaluation. In the algebraic geometry description, even if the moduli space is singular, assuming that semistability implies stability, this should not create rational numbers as the virtual class is anyhow an integral class[68]. As a consequence of this, the Donaldson invariants on the blowup are also integers and indeed these are counted by the integer coefficients of the σ-function expansion. By extending these assumptions to the equivariant setting, this would explain the integrality of the coefficients of expansions of the Painlevé $\mathcal{T}$-function that we conjecturally propose as equivariant Donaldson invariants on the blowup. It would be important then to provide a proof of the integrality of such coefficients. This issue can be also addressed in the AGT dual picture by making use of representation theory of Virasoro algebras [46].

An obvious generalization is to study the higher rank case. For pure super Yang-Mills theory with general gauge group this corresponds to the $\mathcal{T}$-functions of non-autonomous Toda system [21, 22]. The non-equivariant blowup equations for the $SU(N)$ case were studied in [91] were a relation to KdV hierarchy was highlighted and the blow-up factors

[68] M. Kool, private communication.

computed to be the analog of σ functions on higher genus Riemann surfaces. We remark that also higher genus σ functions are known to enjoy Hurwitz integrality (see for example [92]), so one can expect this to be the case also for the $\mathcal{T}$-functions of the non-autonomous Toda system. The Hurwitz integrality of the generating functions of higher rank equivariant Donaldson invariants seems indeed a non trivial prediction and a conjecture to further verify, e.g. against [93, 94].

Another natural extension is to linear $SU(N)^n$ quiver gauge theories whose blowup equations are related to the $\mathcal{T}$-function of isomonodromic deformations on the sphere with $n+3$-punctures [18]. Also circular quivers can be considered by formulating the isomonodromic deformation problem on an elliptic curve [19, 20]. For more general class S theories [11] the corresponding $\mathcal{T}$-functions are the ones of the isomonodromic deformation problem of linear systems with rational coefficients on higher genus Riemann surfaces. In our analysis the blowup equations have been studied in the NS limit connecting them to Painlevé equations. It would be interesting to study them in the full Ω-background, which should correspond to a suitable quantum version of the Painlevé equations. In this context it would be very interesting to compare our results for PI with two-dimensional quantum gravity and possibly find an interpretation of the general Ω-background from this perspective.

In the NS limit, it is known that the gauge theory prepotential $\mathcal{F}_{NS} = \log Z_{NS}$ can be computed in terms of the quantum periods [63, 95–97]. This is defined by a Schrödinger operator obtained with the replacement $y \to -i\epsilon\partial_x$. It is then natural to ask if the expansion of the $\mathcal{T}$ function can be fixed by the invariants of the quantum curve which intuitively will be operators $\hat{g}_2, \hat{g}_3$ acting on the NS wavefunction. This would also require to understand how the contact term gets modified in this case.

Related to the above question, we remark that in the foundational paper [44], Donaldson invariants are obtained from an integral over the Coulomb moduli space of the gauge theory, namely the u-plane integral, see [98–101] for the extension to massive theories. This is proposed as an effective reduction of the gauge theory path integral over the low energy modes and computed by exploiting the implication of the topological nature of the twisted supersymmetric gauge theory. It sounds natural to expect that an equivariant version of this construction exists and that it might reproduce our results.

More recently, the computation of partition functions of AD theories on four manifolds has been considered in [102] along with the question if these define new differential invariants. The results presented in the above paper are given in the form of a u-plane integral, where the relevant contributions to the integrand are computed by assuming that their form follows a universal expression independent on the specific theory, but it is dictated by the underlying SW integrable system. This is consistent with our results as the structure of the SW blow-up factor is universally given by the specific evaluation of the modular invariants and the contact term coefficient. It would be interesting to lift also the results of [102] to an equivariant set-up for toric manifolds.

As a final, more speculative, comment, we observe that the very fact that the blowup factor is a $\mathcal{T}$-function suggests the intriguing possibility that the integrability structure may be not strictly related to supersymmetry but more on the properties of the defects that governs the topological sector of the theory, and as such in principle, it may be extended, to a less-supersymmetric, or even non-supersymmetric setting. Regarding this point we observe that the surface defects we study are exactly the topological defects associated to the magnetic 1-form symmetry of the theory and therefore it is natural to expect that the Painlevé equations may follow from the fusion rules of these protected sector. It would be very interesting to find a derivation of the Hirota bilinear form of the Painlevé equations for the blowup factor by making use of such fusion rules.

III Five-dimensional gauge theory on the blowup

In this part we extend the analysis of the blowup factor we did in the previous sections to 5d $\mathcal{N}=1$ supersymmetric gauge theories on $\mathbb{C}^2 \times S^1$ and q-Painlevé equations. The content of this part is based on the paper [103]. An interesting new result arising from this analysis is an explicit realization of a novel five-dimensional UV completion of AD theories [28, 49] in the Ω-background. One lagrangian description presently available for AD theories is in terms of the supersymmetry enhancement mechanism of [104–107], which describes the AD points as special vacua of four dimensional Lagrangian $\mathcal{N}=1$ theories where supersymmetry gets enhanced to $\mathcal{N}=2$, allowing to compute their index via supersymmetric localization. Our proposal differs from the previous ones as it preserves the full supersymmetry along the whole RG flow. This leads to the possibility of computing the Nekrasov partition function of AD theories from Painlevé $\mathcal{T}$-functions.

One deep relation between the q-Painlevé equations and 5d gauge theories is the coincidence of Seiberg's classification of 5d gauge theories [56] with Sakai's classification[69] of Painlevé equations [110], see figure 8. In this context, our findings lead to a link between the Seiberg's SCFT fixed points in 5d and AD theories through the q-Painlevé renormalisation group flow.

The AD point turns out to arise from the perturbation of a finite coupling point of 5d after a proper limit to 4d is taken. We exemplify this for the simplest AD point H_0, by showing how this arises from the 5d $SU(2)$ $\mathcal{N}=1$ Super Yang-Mills with Chern-Simons level $k=1$ at negative coupling. This particular reduction, expected from Sakai's classification, was studied from the view point of the corresponding cluster integrable system in [28], from that of its SW geometry in [49] and from its BPS spectrum in [111, 112]. One of our aims is to make explicit this construction by following the form of the solution of the RG flow. We perform a detailed analysis of the 4d scalings and find among them a set of flows leading to H_0 theory, see figure 12 for the associated phase diagram. This analysis shows how and when the concrete Hurwitz expansion of the AD BPS partition function arises from a suitable continuous limit of the q-Painlevé $\mathcal{T}$-function. The latter has an integral *q-polynomial Hurwitz expansion*[70] in terms of the vev of the Wilson loop on S^1.

[69]In the table 8 we denote by MN E_n the corresponding Minahan-Nemeschansky 4d SCFTs [108]. The latter are related to difference Painlevé equations [109].

[70]By this we mean that the values of the $\mathcal{T}$-function in q-shifted points, see for example (10.26), are

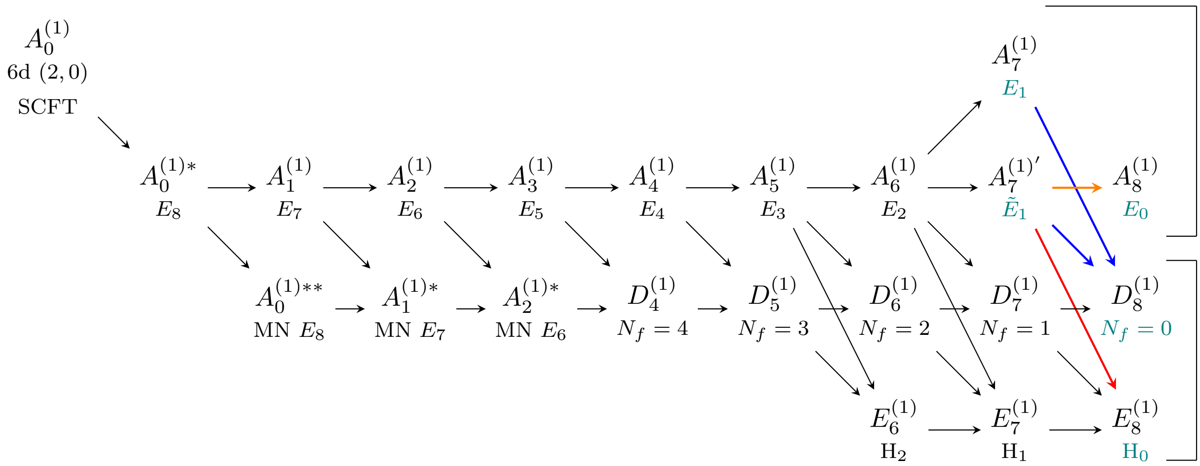

Figure 8. Sakai's classification for Painlevé equations by surface type and corresponding susy theories, where we highlighted the ones we study. The red arrow corresponds to the flow from $\tilde{E}_1$ SCFT (q-PI equation) to 4d AD $\mathrm{H}_0 = (A_1, A_2)$ theory (PI equation). The blue arrows correspond to the 4d geometric engineering limits and the orange one to the flow to E_0 SCFT (local $\mathbb{P}^2$).

To compute the Hurwitz expansions we first extend the present formulation of the Nakajima-Yoshioka (NY) 5d blowup equations by including the Wilson loop contributions [113]. We then take the NS limit of the latter and make use of a 5d version of topological state/operator correspondence to express the $\hat{\mathbb{C}}^2 \times S^1_\beta$ partition function (in presence of co-dimension two observables, see formula (10.12)) in terms of polynomials of the NS Wilson loop vev $\mathbf{U}$ with q-polynomial coefficients, see for example (10.31), (10.32), (10.33). These generalize the Hurwitz expansions for Painlevé equations found in [64–66] to the q-Painlevé case.

For completeness of the analysis we also study the Hurwitz expansion of the q-Painlevé $\mathcal{T}$-function of 5d $SU(2)$ super Yang-Mills with Chern-Simons level $k = 0$ for which we do not find any limit to AD theories. This is expected both from Sakai's table and from topological strings considerations.

As it is well known, $\mathcal{N} = 1$ SCFTs on $\mathbb{C}^2 \times S^1$ can be geometrically engineered from topological strings on local Calabi-Yau (CY) geometries [47]. The theories we study are described by local $\mathbb{F}_{1,0}$ CYs for $k = 1, 0$ respectively. The limits we study on the q-Painlevé $\mathcal{T}$-functions have a non-trivial geometrical interpretation which confirms our findings about the relation to 4d AD points. In particular, as explained in detail in section 12, the AD point H_0 is reached by a two steps procedure. One first performs a limit in the local $\mathbb{F}_1$ geometry, leaving behind a one-parameter deformation of local $\mathbb{P}^2$, see figure 11. The AD point is then reached via a further scaling limit to four dimensions which keeps this deformation parameter finite, see (12.18) and (12.27). It is now clear why the same point cannot be reached from the local $\mathbb{F}_0$ geometry: in this case there is simply no way to deform the geometry to reach[71] the local $\mathbb{P}^2$. From the 5d SCFT viewpoint, the one-parameter

polynomials with coefficients in the ring $\mathbb{Z}[q]$.

[71]This detailed procedure can be better defined in the GLSM description of this geometry and by dis-

deformation of local $\mathbb{P}^2$ corresponds to a negative five dimensional gauge coupling and the corresponding geometry describes a negative massive deformation of Seiberg's $\tilde{E}_1$ SCFT [114–116]. It is indeed by performing a suitable 4d limit of the latter that we reach the H_0 AD point.

The content of this part is the following. In section 10 we review the equivariant topological twist of the 5d gauge theory and the blowup topology changing operator. We discuss the expansion in terms of Wilson loops of the 5d NS blowup partition function and its interpretation in terms of topological operator/state correspondence. In section 11 we compute explicitly the 5d Hurwitz expansion of the q-Painlevé $\mathcal{T}$-function for 5d SYM theory with CS level $k = 0, 1$ and we study its properties in the SW limit. In section 12 we discuss two possible 4d limits of the 5d theory and its Hurwitz expansion. The first one is the standard 4d geometric engineering limit and allows to recover the Hurwitz expansion of PIII_3 corresponding to the 4d pure gauge theory of [64]. The second limit, which arise in the k=1 case only, is the one leading to the Argyres-Douglas SCFT H_0. Finally, in section 13 we discuss some open problems and possible generalizations of our results. In the appendix D, reported in part IV, we display the first coefficients of the Hurwitz expansion studied in section 11.

10 Five-dimensional gauge theory on the blowup

10.1 Topological observables in 5d gauge theory

In this section we will review the equivariant topological twist of 5d $\mathcal{N} = 1$ gauge theory and the construction of the topological observables, focusing for simplicity on the pure theory case.

Consider first 5d SYM theory on flat spacetime $X = \mathbb{C}^2 \times S^1_\beta$ compactified on a circle of radius β. The field content of the theory is given by a 5d vector multiplet $(\varphi, \lambda_\alpha, A_\mu)$ where φ is a real scalar, A_μ is a 5d gauge field and λ_α are gauginos. We want to study the theory in the Ω-background (ϵ_1, ϵ_2). The theory has an Euclidean spacetime rotation symmetry $SO(4)$ and a $\mathcal{R}$-symmetry coming from the symplectic reality condition for the supercharges. We can then do a partial topological twist replacing $SO(4) \simeq SU(2)_L \times SU(2)_R$ with the twisted rotations $SO(4)' = SU(2)_L \times SU(2)'_R$ where $SU(2)'_R = \mathrm{diag}(SU(2)_R \times SU(2)_\mathcal{R})$. After the topological twist, the field content is $(\varphi, A_\mu, \psi_\mu, b^+_{\mu\nu})$ with $\mu = 1, \ldots, 5$. The field ψ_μ is a one-form fermion field which is associated to the exterior derivative of A_μ. We need also to introduce an extra bosonic self-dual 2-form field $H^+_{\mu\nu}$ to close the algebra off-shell. In the topological twisted frame the supercharges become $(Q, G_\mu, Q^+_{\mu\nu})$ and contain a scalar topological charge Q. This can be made equivariant combining it with the action of G_μ and the spacetime symmetry $SO(4)' \times U(1)$ [10]

$$Q_v = Q + v^\mu G_\mu \ , \tag{10.1}$$

cussing the analytic continuation in the FI-parameters before the limits of the negatively coupled theory are taken.

where

$$v = \partial_t + i \sum_{j=1}^{2} (\epsilon_j z_j \partial_{z_j} - \ h.c.) \ , \tag{10.2}$$

is the vector field associated to the Cartan subalgebra $U(1) \times U(1) \subset SO(4)'$ of spacetime rotations and to the circle translations[72]. The action of the charge Q_v is then [117, 118]

$$\begin{aligned} &Q_v A = \psi \ , \quad Q_v \psi = \iota_v F + i D\varphi \ , \quad Q_v \varphi = -i \iota_v \psi \ , \\ &Q_v b^+ = H^+ \ , \quad Q_v H^+ = \mathcal{L}_v^A b^+ + i[\varphi, b^+] \ , \end{aligned} \tag{10.3}$$

where D is the covariant derivative, $\mathcal{L}_v^A = D\iota_v + \iota_v D$ is the covariant Lie derivative, and it can be verified that

$$Q_v^2 = \mathcal{L}_v^A + G_\phi \ , \tag{10.4}$$

where G_ϕ are infinitesimal gauge transformations with respect to the complex scalar $\phi = \varphi + i\iota_v A$. Therefore, the corresponding topological observables are equivariant cohomology classes for the torus action generated by v and which are gauge invariant.

We emphasize that the above twist is a partial topological twist, in the sense that the theory retains a dependence on the circle radius β through the instanton action $\beta \int_{M_4} \mathrm{Tr}\, F \wedge F$. Therefore, the correlation functions of the topological observables given by Q_v-cohomology of the 5d theory reduce to the ones of some susy QM on the circle. Furthermore, although formally a TQFT, the topological twist associated to Q_v captures richer dynamical informations thanks to equivariance.

The twisted supersymmetry algebra $(Q, G_\mu, Q^+_{\mu\nu})$ can be generalized to an arbitrary toric 4-manifold M_4 which admits a toric action of $U(1) \times U(1)$ and we can extend the above construction to any $X = M_4 \times S^1_\beta$.

There are two natural classes of equivariant topological observables which can be defined [119]. The first class is the 5d lift of the 4d local observable $\mathrm{Tr}\, \phi^2$ and corresponds to the circular Wilson loops placed in the fixed points of the toric action of M_4 and wrapped on the compactified circle dimension

$$W_R = \mathrm{Tr}_R\, P \exp \int_{S^1} (\varphi \mathrm{d}t + iA) = \mathrm{Tr}_R\, P \exp \int_{S^1} \phi \, \mathrm{d}t \ , \tag{10.5}$$

where φ is the real scalar in the 5d vector multiplet, P denotes path ordering and R is some representation of the gauge group G, and the associated topological multiplet constructed with the usual descent procedure.

As already discussed in the four-dimensional case, the second class is constructed observing that the transformations (10.3) can be rewritten as the generalized Bianchi identity for the curvature of the universal bundle [54]

$$\mathbf{D}\mathbf{F} = (-Q + D + \iota_v)(F + \psi + i\varphi) = 0 \ , \tag{10.6}$$

and for any ad-invariant polynomial $\mathcal{P}(\mathbf{F})$ of the Lie algebra of the gauge group we have

$$Q\mathcal{P}(\mathbf{F}) = (d + \iota_v)\mathcal{P}(\mathbf{F}) \ . \tag{10.7}$$

[72]The addition of the circle translations is needed to fully localize the path integral to the susy QM on the instanton moduli space.

It then follows that considering an equivariant cohomology class $\Omega \in H_v^\bullet(X)$ from (10.7) the observable

$$O(\Omega, \mathcal{P}) = \int_X \Omega \wedge \mathcal{P}(\mathbf{F}) \ , \tag{10.8}$$

is Q_v-closed.

In four dimensions for $\mathbb{C}^2$ the natural equivariant cohomology class is the equivariant symplectic form $\omega + H$ where H is the moment map and ω is the standard symplectic form of $\mathbb{C}^2$

$$H = \frac{\epsilon_1}{2}|z_1|^2 + \frac{\epsilon_2}{2}|z_1|^2 \ , \quad \omega = \frac{i}{2}\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \frac{i}{2}\mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_2 \ , \quad (d + \iota_v)(\omega + H) = 0 \ . \tag{10.9}$$

which defines the surface observable

$$O^{(4d)}(\Omega, \mathbf{F}^2) = \int_{M_4} (\omega + H) \wedge \mathrm{Tr}\, \mathbf{F}^2 = \int_{M_4} \omega \wedge 2\, \mathrm{Tr} \left(i\varphi F + \frac{1}{2}\psi \wedge \psi \right) + H\, \mathrm{Tr}\, F \wedge F \ . \tag{10.10}$$

For a general toric manifold M_4 the same construction applies patch by patch which are then glued together with the appropriate ϵ_j parameters and sums over fluxes [54].

In 5d we still have $(d + \iota_v)(\omega + H) = 0$, but the surface observable is naturally lifted to a codimension 2 operator. The lift to 5d of the equivariant symplectic form in the odd equivariant cohomology is

$$\Omega = (\omega + H) \wedge \mathrm{d}t - \lambda \ , \tag{10.11}$$

where $\lambda = \frac{i}{2} z_1 \mathrm{d}\bar{z}_1 + \frac{i}{2} z_2 \mathrm{d}\bar{z}_2$ is the 1-form symplectic potential such that $\omega = d\lambda$ and $\iota_v \lambda = H$. The appearance of the locally defined symplectic potential has the following consequences. The 5d lift of the surface observable is

$$\begin{aligned} O(\Omega, \mathbf{F}^2) &= \frac{1}{4\pi} \int_X (\omega \wedge \mathrm{d}t + H\mathrm{d}t - \lambda) \wedge \mathrm{Tr}\, \mathbf{F}^2 = \\ &= \frac{1}{4\pi} \int_X \ \omega \wedge 2\, \mathrm{Tr} \left(i\varphi F + \frac{1}{2}\psi \wedge \psi \right) \wedge dt + (H\mathrm{d}t - \lambda) \wedge \mathrm{Tr}\,(F \wedge F) = \\ &= \frac{1}{4\pi} \int_X \ \omega \wedge 2\, \mathrm{Tr} \left(i\varphi F + \frac{1}{2}\psi \wedge \psi \right) \wedge dt - \omega \wedge CS_3(A) + H\, \mathrm{Tr}\,(F \wedge F) \wedge dt \ , \end{aligned} \tag{10.12}$$

where we integrated by parts the term in λ using $\mathrm{Tr}\, F \wedge F = \mathrm{d}CS_3(A)$ with $CS_3(A)$ the Chern-Simons 3-form

$$CS_3(A) = \mathrm{Tr} \left[\mathrm{d}A \wedge A + \frac{2}{3} A \wedge A \wedge A \right] \ , \tag{10.13}$$

and we normalized by $1/4\pi$ to get the correct normalization for the CS level. Therefore, in 5d equivariance with respect to circle translations requires the addition of a Chern-Simons term. This implies that the physical observable must be exponentiated because the CS term is gauge invariant only if it represents an integral cohomology class [119][73]. This has

[73] In this paper the form of the susy version of the 5D CS term is also given as

$$\frac{1}{24\pi^2} \int_X CS_5(A) + Tr[3i\varphi F \wedge F + 2\psi \wedge \psi \wedge F] \wedge \mathrm{d}t \ ,$$

$$CS_5(A) = \mathrm{Tr} \left[\mathrm{d}A \wedge \mathrm{d}A \wedge A + \frac{3}{2} \mathrm{d}A \wedge A \wedge A \wedge A + \frac{3}{5} A \wedge A \wedge A \wedge A \wedge A \right] \ .$$

the important consequence that the coupling of the exponentiated operator of codimension 2 is discrete, as it is the analogue of the 3D CS level.

10.2 Blowup topology changing operator

Consider the 5d Nekrasov partition function on the 5d spacetime $X = \mathbb{C}^2 \times S^1_\beta$ of a $SU(2)$ gauge theory with Chern-Simons level k

$$Z^{(k)}(a, z, \epsilon_1, \epsilon_2, \beta) = \langle 1 \rangle^{(k)}_{\mathbb{C}^2 \times S^1_\beta} \ , \tag{10.14}$$

where β is the circle radius and $z = \beta^4 \Lambda^4$ is the 5d instanton counting scale. From the point of view of radial quantization, cutting a small "cylinder" $C_\delta(0) = B^4_\delta(0) \times S^1_\beta$, with the bases identified, and centered in the origin of spacetime, we can intepret the partition function $Z^{(k)}$ as the transition amplitude between an "in" state $|C_\delta(0)\rangle$ state given by a path integral on the cylinder and an "out" state $|X \setminus C_\delta(0)\rangle$ corresponding to a path integral on the remaining part of spacetime [45]

$$Z^{(k)}(a, z, \epsilon_1, \epsilon_2, \beta) = \langle out | in \rangle = \langle X \setminus C_\delta(0) | C_\delta(0) \rangle \ . \tag{10.15}$$

Topologically this follows from the identity $\mathbb{C}^2 = \mathbb{C}^2 \# B^4_\delta(0)$ where $\#$ is the connected sum.

Let us now consider the theory on the manifold $X' = \hat{\mathbb{C}}^2 \times S^1_\beta$ where $\hat{\mathbb{C}}^2 = \mathrm{tot}[\mathcal{O}(-1)_{\mathbb{P}^1}]$ is the blowup of $\mathbb{C}^2$ at the origin. This introduces a non-trivial equivariant 2-cycle which is the exceptional divisor $E \simeq \mathbb{C}P^1$. We consider then the codimension 2 defect observable wrapped on $E \times S^1$

$$I(E) \equiv \exp\big(O(\Omega, \mathbf{F}^2)\big) \ , \tag{10.16}$$

where Ω is the equivariant version of the Poincaré dual of E in the sense of (10.11). We define now the generating function of $I(E)$ on X' with first Chern class j and Chern-Simons level k

$$\hat{Z}^{(j,k)}(a, z, \epsilon_1, \epsilon_2, d, \beta) = \left\langle I(E)^{(d+1)} \right\rangle^{(j,k)}_{\hat{\mathbb{C}}^2 \times S^1_\beta} \ . \tag{10.17}$$

The parameter $d \in \mathbb{Z}$ is quantized because from (10.12) it corresponds to the CS level of the CS term contribution appearing in $I(E)$, and we shifted by $r/2 = 1$ to make d symmetric with respect to Serre duality $d \to r/2 - d$, where $r = 2$ is the rank of $U(2)$. The partition function $\hat{Z}^{(j,k)}(d)$ can be expressed in terms of the partition function Z through the Nakajima-Yoshioka (NY) blowup relations (for rank $r = 2$) [120–122]

$$\begin{aligned} &\hat{Z}^{(j,k)}(a, z, \epsilon_1, \epsilon_2, d, \beta) = \\ &= \sum_{n \in \mathbb{Z} + \frac{j}{2}} e^{(d/4 - 1/8)\beta(\epsilon_1 + \epsilon_2)} Z^{(k)}(a + n\epsilon_1, z q_1^{d + k(j-1)/2}, \epsilon_1, \epsilon_2 - \epsilon_1, \beta) \times \\ &\qquad\qquad Z^{(k)}(a + n\epsilon_2, z q_2^{d + k(j-1)/2}, \epsilon_1 - \epsilon_2, \epsilon_2, \beta) \ , \end{aligned} \tag{10.18}$$

where $q_j = e^{\beta \epsilon_j}$. The effect of the observable $I(E)$ is to shift the coupling as $z q_{1,2}^{d + k(j-1)/2}$ in the two patches of $\hat{\mathbb{C}}^2 \times S^1_\beta$.

Topologically, the manifold $X' = \hat{\mathbb{C}}^2 \times S^1_\beta$ is obtained from the Cartesian product of the circle S^1_β times the connected sum $\hat{\mathbb{C}}^2 = \mathbb{C}^2 \# \overline{\mathbb{C}P^2}$. Therefore, cutting a small cylinder $C_\delta(0)$ at the origin, we can interpret the generating function $\hat{Z}^{(j,k)}(d)$ as the transition amplitude

$$\hat{Z}^{(j,k)}(a,z,\epsilon_1,\epsilon_2,d,\beta) = \langle X \setminus C_\delta(0) | \Psi_d \rangle \ , \tag{10.19}$$

where the "in" state $|\Psi_d\rangle$ corresponds to the path integral on $\overline{\mathbb{C}P^2} \times S^1_\beta \setminus C_\delta(0)$ and with $d+1$ insertions of the codimension 2 defect observable $I(E)^{d+1}$

$$|\Psi_d\rangle = I(E)^{(d+1)} \left| \overline{\mathbb{C}P^2} \times S^1_\beta \setminus C_\delta(0) \right\rangle \ . \tag{10.20}$$

Because Q_v-observables are BPS protected we have a 5d version of "operator/state correspondence" where each state $|\psi\rangle$ of the Hilbert space defined on the boundary of the cylinder $C_\delta(0)$ can be replaced by some *line* operator supported on the circle in the origin $\{0\} \times S^1_\beta$, see figure 9.

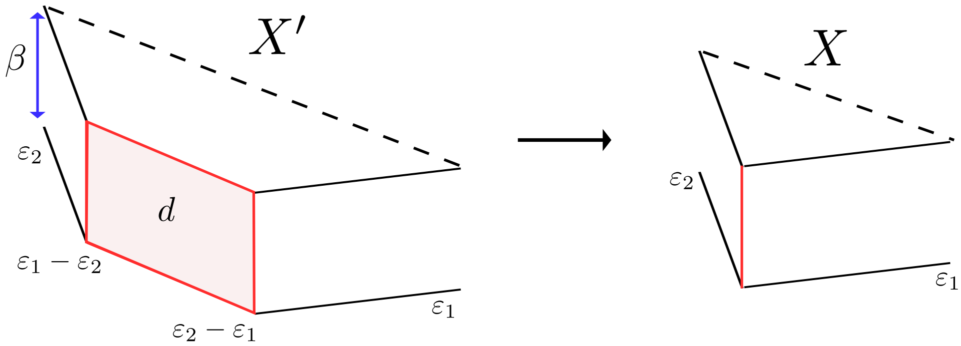

Figure 9. Diagram for the blow up $X' = \hat{\mathbb{C}}^2 \times S^1_\beta$. The red surface is the support $E \times S^1_\beta$ of the codimension 2 operator $I(E)^{d+1}$. The arrow illustrates the 5d topological operator/state correspondence (blow down).

This implies that all topological observables of the 5d theory can be expanded in the basis of Wilson loops W_R defined in (10.5) and by operator/state correspondence the blowup state $|\Psi_d\rangle$ can be expressed as a sum of these Wilson loop operators. We will study the consequences of this in the next sections.

10.3 NS blowup factor

We will focus now on the NS limit $(\epsilon_1, \epsilon_2) \to (\epsilon, 0)$ of the blowup partition function and we will show that it corresponds to the q-Painlevé $\mathcal{T}$-function. In order to do this we define the *blowup factor* as the normalized expectation value

$$\mathcal{B}^{(j,k)}(a,z,\epsilon_1,\epsilon_2,d,\beta) = \frac{\hat{Z}^{(j,k)}(a,z,\epsilon_1,\epsilon_2,d,\beta)}{Z^{(k)}(a,z,\epsilon_1,\epsilon_2,\beta)} \ , \tag{10.21}$$

and we consider the refined genus expansion of the partition function in the Ω-background

$$\log Z(a,z,\epsilon_1,\epsilon_2,\beta) \sim \sum_{g=0}^{+\infty} \sum_{k \in \{\frac{1}{2}\} \cup \mathbb{Z}_{\geq 0}} (-\epsilon_1\epsilon_2)^{g-1} (\epsilon_1+\epsilon_2)^{2k} \mathcal{F}_{g,k}(a,z,\beta) \ , \tag{10.22}$$

where the tree level term $\mathcal{F}_0 \equiv \mathcal{F}_{0,0}$ is the 5d Seiberg-Witten prepotential of the gauge theory. Substituting the expansion (10.22) in the NY blowup relations (10.18) and taking the NS limit $\epsilon_2 \to 0$ with fixed $\epsilon_1 \equiv \epsilon$ the blowup factor (10.22) reads

$$\begin{aligned}
&\mathcal{B}^{(j,k)}_{NS}(a,z,\epsilon,d,\beta) \equiv \lim_{\epsilon_2\to 0} \mathcal{B}^{(j,k)}(a,z,\epsilon,\epsilon_2,d,\beta) = \\
&= q^{d/4-1/8} e^{\alpha - \frac{\beta\gamma(d+k(j-1)/2)}{\epsilon}} \sum_{n\in\mathbb{Z}+\frac{j}{2}} e^{-\frac{n\rho}{\epsilon}} Z^{(k)}_{SD}(a+n\epsilon, zq^{d+k(j-1)/2},\epsilon,\beta) = \\
&= q^{d/4-1/8} e^{\alpha - \frac{\beta\gamma(d+k(j-1)/2)}{\epsilon}} Z^{(j,k)}_{D}(a,\rho, zq^{d+k(j-1)/2},\epsilon,\beta) \ ,
\end{aligned} \tag{10.23}$$

where $q = e^{\beta\epsilon}$ and $Z^{(k)}_{SD}$ is the Nekrasov partition function in the self-dual background $\epsilon_1 = -\epsilon_2 = \epsilon$ and we have defined

$$\rho = \epsilon\frac{\partial}{\partial a}W(a,z,\epsilon,\beta) \ , \quad \alpha = \frac{\partial}{\partial \epsilon}W(a,z,\epsilon,\beta) \ , \quad \gamma = \epsilon z\frac{\partial}{\partial z}W(a,z,\epsilon,\beta) \ , \tag{10.24}$$

and W is the 5d twisted superpotential

$$W(a,z,\epsilon,\beta) = \sum_{k\in\{\frac{1}{2}\}\cup\mathbb{Z}_{\geq 0}} \mathcal{F}_{0,k}(a,z,\beta)\epsilon^{2k-1} \ . \tag{10.25}$$

The NS blowup factor (10.23) has precisely the structure of the Kyiv formula for the q-Painlevé $\mathcal{T}$-function [58] defined in terms of the 5d dual Nekrasov partition function $Z^{(j,k)}_D$. The initial conditions are not generic but, as we explain in the next section, correspond to select a *zero* of the q-Painlevé $\mathcal{T}$-function in the odd sector, see formula (10.32). This implies a time reparametrization where the position of the zero is given by the 5d dimensionless instanton counting scale z and the discrete q-Painlevé time corresponds to the parameter d which counts the insertions of the codimension 2 observable $I(E)$. The effect of the CS level on the time variable is to shift the origin of time $d \to d+k(j-1)/2$. Finally, the normalization of the $\mathcal{T}$-function is fixed by the factor e^{α}. We have then

$$\mathcal{T}^{(j,k)}(zq^{d+k(j-1)/2}) = q^{-d/4+1/8} e^{\frac{\beta\gamma(d+k(j-1)/2)}{\epsilon}} \mathcal{B}^{(j,k)}_{NS}(d) \ . \tag{10.26}$$

10.4 Blowup equations and Wilson loop expansion

As previously discussed, the five-dimensional lift of the topological operator/state correspondence implies that the surface observable $I(E)$ can be expressed as an OPE expansion in terms of the circular Wilson loops around the compactified circle dimension

$$W_R = \mathrm{Tr}_R\, P\exp\int_{S^1}(iA+\varphi \mathrm{d}t) = \mathrm{Tr}_R\, P\exp\int_{S^1}\phi \mathrm{d}t \ , \tag{10.27}$$

which define a basis for the 5d chiral ring. Therefore, we have the following blowup equations

$$\hat{Z}^{(j,k)}(a,z,\epsilon_1,\epsilon_2,d,\beta) = \sum_R B^{(j,k)}_{d,R}(z,q_1,q_2)\langle W_R\rangle_{\mathbb{C}^2\times S^1_\beta} Z^{(k)}(a,z,\epsilon_1,\epsilon_2,\beta) \ .$$

where the sum is over a suitable set of representations of the gauge group G. These equations were conjectured in [113] from topological string arguments and further explored in [123, 124]. They are a generalization of the 5d Nakajima-Yoshioka blowup equations [120] which correspond to the case $|d| < r$ where r is the rank of the gauge group.

For $SU(2)$ all representations are generated by the tensor products $R_n = 2^{\otimes n}$ of the fundamental representation and we will denote with W_n the corresponding Wilson loop. Furthermore, in the NS limit the expectation value of the product of two Wilson loops W_l, W_m factorizes[74] as stated in [125]

$$\langle W_l W_m \rangle_{NS} = \langle W_l \rangle_{NS} \langle W_m \rangle_{NS} \quad \Rightarrow \quad \langle W_n \rangle_{NS} = \langle W_1 \rangle_{NS}^n = \mathbf{U}^n \ . \tag{10.28}$$

where

$$\mathbf{U} \equiv \langle W_1 \rangle_{NS} \ , \tag{10.29}$$

is the expectation value of the fundamental Wilson loop in the NS limit.

Therefore, the blowup factor in the NS limit is an analytic function of $\mathbf{U}$ which depends on the value of the parameter d

$$\mathcal{B}_{NS}^{(j,k)}(a, z, \epsilon, d, \beta) \equiv B_d^{(j,k)}(\mathbf{U}, z, q) \ , \tag{10.30}$$

and has the following structure

$$B_d^{(j,k)}(\mathbf{U}, z, q) \equiv (q^d z)^{\frac{j}{4}} P_d^{(j,k)}(\mathbf{U}, z, q) = (q^d z)^{\frac{j}{4}} \sum_{n=0}^{n_{\max}^{(j,k)}(d)} P_{d,n}^{(j,k)}(z, q) \mathbf{U}^n \ . \tag{10.31}$$

It turns out that for a fixed d only a finite number of representations enter in the rhs of (10.31) and the $P_d^{(j,k)}(\mathbf{U}, z, q)$ are polynomials[75] in the fundamental Wilson loop $\mathbf{U}$ of degree $n_{\max}(d) \sim d^2/4$, as we find for example in (11.11), (11.22).

The polynomials $P_d^{(j,k)}(\mathbf{U}, z, q)$ turn out to be q-polynomials in $\mathbf{U}, z$ with integer coefficients[76]. In particular for $d = 0, 1, -1$ they are given by the NS limit of the standard NY blowup relations [120, 122]

$$\begin{aligned} &P_{-1}^{(0,k)} = 1 \ , \quad P_0^{(0,k)} = 1 \ , \quad P_1^{(0,k)} = 1 \ , \\ &P_{-1}^{(1,k)} = 1 \ , \quad P_0^{(1,k)} = 0 \ , \quad P_1^{(1,k)} = -1 \ . \end{aligned} \tag{10.32}$$

The first non-trivial dependence on the Wilson loop $\mathbf{U}$ appears for $d = \pm 2$ and in the NS limit we have the polynomials

$$P_{\pm 2}^{(0,0)} = 1 - q^{\pm 1} z \ , \quad P_{\pm 2}^{(1,0)} = \mp \mathbf{U} \ , \tag{10.33}$$

$$P_{\pm 2}^{(0,1)} = 1 \ , \quad P_2^{(1,1)} = -\mathbf{U} \ , \quad P_{-2}^{(1,1)} = \mathbf{U} + (q^{-1/2} - q^{1/2}) z \ , \tag{10.34}$$

[74] This is consistent with our findings coming from the analysis of the expansion of the q-Painlevé $\mathcal{T}$-function, see for examples (11.9), (11.20).

[75] We isolated a prefactor $(q^d z)^{\frac{j}{4}}$ because in this way $P_d^{(j,k)}(\mathbf{U}, z, q)$ become polynomials in $\mathbf{U}, q, z$, as will follow from our analysis in section 11.

[76] For CS level $k = 1$ they turn out to be polynomials in $\tilde{z} = zq^3$.

where we observe that we have the following properties[77] under the map $d \to -d,\ q \to q^{-1}$

$$
\begin{aligned}
&\mathbf{U}(a,z,q^{-1}) = \mathbf{U}(a,z,q)\ , && \text{for } k=0\ ,\\
&\tilde{\mathbf{U}}(a,z,q) \equiv \mathbf{U}(a,z,q^{-1}) = \mathbf{U}(a,z,q) + (q^{-1/2}-q^{1/2})z\ , && \text{for } k=1\ ,\\
&B^{(j,0)}_{-d}(\mathbf{U},z,q) = (-1)^j B^{(j,0)}_{d}(\mathbf{U},z,q)\ , \quad B^{(j,1)}_{-d}(\mathbf{U},z,q) = (-1)^j B^{(j,1)}_{d}(\tilde{\mathbf{U}},z,q)\ . && (10.35)
\end{aligned}
$$

The polynomials (10.33) do not appear in [120]. For $k=0$ they were found in the context of topological string theory [113], and for $k=1$ we checked them up to order three in the instanton expansion.

In the following section we will use the relation between the NS blowup factor and the q-Painlevé $\mathcal{T}$-function to compute the OPE coefficients $B^{(j,k)}_d(\mathbf{U},z,q)$ recursively.

From the relation (10.23), this OPE expansion contains the full informations about the self-dual partition function $Z^{(k)}_{SD}$ of the 5d gauge theory and because it is expressed in terms of Wilson loops it can be easily analyzed at any point of the moduli space, including the strongly coupled regime.

As a final comment, let us observe that the scaling of $n_{\max}(d)$ has a possible origin from an effective contact term $T(U)$ which arises when we map the co-dimension two observables $I(E)$, on $\Sigma \times S^1$, in the low-energy effective theory. Indeed, in the 4d case the product of surface observables $O^{(4d)}(\Omega,\mathbf{F}^2)$ in the UV is mapped to the IR one $\tilde{O}^{(4d)}(\Omega,\mathbf{F}^2)$ up to a contact term [44] $O^{(4d)}(\Omega,\mathbf{F}^2)O^{(4d)}(\Omega,\mathbf{F}^2) \to \tilde{O}^{(4d)}(\Omega,\mathbf{F}^2)\tilde{O}^{(4d)}(\Omega,\mathbf{F}^2) + T_{4d}$ given by the local observable (for the 4d pure $SU(2)$ theory)

$$
T_{4d} = \Lambda\partial_\Lambda u = (\Lambda\partial_\Lambda)^2\mathcal{F}_0 = \frac{u}{3} + \frac{\pi^2}{12}\frac{E_2(\tau)}{\omega_1^2}\ , \tag{10.36}
$$

where $\mathcal{F}_0$ is the SW prepotential of the 4d theory and $E_2(\tau)$ is the second Eisenstein series. Similarly in the 5d case we have[78]

$$
\left\langle I(E)^d \right\rangle^{UV}_{\hat{\mathbb{C}}^2\times S^1,NS} = e^{\frac{1}{2}\beta^2 d^2 T}\left\langle \tilde{I}(E)^d \right\rangle^{IR}_{\hat{\mathbb{C}}^2\times S^1,NS}\ . \tag{10.37}
$$

The Chern-Simons level k introduces a further source in the contact term which gets shifted as $d^2T \to d(d+k(j-1))T$. Assuming that the Wilson loop expansion is governed by the UV contact term contribution $T(\mathbf{U}) \sim \log\mathbf{U}$ it therefore follows that the blowup factor scales like $B_d \sim \mathbf{U}^{d^2}$. In the next section we study the NS blowup factor from the q-Painlevé equations and its SW limit and we will check directly the above statement[79].

[77] We checked the formulas for the NS Wilson loop $\mathbf{U}$ under the transformation $q \to q^{-1}$ to order five in the instanton expansion.

[78] Upon exponentiation the Wick contractions introduced by the contact term correspond to a gaussian prefactor in the partition function.

[79] From a UV perspective, in the four-dimensional case the truncation of the OPE in terms of the chiral ring generator $\mathrm{Tr}\,\phi^2$ follows from the counting of zero modes introduced by the surface observable wrapped on the exceptional divisor E and is related to the $U(1)$ R-charge anomaly. We expect that the truncation of the Wilson loop expansion could be derived via a K-theoretic version of the previous analysis studying the theory on the codimension 2 defect $E \times S^1$.

11 Hurwitz expansions of q-Painlevé $\mathcal{T}$-functions

As we observed in the previous section the NS blowup factor is related to the q-Painlevé $\mathcal{T}$-function through (10.26) and from (10.32) we have that it vanishes for $d = 0$ in the odd sector $j = 1$. This is just the five-dimensional analogue of what already observed in [64] where the chiral ring expansion of the 4d NS blowup factor was interpreted as the expansion of the Painlevé $\mathcal{T}$-function around a zero which allows to compute the OPE coefficients recursively from the Painlevé Hirota equation. In the same way, the 5d blowup factor coefficients $B_d^{(j,k)}$ can be determined by studying the expansion of the q-Painlevé $\mathcal{T}$-functions $\mathcal{T}^{(j,k)}(z)$ around a zero z_0 of the odd sector $\mathcal{T}$-function $\mathcal{T}^{(1,k)}(z)$.

The q-Painlevé equation is a q-difference equation for $\mathcal{T}^{(j,k)}(z)$ where z is the time variable and corresponds to the value of the instanton counting scale z which follows a discrete flow $z \to zq$. Once we expand around a zero z_0 we can reparametrize the time variable as

$$z = z_0 q^d \ , \tag{11.1}$$

From the gauge theory point of view the above shift corresponds precisely to the one introduced by the surface observable $I(E)$ as in (10.18). The $\mathcal{T}$-function, being the NS blowup factor $B_d^{(j,k)}$, becomes then a function of the discrete time d and using (10.26) it is possible to rewrite the q-difference equation as a recurrence relation for the NS blowup factor coefficients

$$B_d^{(j,k)} = F(B_{d'}^{(j',k)}) \ , \quad d' < d \ , j' = 0, 1 \ , \tag{11.2}$$

whose explicit form depends on the specific q-Painlevé equation. For the values $d = 0, \pm 1, \pm 2$ the above recurrence is not defined[80]. The corresponding undetermined coefficients $B_d^{(j,k)}$ are *resonances* that parametrize the initial conditions of the the solution and will be fixed using the relations (10.32), (10.33).

As discussed in the previous section the structure of the solution will be given by polynomials in $\mathbf{U}$ (10.31). A non-trivial result which we just checked numerically is that these polynomials have coefficients in the ring $\mathbb{Z}[q]$. From the gauge theory point of view this should correspond to the fact that the BPS states introduced by the defect in the exceptional divisor can be reorganized in terms of Wilson loop BPS states and $B_{n,d}^{(j,k)}$ count the degeneracy of these states for some fixed representation $2^{\otimes n}$ and number of insertions d. In the following we will study in detail the equation

$$\mathcal{T}^{(j,k)}(qz)\mathcal{T}^{(j,k)}(q^{-1}z) = \mathcal{T}^{(j,k)}(z)^2 - z^{1/2}\mathcal{T}^{(j+1,k)}\left(q^{k/2}z\right)\mathcal{T}^{(j-1,k)}\left(q^{-k/2}z\right) \ , \ j \in \mathbb{Z}_2 \ , \tag{11.3}$$

associated to 5d $\mathcal{N} = 1$ $SU(2)$ pure gauge theory with Chern-Simons (CS) level $k = 0, 1$ and corresponding to q-PIII$_3$ and q-PI equation respectively [58]. These equations come from the cluster algebra associated to the mutations of the 5d BPS quiver [27, 28, 58]. This quiver can be also constructed starting from the Newton polygon associated to the mirror geometry.

[80]This is true for the odd sector $j = 1$. The other sector is then completely determined.

11.1 q-PIII$_3$ alias $N_f = 0, k = 0$

We start from the q-PIII$_3$ equation which corresponds to 5d pure gauge theory with CS level $k = 0$. The Hirota equation (11.3) becomes

$$\mathcal{T}^{(j,0)}(qz)\mathcal{T}^{(j,0)}(q^{-1}z) = \mathcal{T}^{(j,0)}(z)^2 - z^{1/2}\mathcal{T}^{(j+1,0)}(z)\,\mathcal{T}^{(j-1,0)}(z) \ . \tag{11.4}$$

We expand now around a zero defining the map given by (10.26)[81]

$$z \to zq^d \ , \quad B_d^{(j,0)} = \mathcal{T}^{(j,0)}(zq^d) \ . \tag{11.5}$$

where we redefined the time variable so that z now parametrizes the position of the zero, and we can rewrite the equation (11.4) as

$$B_{d+1}^{(j,0)}B_{d-1}^{(j,0)} = B_d^{(j,0)}B_d^{(j,0)} - z^{1/2}q^{d/2}B_d^{(j+1,0)}B_d^{(j-1,0)} \ . \tag{11.6}$$

This gives the bilateral recurrence relation

$$B_{d+1}^{(j,0)} = \frac{1}{B_{d-1}^{(j,0)}}\left(B_d^{(j,0)}B_d^{(j,0)} - z^{1/2}q^{d/2}B_d^{(j+1,0)}B_d^{(j-1,0)}\right) \ , \quad d \geq 0 \ , \tag{11.7}$$

$$B_{d-1}^{(j,0)} = \frac{1}{B_{d+1}^{(j,0)}}\left(B_d^{(j,0)}B_d^{(j,0)} - z^{1/2}q^{d/2}B_d^{(j+1,0)}B_d^{(j-1,0)}\right) \ , \quad d \leq 0 \ . \tag{11.8}$$

where the initial conditions are given by the NY blowup equations (10.32) and in particular $B_0^{(1,0)} = 0$, as we expand around a zero. This is valid only if $B_{d\mp1}^{(j,0)} \neq 0$ which is true for $d \neq \pm1$ or $j \neq 1$. Therefore, the relations (11.7) for $j = 1, d = \pm1$ do not determine the resonant coefficients $B_{\pm2}^{(1,0)}$. As it is clear from (10.33) the resonances $B_{\pm2}^{(1,0)}$ carry the dependence of the blowup factor on the Wilson loop $\mathbf{U}$. Together, the knowledge of the coefficients $B_d^{(j,0)}$ for $d = 0, \pm1, \pm2$ fix then completely the solution[82]. We observe that the recurrence (11.7) contains some denominators, but when we compute the solution they all cancel and the coefficients are polynomials. This non-trivial cancellation is due to the Laurent phenomenon of the cluster algebra in which the new cluster variables are Laurent polynomials of the original ones [126, 127].

Applying the above recurrence, we can compute all the coefficients $B_d^{(j,k)}$ which have the following form

$$B_d^{(j,0)}(\mathbf{U}, z, q) = (q^d z)^{\frac{j}{4}} P_d^{(j,0)}(\mathbf{U}, z, q) = (q^d z)^{\frac{j}{4}} \sum_{n=0}^{n_{\max}^{(j,0)}(d)} P_{d,n}^{(j,0)}(z, q)\mathbf{U}^n \ , \tag{11.9}$$

$$B_{-d}^{(j,0)}(\mathbf{U}, z, q) = (-1)^j B_d^{(j,0)}(\mathbf{U}, z, q^{-1}) \ , \tag{11.10}$$

[81] In (10.26) the blowup factor differs from the $\mathcal{T}$-function by a "gauge" prefactor $\exp(ad)$. However, the equation (11.3) is invariant under these gauge transformations and we can simply redefine the $\mathcal{T}$-function reabsorbing the gauge prefactor.

[82] These coefficients are not all independent but are sufficient to fix the solution uniquely.

where $P_d^{(j,k)}(\mathbf{U},z,q)$ turn out to be polynomials in $z,\mathbf{U},q$ with integer coefficients[83]. We observed numerically that the maximum power $n_{\max}^{(j,0)}(d)$ of $\mathbf{U}$ which appears in the polynomial $P_d^{(j,k)}$ is

$$n_{\max}^{(j,0)}(d) = \left\lfloor \frac{d^2}{4} \right\rfloor - \chi_4(d+2-2j) \ . \tag{11.11}$$

where

$$\chi_p(d) = 1 \text{ if } d = 0 \bmod p \ (d \neq 0) \text{ and } \chi_p(d) = 0 \text{ otherwise} \ . \tag{11.12}$$

Furthermore, we have the following selection rule

$$P_{d,n}^{(j,0)}(z,q) = 0 \ , \qquad \text{if } n = 1 - j(d+1) \mod 2 \ . \tag{11.13}$$

For illustration, the polynomials for $0 \leq d \leq 6$ are reported in appendix (D.1) and they are in agreement with the topological string result (110) in [113] in the NS limit $q_2 \to 1, q_1 = q$.

Finally, we observe that the solution is particularly simple for $\mathbf{U} = 0$ for which we have the closed formula

$$B_d^{(j,0)}(U=0,z,q) = (-1)^{j(\lfloor d/2 \rfloor -1)} (zq^d)^{j/4} \frac{1+(-1)^{j(d+1)}}{2} \prod_{l=0}^{\lfloor \frac{d}{2} \rfloor -1} (1-q^{2l+1}z)^{d-1-2l} \ . \tag{11.14}$$

This should be related to the algebraic solution of q-PIII$_3$ studied in [128, 129].

11.2 q-PI alias $N_f = 0, k = 1$

We consider now q-PI equation which is associated to 5d pure gauge theory with CS level $k = 1$. The Hirota equation (11.3) reduces to

$$\mathcal{T}^{(j,1)}(qz)\mathcal{T}^{(j,1)}(q^{-1}z) = \mathcal{T}^{(j,1)}(z)^2 - z^{1/2}\mathcal{T}^{(j+1,1)}\left(q^{1/2}z\right)\mathcal{T}^{(j-1,1)}\left(q^{-1/2}z\right) \ , \ j \in \mathbb{Z}_2 \ . \tag{11.15}$$

From (10.26) we have the following map

$$z \to zq^d \ , \quad B_d^{(j,1)} = \mathcal{T}^{(j,1)}(zq^{d+(j-1)/2}) \ , \tag{11.16}$$

and evaluating (11.13) at $zq^{(j-1)/2}$ we obtain

$$B_{d+1}^{(j,1)} B_{d-1}^{(j,1)} = B_d^{(j,1)} B_d^{(j,1)} - z^{1/2} q^{d/2+(j-1)/4} B_d^{(j+1,1)} B_{d-(-1)^j}^{(j-1,1)} \ . \tag{11.17}$$

From (11.17) we get the following recurrence relations[84]

$$B_{d+1}^{(0,1)} = \frac{1}{B_{d-1}^{(0,1)}} (B_d^{(0,1)} B_d^{(0,1)} - z^{1/2} q^{d/2-1/4} B_d^{(1,1)} B_{d-1}^{(1,1)}) \ , \tag{11.18}$$

$$B_{d+1}^{(1,1)} = \frac{1}{B_{d-1}^{(1,1)}} (B_d^{(1,1)} B_d^{(1,1)} - z^{1/2} q^{d/2} B_d^{(0,1)} B_{d+1}^{(0,1)}) \ . \tag{11.19}$$

---

[83] We checked this numerically to high order in d but we don't have a proof.

[84] With our conventions there is some asymmetry between $j = 0, 1$. This will be convenient in the following, when we will take the strong coupling 4d limit.

And again the coefficients $B_{\pm 2}^{(1,1)}$ are resonances and we need their value to fix completely the solution. These are again fixed by (10.33). The solution has the following structure

$$B_d^{(j,1)}(\mathbf{U},z,q) = (zq^d)^{j/4} P_d^{(j,1)}(\mathbf{U},z,q) = (zq^d)^{j/4} \sum_{n=0}^{n_{\max}^{(j,1)}(d)} P_{d,n}^{(j,1)}(zq^{3/2},q)\mathbf{U}^n \ , \tag{11.20}$$

$$B_{-d}^{(j,1)}(\mathbf{U},z,q) = (-1)^j B_d^{(j,1)}(\tilde{\mathbf{U}},z,q^{-1}) \ , \quad \tilde{\mathbf{U}} = \mathbf{U} + (q^{-1/2} - q^{1/2})z \ , \tag{11.21}$$

and again $P_d^{(j,1)}(\mathbf{U},z,q)$ turn out to be integer q-polynomials in $\mathbf{U}, \tilde{z} = zq^{3/2}$ and numerically the maximum power $n_{\max}^{(j,1)}(d)$ of $\mathbf{U}$ turns out to be

$$n_{\max}^{(j,1)}(d) = \left\lfloor \frac{d(d+j-1)}{4} \right\rfloor - j\chi_4(d) \ . \tag{11.22}$$

In this case there is no selection rule like (11.13). The first polynomials $P_d^{(j,1)}$ are reported in appendix (D.2).

11.3 Blowup factor in the SW limit

To understand better some features of the Wilson loop Hurwitz expansion, let us now study the SW limit $\epsilon \to 0$ of the NS blowup factor (10.23). Substituting the genus expansion (10.22) in (10.23) and following the same steps of [64] we arrive at the following result

$$\begin{aligned} \mathcal{B}_{SW}^{(j,k)}(a,z,d,\beta) &= \lim_{\epsilon\to 0} \mathcal{B}_{NS}^{(j,k)}(a,z,\epsilon,d,\beta) = \\ &= -e^{\mathcal{F}_{1,0}+\mathcal{F}_{0,1}} \exp\left[\frac{1}{2}\beta^2(d+k(j-1)/2)^2 T\right] \theta_{4-3j}\left(h\beta(d+k(j-1)/2)\middle|\tau\right) \ , \end{aligned} \tag{11.23}$$

where

$$\tau(a,z,\beta) = \frac{1}{2\pi i}\frac{\partial^2 \mathcal{F}_0}{\partial a^2}(a,z,\beta) \ , \quad h(a,z,\beta) = \frac{1}{2i} z \frac{\partial^2 \mathcal{F}_0}{\partial z \partial a} \ , \quad T(a,z,\beta) = (z\partial_z)^2 \mathcal{F}_0(a,z,\beta) \ . \tag{11.24}$$

and we identify T with the contact term introduced by the 5d codimension 2 observable $I(E)$ that we discussed in subsection 10.4. is the IR coupling and we used the Jacobi theta functions

$$\theta_1(x|\tau) = -\sum_{n\in\mathbb{Z}+\frac{1}{2}} (-1)^n e^{i\pi\tau n^2 + 2nix} \ , \quad \theta_4(x|\tau) = \sum_{n\in\mathbb{Z}} (-1)^n e^{i\pi\tau n^2+2nix} \ . \tag{11.25}$$

We can now check the contact term scaling behaviour discussed in the end of the previous section. For simplicity we consider the case $k=0$. To do this we compare the explicit expression of the blowup factor (11.23) with its expression in terms of polynomials in the SW Wilson loop vev U

$$B_{d,SW}^{(j,k)}(U,z) = \lim_{q\to 1} B_d^{(j,k)}(\mathbf{U},z,q) \ , \quad \mathcal{B}_{SW}^{(j,k)}(a,z,d,\beta) = B_{d,SW}^{(j,k)}(U,z) \tag{11.26}$$

In particular from (10.31), (10.32), (10.33) in the limit $q \to 1$ for $k = 0$ we have the following equations

$$B^{(0,0)}_{0,SW}(U,z) = 1 \ , \quad B^{(0,0)}_{1,SW}(U,z) = 1 \ , \quad B^{(1,0)}_{1,SW}(U,z) = -z^{1/4} \ , \quad B^{(1,0)}_{2,SW}(U,z) = -z^{1/4}U \ , \tag{11.27}$$

and substituting the explicit expression (11.23) we get

$$d = 0: \quad e^{\mathcal{F}_{1,0}+\mathcal{F}_{0,1}} = -\frac{1}{\theta_4(0|\tau)} \ , \tag{11.28}$$

$$d = 1: \quad e^{-\frac{1}{2}\beta^2 T} = \frac{\theta_4\left(\beta h\,|\tau\right)}{\theta_4\left(0\,|\tau\right)} = -z^{-1/4}\frac{\theta_1\left(\beta h\,|\tau\right)}{\theta_4\left(0\,|\tau\right)} \ , \tag{11.29}$$

$$d = 2: \quad U = -z^{-1/4}e^{2\beta^2 T}\frac{\theta_1(2\beta h|\tau)}{\theta_4(0|\tau)} \ , \tag{11.30}$$

which implies

$$z^{1/4} = -\frac{\theta_1\left(\beta h\,|\tau\right)}{\theta_4\left(\beta h\,|\tau\right)} \ , \quad U = \frac{\theta_4^3\left(0\,|\tau\right)}{\theta_4^3\left(\beta h\,|\tau\right)}\frac{\theta_1(2\beta h|\tau)}{\theta_1\left(\beta h|\tau\right)} \ , \quad e^{-\frac{1}{2}\beta^2 T} = \frac{\theta_4\left(\beta h\,|\tau\right)}{\theta_4\left(0\,|\tau\right)} \ , \tag{11.31}$$

and

$$e^{\frac{1}{2}\beta^2 T} = U^{1/4}\left(\frac{\theta_1(2\beta h|\tau)}{\theta_1\left(\beta h\,|\tau\right)}\frac{\theta_4\left(\beta h\,|\tau\right)}{\theta_4(0|\tau)}\right)^{-1/4} \tag{11.32}$$

which is the five-dimensional analogue of the 4d contact term equation

$$T_{4d} = \Lambda\partial_\Lambda u = (\Lambda\partial_\Lambda)^2\mathcal{F}_0 = \frac{u}{3} + \frac{\pi^2}{12}\frac{E_2(\tau)}{\omega_1^2} \ , \tag{11.33}$$

and reduces to it in the limit $\beta \to 0$ (at second order in β). Using the above relations the blowup factor for $j = 1, k = 0$ reads

$$\mathcal{B}^{(j,0)}_{SW}(a,z,d,\beta) = B^{(j,0)}_{d,SW}(U,z) = U^{d^2/4}\left(\frac{\theta_1(2\beta h|\tau)}{\theta_1\left(\beta h\,|\tau\right)}\frac{\theta_4\left(\beta h\,|\tau\right)}{\theta_4(0|\tau)}\right)^{-d^2/4}\frac{\theta_{4-3j}\left(\beta d h\,|\tau\right)}{\theta_4(0|\tau)} \ , \tag{11.34}$$

this is manifestly modular invariant if rewritten in terms of Weierstrass σ-function as

$$\mathcal{B}^{(j,0)}_{SW}(a,z,d,\beta) = B^{(j,0)}_{d,SW}(U,z) = U^{d^2/4}\left(\frac{\sigma(2\beta)}{\sigma\left(\beta\right)}\frac{\sigma\left(\beta+\omega_2\right)}{\sigma(\omega_2)}\right)^{-d^2/4}\frac{\sigma\left(\beta d + \omega_2(1-j)\right)}{\sigma(\omega_2)} \ , \tag{11.35}$$

where the Weierstrass σ-function is

$$\sigma(s;\omega_1,\omega_2) = \frac{2\omega_1}{\pi}e^{\frac{\pi^2}{6}\frac{E_2(\tau)}{4\omega_1^2}s^2}\frac{\theta_1\left(\frac{\pi s}{2\omega_1}\right)}{\theta_1'(0)} \ , \quad \theta_1'(0) = 2\eta(\tau)^3 \ , \quad \omega_1 = \frac{1}{2h} \ , \quad \omega_2 = \frac{\tau}{2h} \ . \tag{11.36}$$

which suggests a quadratic scaling $\mathcal{B}^{(j,0)}_{SW}(a,z,d,\beta) \sim U^{d^2/4}$ as previously discussed[85] and from (11.28) we see that this is given precisely by the contribution of the contact term.

[85] A similar analysis can be done for the case $k = 1$. The results are in agreement with the autonomous limit $q \to 1$ of q-PI Hurwitz expansion that we found in 11.2.

We observe that the actual scaling cannot be $d^2/4$ because in general it is rational, which violates holomorphicity in U, and in principle it can be modified by the contributions coming from the θ-functions in (11.34). This is indeed the case, and the effect of this is precisely to make the maximal power of U integer with the addition of a suitable function of d which depends on the sector and does not change the qualitative quadratic growing.

To verify this statement we can estimate the maximal power of U appearing in $B^{(j,0)}_{d,SW}$ by analyzing the following limit of the SW parameters

$$\beta h = \frac{\tau}{4} + x, \qquad \tau \to i\infty \, , \tag{11.37}$$

which gives the following asymptotics of the theta functions

$$\begin{aligned} \theta_1(\beta h|\tau) &= ie^{-\pi x} + O(e^{\frac{\pi i \tau}{2}}), \qquad \theta_4(\beta h|\tau) = 1 + O(e^{\frac{\pi i \tau}{2}}), \\ \theta_1(2\beta h|\tau) &= e^{2\pi x}e^{-\frac{\pi i \tau}{4}} + O(e^{\frac{3\pi i \tau}{4}}), \qquad \theta_4(0|\tau) = 1 + O(e^{\pi i \tau}) \, . \end{aligned} \tag{11.38}$$

In this limit the parameters U and z have the following asymptotics

$$z^{1/4} \sim -ie^{-\pi x}, \qquad U \sim -ie^{-\frac{\pi i \tau}{4}} e^{\pi x} \, , \tag{11.39}$$

which means that the limit (11.37) corresponds to $U \to \infty$ with z fixed and therefore in this limit we can directly see the polynomial degree in U.

The limit of the blow-up factor can be done with a saddle-point approximation of the θ series (11.25) and it gives

$$B^{(j,0)}_{d,SW}(U,z) = \frac{\theta_4(0|\tau)^{d^2}}{\theta_4(\beta h|\tau)^{d^2}} \frac{\theta_1(\beta d h|\tau)}{\theta_4(0|\tau)} \sim \theta_{4-3j}(\beta d h|\tau) \sim e^{i\pi\tau f_j(d)}. \tag{11.40}$$

where $f_j(d)$ is the value at the saddle-point

$$f_j(d) = \min_{n\in\mathbb{Z}} \left((n + d/4 + j/2)^2 - d^2/16 \right) \, . \tag{11.41}$$

From the previous asymptotic analysis we obtain that the leading order of U in $B^{(j,0)}_{d,SW}(U,z)$ is $U^{n^{(j,0)}_{\max}(d)}$ with

$$n^{(j,0)}_{\max}(d) = \frac{d^2}{4} - \min_{n\in\mathbb{Z}}(2n + d/2 + j)^2 = \left\lfloor \frac{d^2}{4} \right\rfloor - \chi_4(d + 2 - 2j) \, , \tag{11.42}$$

which is in agreement with the q-PIII$_3$ result (11.11), in particular, the maximal power of U is not modified by the Ω-background corrections.

Expansion around the 5d superconformal point The SW blowup factor (11.34) has an interesting structure in the strongly coupled superconformal point $z = 1$. In this case we have

$$B^{(j,0)}_{d} = (-1)^{\lfloor \frac{d}{4} \rfloor + j} U^{\left\lfloor \frac{d^2}{4} \right\rfloor} (1 - \chi_4(d + 2 - 2j)) \, , \tag{11.43}$$

which gives the scaling $n^{(j,0)}_{\max}(d) = \left\lfloor \frac{d^2}{4} \right\rfloor - \chi_4(d+2-2j)$ as observed in the q-Painlevé case (11.11).

It is natural to ask if there are other points in the moduli space where the blowup factor has this simple structure. In the following we will find these special points studying the SW blowup factor from the autonomous limit $q \to 1$ of the q-Painlevé equations[86].

11.3.1 Special points for $k = 0$

For local $\mathbb{F}_0$ ($k = 0$) the SW curve is

$$e^p + e^{-p} + e^x + ze^{-x} + U = 0 \ . \tag{11.44}$$

The corresponding q-PIII$_3$ equation (11.6) in the autonomous, i.e. SW, limit $q \to 1$ is

$$B^{(j,0)}_{d+1,SW} B^{(j,0)}_{d-1,SW} = B^{(j,0)}_{d,SW} B^{(j,0)}_{d,SW} - z^{1/2} B^{(j+1,0)}_{d,SW} B^{(j-1,0)}_{d,SW} \ . \tag{11.45}$$

To find the special points we consider the ansatz

$$B^{(j,0)}_{d,SW} = z^{j/4} b^{(j)}_d U^{\left\lfloor \frac{d^2}{p} \right\rfloor} \ , \quad b^{(j)}_d = 0, \pm 1 \ , \tag{11.46}$$

where p is a positive integer and $b^{(j)}_d$ is p-periodic/anti-periodic

$$b^{(j)}_{d+p} = s_j b^{(j)}_d \ , \quad s_j = \pm 1 \ . \tag{11.47}$$

The NY blowup equations (10.32) fix the initial conditions

$$\begin{aligned} b^{(0,0)}_{-1} &= 1 \ , \quad b^{(0,0)}_0 = 1 \ , \quad b^{(0,0)}_1 = 1 \ , \\ b^{(1,0)}_{-1} &= 1 \ , \quad b^{(1,0)}_0 = 0 \ , \quad b^{(1,0)}_1 = -1 \ , \end{aligned} \tag{11.48}$$

and from (10.33) we have the condition

$$B^{(1,0)}_2 = -z^{1/4} U = z^{1/4} b^{(1,1)}_2 U^{\left\lfloor \frac{4}{p} \right\rfloor} \quad \Rightarrow \quad b^{(1,1)}_2 = -1 \ , \quad \left\lfloor \frac{4}{p} \right\rfloor = 1 \quad \Rightarrow \quad p = 3, 4 \ . \tag{11.49}$$

Substituting the ansatz (11.46) in (11.45) we get

$$b^{(0,0)}_{d+1} b^{(0,0)}_{d-1} U^{\Delta_p(d)} = b^{(0,0)}_d b^{(0,0)}_d - z b^{(1,0)}_d b^{(1,0)}_d \ , \tag{11.50}$$

$$b^{(1,0)}_{d+1} b^{(1,0)}_{d-1} U^{\Delta_p(d)} = b^{(1,0)}_d b^{(1,0)}_d - b^{(0,0)}_d b^{(0,0)}_d \ . \tag{11.51}$$

where

$$\Delta_p(d) = \left\lfloor \frac{(d+1)^2}{p} \right\rfloor + \left\lfloor \frac{(d-1)^2}{p} \right\rfloor - 2 \left\lfloor \frac{d^2}{p} \right\rfloor \ , \tag{11.52}$$

which is p-periodic and for $p \geq 3$ takes values in $\{0, 1, -1\}$

$$\Delta_p(d+p) = \Delta_p(d) \ , \quad \Delta_p(d) = 0, \pm 1 \text{ for } p \geq 3 \tag{11.53}$$

From (11.48) and the equation (11.50) for $d = 1$ we get the condition

$$b^{(0,0)}_2 U^{\Delta_p(1)} = 1 - z \quad \Rightarrow \quad z = 1 - b^{(0,0)}_2 U^{\Delta_p(1)} \ . \tag{11.54}$$

[86]In principle one can consider also the case with Ω-background $q \neq 1$ but we didn't find any simple structure in this case.

We can choose $b_2^{(0,0)} = 0, \pm 1$ and we have $\Delta_p(1) = 1$ for $p = 3, 4$ so the only possible special points are

$$z = 1 \ , \quad z = 1 \pm U \ , \tag{11.55}$$

which can have $\mathbb{Z}_p$ symmetry with $p = 3, 4$. To fix the period we have to verify if we get $b_p^{(j,0)} = s_j b_0^{(j,0)}$, $b_{p-1}^{(j,0)} = s_j b_{-1}^{(j,0)}$ for some p[87]. By a direct check we find that $z = 1$ correspond to the $\mathbb{Z}_4$ symmetric solution (11.43)

$$B_d^{(j,0)} = (-1)^{\lfloor \frac{d}{4} \rfloor + j} U^{\left\lfloor \frac{d^2}{4} \right\rfloor} (1 - \chi_4(d + 2 - 2j)) \ , \tag{11.56}$$

therefore, we recover the superconformal point that we previously discussed. In the same way, we find that the two points $z = 1 \pm U$ correspond to the $\mathbb{Z}_3$ symmetric solutions[88]

$$B_d^{(0,0)} = U^{\left\lfloor \frac{d^2}{3} \right\rfloor} \ , \quad B_d^{(1,0)} = -(1 \pm U)^{1/4} (-1)^{\lfloor \frac{d}{3} \rfloor} U^{\left\lfloor \frac{d^2}{3} \right\rfloor} (1 - \chi_3(d)) \ . \tag{11.57}$$

11.3.2 Special points for $k = 1$

For local $\mathbb{F}_1$ ($k = 1$) the SW curve is

$$e^p + e^{-p+x} + e^x + z e^{-x} + U = 0 \ , \tag{11.58}$$

and the corresponding q-PI equation (11.17) in the autonomous limit $q \to 1$ reads

$$B_{d+1,SW}^{(j,1)} B_{d-1,SW}^{(j,1)} = B_{d,SW}^{(j,1)} B_{d,SW}^{(j,1)} - z^{1/2} B_{d,SW}^{(j+1,1)} B_{d-(-1)^j,SW}^{(j-1,1)} \ . \tag{11.59}$$

In this case we consider the ansatz

$$B_d^{(0,1)} = b_d^{(0,1)} U^{\left\lfloor \frac{d(d-1)}{p} \right\rfloor} \ , \quad b_d^{(0,1)} = 0, \pm 1 \ , \tag{11.60}$$

$$B_d^{(1,1)} = (q^d z)^{1/4} b_d^{(1,1)} U^{\mathrm{nint}(\frac{d^2}{p})} \ , \quad b_d^{(1,1)} = 0, \pm 1 \ , \tag{11.61}$$

where $\mathrm{nint}(x)$ is the nearest integer function[89]and the initial conditions (10.32)

$$\begin{aligned} b_{-1}^{(0,1)} &= 1 \ , \quad b_0^{(0,1)} = 1 \ , \quad b_1^{(0,1)} = 1 \ , \\ b_{-1}^{(1,1)} &= 1 \ , \quad b_0^{(1,1)} = 0 \ , \quad b_1^{(1,1)} = -1 \ , \end{aligned} \tag{11.62}$$

and (10.33)

$$B_2^{(1,1)} = -z^{1/4} U = z^{1/4} b_2^{(1,1)} U^{\mathrm{nint}(\frac{4}{p})} \quad \Rightarrow \quad b_2^{(1,1)} = -1 \ , \quad \mathrm{nint}\left(\frac{4}{p}\right) = 1 \quad \Rightarrow \quad p = 3, 4, 5, 6, 7 \ . \tag{11.63}$$

[87]Indeed, from (11.50), (11.51), we have an autonomous recurrence and $b_{d+1}^{(j,0)}$ has the same sign of $b_{d-1}^{(j,0)}$. Therefore, $b_d^{(j,0)}$ will be periodic/anti-periodic if at some point we go back to the initial values up to the sign s_j. We also observe for the point $z = 1$ we have $b_2^{(0,0)} = 0$ and the recurrence relation is ill-defined and should be regularized. This is done setting $b_2^{(0,0)} = \delta$ and taking the limit $\delta \to 0$ at the end.

[88]It seems that some structure survives if we choose $q^m = 1$ for some integer m. For instance for $q^4 = 1$ some of the zeros survive and the coefficients have a simple structure $B_d \sim U^n((-4 - 4i) + U^2)^l)$.

[89]For $p = 3, 4$ we have $\mathrm{nint}(\frac{d^2}{p}) = \left\lfloor \frac{d^2}{p} \right\rfloor$.

and the equation (11.59) becomes

$$b_{d+1}^{(0,1)}b_{d-1}^{(0,1)}U^{\Delta'_p(d)} = b_d^{(0,1)}b_d^{(0,1)} - zb_d^{(1,1)}b_{d-1}^{(1,1)}U^{\Delta''_p(d)} \ , \tag{11.64}$$

$$b_{d+1}^{(1,1)}b_{d-1}^{(1,1)}U^{\mathrm{nint}\left(\frac{(d+1)^2}{p}\right)+\mathrm{nint}\left(\frac{(d-1)^2}{p}\right)} = U^{2\mathrm{nint}\left(\frac{d^2}{p}\right)}b_d^{(1,1)}b_d^{(1,1)} - b_d^{(0,1)}b_{d+1}^{(0,1)}U^{\left\lfloor\frac{(d+1)d}{p}\right\rfloor+\left\lfloor\frac{d(d-1)}{p}\right\rfloor} \ , \tag{11.65}$$

where we defined

$$\Delta'_p(d) = \left\lfloor\frac{(d+1)d}{p}\right\rfloor + \left\lfloor\frac{(d-1)(d-2)}{p}\right\rfloor - 2\left\lfloor\frac{d(d-1)}{p}\right\rfloor \ , \tag{11.66}$$

$$\Delta''_p(d) = \mathrm{nint}\left(\frac{d^2}{p}\right) + \mathrm{nint}\left(\frac{(d-1)^2}{p}\right) - 2\left\lfloor\frac{d(d-1)}{p}\right\rfloor \ , \tag{11.67}$$

that are again p-periodic. For $d=1$ we have $\Delta'_p(1) = \Delta''_p(1) = 0$ for $p \geq 3$ and the equation (11.64) gives

$$b_2^{(0,1)} = 1 \ , \tag{11.68}$$

and for $d=2$ we have the condition

$$b_3^{(0,1)}U^{\Delta'_p(2)} = 1 - zU^{\Delta''_p(2)} \quad \Rightarrow \quad z = U^{-\Delta''_p(2)} - b_3^{(0,1)}U^{\Delta'_p(2)-\Delta''_p(2)} \tag{11.69}$$

We can choose $b_3^{(0,1)} = 0, \pm 1$ and we have

$$\Delta'_p(2) = \begin{cases} 2 & \text{if } p=3 \\ 1 & \text{if } p=4,5,6 \\ 0 & \text{if } p>6 \end{cases} \quad , \quad \Delta''_p(2) = \begin{cases} 1 & \text{if } p=3,4,5 \\ 0 & \text{if } p>5 \ , \end{cases} \tag{11.70}$$

so the possible special points are

$$z = U^{-1} \ , \quad z = U^{-1} \pm U \quad \text{for } p=3 \ , \tag{11.71}$$

$$z = U^{-1} \ , \quad z = U^{-1} \pm 1 \quad \text{for } p=4,5 \tag{11.72}$$

$$z = 1 \ , \quad z = 1 \pm U \ , \quad \text{for } p=6 \ , \tag{11.73}$$

$$z = 0,1,2 \quad \text{for } p>6 \ . \tag{11.74}$$

Again we need to verify if the recurrence closes at some p, $b_p^{(j,1)} = s_j b_0^{(j,1)}$, $b_{p-1}^{(j,1)} = s_j b_{-1}^{(j,1)}$. This happens only for the point $z = U^{-1} - U$ which gives the $\mathbb{Z}_3$ symmetric solution

$$B_d^{(0,1)} = U^{\left\lfloor\frac{d(d-1)}{3}\right\rfloor} \ , \quad B_d^{(1,1)} = -(U^{-1}-U)^{1/4}(-1)^{\lfloor\frac{d}{3}\rfloor}U^{\mathrm{nint}\left(\frac{d^2}{3}\right)}(1-\chi_3(d)) \ . \tag{11.75}$$

and the point $z = U^{-1}$ which corresponds to the $\mathbb{Z}_5$ symmetric solution

$$B_d^{(0,1)} = (-1)^{\lfloor\frac{d}{5}\rfloor}U^{\left\lfloor\frac{d(d-1)}{5}\right\rfloor}(1-\chi_5(d+3)) \ , \quad B_d^{(1,1)} = -U^{-1/4}(-1)^{\lfloor\frac{d}{5}\rfloor}U^{\mathrm{nint}\left(\frac{d^2}{5}\right)}(1-\chi_5(d)) \ . \tag{11.76}$$

12 Limits to four-dimensional theories

In this section we study the four-dimensional reduction of 5d SYM with CS level $k = 0, 1$ and there are two possible limits we can consider.

The UV completions of the 5d $\mathcal{N} = 1$ SYM with CS level $k = 0$ and $k = 1$ are the so-called E_1 and $\tilde{E}_1$ SCFTs [56]. In both cases, this SCFT has a single real mass deformation $M_{UV} = 1/g_5^2$, which can be positive or negative, related to the instanton mass i.e. the 5d UV bare coupling. The relation with the 5d instanton counting parameter z is

$$z = e^{-\frac{8\pi^2\beta}{g_5^2}} \,. \tag{12.1}$$

The first limit we can take is the standard geometric engineering limit where we integrate out the massive KK modes keeping the combination $\Lambda^4 = \beta^{-4} z$ finite, so we are in a weakly coupled phase of the 5d theory. In this way we recover 4d $\mathcal{N} = 2\, SU(2)$ pure SYM with coupling Λ. Notice that this limit is independent on the value of the CS level k.

The other limit we consider is obtained by doing a dimensional reduction $\beta \to 0$ at *finite* 5d coupling z. This gives a non trivial result only for $k = 1$, as in the $k = 0$ theory this limiting procedure leads to a purely topological theory as the RGE become trivial. This is explained in full detail from this view point in subsection 12.2.

Indeed, in the case of E_1 SCFT, for $M_{UV} > 0$ the theory flows to the 5d pure theory with $k = 0$ which is weakly coupled. For a negative mass deformation $M_{UV} < 0$ we obtain a physically equivalent theory and again we flow to the weakly coupled 5d $SU(2)$ SYM theory with $k = 0$ where instanton particles and W-bosons are just exchanged. The new parameters become [114]

$$\tilde{m}_I = m_W, \quad \tilde{m}_W = m_I \iff \tilde{M}_{UV} = 1/\tilde{g}_5^2 = -1/g_5^2 = -M_{UV} \ , \quad \tilde{a} = a + 1/g_5^2 \ . \tag{12.2}$$

In the (p, q)-web construction, for $M_{UV} < 0$ the roles of D5 branes and NS5 branes are inverted, see figure 10 where the horizontal lines represent D5 branes and the vertical ones represent NS5 branes. This means that for $k = 0$ there is no strongly coupled phase.

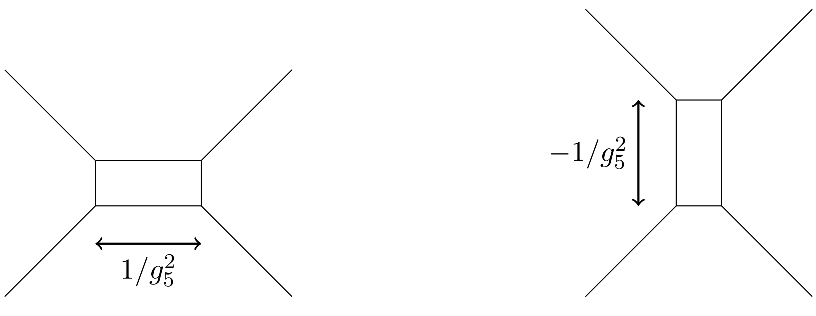

Figure 10. (p, q)-web for 5d SYM with CS level $k = 0$. For negative coupling the roles of instantons and W-bosons are exchanged.

The situation is drastically different in the case of $\tilde{E}_1$ SCFT. In this case the flow triggered by the mass deformation depends on the sign of M_{UV}. For $M_{UV} > 0$ the theory flows to 5d SYM with CS level $k = 1$ so we land in a weakly coupled gauge theory phase. For $M_{UV} < 0$ instead the theory flows to the non-lagrangian E_0 SCFT which has no flavour symmetry and corresponds to a deformed version of the local $\mathbb{P}^2$ geometry, obtained from the local $\mathbb{F}_1$ geometry, (see figure 11), the undeformed one being reached in the limit $M_{UV} \to -\infty \iff z \to \infty$. We will see that in this strongly coupled region, at finite coupling z, the theory admits a non trivial 4d limit to the AD theory H_0.

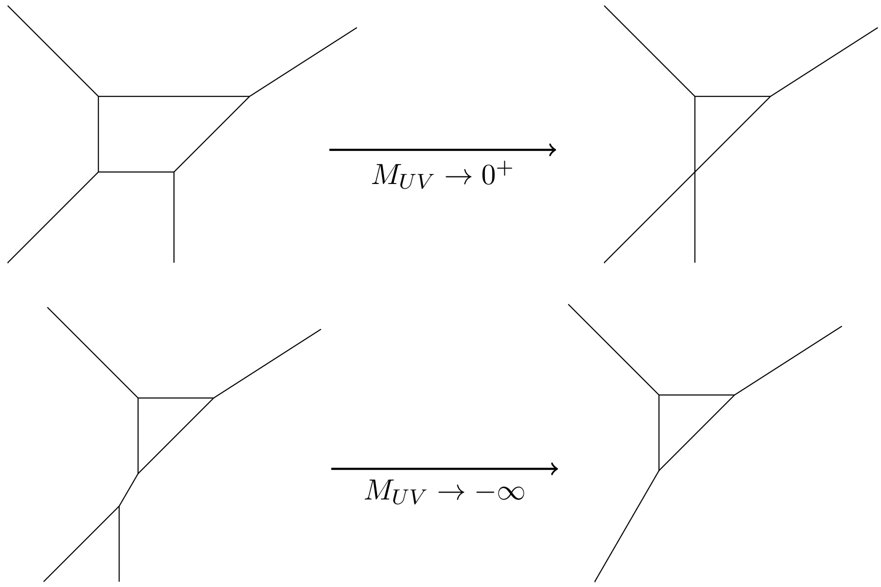

Figure 11. (p, q)-web for 5d SYM with CS level $k = 1$ for $M_{UV} > 0$ (upper part) and $M_{UV} < 0$ (lower part). In the limit $M_{UV} \to -\infty$ we obtain the E_0 theory (local $\mathbb{P}^2$ geometry).

At the level of the q-Painlevé $\mathcal{T}$-function this difference arises from an extra symmetry $z \to 1/z$ of the equation (11.3) for $k = 0$ which is absent for $k = 1$ [130].

We want now to analyze in detail these limits at the level of the RGEs.

12.1 Geometric engineering limit

For any k we can recover the standard $4d$ limit to pure $SU(2)$ gauge theory and the corresponding NS blowup factor by the geometric engineering limit corresponding to Kaluza-Klein (KK) reduction on the circle direction. More precisely, in the 4d limit we scale

$$\beta = \frac{\log q}{\epsilon} \to 0 \iff q \to 1 \; , \tag{12.3}$$

keeping fixed the 4d instanton scale Λ^4

$$z = \beta^4 \Lambda^4 = \frac{(\log q)^4}{\epsilon^4} \Lambda^4 = \frac{(q-1)^4}{\epsilon^4} \Lambda^4 + O((q-1)^5) \; . \tag{12.4}$$

This limit corresponds to one in which the 4d UV theory is weakly coupled. Indeed, the UV coupling $g_4(1/\beta)$ of the 4d theory at the scale of the circle radius $\mu = 1/\beta$ given by RG matching is

$$\frac{1}{g_4^2(1/\beta)} = \frac{\beta}{g_5^2} \; , \tag{12.5}$$

and the RG invariant scale Λ is then

$$\Lambda^4 = \frac{1}{\beta^4} e^{-\frac{8\pi^2}{g_4^2(1/\beta)}} \ . \tag{12.6}$$

Therefore, in the limit $\beta \to 0$ at fixed Λ we have $g_4^2(1/\beta) \to 0$ in agreement with 4d asymptotic freedom. From a 5d point of view in the limit $\beta \to 0$ we send the ratio between the UV mass deformation of the 5d SCFT $M_{UV} = 1/g_5^2$ and the KK mass scale $M_{KK} = 1/\beta$ to infinity, $M_{UV}/M_{KK} \to +\infty$, therefore instanton particles become heavy and are integrated out.

In the limit we are considering we scale also the Wilson loop $\mathbf{U}$ as[90]

$$\mathbf{U} = 2 + \left(\mathbf{u} - \frac{\epsilon^2}{4}\right)\beta^2 + O(\beta^3) = 2 + \left(\frac{\mathbf{u}}{\epsilon^2} - \frac{1}{4}\right)(q-1)^2 + O((q-1)^3) \ . \tag{12.7}$$

The finite continuous 4d surface observable parameter s is obtained by sending $d \to \infty$ as

$$d = \frac{\epsilon s}{q-1} \tag{12.8}$$

and we define the NS 4d blowup factor[91] as

$$\mathcal{B}_{NS}^{(j)}(s) = \lim_{\beta \to 0} q^{-d/4} \mathcal{B}_{NS}^{(j,k)}\left(\frac{\epsilon s}{q-1}, \beta\right) = \left\langle e^{sO^{(4d)}(\Omega, \mathbf{F}^2)}\right\rangle_{\hat{\mathbb{C}}^2, NS} \ , \tag{12.9}$$

where $O^{(4d)}(\Omega, \mathbf{F}^2)$ is the 4d surface observable (10.10) inserted in the exceptional divisor E. We now consider the expansion

$$\begin{aligned} B_{d+m}^{(j,k)} B_{d-m}^{(j,k)} &= \mathcal{B}_{NS}^{(j,k)}(d+m, \beta)\, \mathcal{B}_{NS}^{(j,k)}(d-m, \beta) = e^{mD}(\mathcal{B}_{NS}^{(j,k)}, \mathcal{B}_{NS}^{(j,k)}) = \\ &= \sum_{n=0}^{\infty} \frac{m^n}{n!} D_d^{(n)}\left(\mathcal{B}_{NS}^{(j,k)}, \mathcal{B}_{NS}^{(j,k)}\right) = \sum_{n=0}^{\infty} \frac{m^n}{n!}\left(\frac{q-1}{\epsilon}\right)^n D_s^{(n)}\left(\mathcal{B}_{NS}^{(j,k)}, \mathcal{B}_{NS}^{(j,k)}\right) \ . \end{aligned} \tag{12.10}$$

In the limit $q \to 1$ both the Hirota equations (11.6) and (11.17) become

$$D_s^{(2)}(\mathcal{B}_{NS}^{(j)}, \mathcal{B}_{NS}^{(j)}) = -2e^{\frac{\epsilon s}{2}} \Lambda^2\, \mathcal{B}_{NS}^{(j-1)} \mathcal{B}_{NS}^{(j+1)} \ , \tag{12.11}$$

which is the differential Hirota Painlevé III_3 equation in Toda form, where Λ^4 parametrizes the position of the zero in the odd sector $j = 1$. The surface observable parameter s shifts the 4d coupling Λ^4 and the Hurwitz expansion is obtained expanding around $s = 0$ [64]

$$\Lambda_{\epsilon s}^4 = \Lambda^4 e^{\epsilon s} \ , \quad \mathcal{B}_{NS}^{(j)}(s) = 1 - j + \sum_{n=0}^{+\infty} c_n^{PIII_3,(j)} \frac{s^{n+1}}{(n+1)!} \ , \tag{12.12}$$

[90]This should correspond to the expansion in β of the Wilson loop observable

$$\left\langle \mathrm{Tr}_2\, P\text{-}\exp \int_{S^1_\beta} \phi(x,t)\mathrm{d}t \right\rangle_{\mathbb{C}^2 \times S^1_\beta, NS} = \left\langle \mathrm{Tr}_2 \left(1 + \beta\phi(x) + \frac{1}{2}\beta^2\phi^2(x) + O(\beta^3)\right)\right\rangle_{\mathbb{C}^2, NS} \ ,$$

with

$$\int_{S^1_\beta} \phi(x,t)\mathrm{d}t = \beta\phi(x) \ , \quad \mathrm{Tr}_2\, \phi(x) = 0 \ ,$$

and $\phi(x)$ the 4d complex scalar.

[91]The 4d limit is independent on the value of the CS level k. We also observe, from (10.18), that the 4d and 5d blowup factors differ by a "gauge prefactor" $q^{-d/4}$.

where the coefficients[92] are integer polynomials in $\mathbf{u}$, Λ^4 and ϵ.

Finally, the Hurwitz expansion around $s = 0$ is easily recovered observing that the $q \to 1$ limit of the discrete derivative D_q at $d = 0$ gives the continuous derivative ∂_s evaluated at $s = 0$

$$c_n^{PIII_3,(j)} = \epsilon^n \lim_{q\to 1} D_q^n (q^{-d/4} B_d^{(j,k)}(\mathbf{U}, z, q))\Big|_{d=0} = \partial_s^n \mathcal{B}_{NS}^{(j)}(\mathbf{u}, \Lambda, s)\Big|_{s=0} , \tag{12.13}$$

$$D_q B_d^{(j,k)}(z) \equiv \frac{B_{d+1}^{(j,k)}(z) - B_d^{(j,k)}(z)}{q-1} . \tag{12.14}$$

12.2 Strongly coupled 4d limit at negative coupling

We consider now the 4d limit to H_0 AD theory. This is achieved considering a double scaling limit of the $k = 1$ theory in which we keep finite the UV coupling $z \leftrightarrow 1/g_4^2(1/\beta)$, i.e. the mass ratio $M_{UV}/M_{KK} = \beta/g_5^2 = -\log z$ is kept fixed in the limit $\beta \to 0$.

To get the limit to the AD theory we have to tune the coupling z and the Wilson loop $\mathbf{U}$ to some special finite values. The deformations of the theory at this point have exactly the correct fractional scaling dimensions of the AD SCFT. In the following we will illustrate in detail how this scaling arises from the q-Painlevé equation. Furthermore, we find that the first Chern class j cannot be kept fixed in the flow of the $5d$ surface observable to the IR. At the level of the equations this means that we actually have to consider an alternating dynamics where we jump from $\mathcal{T}^{(0,1)}$ to $\mathcal{T}^{(1,1)}$ depending on the value of d.

Precisely, we now set $k = 1$ and we define the “twisted” NS blowup factor

$$\tilde{\mathcal{B}}_{NS}(a, z, \epsilon, d, \beta) = \tilde{B}_d(\mathbf{U}, z, q) \equiv \begin{cases} B_{d/2}^{(1,1)}(\mathbf{U}, z, q) & \text{if } d \text{ even} , \\ B_{d/2+1/2}^{(0,1)}(\mathbf{U}, z, q) & \text{if } d \text{ odd} , \end{cases} \tag{12.15}$$

and we can rewrite the equation (11.17) in terms[93] of $\tilde{B}_d$

$$\tilde{B}_{d+2}\tilde{B}_{d-2} = \tilde{B}_d\tilde{B}_d - z^{1/2}q^{d/4}\tilde{B}_{d+1}\tilde{B}_{d-1} . \tag{12.16}$$

We want now to recover the AD theory which corresponds to the PI equation

$$D_s^{(4)}(\mathcal{B}_{NS}, \mathcal{B}_{NS}) + 2\epsilon s \mathcal{B}_{NS}^2 - g_2 \mathcal{B}_{NS}^2 = 0 . \tag{12.17}$$

To do this we scale the parameters of the theory in the limit $q \to 1$ as

$$d = -\delta^{-1}s , \quad z = z_0\left(1 + \frac{g_2}{4}\delta^p\right) , \quad \delta = \left(\frac{q-1}{\epsilon}\right)^\Delta , \tag{12.18}$$

The scaling dimensions Δ and $\Delta' = p\Delta$ correspond precisely to the ones of the surface observable parameter s and of the gauge coupling g_2 of the 4d theory we reach in the limit. We define now

$$\tilde{\mathcal{B}}'_{NS}(d, \beta) = \kappa^{-d^2}\tilde{B}_d , \tag{12.19}$$

[92]For the first values of the coefficients see appendix E.3.

[93]The equation (12.16) corresponds at even times to (11.17) for $j = 1$ and at odd times to (11.17) for $j = 0$.

and the 4d blowup factor

$$\tilde{\mathcal{B}}_{NS}^{(4d)}(s) = \lim_{\beta\to 0} \tilde{\mathcal{B}}'_{NS}(d,\beta) \ . \tag{12.20}$$

As we will show below, the blowup factor $\tilde{\mathcal{B}}_{NS}^{(4d)}(s)$ is precisely the one of AD theory H_0. The gaussian prefactor contribution κ in (12.19) is the contribution to the contact term $T(\mathbf{U})$ (10.37) of the dof that decouple in the AD limit.

To take the limit we rewrite

$$\begin{aligned} &\tilde{B}_{d+m}\tilde{B}_{d-m} = \kappa^{2d^2+2m^2}\tilde{\mathcal{B}}'_{NS}(d+m,\beta)\,\tilde{\mathcal{B}}'_{NS}(d-m,\beta) \ , \\ &\tilde{\mathcal{B}'}_{NS}(d+m,\beta)\,\tilde{\mathcal{B}'}_{NS}(d-m,\beta) = e^{mD}(\tilde{\mathcal{B}}'_{NS},\tilde{\mathcal{B}}'_{NS}) = \\ &= \sum_{n=0}^{\infty}\frac{m^n}{n!}D_d^{(n)}\left(\tilde{\mathcal{B}}'_{NS},\tilde{\mathcal{B}}'_{NS}\right) = \sum_{n=0}^{\infty}\frac{m^n}{n!}\delta^n D_s^{(n)}\left(\tilde{\mathcal{B}}'_{NS},\tilde{\mathcal{B}}'_{NS}\right) \ , \end{aligned} \tag{12.21}$$

where $D_d^{(n)}\left(\tilde{\mathcal{B}}_{NS},\tilde{\mathcal{B}}_{NS}\right)$ is the n-th Hirota derivative and where in the last step we changed variable $d \to s$.

Rewriting the equation (12.16) in terms of $\tilde{\mathcal{B}}'_{NS}$ and dividing by κ^{2d^2+8} we get

$$\tilde{\mathcal{B}}'_{NS}(d+2,\beta)\,\tilde{\mathcal{B}}'_{NS}(d-2,\beta) = \kappa^{-8}\tilde{\mathcal{B}}'_{NS}(d,\beta)^2 - z^{1/2}q^{d/4}\kappa^{-6}\tilde{\mathcal{B}}'_{NS}(d+1,\beta)\,\tilde{\mathcal{B}}'_{NS}(d-1,\beta) \ . \tag{12.22}$$

and expanding the equation (12.22) for $q \to 1$ and using (12.18), (12.21) we obtain[94]

$$\begin{aligned} &\left(1 + z_0^{1/2}\kappa^{-6} - \kappa^{-8}\right)\tilde{\mathcal{B}}'_{NS}(s)^2 + \\ &\left(2 + \frac{1}{2}z_0^{1/2}\kappa^{-6}\right) D_s^{(2)}\left(\tilde{\mathcal{B}}'_{NS},\tilde{\mathcal{B}}'_{NS}\right)\delta^2 + \\ &\frac{1}{24}\left(16 + z_0^{1/2}\kappa^{-6}\right) D_s^{(4)}\left(\tilde{\mathcal{B}}'_{NS},\tilde{\mathcal{B}}'_{NS}\right)\delta^4 + \\ &\frac{1}{8}g_2 z_0^{1/2}\kappa^{-6}\tilde{\mathcal{B}}'_{NS}(s)^2\delta^p - \frac{1}{4}\epsilon s z_0^{1/2}\kappa^{-6}\tilde{\mathcal{B}}'_{NS}(s)^2\delta^{\frac{1}{\Delta}-1} + \dots \ , \end{aligned} \tag{12.23}$$

where $\dots$ are subleading terms which do not contribute in the limit $q \to 1$. The dynamics of the 4d theory is determined by the first non-zero term in the δ expansion and depends on the values of the parameters Δ, p, z_0, κ.

Looking at the leading contribution in δ in the equation (12.23) as we move in the parameter space we can construct a full phase diagram, which is reported in figure 12. For a generic value of the coupling[95] z_0 the theory is trivial ($\mathcal{T} = 0$), except for $p \geq 2$ together with $\Delta = 1$ or $\Delta \leq 1/3$ where a gapped phase arises, shown as dashed line and dashed region in figure 12. In this phase the only surviving contribution is the one of the topological defect coming from some contact term ($\mathcal{T} \sim e^{f(s)}$ with $f(s)$ some elementary function).

For a special value of z_0 instead we obtain the AD theory. The corresponding PI equation can appear only at order δ^4 therefore the contributions at lower orders must

[94]Notice that $D_s^{(n)}\left(\tilde{\mathcal{B}}'_{NS},\tilde{\mathcal{B}}'_{NS}\right)$ is vanishing for n odd.

[95]We can always fix κ in terms of z_0 to cancel the zero order contribution in (12.23) which otherwise always trivializes the dynamics.

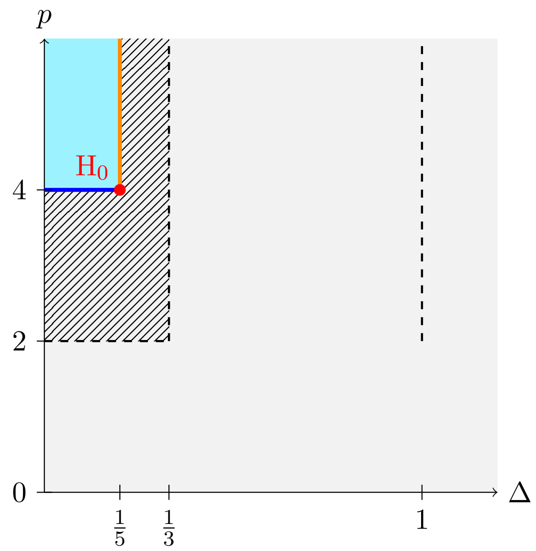

Figure 12. Phase diagram for the 4d scaling limits. The grey region corresponds to the trivial theory $\mathcal{T} = 0$ and the dashed regions correspond to gapped theories ($\mathcal{T} \sim e^{f(s)}$) that we obtain only for generic values of z_0. The red point corresponds to the H_0 AD SCFT, the orange line to H_0 with no coupling deformation and the blue line and the light blue region are the SW limits of the previous two.

cancel and the extra terms at order $\delta^p, \delta^{\frac{1}{\Delta}-1}$ must be of order δ^4 . This happens if we have

$$\kappa^8 = -\frac{1}{3} \; , \quad z_0^{1/2}\kappa^{-6} = -4 \quad , \quad \Delta = \frac{1}{5} \; , \quad p = 4 \Rightarrow \Delta' = \frac{4}{5} \; , \tag{12.24}$$

which reproduce the correct scaling dimensions of the H_0 theory operators. In particular, we have

$$|\kappa| = \left(\frac{1}{3}\right)^{1/8} \; , \quad |z_0| = 16\left(\frac{1}{3}\right)^{3/2} > 1 \; , \tag{12.25}$$

therefore the AD point appear in the negative coupling phase of the $k = 1$ theory corresponding to a deformation of the local $\mathbb{P}^2$ geometry.

Using this, in the limit $q \to 1$ the equation (12.23) becomes exactly the quartic differential Hirota bilinear equation for the AD blowup factor

$$D_s^{(4)}\left(\tilde{\mathcal{B}}_{NS}, \tilde{\mathcal{B}}_{NS}\right) + 2\epsilon s \tilde{\mathcal{B}}_{NS}^2 - g_2 \tilde{\mathcal{B}}_{NS}^2 = 0 \; , \tag{12.26}$$

which corresponds to the equation for the PI $\mathcal{T}$-function. If the deformation g_2 scales too fast ($p > 4$) it is irrelevant in the limit and we obtain the AD theory where only the Coulomb branch operator deformation is present i.e. effectively $g_2 = 0$. If instead the surface observable, scales too slow $\Delta < 1/5$ then the radius β scales faster than the Ω-background parameter ϵ and it washes it out leading just to the SW limit of the theory, the $\mathcal{T}$-function is given then by (11.23).

We observe that this behaviour is possible precisely because of the presence of the CS term which introduces the extra shift in the term $\tilde{\mathcal{B}}'_{NS}(d+1,\beta)\,\tilde{\mathcal{B}}'_{NS}(d-1,\beta)$. This breaks the symmetry $z \to 1/z$ and gives an extra $D_s^{(2)}\left(\tilde{\mathcal{B}}'_{NS}, \tilde{\mathcal{B}}'_{NS}\right)$ contribution that allows for the cancellation of the δ^2 term. Conversely, for $k = 0$ no cancellation arises and we don't find any point with non-trivial dynamics as expected from the analysis of the geometries we discussed before.

We can also take the limit directly at the level of the $\mathcal{T}$-function scaling the Wilson loop parameter as follows[96]

$$\mathbf{U} = -\frac{2i}{3\sqrt{3}}\left(1 - \frac{g_2}{2}\delta^4 - 4\epsilon\delta^5 - \frac{3}{2}\mathbf{g_3}\delta^6\right) , \tag{12.27}$$

where g_2 is the AD coupling and $\mathbf{g_3} = -2\mathbf{u}$ is the NS modulus corresponding to the Coulomb branch operator, reproduces the correct scaling dimension $6/5$ of H_0 theory and also contain some ϵ corrections which are due to the NS quantum SW geometry[97].

The shifted coupling of AD theory is additive $g_2(s) = g_2 + \epsilon^{1/5}s$ and the surface observable parameter has fractional dimension $[s] = -1/5$. The Hurwitz expansion[98] around $s = 0$

$$\tilde{\mathcal{B}}_{NS}(s) = \sum_{n=0}^{+\infty} c_n^{PI}\frac{s^{n+1}}{(n+1)!} , \tag{12.28}$$

can be recovered taking the continuous limit of the "fractional" discrete derivatives $\tilde{D}_q^n$ of the 5d blowup factor

$$c_n^{PI} = \epsilon^{n/5}\lim_{q\to 1}\tilde{D}_q^n\left[\left(-\frac{1}{3}\right)^{-d^2/8}\tilde{B}_d(\mathbf{U}, z, q)\right]\Bigg|_{d=0} = \partial_s^n\tilde{\mathcal{B}}_{NS}(\mathbf{u}, g_2, s)\Bigg|_{s=0} , \tag{12.29}$$

$$\tilde{D}_q\tilde{B}_d(z) \equiv \frac{\tilde{B}_{d+1}(z) - \tilde{B}_d(z)}{(q-1)^{1/5}} . \tag{12.30}$$

In this way we recover the 4d $\mathcal{T}$-function, therefore the self-dual partition function, of the AD theory.

13 Conclusions and future perspectives

In this part we discussed the lift of the Hurwitz chiral ring expansion of the blowup factor to five-dimensional theories on $\mathbb{R}^4 \times S^1_\beta$. The local observable NS vev $\mathbf{u} = \langle \mathrm{Tr}\,\phi^2\rangle_{NS}$ of the 4d theory is promoted to the expectation value of the fundamental Wilson loop $\mathbf{U} \equiv \langle W_1\rangle_{NS}$ wrapped along the circle S^1_β. To obtain the equivariance with respect to circle translation the 5d lift of the blowup surface defect gets modified by a 3d CS term

$$I(E) = \exp\left[\frac{1}{4\pi}\int_X \omega\wedge 2\,\mathrm{Tr}\left(i\varphi F + \frac{1}{2}\psi\wedge\psi\right)\wedge dt - \omega\wedge CS_3(A) + H\,\mathrm{Tr}\left(F\wedge F\right)\wedge dt\right] , \tag{13.1}$$

[96]This particular scaling is obtained matching the solution with the Hurwitz expansion of the 4d AD theory and should be understood as the RG flow of the Wilson loop observable to the IR AD point.

[97]In the SW limit $\epsilon \to 0$, the same scalings of z, U were found in [49] studying the SW U-plane of 5d susy gauge theories. This is consistent with the autonomous limit of the q-Painlevé equation which corresponds to the SW theory.

[98]See appendix E.4 for the first values of the coefficients.

and the corresponding coupling d of the defect, being the analogue of the CS level, becomes an integer. The structure of the blowup factor

$$\left\langle I(E)^{(d+1)}\right\rangle^{(j,k)}_{\hat{\mathbb{C}}^2\times S^1_\beta} = B^{(j,k)}_d(\mathbf{U},z,q) = (q^d z)^{\frac{j}{4}} P^{(j,k)}_d(\mathbf{U},z,q) = (q^d z)^{\frac{j}{4}} \sum_{n=0}^{n^{(j,k)}_{\max}(d)} P^{(j,k)}_{d,n}(z,q)\mathbf{U}^n \,. \tag{13.2}$$

is given, up to a normalization prefactor, by a *polynomial* $P^{(j,k)}_d(\mathbf{U},z,q)$ in $\mathbf{U}$ of degree $n_{max}(d) \sim \lfloor d^2/4 \rfloor$ with coefficients given by integer polynomials in $q = e^{\beta\epsilon}$ and z.

Using the Painlevé-gauge correspondence in the NS limit we showed that the blowup factor of the 5d gauge theory is the $\mathcal{T}$-function of the corresponding discrete q-Painlevé equation. This allows to compute explicitly the chiral ring expansion recursively, using the discrete q-Painlevé flow.

As in the 4d case, the Wilson loop chiral ring expansion, given by the solution of the q-Painlevé equastion, can be easily studied in the strongly coupled regions of the moduli space. Using this fact we studied the chiral ring expansion for the 5d SYM theory with CS level $k = 0, 1$ and we showed that for $k = 1$, corresponding to the $\tilde{E}_1$ 5d SCFT, at negative *finite* coupling we can take an alternative 4d limit to the non-lagrangian H_0 theory. Interestingly, this limit can be taken explicitly at the level of the chiral ring expansion giving full control of the renormalization group flow of the supersymmetric partition function of the gauge theory.

We conclude this part with some open questions and some possible generalizations of the results we discussed.

In this part we lifted the evidence of the Hurwitz $\mathbb{Z}$-integrality of the expansion coefficients of Painlevé $\mathcal{T}$-function around its zeroes[99] [64–66] that we discussed in part II to that of the $\mathbb{Z}[q]$-integrality of the q-Painlevé $\mathcal{T}$-function. It is an open problem for us to give a fundamental proof of this fact. Geometrically, this should correspond to (K-theoretic) equivariant Donaldson invariants on the blow-up. This could be possibly extended to the $\hat{\hat{E}}_8$ theory in 6d on an elliptic curve to be interpreted as an elliptic version of Donaldson theory. As we already observed, our expansion takes a very simple form at $\mathbf{U} = 0$, see formula (11.14). It will be nice to compare our results with the Donaldson-Thomas invariants discussed in [129].

The first generalization of our result is to consider other realizations of AD theories as 4d limits of 5d gauge theories. From Sakai's classification of (q-)Painlevé equations one expects that analogue limits can be identified from the 5d $SU(2)$ theories with N_f fundamentals to the AD points of the 4d SU(2) theories with $N_f + 1$ fundamentals (see figure 8). Geometrically this should be matched by an appropriate deformation of the corresponding del Pezzo geometries, see [28, 128] for the $N_f = 2$ case and [49] for a general description in terms on the classification of rational elliptic curves.

In section 7, we used the modular properties of the Hurwitz expansion of the Painlevé $\mathcal{T}$-function to derive the holomorphic anomaly equations for the topological string that

[99]Strictly speaking this is valid for the odd sector $j = 1$. The initial value for the expansion in the even sector $j = 0$ is $\mathcal{T} = 1$ but the integrality properties still hold.

engineers the gauge theory and to construct non-perturbative background independent completion of the corresponding partition function. We expect that this construction gets naturally lifted to the 5d setting where a deeper understanding of the full generating function of the Wilson loop $\mathbf{U}$ is required. This should give the holomorphic anomaly equations of [113] and their non-perturbative solutions. An evidence in this direction is the manifest modular invariance of the blowup factor in the SW limit, that we observed in formula (11.35). As topological string/spectral theory duality suggests [33, 131], the q-Painlevé $\mathcal{T}$-function can be written as the spectral determinant of the quantum mirror curve. Regarding this point it may be very interesting to give a spectral theory interpretation of the Hurwitz expansion in terms of truncated spectral determinants.

As we already observed in the four-dimensional case, it turns out from [44, 99] that the $SU(2)$ blowup partition function in terms of the IR Coulomb modulus can be derived from the u-plane integral of the low-energy theory. This can be promoted to the 5d theory in the form of a U-plane integral [49] with U being the Wilson loop vev in the SW limit. Our results suggest the existence of an equivariant extension of the above constructions.

A remarkable feature of the 5d UV completion of the AD theories is that the 5d Nekrasov partition function has positive radius of convergence in the instanton counting parameter z [132]. It would be interesting to see if an alternative expansion of the partition function around the finite coupling point z_0 holds. This will be an interesting perspective to further understand the UV completion we propose for the AD theories.

Finally, the structure of the Wilson loop expansion presents some peculiar features such as q-integrality and truncation at some maximal power $n_{\max}(d)$ in the Wilson loop variable $\mathbf{U}$. One could ask whether there is a representation theoretic meaning of these properties. Indeed, if we consider insertions of operators in the poles of the exceptional sphere E, the chiral ring expansion resembles the structure of fusion rules of generalized symmetries, deformed by the presence of a defect

$$1 \times 1 = 1 \to 1 \times_d 1 = \sum_{n=0}^{n_{\max}(d)} B_{d,n} W_n \ ,$$

that can be also generalized to the one in presence of Wilson loops inserted in the poles of E [123]

$$W_k \times W_l = W_{k+l} \to W_n \times_d W_m = \sum_{l=0}^{n_{\max,kl}(d)} (B_d)^n_{kl} W_n \ .$$

In particular, we observe that the structure constants $(B_d)^n_{kl}$ satisfy an associativity property which arises from a double blowup consistency condition. It would be interesting to investigate this fusion algebra more in detail. The integrability properties of the structure constants $(B_d)^n_{kl}$ should be related to a hierarchy associated to q-Painlevé equations.

IV Appendix

A Weierstrass elliptic functions

In this section we review the theory of Weierstrass elliptic functions, we will follow the conventions of [133]. Consider the lattice $\Lambda = \{2n\omega_1 + 2m\omega_2 |\ n, m \in \mathbb{Z}\}$ generated by the half-periods $\omega_1, \omega_2 \in \mathbb{C}$ which are linearly independent in $\mathbb{R}$. The Weierstrass elliptic function is

$$\wp(z|\omega_1,\omega_2) \equiv \wp(z) \equiv \frac{1}{z^2} + \sum_{w\in\Lambda\setminus\{0\}} \frac{1}{(z-w)^2} - \frac{1}{w^2} \ . \tag{A.1}$$

This function has a double pole in each point of the lattice Λ and is doubly periodic with periods $2\omega_1, 2\omega_2$

$$\wp(z+2\omega_1) = \wp(z+2\omega_2) = \wp(z) \ . \tag{A.2}$$

It is useful to define also the Eisenstein series

$$G_{2k}(\omega_1,\omega_2) = \sum_{w\in\Lambda\setminus\{0\}} \frac{1}{w^{2k}} \ . \tag{A.3}$$

If we normalize the lattice using the generators $(1,\tau)$, $\tau = \omega_2/\omega_1$, then the Eisenstein series can be seen as function of the modular parameter τ

$$G_{2k}(\tau) = \sum_{(n,m)\neq 0,0} \frac{1}{(n+m\tau)^{2k}} \ , \quad G_{2k}(\omega_1,\omega_2) = (2\omega_1)^{-2k} G_{2k}(\tau) \ . \tag{A.4}$$

For $2k > 2$ G_{2k} is a modular form of weight $2k$, which means that under a modular transformation of τ it transforms as

$$G_{2k}(A\tau) = (c\tau+d)^{2k} G_{2k}(\tau) \ , \quad A\tau = \frac{a\tau+b}{c\tau+d} \ , \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{A.5}$$

for $2k = 2$ instead $G_2(\tau)$ is not a modular form, but a quasi-modular form, which means that under a modular transformation it get also a shift

$$G_2(A\tau) = (c\tau+d)^2 \left(G_2(\tau) - \frac{2\pi i c}{c\tau+d} \right) \ . \tag{A.6}$$

It turns out that the Eisenstein series G_4, G_6 are the generators for the space of the modular forms with respect to $SL(2,\mathbb{Z})$. Therefore any G_{2k} can be written as a polynomial in G_4, G_6. Finally, it is useful also to define the normalized[100] Eisenstein series

$$E_{2k}(\tau) = \frac{G_{2k}(\tau)}{2\zeta(2k)}, \tag{A.7}$$

where $\zeta(s)$ is the Riemann Zeta function. The Eisenstein series are strictly related to the Weierstrass elliptic function. Precisely, defining the modular invariants $g_2 = 60G_4$, $g_3 = 140G_6$, it turns out that the Weierstrass $\wp$-function satisfies the following differential equation

$$\dot{\wp}^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3 \ , \tag{A.8}$$

which gives a parametrization for an elliptic curve in the Weierstrass form

$$y^2 = 4x^3 - g_2 x - g_3 \ , \tag{A.9}$$

where $\dot{\wp} = \partial_z \wp$. Another form of the differential equation (A.8), which is useful in connection with the study of the $\mathcal{T}$-function, is obtained taking the derivative[101] of (A.8)

$$\ddot{\wp}(z) = 6\wp^2(z) - \frac{g_2}{2} \ . \tag{A.10}$$

We can express the $\wp$-function also as a function of the invariants g_2, g_3, in this case[102] we denote it as $\wp(z; g_2, g_3)$.

From the $\wp$-function we can define the Weierstrass Zeta function $\zeta_W(z; g_2, g_3)$ as

$$-\frac{\mathrm{d}}{\mathrm{d}z}\zeta_W(z; g_2, g_3) \equiv \wp(z; g_2, g_3) \ . \tag{A.11}$$

The function $\zeta_W(z; g_2, g_3) \equiv \zeta_W(z)$ is quasi-periodic in the following sense

$$\zeta_W(z + 2\omega_1) = \zeta_W(z) + 2\eta_1 \ , \quad \zeta_W(z + 2\omega_2) = \zeta_W(z) + 2\eta_2 \ , \tag{A.12}$$

the constants η_1, η_2 are the half-quasi-periods and are given by

$$\eta_1 = \zeta_W(\omega_1), \quad \eta_2 = \zeta_W(\omega_2) \ , \tag{A.13}$$

which follows from the periodicity of $\wp$ and from the fact that ζ_W is a odd function. Finally, the quasi-period η_1 is directly related to the second Eisenstein series

$$\eta_1 = \frac{\pi^2}{6}\frac{E_2(\tau)}{2\omega_1} \ . \tag{A.14}$$

From the Weierstrass Zeta function we can define the Weierstrass σ-function

$$\frac{\mathrm{d}}{\mathrm{d}z}\log\sigma(z; g_2, g_3) = \zeta_W(z; g_2, g_3) \ . \tag{A.15}$$

[100]The normalization is chosen in such a way that the first coefficient of the Fourier series of G_{2k} in terms of the nome $q = \exp(2\pi i\tau)$ is 1.

[101]Notice that in this way the parameter g_3 becomes an integration constant.

[102]It can be shown that the elliptic invariants g_2, g_3 and the half-periods ω_1, ω_2 are equivalent data so we can always perform the change of variables $\omega_i(g_2, g_3), i = 1, 2$.

The σ-function is an entire function with simple zeros at the lattice points and it is related to the Jacobi theta functions. Precisely, the following relation holds

$$\sigma(z;\omega_1,\omega_2) = \frac{2\omega_1}{\pi} e^{\frac{\eta_1 z^2}{2\omega_1}} \frac{\theta_1(v)}{\theta_1'(0)} = \frac{2\omega_1}{\pi} e^{\frac{\pi^2}{6}\frac{E_2(\tau)}{4\omega_1^2}z^2} \frac{\theta_1(v)}{\theta_1'(0)} \ , \quad \theta_1'(0) = 2\eta(\tau)^3 \ , \tag{A.16}$$

where $v = \pi z/2\omega_1$, $\eta(\tau)$ is the Dedekind eta function, and θ_1 is the theta function defined as

$$\theta_1(z,q) \equiv \theta_1(z) \equiv -\sum_{n\in\mathbb{Z}+\frac{1}{2}} (-1)^n q^{n^2} e^{2niz} \ . \tag{A.17}$$

In the above expression we have defined the nome $q = \exp(i\pi\tau)$, where $\tau = \omega_2/\omega_1$ is the modular parameter (with $\operatorname{Im}\tau > 0$) and where the variable z is the argument of the theta function. For convenience we define also the theta constants

$$\theta_2(\tau) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} e^{i\pi\tau n^2} \ , \quad \theta_3(\tau) = \sum_{n\in\mathbb{Z}} e^{i\pi\tau n^2}, \quad \theta_4(\tau) = \sum_{n\in\mathbb{Z}} (-1)^n e^{i\pi\tau n^2} \ . \tag{A.18}$$

The Weierstrass σ-function satisfies the Hirota bilinear equation[103]

$$D_z^{(4)}(\sigma,\sigma) - g_2\sigma^2(z) = 0 \ , \tag{A.19}$$

which follows expressing $\wp$ in terms of σ in the differential equation (A.10).

Finally, the σ-function admits the following Hurwitz expansion around the origin

$$\sigma(z;g_2,g_3) = \sum_{m,n=0}^{\infty} a_{mn} \left(\frac{g_2}{2}\right)^m (2g_3)^n \frac{z^{4m+6n+1}}{(4m+6n+1)!} \ , \tag{A.20}$$

where $a_{00} = 1$, $a_{mn} = 0$ if some index is negative, the other coefficients are given by the following recurrence relation

$$a_{mn} = 3(m+1)a_{m+1,n+1} + \frac{16}{3}(n+1)a_{m-2,n+1} - \frac{1}{3}(2m+3n-1)(4m+6n-1)a_{m-1,n} \ , \tag{A.21}$$

and it can be shown that they are all integers[104]. We can rewrite the σ-function as

$$\sigma(z;g_2,g_3) = \sum_{n=0}^{+\infty} c_n^\sigma \frac{z^{n+1}}{(n+1)!} \ , \tag{A.22}$$

where we defined the coefficients

$$c_n^\sigma \equiv c_n^\sigma(g_2,g_3) = \sum_{4k+6l+1=n} a_{kl} \left(\frac{g_2}{2}\right)^k (2g_3)^l \ , \tag{A.23}$$

and because a_{nm} are integers we have that c_n^σ are polynomials with integers coefficients, $c_n^\sigma \in \mathbb{Z}[g_2/2, 2g_3]$.

[103] As said previously g_3 appears in (A.10) as an integration constant. In the Hirota bilinear equation for σ this corresponds exactly to the resonant coefficient c_6^σ.

[104] In general they can be negative. See [72].

B Computation of elliptic invariants and contact term for PVI

In this section we compute the elliptic invariants and the contact term of $N_f = 4$ theory (PVI) in the massless case. In the massless case the prepotential for PVI is

$$\mathcal{F}_0 = i\pi\tau a^2 \ , \tag{B.1}$$

where τ is the IR coupling. The IR coupling τ is related to the UV coupling τ_0 as follows

$$q = \exp(i\pi\tau_0) = \frac{1}{16}\frac{\theta_2^4(\tau)}{\theta_3^4(\tau)} = k^2(\tau) \ , \tag{B.2}$$

and we compute the contact term as

$$\tilde{T} = -q\frac{\partial}{\partial q}u = -\left(q\frac{\partial}{\partial q}\right)^2 \mathcal{F}_0 \ . \tag{B.3}$$

To do the computation we use the following identities[105] from the theory of elliptic functions

$$i\pi q\frac{\partial}{\partial q}\tau = \theta_4^{-4}(\tau) \ , \tag{B.4}$$

$$\frac{\partial}{\partial\tau}\log\theta_4(\tau) = \frac{i\pi}{12}(E_2(\tau) - \theta_2(\tau)^4 - \theta_3(\tau)^4) \ , \tag{B.5}$$

$$\frac{\theta_2^4(\tau)}{\theta_4^4(\tau)} = \frac{q}{1-q} \ , \quad \frac{\theta_3^4(\tau)}{\theta_4^4(\tau)} = \frac{1}{1-q} \ , \tag{B.6}$$

where $E_2(\tau)$ is the second Eisenstein series. Using (B.4) we have

$$u = q\frac{\partial}{\partial q}\mathcal{F}_0 = i\pi q\frac{\partial\tau}{\partial q}a^2 = \theta_4^{-4}(\tau)a^2 \ , \quad \frac{i\pi}{\omega_1} = \frac{\partial u}{\partial a} = 2\theta_4^{-4}(\tau)a \ , \tag{B.7}$$

and

$$\tilde{T} = -q\frac{\partial}{\partial q}u = -a^2 q\frac{\partial\tau}{\partial q}\frac{\partial}{\partial\tau}\theta_4^{-4}(\tau) = \frac{4a^2}{i\pi}(\theta_4^{-4}(\tau))^2\frac{\partial}{\partial\tau}\log\theta_4(\tau) \ . \tag{B.8}$$

By using (B.5),(B.6) and (B.7) we get

$$\tilde{T} = \frac{a^2}{3}(\theta_4^{-4}(\tau))^2(E_2(\tau) - \theta_2(\tau)^4 - \theta_3(\tau)^4) = -\frac{\pi^2}{12}\frac{E_2(\tau)}{\omega_1^2} - \frac{u}{3}\frac{1+q}{1-q} \ , \tag{B.9}$$

i.e.

$$T = \tilde{T} + \frac{\pi^2}{12}\frac{E_2(\tau)}{\omega_1^2} = -\frac{u}{3}\frac{1+q}{1-q} \ , \tag{B.10}$$

which agrees with the PVI result (8.32) in the massless case. Consider the elliptic invariants

$$\begin{aligned} g_2(\omega_1,\omega_2) &= 60G_4(\omega_1,\omega_2) = \frac{4}{3}\left(\frac{\pi}{2\omega_1}\right)^4 E_4(\tau) = \frac{4}{3}\left(\frac{1}{2i}\frac{\partial u}{\partial a}\right)^4 E_4(\tau) \ , \\ g_3(\omega_1,\omega_2) &= 140G_6(\omega_1,\omega_2) = \frac{8}{27}\left(\frac{\pi}{2\omega_1}\right)^6 E_6(\tau) = \frac{8}{27}\left(\frac{1}{2i}\frac{\partial u}{\partial a}\right)^6 E_6(\tau) \ . \end{aligned} \tag{B.11}$$

[105] See [134] appendix A.

We consider now the following identities[106]

$$E_4(\tau) = \frac{1}{2}(\theta_2^8(\tau) + \theta_3^8(\tau) + \theta_4^8(\tau)) \ , \tag{B.12}$$

$$E_6(\tau) = \frac{1}{2}(\theta_2(\tau)^4 + \theta_3(\tau)^4)(\theta_3(\tau)^4 + \theta_4(\tau)^4)(\theta_4(\tau)^4 - \theta_2(\tau)^4) \ , \tag{B.13}$$

and using (B.6) and (B.11) we obtain

$$g_2(\omega_1,\omega_2) = \frac{4}{3}\frac{(q^2-q+1)}{(q-1)^2}u^2 \ , \quad g_3(\omega_1,\omega_2) = \frac{4}{27}\frac{(2-3q-3q^2+2q^3)}{(q-1)^3}u^3 \ , \tag{B.14}$$

which again agree with (8.32).

C Details on the derivation of the holomorphic anomaly equations

C.1 Equivalence of holomorphic anomaly equations with and without sources

We want to find the relation between the holomorphic anomaly equations for the SD generating function $\mathcal{F}^{x,s}_{\rm st}(a,\Lambda)$ and the SD free energy $\widehat{\mathcal{F}}(a,\Lambda)$. To do this we observe that

$$\mathcal{F}^{x,s}_{\rm st}(a,\Lambda) = \widehat{\mathcal{F}}(a+\epsilon x, \Lambda e^{\epsilon s}) + \mathcal{F}^{x,s}_{0,\rm st}(a,\Lambda) \Rightarrow e^{\mathcal{F}^{x,s}_{\rm st}(a,\Lambda)} = e^{\mathcal{F}^{x,s}_{0,\rm st}(a,\Lambda)}e^{\epsilon x\partial_a + \epsilon s\Lambda\partial\Lambda}e^{\widehat{\mathcal{F}}(a,\Lambda)} \ . \tag{C.1}$$

We want to prove the following relation

$$\left(D_{E_2} + \frac{\epsilon^2}{24}\partial_a^2\right)e^{\widehat{\mathcal{F}}(a,\Lambda)} = e^{-\epsilon x\partial_a - \epsilon s\Lambda\partial_\Lambda}e^{-\mathcal{F}^{x,s}_{0,\rm st}(a,\Lambda)}\left(D_{E_2} + \frac{\partial_x^2}{24}\right)e^{\mathcal{F}^{x,s}_{0,\rm st}(a,\Lambda)}e^{\epsilon x\partial_a + \epsilon s\Lambda\partial_\Lambda}e^{\widehat{\mathcal{F}}(a,\Lambda)} \ , \tag{C.2}$$

where[107]

$$\widehat{\mathcal{F}}(a,\Lambda) = \mathcal{F}(a,\Lambda,\epsilon) - \frac{1}{\epsilon^2}\mathcal{F}_0(a,\Lambda) \ , \tag{C.3}$$

$$\mathcal{F}^{x,s}_{0,\rm st}(a,\Lambda) = \frac{1}{\epsilon^2}\mathcal{F}_0(a+\epsilon x,\Lambda_{\epsilon s}) - \frac{1}{\epsilon^2}\mathcal{F}_0(a,\Lambda) - \frac{x}{\epsilon}\partial_a\mathcal{F}_0(a,\Lambda) - \frac{x^2}{2}\partial_a^2\mathcal{F}_0(a,\Lambda) \ , \tag{C.4}$$

and $\Lambda_{\epsilon s} = \Lambda\exp(\epsilon s)$. We recall that we have the following algebra of vector fields (7.20)[108]

$$\begin{aligned}
[\partial_a, \Lambda\partial_\Lambda] &= 0 \ , \\
[D_{E_2}, \partial_a] &= \frac{1}{12}\partial_a^3\mathcal{F}_0(a,\Lambda)\widehat{d} \ , \\
[D_{E_2}, \Lambda\partial_\Lambda] &= \frac{1}{12}\Lambda\partial_\Lambda\partial_a^2\mathcal{F}_0(a,\Lambda)\widehat{d} - \frac{1}{12}\Lambda\partial_\Lambda\partial_a\mathcal{F}_0(a,\Lambda)\partial_a \ , \\
[\widehat{d}, \Lambda\partial_\Lambda] = 0 \ , \qquad [\widehat{d}, \partial_a] &= -\partial_a \ , \qquad [\widehat{d}, D_{E_2}] = -2D_{E_2} \ ,
\end{aligned} \tag{C.5}$$

[106] See [134] appendix F.

[107] In this appendix we omit the explicit dependence of the free energies on ϵ for simplicity.

[108] In this section we remove ∂'_τ terms in the commutators, since this algebra will act only on the stable expressions that do not have an explicit τ dependence.

with the following action on the amplitudes

$$D_{E_2}\Lambda\partial_\Lambda\mathcal{F}_0(a,\Lambda)=0\ ,\quad D_{E_2}\partial_a^3\mathcal{F}_0(a,\Lambda)=0\ ,\quad \widehat{d}\mathcal{F}_{g,n}(a,\Lambda)=-\frac{1}{2}\delta_{g,1}\delta_{n,0}\ ,\ g+n>0,$$
$$D_{E_2}\mathcal{F}_{1,0}(a,\Lambda)=0\ ,\quad D_{E_2}\mathcal{F}_{0,1}(a,\Lambda)=0\ . \tag{C.6}$$

To prove the result we need to compute the conjugation which appears in (C.2). We consider first the conjugation by $\exp\Big(\mathcal{F}^{x,s}_{0,\mathrm{st}}(a,\Lambda)\Big)$

$$e^{-\mathcal{F}^{x,s}_{0,\mathrm{st}}(a,\Lambda)}\left(D_{E_2}+\frac{\partial_x^2}{24}\right)e^{\mathcal{F}^{x,s}_{0,\mathrm{st}}(a,\Lambda)}= \tag{C.7}$$
$$=D_{E_2}+D_{E_2}\mathcal{F}^{x,s}_{0,\mathrm{st}}(a,\Lambda)+\frac{1}{24}\partial_x^2+\frac{1}{12}\partial_x\mathcal{F}^{x,s}_{0,\mathrm{st}}(a,\Lambda)\partial_x+\frac{1}{24}\partial_x^2\mathcal{F}^{x,s}_{0,\mathrm{st}}(a,\Lambda)+\frac{1}{24}\Big(\partial_x\mathcal{F}^{x,s}_{0,\mathrm{st}}(a,\Lambda)\Big)^2\ . \tag{C.8}$$

Using the algebra (C.5) and the conditions (C.6) we obtain (see next subsection C.2)

$$D_{E_2}\mathcal{F}^{x,s}_{0,\mathrm{st}}(a,\Lambda)=-\frac{1}{24}\Big(\partial_x\mathcal{F}^{x,s}_{0,\mathrm{st}}(a,\Lambda)\Big)^2\ , \tag{C.9}$$

which implies

$$e^{-\mathcal{F}^{x,s}_{0,\mathrm{st}}(a,\Lambda)}\left(D_{E_2}+\frac{\partial_x^2}{24}\right)e^{\mathcal{F}^{x,s}_{0,\mathrm{st}}(a,\Lambda)}=D_{E_2}+\frac{1}{24}\partial_x^2+\frac{1}{12}\partial_x\mathcal{F}^{x,s}_{0,\mathrm{st}}(a,\Lambda)\partial_x+\frac{1}{24}\partial_x^2\mathcal{F}^{x,s}_{0,\mathrm{st}}(a,\Lambda)\ . \tag{C.10}$$

We consider now the conjugation by the shift operator $\exp(\epsilon x\partial_a+\epsilon s\Lambda\partial_\Lambda)$ of (C.10). We have

$$e^{-\epsilon x\partial_a-\epsilon s\Lambda\partial_\Lambda}\partial_x e^{\epsilon x\partial_a+\epsilon s\Lambda\partial_\Lambda}=\partial_x+\epsilon\partial_a\ , \tag{C.11}$$

and

$$\begin{aligned}&e^{-\epsilon x\partial_a-\epsilon s\Lambda\partial_\Lambda}\partial_x\mathcal{F}^{x,s}_{0,\mathrm{st}}(a,\Lambda)e^{\epsilon x\partial_a+\epsilon s\Lambda\partial_\Lambda}=\\&\qquad=\frac{1}{\epsilon}\left(\partial_a\mathcal{F}_0(a,\Lambda)-\partial_a\mathcal{F}_0(a-\epsilon x,\Lambda_{-\epsilon s})-\epsilon x\partial_a^2\mathcal{F}_0(a-\epsilon x,\Lambda_{-\epsilon s})\right),\\&e^{-\epsilon x\partial_a-\epsilon s\Lambda\partial_\Lambda}\partial_x^2\mathcal{F}^{x,s}_{0,\mathrm{st}}(a,\Lambda)e^{\epsilon x\partial_a+\epsilon s\Lambda\partial_\Lambda}=\partial_a^2\mathcal{F}_0(a,\Lambda)-\partial_a^2\mathcal{F}_0(a-\epsilon x,\Lambda_{-\epsilon s})\ .\end{aligned} \tag{C.12}$$

For D_{E_2} we define

$$D^{x,s}_{E_2}=e^{-\epsilon x\partial_a-\epsilon s\Lambda\partial_\Lambda}D_{E_2}e^{\epsilon x\partial_a+\epsilon s\Lambda\partial_\Lambda}\ , \tag{C.13}$$

and taking the derivatives with respect to x we obtain

$$\partial_x(D^{x,s}_{E_2})=\frac{\epsilon}{12}e^{-\epsilon x\partial_a-\epsilon s\Lambda\partial_\Lambda}\partial_a^3\mathcal{F}_0(a,\Lambda)\widehat{d}e^{\epsilon x\partial_a+\epsilon s\Lambda\partial_\Lambda}\ . \tag{C.14}$$

From the algebra (C.5) we have

$$e^{-\epsilon x\partial_a-\epsilon s\Lambda\partial_\Lambda}\widehat{d}e^{\epsilon x\partial_a+\epsilon s\Lambda\partial_\Lambda}=\hat{d}-\epsilon x\partial_a\ , \tag{C.15}$$

then (C.14) reads

$$\partial_x\left(D^{x,s}_{E_2}\right)=\frac{\epsilon}{12}\partial_a^3\mathcal{F}_0(a-\epsilon x,\Lambda_{-\epsilon s})\left(\widehat{d}-\epsilon x\partial_a\right)\ . \tag{C.16}$$

Analogously for the s derivative we have

$$\partial_s \left(D_{E_2}^{x,s} \right) = \frac{\epsilon}{12} \Lambda_{-\epsilon s} \partial_\Lambda \partial_a^2 \mathcal{F}_0(a - \epsilon x, \Lambda_{-\epsilon s}) \left(\widehat{d} - \epsilon x \partial_a \right) - \frac{\epsilon}{12} \Lambda_{-\epsilon s} \partial_\Lambda \partial_a \mathcal{F}_0(a - \epsilon x, \Lambda_{-\epsilon s}) \partial_a \ , \tag{C.17}$$

and integrating (C.16) and (C.17) in x and s we obtain

$$\begin{aligned} D_{E_2}^{x,s} = -\frac{1}{12} \partial_a^2 \mathcal{F}_0(a - \epsilon x, \Lambda_{-\epsilon s}) \left(\widehat{d} - \epsilon x \partial_a \right) + \frac{1}{12} \partial_a^2 \mathcal{F}_0(a, \Lambda) \widehat{d} \\ + \frac{1}{12} \left(\partial_a \mathcal{F}_0(a - \epsilon x, \Lambda_{-\epsilon s}) - \partial_a \mathcal{F}_0(a, \Lambda) \right) \partial_a + D_{E_2} \ . \end{aligned} \tag{C.18}$$

Finally, combining (C.18) with (C.11), (C.12) we obtain

$$\begin{aligned} & e^{-\epsilon x \partial_a - \epsilon s \Lambda \partial_\Lambda} \left(D_{E_2} + \frac{1}{24} \partial_x^2 + \frac{1}{12} \partial_x \mathcal{F}_{0,\mathrm{st}}^{x,s}(a, \Lambda) \partial_x + \frac{1}{24} \partial_x^2 \mathcal{F}_{0,\mathrm{st}}^{x,s}(a, \Lambda) \right) e^{\epsilon x \partial_a + \epsilon s \Lambda \partial_\Lambda} e^{\widehat{\mathcal{F}}(a,\Lambda)} = \\ & = \left(D_{E_2} + \frac{\epsilon^2}{24} \partial_a^2 + \frac{1}{24} \left(\partial_a^2 \mathcal{F}_0(a, \Lambda) - \partial_a^2 \mathcal{F}_0(a - \epsilon x, \Lambda_{-\epsilon s}) \right) (2\widehat{d} + 1) \right) e^{\widehat{\mathcal{F}}(a,\Lambda)} \ , \end{aligned} \tag{C.19}$$

where we used $\partial_x \widehat{\mathcal{F}}(a, \Lambda) = 0$. From (C.6) the last term in (C.19) vanishes and we obtain

$$e^{-\epsilon x \partial_a - \epsilon s \Lambda \partial_\Lambda} e^{-\mathcal{F}_{0,\mathrm{st}}^{x,s}(a,\Lambda)} \left(D_{E_2} + \frac{\partial_x^2}{24} \right) e^{\mathcal{F}_{0,\mathrm{st}}^{x,s}(a,\Lambda)} e^{\epsilon x \partial_a + \epsilon s \Lambda \partial_\Lambda} e^{\widehat{\mathcal{F}}(a,\Lambda)} = \left(D_{E_2} + \frac{\epsilon^2}{24} \partial_a^2 \right) e^{\widehat{\mathcal{F}}(a,\Lambda)} \ . \tag{C.20}$$

C.2 Holomorphic anomaly at genus zero

In this subsection we want to derive the holomorphic anomaly equation for genus zero (C.9). To do this it is convenient to consider first the E_2 derivative of the derivatives of $\mathcal{F}_{0,\mathrm{st}}^{x,s}(a, \Lambda)$ in order to apply (C.6). We have

$$\begin{aligned} \partial_x^3 D_{E_2} \mathcal{F}_{0,\mathrm{st}}^{x,s}(a, \Lambda) = D_{E_2} \partial_x^3 \mathcal{F}_{0,\mathrm{st}}^{x,s}(a, \Lambda) = \epsilon D_{E_2} \partial_a^3 \mathcal{F}_0(a + \epsilon x, \Lambda_{\epsilon s}) = \\ = \epsilon D_{E_2} e^{\epsilon x \partial_a + \epsilon s \Lambda \partial \Lambda} \partial_a^3 \mathcal{F}_0(a, \Lambda) = \epsilon e^{\epsilon x \partial_a + \epsilon s \Lambda \partial \Lambda} D_{E_2}^{x,s} \partial_a^3 \mathcal{F}_0(a, \Lambda) \ . \end{aligned} \tag{C.21}$$

and using (C.18) and (C.6) we obtain

$$\begin{aligned} \partial_x^3 D_{E_2} \mathcal{F}_{0,\mathrm{st}}^{x,s}(a, \Lambda) = & \frac{\epsilon}{12} e^{\epsilon x \partial_a + \epsilon s \Lambda \partial_\Lambda} \Big(- \partial_a^2 \mathcal{F}_0(a - \epsilon x, \Lambda_{-\epsilon s}) \left(\widehat{d} - \epsilon x \partial_a \right) + \partial_a^2 \mathcal{F}_0(a, \Lambda) \widehat{d} + \\ & + \left(\partial_a \mathcal{F}_0(a - \epsilon x, \Lambda_{-\epsilon s}) - \partial_a \mathcal{F}_0(a, \Lambda) \right) \partial_a + 12 D_{E_2} \Big) \partial_a^3 \mathcal{F}_0(a, \Lambda) = \\ = & \frac{\epsilon}{12} e^{\epsilon x \partial_a + \epsilon s \Lambda \partial_\Lambda} \Big(- \partial_a^2 \mathcal{F}_0(a - \epsilon x, \Lambda_{-\epsilon s}) (-3 - \epsilon x \partial_a) - 3 \partial_a^2 \mathcal{F}_0(a, \Lambda) \\ & + \left(\partial_a \mathcal{F}_0(a - \epsilon x, \Lambda_{-\epsilon s}) - \partial_a \mathcal{F}_0(a, \Lambda) \right) \partial_a \Big) \partial_a^3 \mathcal{F}_0(a, \Lambda) = \\ = & \frac{\epsilon}{12} \Big(3 \partial_a^2 \mathcal{F}_0(a, \Lambda) \partial_a^3 \mathcal{F}_0(a + \epsilon x, \Lambda_{\epsilon s}) + \epsilon x \partial_a^2 \mathcal{F}_0(a, \Lambda) \partial_a^4 \mathcal{F}_0(a + \epsilon x, \Lambda_{\epsilon s}) \\ & - 3 \partial_a^2 \mathcal{F}_0(a + \epsilon x, \Lambda_{\epsilon s}) \partial_a^3 \mathcal{F}_0(a + \epsilon x, \Lambda_{\epsilon s}) + \partial_a \mathcal{F}_0(a, \Lambda) \partial_a^4 \mathcal{F}_0(a + \epsilon x, \Lambda_{\epsilon s}) \\ & - \partial_a \mathcal{F}_0(a + \epsilon x, \Lambda_{\epsilon s}) \partial_a^4 \mathcal{F}_0(a + \epsilon x, \Lambda_{\epsilon s}) \Big) = \\ = & \frac{1}{12 \epsilon^2} \partial_x^3 \left(-\frac{1}{2} \partial_a \mathcal{F}_0(a + \epsilon x, \Lambda_{\epsilon s})^2 + \epsilon x \partial_a^2 \mathcal{F}_0(a, \Lambda) \partial_a \mathcal{F}_0(a + \epsilon x, \Lambda_{\epsilon s}) + \partial_a \mathcal{F}_0(a, \Lambda) \partial_a \mathcal{F}_0(a + \epsilon x, \Lambda_{\epsilon s}) \right) \ . \end{aligned} \tag{C.22}$$

Analogously, for the s derivative we have

$$
\begin{aligned}
\partial_s D_{E_2}\mathcal{F}^{x,s}_{0,\text{st}}(a,\Lambda) &= \frac{1}{\epsilon}e^{\epsilon x\partial_a+\epsilon s\Lambda\partial\Lambda}D^{x,s}_{E_2}\Lambda\partial_\Lambda\mathcal{F}_0(a,\Lambda) \;= \\
&= \frac{1}{12\epsilon}e^{\epsilon x\partial_a+\epsilon s\Lambda\partial_\Lambda}\Big(-\partial_a^2\mathcal{F}_0(a-\epsilon x,\Lambda_{-\epsilon s})\left(\widehat{d}-\epsilon x\partial_a\right)+\partial_a^2\mathcal{F}_0(a,\Lambda)\widehat{d} \\
&\quad +\left(\partial_a\mathcal{F}_0(a-\epsilon x,\Lambda_{-\epsilon s})-\partial_a\mathcal{F}_0(a,\Lambda)\right)\partial_a+12D_{E_2}\Big)\Lambda\partial_\Lambda\mathcal{F}_0(a,\Lambda) = \\
= \frac{1}{12\epsilon}&e^{\epsilon x\partial_a+\epsilon s\Lambda\partial_\Lambda}\Big(\epsilon x\partial_a^2\mathcal{F}_0(a-\epsilon x,\Lambda_{-\epsilon s})+\partial_a\mathcal{F}_0(a-\epsilon x,\Lambda_{-\epsilon s})-\partial_a\mathcal{F}_0(a,\Lambda)\Big)\Lambda\partial_\Lambda\partial_a\mathcal{F}_0(a,\Lambda) = \\
&= \frac{1}{12\epsilon}\Big(\epsilon x\partial_a^2\mathcal{F}_0(a,\Lambda)+\partial_a\mathcal{F}_0(a,\Lambda)-\partial_a\mathcal{F}_0(a+\epsilon x,\Lambda_{\epsilon s})\Big)\Lambda\partial_\Lambda\partial_a\mathcal{F}_0(a+\epsilon x,\Lambda_{\epsilon s}) = \\
= \frac{1}{12\epsilon^2}\partial_s&\left(-\frac{1}{2}\partial_a\mathcal{F}_0(a+\epsilon x,\Lambda_{\epsilon s})^2+\epsilon x\partial_a^2\mathcal{F}_0(a,\Lambda)\partial_a\mathcal{F}_0(a+\epsilon x,\Lambda_{\epsilon s})+\partial_a\mathcal{F}_0(a,\Lambda)\partial_a\mathcal{F}_0(a+\epsilon x,\Lambda_{\epsilon s})\right)\,.
\end{aligned}
\tag{C.23}
$$

Integrating back we find

$$
\begin{aligned}
D_{E_2}\mathcal{F}^{x,s}_{0,\text{st}}(a,\Lambda) &= -\frac{1}{24\epsilon^2}\partial_a\mathcal{F}_0(a+\epsilon x,\Lambda_{\epsilon s})^2 \\
&\quad+\frac{1}{12\epsilon}x\partial_a^2\mathcal{F}_0(a,\Lambda)\partial_a\mathcal{F}_0(a+\epsilon x,\Lambda_{\epsilon s})+\frac{1}{12\epsilon^2}\partial_a\mathcal{F}_0(a,\Lambda)\partial_a\mathcal{F}_0(a+\epsilon x,\Lambda_{\epsilon s}) \\
&\quad-\frac{1}{24\epsilon^2}\partial_a\mathcal{F}_0(a,\Lambda)^2-\frac{1}{12\epsilon}x\partial_a\mathcal{F}_0(a,\Lambda)\partial_a^2\mathcal{F}_0(a,\Lambda)-\frac{x^2}{24}\partial_a^2\mathcal{F}_0(a,\Lambda)^2 = \\
&= -\frac{1}{24\epsilon^2}\left(\partial_a\mathcal{F}_0(a+\epsilon x,\Lambda_{\epsilon s})-\partial_a\mathcal{F}_0(a,\Lambda)-\epsilon x\partial_a^2\mathcal{F}_0(a,\Lambda)\right)^2 = -\frac{1}{24}\left(\partial_x\mathcal{F}^{x,s}_{0,\text{st}}(a,\Lambda)\right)^2\,,
\end{aligned}
\tag{C.24}
$$

therefore we obtain

$$
D_{E_2}\mathcal{F}^{x,s}_{0,\text{st}}(a,\Lambda) = -\frac{1}{24}\left(\partial_x\mathcal{F}^{x,s}_{0,\text{st}}(a,\Lambda)\right)^2\,. \tag{C.25}
$$

C.3 Proof of decoupling of the NS and SD equations

We have the equation (7.75)

$$
\begin{aligned}
&\left(D_{E_2}\mathcal{F}^{x,s}_{\text{st}}(a,\Lambda)+\frac{1}{24}\partial_x^2\mathcal{F}^{x,s}_{\text{st}}(a,\Lambda)+\frac{1}{24}\left(\partial_x\mathcal{F}^{x,s}_{\text{st}}(a,\Lambda)\right)^2\right) \\
&-\left(\frac{\rho_{\text{st}}}{12}+\frac{D_{E_2}\mathbf{u}}{\partial_a\mathbf{u}}\right)\frac{1}{\epsilon}\partial_x\mathcal{F}^{x,s}_{\text{st}}(a,\Lambda) \\
&-\frac{x}{\epsilon}\left(D_{E_2}\rho_{\text{st}}-\frac{D_{E_2}\mathbf{u}}{\partial_a\mathbf{u}}\partial_a\rho_{\text{st}}\right)+\left(D_{E_2}\partial_\epsilon W_{\text{st}}+\frac{(\rho_{\text{st}})^2}{24\epsilon^2}-\frac{D_{E_2}\mathbf{u}}{\partial_a\mathbf{u}}\partial_a\partial_\epsilon W_{\text{st}}\right)=0\,,
\end{aligned}
\tag{C.26}
$$

and we want to decouple the SD and NS contributions. To do this we first rewrite

$$
\begin{aligned}
&\frac{\rho_{\text{st}}}{12}+\frac{D_{E_2}\mathbf{u}}{\partial_a\mathbf{u}} = \frac{1}{\partial_a\mathbf{u}}\left(\frac{1}{12}\rho_{\text{st}}\partial_a\mathbf{u}+D_{E_2}\mathbf{u}\right) = \frac{\epsilon\Lambda\partial_\Lambda}{\partial_a\mathbf{u}}\left(\frac{\epsilon}{24}(\partial_aW_{\text{st}})^2+D_{E_2}W_{\text{st}}\right) \\
&\Rightarrow \frac{D_{E_2}\mathbf{u}}{\partial_a\mathbf{u}} = -\frac{\rho_{\text{st}}}{12}+\frac{\epsilon\Lambda\partial_\Lambda}{\partial_a\mathbf{u}}\left(D_{E_2}W_{\text{st}}+\frac{\epsilon}{24}(\partial_aW_{\text{st}})^2\right)\,,
\end{aligned}
\tag{C.27}
$$

and substituting in (C.26) we obtain

$$
\begin{aligned}
&\left(D_{E_2}\mathcal{F}^{x,s}_{\rm st}(a,\Lambda)+\frac{1}{24}\partial_x^2\mathcal{F}^{x,s}_{\rm st}(a,\Lambda)+\frac{1}{24}\left(\partial_x\mathcal{F}^{x,s}_{\rm st}(a,\Lambda)\right)^2\right)\\
&-\left(x\partial_a-\partial_\epsilon+\frac{\partial_x\mathcal{F}^{x,s}_{\rm st}(a,\Lambda)-x\partial_a\rho_{\rm st}+\epsilon\partial_a\partial_\epsilon W_{\rm st}}{\partial_a\mathbf{u}}\Lambda\partial_\Lambda\right)\times\\
&\times\left(D_{E_2}W_{\rm st}+\frac{\epsilon}{24}(\partial_a W_{\rm st})^2\right)=0\ .
\end{aligned}\tag{C.28}
$$

We observe now that the SD contributions are always evaluated at the shifted scales ($y=\epsilon x+a,\Lambda'=\Lambda_{\epsilon s}=\Lambda\exp(\epsilon s)$), the NS contributions instead are evaluated at the unshifted scales (a,Λ). To decouple the previous equation it is then useful to rewrite it as a function of the SD and NS scales. Using (C.1), (C.2) we have

$$
\begin{aligned}
&e^{-\mathcal{F}^{x,s}_{\rm st}(a,\Lambda)}\left(D_{E_2}+\frac{\partial_x^2}{24}\right)e^{\mathcal{F}^{x,s}_{\rm st}(a,\Lambda)}=\\
&=e^{-\mathcal{F}^{x,s}_{\rm st}(a,\Lambda)}\left(D_{E_2}+\frac{\partial_x^2}{24}\right)e^{\mathcal{F}^{x,s}_{0,\rm st}(a,\Lambda)}e^{\epsilon x\partial_a+s\Lambda\partial\Lambda}e^{\widehat{\mathcal{F}}(a,\Lambda)}=\\
&=e^{-\mathcal{F}^{x,s}_{\rm st}(a,\Lambda)}e^{\mathcal{F}^{x,s}_{0,\rm st}(a,\Lambda)}e^{\epsilon x\partial_a+s\Lambda\partial\Lambda}\left(D_{E_2}+\frac{\epsilon^2}{24}\partial_a^2\right)e^{\widehat{\mathcal{F}}(a,\Lambda)}=\\
&=e^{\epsilon x\partial_a+s\Lambda\partial\Lambda}e^{-\widehat{\mathcal{F}}(a,\Lambda)}\left(D_{E_2}+\frac{\epsilon^2}{24}\partial_a^2\right)e^{\widehat{\mathcal{F}}(a,\Lambda)}=\\
&=D_{E_2}\widehat{\mathcal{F}}(y,\Lambda')+\frac{1}{24}\partial_y^2\widehat{\mathcal{F}}(y,\Lambda')+\frac{1}{24}\left(\partial_y\widehat{\mathcal{F}}(y,\Lambda')\right)^2\ ,
\end{aligned}\tag{C.29}
$$

and we can also rewrite

$$
\partial_x\mathcal{F}^{x,s}_{\rm st}(a,\Lambda)-x\partial_a\rho_{\rm st}+\epsilon\partial_a\partial_\epsilon W_{\rm st}=\epsilon\partial_y\mathcal{F}(y,\Lambda')-x\partial_a\rho+\epsilon\partial_a\partial_\epsilon W\ . \tag{C.30}
$$

Then (C.28) becomes

$$
\begin{aligned}
&\left(D_{E_2}\widehat{\mathcal{F}}(y,\Lambda')+\frac{\epsilon^2}{24}\partial_y^2\widehat{\mathcal{F}}(y,\Lambda')+\frac{\epsilon^2}{24}\left(\partial_y\widehat{\mathcal{F}}(y,\Lambda')\right)^2\right)\\
&-\left(\left(\frac{y-a}{\epsilon}\right)\partial_a-\partial_\epsilon+\frac{\epsilon\partial_y\mathcal{F}(y,\Lambda')-(y-a)\partial_a\rho/\epsilon+\epsilon\partial_a\partial_\epsilon W}{\partial_a\mathbf{u}}\Lambda\partial_\Lambda\right)\times\\
&\times\left(D_{E_2}W_{\rm st}+\frac{\epsilon}{24}(\partial_a W_{\rm st})^2\right)=0\ .
\end{aligned}\tag{C.31}
$$

Taking the second derivative with respect to y we get

$$
\begin{aligned}
&\partial_y^2\left(D_{E_2}\widehat{\mathcal{F}}(y,\Lambda')+\frac{\epsilon^2}{24}\partial_y^2\widehat{\mathcal{F}}(y,\Lambda')+\frac{\epsilon^2}{24}\left(\partial_y\widehat{\mathcal{F}}(y,\Lambda')\right)^2\right)=\\
&=\epsilon\frac{\partial_y^3\mathcal{F}(y,\Lambda')}{\partial_a\mathbf{u}}\Lambda\partial_\Lambda\left(D_{E_2}W_{\rm st}+\frac{\epsilon}{24}(\partial_a W_{\rm st})^2\right)
\end{aligned}\tag{C.32}
$$

and the first derivative in Λ' gives

$$
\begin{aligned}
&\partial_{\Lambda'}\left(D_{E_2}\widehat{\mathcal{F}}(y,\Lambda')+\frac{\epsilon^2}{24}\partial_y^2\widehat{\mathcal{F}}(y,\Lambda')+\frac{\epsilon^2}{24}\left(\partial_y\widehat{\mathcal{F}}(y,\Lambda')\right)^2\right)=\\
&=\epsilon\frac{\partial_{\Lambda'}\partial_y\mathcal{F}(y,\Lambda')}{\partial_a\mathbf{u}}\Lambda\partial_\Lambda\left(D_{E_2}W_{\rm st}+\frac{\epsilon}{24}(\partial_a W_{\rm st})^2\right)\ .
\end{aligned}\tag{C.33}
$$

From this equations it follows that

$$\Lambda\partial_\Lambda\left(D_{E_2}W_{\rm st}+\frac{\epsilon}{24}(\partial_a W_{\rm st})^2\right)=c_1\partial_a\mathbf{u}=\epsilon c_1\Lambda\partial_\Lambda\partial_a W\ ,\tag{C.34}$$

$$\left(D_{E_2}\widehat{\mathcal{F}}(y,\Lambda')+\frac{\epsilon^2}{24}\partial_y^2\widehat{\mathcal{F}}(y,\Lambda')+\frac{\epsilon^2}{24}\left(\partial_y\widehat{\mathcal{F}}(y,\Lambda')\right)^2\right)-\epsilon c_1\partial_y\mathcal{F}(y,\Lambda')=\frac{c_2}{\epsilon}y+c_3\ ,\tag{C.35}$$

for some dimensionless[109] constants c_1,c_2,c_3. Integrating (C.34) we get

$$\left(D_{E_2}W_{\rm st}+\frac{\epsilon}{24}(\partial_a W_{\rm st})^2\right)=c_1\rho+f(a,\epsilon)\ ,\tag{C.36}$$

for some function $f(a,\epsilon)$. Substituting in (C.31) we obtain

$$\frac{c_2}{\epsilon}y+c_3+c_1\partial_a W-\left(\left(\frac{y-a}{\epsilon}\right)\partial_a-\partial_\epsilon\right)f(a,\epsilon)=0\ ,\tag{C.37}$$

whose coefficients in y give the two equations

$$\partial_a f(a,\epsilon)=c_2\ \Rightarrow f(a,\epsilon)=c_2 a+f_0\epsilon\ ,\tag{C.38}$$

$$(c_3+f_0)\epsilon+c_1\rho+ac_2=0\ ,\tag{C.39}$$

for some dimensionless constant f_0. We have now two cases: if $c_1=0$ then from (C.39) we get $c_2=0, c_3=-f_0$ and we obtain the following equations

$$D_{E_2}W_{\rm st}+\frac{\epsilon}{24}(\partial_a W_{\rm st})^2=\epsilon f_0\ ,\tag{C.40}$$

$$D_{E_2}\widehat{\mathcal{F}}(y,\Lambda')+\frac{\epsilon^2}{24}\partial_y^2\widehat{\mathcal{F}}(y,\Lambda')+\frac{\epsilon^2}{24}\left(\partial_y\widehat{\mathcal{F}}(y,\Lambda')\right)^2=-f_0\ ,\tag{C.41}$$

in the limit $\epsilon\to 0$ we have $\widehat{\mathcal{F}}(y,\Lambda')\to\mathcal{F}_{1,0}(y,\Lambda')$ and (C.41) becomes

$$D_{E_2}\mathcal{F}_{1,0}(y,\Lambda')=-f_0\ ,\tag{C.42}$$

from (C.6) we have that $\mathcal{F}_{1,0}(y,\Lambda')$ is modular therefore we need $f_0=0$ and (C.40), (C.41) become exactly the holomorphic anomaly equations for NS and SD respectively.

The case $c_1\neq 0$ in unphysical. Indeed, from (C.39) we get

$$\rho=-\frac{c_2}{c_1}a-\frac{c_3+f_0}{c_1}\epsilon\ .\tag{C.43}$$

and integrating in a we obtain

$$W=-\frac{1}{2\epsilon}\frac{c_2}{c_1}a^2-\frac{c_3+f_0}{c_1}a+\Lambda w\left(\frac{\Lambda}{\epsilon}\right)\ .\tag{C.44}$$

for some dimensionless function $w(\Lambda/\epsilon)$. In particular in the SW limit $\epsilon\to 0$ we have

$$\lim_{\epsilon\to 0}\epsilon w\left(\frac{\Lambda}{\epsilon}\right)=w_0\Lambda\ ,\quad \mathcal{F}_0(a,\Lambda)=\lim_{\epsilon\to 0}\epsilon W=-\frac{1}{2}\frac{c_2}{c_1}a^2+\Lambda^2 w_0\ ,\tag{C.45}$$

for some dimensionless constant w_0. The prepotential in (C.45) is unphysical because the IR coupling τ_{SW} is independent on the UV coupling Λ. Indeed, in this case the structure constants will be trivial $\partial_a^3\mathcal{F}_0(a,\Lambda)$=0.

[109]In the theories with flavour in principle these constants can actually be functions of dimensionless combinations of the mass parameters but we can rule out this dependence because it will produce extra terms in the holomorphic anomaly equations which are inconsistent with the limit $\epsilon\to 0$.

D The first coefficients of the q-Painlevé Hurwitz expansions

In this appendix we give the coefficients of the Hurwitz expansions studied in section 11. We computed them up to the order $d = 15$ but for simplicity we report only the first coefficients ($0 \leq d \leq 6$). We recall that the NS blowup factor (10.31) has the following structure

$$\mathcal{B}_{NS}^{(j,k)}(a,z,\epsilon,d,\beta) \equiv B_d^{(j,k)}(\mathbf{U},z,q) = (q^d z)^{\frac{j}{4}} P_d^{(j,k)}(\mathbf{U},z,q) = (q^d z)^{\frac{j}{4}} \sum_{n=0}^{n_{\max}^{(j,k)}(d)} P_{d,n}^{(j,k)}(z,q)\mathbf{U}^n \ . \tag{D.1}$$

where j is the first Chern class of the blowup, k is the Chern-Simons level and d is the number of insertions of the codimension 2 observable $I(E)$, and we have

$$\begin{aligned}
&\mathbf{U}(a,z,q^{-1}) = \mathbf{U}(a,z,q) \ , && \text{for } k=0 \ , \\
&\tilde{\mathbf{U}}(a,z,q) \equiv \mathbf{U}(a,z,q^{-1}) = \mathbf{U}(a,z,q) + (q^{-1/2} - q^{1/2})z \ , && \text{for } k=1 \ , \\
&B_{-d}^{(j,0)}(\mathbf{U},z,q) = (-1)^j B_d^{(j,0)}(\mathbf{U},z,q) \ , \quad B_{-d}^{(j,1)}(\mathbf{U},z,q) = (-1)^j B_d^{(j,1)}(\tilde{\mathbf{U}},z,q) \ .
\end{aligned} \tag{D.2}$$

In the following we report the q-polynomials $P_d^{(j,k)}(\mathbf{U},z,q)$ in the NS Wilson loop variable $\mathbf{U} = \langle W_1 \rangle_{NS}$ and the coupling z.

D.1 q-PIII$_3$

In this subsection we report the $P_d^{(j,0)}$ for the recursion (11.7) of q-PIII$_3$ ($k=0$) in the two sectors $j=0,1$. We find $n_{\max}^{(j,0)}(d) = \left\lfloor \frac{d^2}{4} \right\rfloor - \chi_4(d+2-2j)$

$$P_0^{(0,0)} = 1\ ,$$

$$P_1^{(0,0)} = 1\ ,$$

$$P_2^{(0,0)} = 1 - qz\ ,$$

$$\begin{aligned} P_3^{(0,0)} = {} & (1-2qz+q^2z^2)- \\ & \mathbf{U}^2 q^2 z\ , \end{aligned}$$

$$\begin{aligned} P_4^{(0,0)} = {} & 1+(-3q-q^3)z+(3q^2+3q^4)z^2+(-q^3-3q^5)z^3+q^6z^4+ \\ & \mathbf{U}^2\big[(-2q^2+2q^3)z+(2q^3-2q^4)z^2\big]- \\ & \mathbf{U}^4 q^3 z\ , \end{aligned}$$

$$\begin{aligned} P_5^{(0,0)} = {} & 1+(-4q-2q^3)z+(6q^2+8q^4+q^6)z^2+(-4q^3-12q^5-4q^7)z^3+(q^4+8q^6+6q^8)z^4+ \\ & \quad (-2q^7-4q^9)z^5+q^{10}z^6+ \\ & \mathbf{U}^2\big[(-3q^2+4q^3-4q^4)z+(6q^3-8q^4+10q^5+4q^6)z^2+(-3q^4+4q^5-8q^6-8q^7-3q^8)z^3+ \\ & \quad (2q^7+4q^8+6q^9)z^4-3q^{10}z^5\big]+ \\ & \mathbf{U}^4\big[(-2q^3+4q^4)z+(3q^4-4q^5-2q^6)z^2+q^{10}z^4\big]- \\ & \mathbf{U}^6 q^4 z\ , \end{aligned}$$

$$\begin{aligned} P_6^{(0,0)} = {} & 1+(-5q-3q^3-q^5)z+(10q^2+15q^4+8q^6+3q^8)z^2+(-10q^3-30q^5-25q^7-16q^9-3q^{11})z^3+ \\ & \quad (5q^4+30q^6+40q^8+35q^{10}+15q^{12}+q^{14})z^4+(-q^5-15q^7-35q^9-40q^{11}-30q^{13}-5q^{15})z^5+ \\ & \quad (3q^8+16q^{10}+25q^{12}+30q^{14}+10q^{16})z^6+(-3q^{11}-8q^{13}-15q^{15}-10q^{17})z^7+ \\ & \quad (q^{14}+3q^{16}+5q^{18})z^8-q^{19}z^9+ \\ & \mathbf{U}^2\big[(-4q^2+6q^3-8q^4+6q^5)z+(12q^3-18q^4+30q^5-14q^6-2q^7-8q^8)z^2+ \\ & \quad (-12q^4+18q^5-42q^6+6q^7-2q^8+22q^9+4q^{10}+6q^{11})z^3+ \\ & \quad (4q^5-6q^6+26q^7+6q^8+18q^9-18q^{10}-6q^{11}-26q^{12}+6q^{13}-4q^{14})z^4+ \\ & \quad (-6q^8-4q^9-22q^{10}+2q^{11}-6q^{12}+42q^{13}-18q^{14}+12q^{15})z^5+ \\ & \quad (8q^{11}+2q^{12}+14q^{13}-30q^{14}+18q^{15}-12q^{16})z^6+(-6q^{14}+8q^{15}-6q^{16}+4q^{17})z^7\big]+ \\ & \mathbf{U}^4\big[(-3q^3+8q^4-11q^5)z+(9q^4-20q^5+21q^6+12q^7+8q^8)z^2+ \\ & \quad (-6q^5+12q^6-13q^7-20q^8-22q^9-8q^{10}-3q^{11})z^3+ \\ & \quad (3q^8+8q^9+22q^{10}+20q^{11}+13q^{12}-12q^{13}+6q^{14})z^4+ \\ & \quad (-8q^{11}-12q^{12}-21q^{13}+20q^{14}-9q^{15})z^5+(11q^{14}-8q^{15}+3q^{16})z^6\big]+ \\ & \mathbf{U}^6\big[(-2q^4+6q^5)z+(4q^5-6q^6-4q^7-2q^8)z^2+(2q^{11}+4q^{12}+6q^{13}-4q^{14})z^4+ \\ & \quad (-6q^{14}+2q^{15})z^5\big]+ \\ & \mathbf{U}^8(-q^5z+q^{14}z^4)\ , \end{aligned}$$

$$P_0^{(1,0)} = 0\ ,$$

$$P_1^{(1,0)} = -1\ ,$$

$$P_2^{(1,0)} = -\mathbf{U}\ ,$$

$$\begin{aligned}P_3^{(1,0)} = {} & 1 - 2qz + q^2z^2 - \\ & \mathbf{U}^2\ ,\end{aligned}$$

$$\begin{aligned}P_4^{(1,0)} = {} & \mathbf{U}\big[2 + (-4q - 2q^2)z + (2q^2 + 4q^3)z^2 - 2q^4z^3\big] + \\ & \mathbf{U}^3\big[-1 + q^4z^2\big]\ ,\end{aligned}$$

$$\begin{aligned}P_5^{(1,0)} = {} & -1 + (4q + 2q^3)z + (-6q^2 - 8q^4 - q^6)z^2 + (4q^3 + 12q^5 + 4q^7)z^3 + (-q^4 - 8q^6 - 6q^8)z^4 + \\ & \quad (2q^7 + 4q^9)z^5 - q^{10}z^6 + \\ & \mathbf{U}^2\big[3 + (-6q - 4q^2 - 2q^3)z + (3q^2 + 8q^3 + 8q^4 - 4q^5 + 3q^6)z^2 + (-4q^4 - 10q^5 + 8q^6 - 6q^7)z^3 + \\ & \quad (4q^6 - 4q^7 + 3q^8)z^4\big] + \\ & \mathbf{U}^4\big[-1 + (2q^4 + 4q^5 - 3q^6)z^2 + (-4q^6 + 2q^7)z^3\big] + \\ & \mathbf{U}^6q^6z^2\ ,\end{aligned}$$

$$\begin{aligned}P_6^{(1,0)} = {} & \mathbf{U}\big[-3 + (12q + 4q^2 + 4q^3 + 4q^4)z + (-18q^2 - 16q^3 - 16q^4 - 24q^5 - 2q^6 - 8q^7)z^2 + \\ & \quad (12q^3 + 24q^4 + 24q^5 + 56q^6 + 8q^7 + 36q^8 + 4q^9 + 4q^{10})z^3 + \\ & \quad (-3q^4 - 16q^5 - 16q^6 - 64q^7 - 12q^8 - 64q^9 - 16q^{10} - 16q^{11} - 3q^{12})z^4 + \\ & \quad (4q^6 + 4q^7 + 36q^8 + 8q^9 + 56q^{10} + 24q^{11} + 24q^{12} + 12q^{13})z^5 + \\ & \quad (-8q^9 - 2q^{10} - 24q^{11} - 16q^{12} - 16q^{13} - 18q^{14})z^6 + (4q^{12} + 4q^{13} + 4q^{14} + 12q^{15})z^7 - 3q^{16}z^8\big] + \\ & \mathbf{U}^3\big[4 + (-8q - 6q^2 - 4q^3 - 2q^4)z + (4q^2 + 12q^3 + 14q^4 + 4q^5 - 10q^6 + 20q^7 - 8q^8)z^2 + \\ & \quad (-6q^4 - 16q^5 - 10q^6 + 40q^7 - 50q^8 + 16q^9 + 6q^{10})z^3 + \\ & \quad (6q^6 + 16q^7 - 50q^8 + 40q^9 - 10q^{10} - 16q^{11} - 6q^{12})z^4 + \\ & \quad (-8q^8 + 20q^9 - 10q^{10} + 4q^{11} + 14q^{12} + 12q^{13} + 4q^{14})z^5 + \\ & \quad (-2q^{12} - 4q^{13} - 6q^{14} - 8q^{15})z^6 + 4q^{16}z^7\big] + \\ & \mathbf{U}^5\big[-1 + (3q^4 + 8q^5 + 2q^6 - 16q^7 + 12q^8)z^2 + (-8q^6 - 16q^7 + 32q^8 - 16q^9 - 8q^{10})z^3 + \\ & \quad (12q^8 - 16q^9 + 2q^{10} + 8q^{11} + 3q^{12})z^4 - q^{16}z^6\big] + \\ & \mathbf{U}^7\big[(2q^6 + 4q^7 - 6q^8)z^2 + (-6q^8 + 4q^9 + 2q^{10})z^3\big] + \\ & \mathbf{U}^9q^8z^2\ ,\end{aligned}$$

D.2 q-PI

In this subsection we report the $P_d^{(j,1)}$ for the recursion (11.18) of q-PI ($k=1$) in the two sectors $j=0,1$. We denote $\widetilde{z}=zq^{3/2}$ and we find $n_{\max}^{(j,1)}(d)=\left\lfloor\frac{d(d+j-1)}{4}\right\rfloor-j\chi_4(d)$

$$P_0^{(0,1)}=1\ ,$$

$$P_1^{(0,1)}=1\ ,$$

$$P_2^{(0,1)}=1\ ,$$

$$\begin{aligned}P_3^{(0,1)}=&\,1-\\&\mathbf{U}\widetilde{z}\ ,\end{aligned}$$

$$\begin{aligned}P_4^{(0,1)}=&\,1+\\&\mathbf{U}(-2+q)\widetilde{z}+\\&\mathbf{U}^2(1-q)\widetilde{z}^2-\\&\mathbf{U}^3q\widetilde{z}\ ,\end{aligned}$$

$$\begin{aligned}P_5^{(0,1)}=&\,1+(q^2-q^3)\widetilde{z}^2+\\&\mathbf{U}\big[(-3+2q-2q^2)\widetilde{z}+(-2q^2+2q^3)\widetilde{z}^3\big]+\\&\mathbf{U}^2\big[(3-4q+2q^2+2q^3)\widetilde{z}^2+(q^2-q^3)\widetilde{z}^4\big]+\\&\mathbf{U}^3\big[(-2q+3q^2)\widetilde{z}+(-1+2q-2q^3)\widetilde{z}^3\big]+\\&\mathbf{U}^4(2q-2q^2-q^3)\widetilde{z}^2-\\&\mathbf{U}^5q^2\widetilde{z}\ ,\end{aligned}$$

$$\begin{aligned}P_6^{(0,1)}=&\,1+(2q^2-3q^3+q^4)\widetilde{z}^2+(q^3-2q^4+2q^6-q^7)\widetilde{z}^4\ ,\\&\mathbf{U}\big[(-4+3q-4q^2+2q^3)\widetilde{z}+(-6q^2+6q^3+4q^5-4q^6)\widetilde{z}^3+(-3q^3+6q^4-6q^6+3q^7)\widetilde{z}^5\big]+\\&\mathbf{U}^2\big[(6-9q+9q^2+5q^3-4q^4-4q^5)\widetilde{z}^2+(6q^2-3q^3-12q^5+7q^6+2q^7)\widetilde{z}^4+\\&\quad(3q^3-6q^4+6q^6-3q^7)\widetilde{z}^6\big]+\\&\mathbf{U}^3\big[(-3q+6q^2-7q^3)\widetilde{z}+(-4+9q-6q^2-13q^3+2q^4+5q^5+6q^6)\widetilde{z}^3+\\&\quad(-2q^2-4q^4+12q^5-2q^6-4q^7)\widetilde{z}^5+(-q^3+2q^4-2q^6+q^7)\widetilde{z}^7\big]+\end{aligned}$$

$$\mathbf{U}^4\big[(6q-12q^2+5q^3+6q^4+4q^5)\widetilde{z}^2+(1-3q+q^2+6q^3+2q^4+2q^5-8q^6-q^7)\widetilde{z}^4+$$
$$(3q^4-4q^5-q^6+2q^7)\widetilde{z}^6\big]+$$
$$\mathbf{U}^5\big[(-2q^2+5q^3)\widetilde{z}+(-3q+6q^2+q^3-4q^4-4q^5-2q^6)\widetilde{z}^3+(-3q^5+2q^6+q^7)\widetilde{z}^5\big]+$$
$$\mathbf{U}^6\big[(3q^2-3q^3-2q^4-q^5)\widetilde{z}^2+q^6\widetilde{z}^4\big]-$$
$$\mathbf{U}^7q^3\widetilde{z}\ ,$$

$$P_0^{(1,1)}=0\ ,$$

$$P_1^{(1,1)}=-1\ ,$$

$$P_2^{(1,1)}=-\mathbf{U}\ ,$$

$$P_3^{(1,1)}=1-$$
$$\mathbf{U}\widetilde{z}-$$
$$\mathbf{U}^2\ ,$$

$$P_4^{(1,1)}=(-1+q)\widetilde{z}+$$
$$\mathbf{U}\big[2+(2-2q)\widetilde{z}^2\big]+$$
$$\mathbf{U}^2\big[(-2-q)\widetilde{z}+(-1+q)\widetilde{z}^3\big]+$$
$$\mathbf{U}^3\big[-1+q\widetilde{z}^2\big]\ ,$$

$$P_5^{(1,1)}=-1+(1-2q+q^3)\widetilde{z}^2+$$
$$\mathbf{U}\big[(q+2q^2)\widetilde{z}+(-3+6q-4q^3+q^4)\widetilde{z}^3\big]+$$
$$\mathbf{U}^2\big[3+(3+q-8q^2+q^3)\widetilde{z}^2+(3-6q+5q^3-2q^4)\widetilde{z}^4\big]+$$
$$\mathbf{U}^3\big[(-3-2q-q^2)\widetilde{z}+(-2-5q+10q^2-2q^4)\widetilde{z}^3+(-1+2q-2q^3+q^4)\widetilde{z}^5\big]+$$
$$\mathbf{U}^4\big[-1+(2q+4q^2-3q^3)\widetilde{z}^2+(3q-4q^2-q^3+2q^4)\widetilde{z}^4\big]+$$
$$\mathbf{U}^5(-3q^2+2q^3+q^4)\widetilde{z}^3+$$
$$\mathbf{U}^6q^3\widetilde{z}^2\ ,$$

$$
\begin{aligned}
P_6^{(1,1)} = &(2-2q+q^2-q^3)\tilde{z}+(-1+3q-q^2-2q^3+2q^4-3q^5+2q^6)\tilde{z}^3+(q^5-2q^6+2q^8-q^9)\tilde{z}^5+\\
&\mathbf{U}\big[-3+(-4+4q-10q^2+10q^3-4q^4+4q^5)\tilde{z}^2+\\
&\quad(4-12q+4q^2+10q^3-8q^4+4q^5-4q^6+6q^7-4q^8)\tilde{z}^4+(-3q^5+6q^6-6q^8+3q^9)\tilde{z}^6\big]+\\
&\mathbf{U}^2\big[(3+3q+3q^2+3q^3)\tilde{z}+(-5q+36q^2-31q^3+3q^4+3q^5-2q^6-4q^7)\tilde{z}^3+\\
&\quad(-6+18q-6q^2-18q^3+12q^4+3q^5+6q^6-18q^7+6q^8+3q^9)\tilde{z}^5+(3q^5-6q^6+6q^8-3q^9)\tilde{z}^7\big]+\\
&\mathbf{U}^3\big[4+(3-8q^2-20q^3+16q^4-10q^5)\tilde{z}^2+\\
&\quad(4+11q-58q^2+37q^3+18q^4-15q^5-6q^6-q^7+10q^8)\tilde{z}^4+\\
&\quad(4-12q+4q^2+14q^3-8q^4-6q^5-8q^6+18q^7-6q^9)\tilde{z}^6+(-q^5+2q^6-2q^8+q^9)\tilde{z}^8\big]+\\
&\mathbf{U}^4\big[(-4-3q-2q^2-q^3)\tilde{z}+(-3-9q-3q^2+55q^3-35q^4-2q^5+4q^6+8q^7)\tilde{z}^3+\\
&\quad(-2-13q+43q^2-16q^3-29q^4+8q^5+12q^6+14q^7-14q^8-3q^9)\tilde{z}^5+\\
&\quad(-1+3q-q^2-4q^3+2q^4+2q^5+4q^6-6q^7-2q^8+3q^9)\tilde{z}^7\big]+\\
&\mathbf{U}^5\big[-1+(3q+8q^2+2q^3-16q^4+12q^5)\tilde{z}^2+\\
&\quad(6q+18q^2-56q^3+16q^4+22q^5+4q^6-8q^7-8q^8)\tilde{z}^4+\\
&\quad(5q-12q^2+q^3+12q^4-4q^6-9q^7+4q^8+3q^9)\tilde{z}^6\big]+\\
&\mathbf{U}^6\big[(-6q^2-13q^3+31q^4-5q^5-6q^6-5q^7)\tilde{z}^3+(-10q^2+18q^3+3q^4-11q^5-6q^6+6q^8+q^9)\tilde{z}^5\big]+\\
&\mathbf{U}^7\big[(2q^3+4q^4-6q^5)\tilde{z}^2+(10q^3-12q^4-5q^5+2q^6+4q^7+2q^8)\tilde{z}^4\big]+\\
&\mathbf{U}^8(-5q^4+3q^5+2q^6+q^7)\tilde{z}^3+\\
&\mathbf{U}^9 q^5\tilde{z}^2 \ ,
\end{aligned}
$$

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E Coefficients of the Painlevé Hurwitz expansions

In this extra appendix we collect some numerical evidence of the integrality of the polynomials in the Hurwitz expansions of the Painlevé $\mathcal{T}$-functions (4d gauge theories) in some trial cases. Decomposing the numerical coefficients in prime factors we could not recognize any regularity to further reduce them in an obvious way. We computed many more coefficient polynomials evidencing their integrality, but we do not report them for brevity. For notational simplicity *in this appendix we denote all symbols without boldface.* We refer to section 8 for the corresponding assignments. We also refer to section 8 for the explicit expressions of the invariants g_2, g_3, T.

E.1 Hurwitz expansion for PVI

In this subsection we denote $c_n = c_n^{PVI}$ from (8.34).

(Massive case: $\tilde{e}_2 = e_2/4$, $\tilde{e}_4 = e_4/2$, $\tilde{\epsilon} = \epsilon/2$)[110]

$c_0 = 1$

$c_1 = 0$

$c_2 = \tilde{e}_2 q^2 + u - q^2 u - \tilde{\epsilon}^2 + 2q\tilde{\epsilon}^2 - q^2\tilde{\epsilon}^2$

$c_3 = -8\tilde{e}_2 q^2\tilde{\epsilon} - 8qu\tilde{\epsilon} + 8q^2u\tilde{\epsilon}$

$c_4 = 2p_4 q - \tilde{e}_4 q^2 - 3p_4 q^2 + \tilde{e}_4 q^3 + p_4 q^3 + \tilde{e}_2^2 q^4 + 6\tilde{e}_2 q^2 u - 4\tilde{e}_2 q^3 u - 2\tilde{e}_2 q^4 u + u^2 + 2qu^2 - 6q^2u^2 + 2q^3u^2 +$

[110] In the case of PVI the natural basis is not given by the elliptic invariants g_2, g_3 and the contact term T due to the presence of the adimensional coupling q which introduces a more complicated dependence. Nevertheless, in the SW limit the expansion can be reorganized in this basis because of the universality of the structure of the SW blowup factor.

$q^4u^2+46\tilde{e}_2q^2\tilde{\epsilon}^2+28\tilde{e}_2q^3\tilde{\epsilon}^2-2\tilde{e}_2q^4\tilde{\epsilon}^2-2u\tilde{\epsilon}^2+28qu\tilde{\epsilon}^2-28q^3u\tilde{\epsilon}^2+2q^4u\tilde{\epsilon}^2+\tilde{\epsilon}^4-12q\tilde{\epsilon}^4+22q^2\tilde{\epsilon}^4-12q^3\tilde{\epsilon}^4+q^4\tilde{\epsilon}^4$

$c_5=-8p_4q\tilde{\epsilon}+16\tilde{e}_4q^2\tilde{\epsilon}-8p_4q^2\tilde{\epsilon}-12\tilde{e}_4q^3\tilde{\epsilon}+20p_4q^3\tilde{\epsilon}-24\tilde{e}_2^2q^4\tilde{\epsilon}-4\tilde{e}_4q^4\tilde{\epsilon}-4p_4q^4\tilde{\epsilon}-88\tilde{e}_2q^2u\tilde{\epsilon}+24\tilde{e}_2q^3u\tilde{\epsilon}+64\tilde{e}_2q^4u\tilde{\epsilon}-32qu^2\tilde{\epsilon}+32q^2u^2\tilde{\epsilon}+32q^3u^2\tilde{\epsilon}-32q^4u^2\tilde{\epsilon}-232\tilde{e}_2q^2\tilde{\epsilon}^3-496\tilde{e}_2q^3\tilde{\epsilon}^3-40\tilde{e}_2q^4\tilde{\epsilon}^3-40qu\tilde{\epsilon}^3-264q^2u\tilde{\epsilon}^3+264q^3u\tilde{\epsilon}^3+40q^4u\tilde{\epsilon}^3+32q\tilde{\epsilon}^5-32q^2\tilde{\epsilon}^5-32q^3\tilde{\epsilon}^5+32q^4\tilde{\epsilon}^5$

$c_6=6e_6q^2-12e_6q^3+24\tilde{e}_2p_4q^3-24\tilde{e}_2\tilde{e}_4q^4+6e_6q^4-36\tilde{e}_2p_4q^4+24\tilde{e}_2\tilde{e}_4q^5+12\tilde{e}_2p_4q^5+64\tilde{e}_2^3q^6-6p_4qu+30\tilde{e}_4q^2u+3p_4q^2u-84\tilde{e}_4q^3u+30p_4q^3u-240\tilde{e}_2^2q^4u+72\tilde{e}_4q^4u-48p_4q^4u+432\tilde{e}_2^2q^5u-12\tilde{e}_4q^5u+24p_4q^5u-144\tilde{e}_2^2q^6u-6\tilde{e}_4q^6u-3p_4q^6u-48\tilde{e}_2^2q^7u+156\tilde{e}_2q^2u^2-528\tilde{e}_2q^3u^2+564\tilde{e}_2q^4u^2-96\tilde{e}_2q^5u^2-156\tilde{e}_2q^6u^2+48\tilde{e}_2q^7u^2+12\tilde{e}_2q^8u^2-u^3-3qu^3+30q^2u^3-62q^3u^3+36q^4u^3+36q^5u^3-62q^6u^3+30q^7u^3-3q^8u^3-q^9u^3+72p_4q\tilde{\epsilon}^2-1128\tilde{e}_4q^2\tilde{\epsilon}^2+916p_4q^2\tilde{\epsilon}^2-424\tilde{e}_4q^3\tilde{\epsilon}^2-636p_4q^3\tilde{\epsilon}^2+23360\tilde{e}_2^2q^4\tilde{\epsilon}^2+1480\tilde{e}_4q^4\tilde{\epsilon}^2-388p_4q^4\tilde{\epsilon}^2+7040\tilde{e}_2^2q^5\tilde{\epsilon}^2+72\tilde{e}_4q^5\tilde{\epsilon}^2+36p_4q^5\tilde{\epsilon}^2-192\tilde{e}_2^2q^6\tilde{\epsilon}^2-13792\tilde{e}_2q^2u\tilde{\epsilon}^2+4832\tilde{e}_2q^3u\tilde{\epsilon}^2+27712\tilde{e}_2q^4u\tilde{\epsilon}^2-14656\tilde{e}_2q^5u\tilde{\epsilon}^2-4192\tilde{e}_2q^6u\tilde{\epsilon}^2+96\tilde{e}_2q^7u\tilde{\epsilon}^2-12u^2\tilde{\epsilon}^2+536qu^2\tilde{\epsilon}^2-416q^2u^2\tilde{\epsilon}^2-2968q^3u^2\tilde{\epsilon}^2+5720q^4u^2\tilde{\epsilon}^2-2968q^5u^2\tilde{\epsilon}^2-416q^6u^2\tilde{\epsilon}^2+536q^7u^2\tilde{\epsilon}^2-12q^8u^2\tilde{\epsilon}^2+62656\tilde{e}_2q^2\tilde{\epsilon}^4+367872\tilde{e}_2q^3\tilde{\epsilon}^4+182400\tilde{e}_2q^4\tilde{\epsilon}^4+1280\tilde{e}_2q^5\tilde{\epsilon}^4+192\tilde{e}_2q^6\tilde{\epsilon}^4-48u\tilde{\epsilon}^4-272qu\tilde{\epsilon}^4-29616q^2u\tilde{\epsilon}^4+29936q^3u\tilde{\epsilon}^4+29936q^4u\tilde{\epsilon}^4-29616q^5u\tilde{\epsilon}^4-272q^6u\tilde{\epsilon}^4-48q^7u\tilde{\epsilon}^4-64\tilde{\epsilon}^6-4224q\tilde{\epsilon}^6-23488q^2\tilde{\epsilon}^6+55552q^3\tilde{\epsilon}^6-23488q^4\tilde{\epsilon}^6-4224q^5\tilde{\epsilon}^6-64q^6\tilde{\epsilon}^6$

$c_7=-96e_6q^2\tilde{\epsilon}+96e_6q^3\tilde{\epsilon}-288\tilde{e}_2p_4q^3\tilde{\epsilon}+448\tilde{e}_2\tilde{e}_4q^4\tilde{\epsilon}+96e_6q^4\tilde{\epsilon}-96\tilde{e}_2\tilde{e}_4q^5\tilde{\epsilon}-96e_6q^5\tilde{\epsilon}+448\tilde{e}_2p_4q^5\tilde{\epsilon}-1056\tilde{e}_2^3q^6\tilde{\epsilon}-352\tilde{e}_2\tilde{e}_4q^6\tilde{\epsilon}-160\tilde{e}_2p_4q^6\tilde{\epsilon}-1008\tilde{e}_2^3q^7\tilde{\epsilon}+160p_4qu\tilde{\epsilon}-400\tilde{e}_4q^2u\tilde{\epsilon}-64p_4q^2u\tilde{\epsilon}+736\tilde{e}_4q^3u\tilde{\epsilon}-624p_4q^3u\tilde{\epsilon}+3536\tilde{e}_2^2q^4u\tilde{\epsilon}+176\tilde{e}_4q^4u\tilde{\epsilon}+528p_4q^4u\tilde{\epsilon}-2688\tilde{e}_2^2q^5u\tilde{\epsilon}-864\tilde{e}_4q^5u\tilde{\epsilon}+320p_4q^5u\tilde{\epsilon}-4640\tilde{e}_2^2q^6u\tilde{\epsilon}+256\tilde{e}_4q^6u\tilde{\epsilon}-368p_4q^6u\tilde{\epsilon}+3024\tilde{e}_2^2q^7u\tilde{\epsilon}+96\tilde{e}_4q^7u\tilde{\epsilon}+48p_4q^7u\tilde{\epsilon}+768\tilde{e}_2^2q^8u\tilde{\epsilon}-2816\tilde{e}_2q^2u^2\tilde{\epsilon}+6512\tilde{e}_2q^3u^2\tilde{\epsilon}-544\tilde{e}_2q^4u^2\tilde{\epsilon}-7904\tilde{e}_2q^5u^2\tilde{\epsilon}+4128\tilde{e}_2q^6u^2\tilde{\epsilon}+1776\tilde{e}_2q^7u^2\tilde{\epsilon}-960\tilde{e}_2q^8u^2\tilde{\epsilon}-192\tilde{e}_2q^9u^2\tilde{\epsilon}+32u^3\tilde{\epsilon}+96qu^3\tilde{\epsilon}-464q^2u^3\tilde{\epsilon}+336q^3u^3\tilde{\epsilon}+592q^4u^3\tilde{\epsilon}-1120q^5u^3\tilde{\epsilon}+384q^6u^3\tilde{\epsilon}+496q^7u^3\tilde{\epsilon}-432q^8u^3\tilde{\epsilon}+64q^9u^3\tilde{\epsilon}+16q^{10}u^3\tilde{\epsilon}-896p_4q\tilde{\epsilon}^3+16688\tilde{e}_4q^2\tilde{\epsilon}^3-13616p_4q^2\tilde{\epsilon}^3+24624\tilde{e}_4q^3\tilde{\epsilon}^3-5360p_4q^3\tilde{\epsilon}^3-372432\tilde{e}_2^2q^4\tilde{\epsilon}^3-17232\tilde{e}_4q^4\tilde{\epsilon}^3+15632p_4q^4\tilde{\epsilon}^3-483216\tilde{e}_2^2q^5\tilde{\epsilon}^3-23056\tilde{e}_4q^5\tilde{\epsilon}^3+4688p_4q^5\tilde{\epsilon}^3-108144\tilde{e}_2^2q^6\tilde{\epsilon}^3-1024\tilde{e}_4q^6\tilde{\epsilon}^3-448p_4q^6\tilde{\epsilon}^3+3024\tilde{e}_2^2q^7\tilde{\epsilon}^3+227904\tilde{e}_2q^2u\tilde{\epsilon}^3+148288\tilde{e}_2q^3u\tilde{\epsilon}^3-517920\tilde{e}_2q^4u\tilde{\epsilon}^3-220576\tilde{e}_2q^5u\tilde{\epsilon}^3+298208\tilde{e}_2q^6u\tilde{\epsilon}^3+65632\tilde{e}_2q^7u\tilde{\epsilon}^3-1536\tilde{e}_2q^8u\tilde{\epsilon}^3+144u^2\tilde{\epsilon}^3-6704qu^2\tilde{\epsilon}^3+304q^2u^2\tilde{\epsilon}^3+50288q^3u^2\tilde{\epsilon}^3-47888q^4u^2\tilde{\epsilon}^3-41808q^5u^2\tilde{\epsilon}^3+55824q^6u^2\tilde{\epsilon}^3-1968q^7u^2\tilde{\epsilon}^3-8384q^8u^2\tilde{\epsilon}^3+192q^9u^2\tilde{\epsilon}^3-989952\tilde{e}_2q^2\tilde{\epsilon}^5-6832848\tilde{e}_2q^3\tilde{\epsilon}^5-8737600\tilde{e}_2q^4\tilde{\epsilon}^5-2905312\tilde{e}_2q^5\tilde{\epsilon}^5-23104\tilde{e}_2q^6\tilde{\epsilon}^5-3024\tilde{e}_2q^7\tilde{\epsilon}^5+816u\tilde{\epsilon}^5+5568qu\tilde{\epsilon}^5+499232q^2u\tilde{\epsilon}^5+6032q^3u\tilde{\epsilon}^5-969104q^4u\tilde{\epsilon}^5-26144q^5u\tilde{\epsilon}^5+477760q^6u\tilde{\epsilon}^5+5072q^7u\tilde{\epsilon}^5+768q^8u\tilde{\epsilon}^5+1008\tilde{\epsilon}^7+67664q\tilde{\epsilon}^7+439920q^2\tilde{\epsilon}^7-508592q^3\tilde{\epsilon}^7-508592q^4\tilde{\epsilon}^7+439920q^5\tilde{\epsilon}^7+67664q^6\tilde{\epsilon}^7+1008q^7\tilde{\epsilon}^7$

(Massless case: $\tilde{\epsilon}=\epsilon/2$)

$c_0=1$

$c_1=0$

$c_2=u-q^2u-\tilde{\epsilon}^2+2q\tilde{\epsilon}^2-q^2\tilde{\epsilon}^2$

$c_3=-8qu\tilde{\epsilon}+8q^2u\tilde{\epsilon}$

$c_4=u^2+2qu^2-6q^2u^2+2q^3u^2+q^4u^2-2u\tilde{\epsilon}^2+28qu\tilde{\epsilon}^2-28q^3u\tilde{\epsilon}^2+2q^4u\tilde{\epsilon}^2+\tilde{\epsilon}^4-12q\tilde{\epsilon}^4+22q^2\tilde{\epsilon}^4-12q^3\tilde{\epsilon}^4+q^4\tilde{\epsilon}^4$

$c_5=-32qu^2\tilde{\epsilon}+32q^2u^2\tilde{\epsilon}+32q^3u^2\tilde{\epsilon}-32q^4u^2\tilde{\epsilon}-40qu\tilde{\epsilon}^3-264q^2u\tilde{\epsilon}^3+264q^3u\tilde{\epsilon}^3+40q^4u\tilde{\epsilon}^3+32q\tilde{\epsilon}^5-32q^2\tilde{\epsilon}^5-32q^3\tilde{\epsilon}^5+32q^4\tilde{\epsilon}^5$

$c_6=-u^3-3qu^3+30q^2u^3-62q^3u^3+36q^4u^3+36q^5u^3-62q^6u^3+30q^7u^3-3q^8u^3-q^9u^3-12u^2\tilde{\epsilon}^2+536qu^2\tilde{\epsilon}^2-416q^2u^2\tilde{\epsilon}^2-2968q^3u^2\tilde{\epsilon}^2+5720q^4u^2\tilde{\epsilon}^2-2968q^5u^2\tilde{\epsilon}^2-416q^6u^2\tilde{\epsilon}^2+536q^7u^2\tilde{\epsilon}^2-12q^8u^2\tilde{\epsilon}^2-48u\tilde{\epsilon}^4-272qu\tilde{\epsilon}^4-29616q^2u\tilde{\epsilon}^4+29936q^3u\tilde{\epsilon}^4+29936q^4u\tilde{\epsilon}^4-29616q^5u\tilde{\epsilon}^4-272q^6u\tilde{\epsilon}^4-48q^7u\tilde{\epsilon}^4-64\tilde{\epsilon}^6-4224q\tilde{\epsilon}^6-23488q^2\tilde{\epsilon}^6+55552q^3\tilde{\epsilon}^6-23488q^4\tilde{\epsilon}^6-4224q^5\tilde{\epsilon}^6-64q^6\tilde{\epsilon}^6$

$c_7=32u^3\tilde{\epsilon}+96qu^3\tilde{\epsilon}-464q^2u^3\tilde{\epsilon}+336q^3u^3\tilde{\epsilon}+592q^4u^3\tilde{\epsilon}-1120q^5u^3\tilde{\epsilon}+384q^6u^3\tilde{\epsilon}+496q^7u^3\tilde{\epsilon}-432q^8u^3\tilde{\epsilon}+64q^9u^3\tilde{\epsilon}+16q^{10}u^3\tilde{\epsilon}+144u^2\tilde{\epsilon}^3-6704qu^2\tilde{\epsilon}^3+304q^2u^2\tilde{\epsilon}^3+50288q^3u^2\tilde{\epsilon}^3-47888q^4u^2\tilde{\epsilon}^3-41808q^5u^2\tilde{\epsilon}^3+55824q^6u^2\tilde{\epsilon}^3-1968q^7u^2\tilde{\epsilon}^3-8384q^8u^2\tilde{\epsilon}^3+192q^9u^2\tilde{\epsilon}^3+816u\tilde{\epsilon}^5+5568qu\tilde{\epsilon}^5+499232q^2u\tilde{\epsilon}^5+6032q^3u\tilde{\epsilon}^5-969104q^4u\tilde{\epsilon}^5-$

$26144q^5u\tilde\epsilon^5+477760q^6u\tilde\epsilon^5+5072q^7u\tilde\epsilon^5+768q^8u\tilde\epsilon^5+1008\tilde\epsilon^7+67664q\tilde\epsilon^7+439920q^2\tilde\epsilon^7-508592q^3\tilde\epsilon^7-508592q^4\tilde\epsilon^7+$
$439920q^5\tilde\epsilon^7+67664q^6\tilde\epsilon^7+1008q^7\tilde\epsilon^7$

$c_8=-23u^4-96qu^4+500q^2u^4-504q^3u^4-450q^4u^4+1380q^5u^4-1048q^6u^4-132q^7u^4+697q^8u^4-$
$372q^9u^4+36q^{10}u^4+12q^{11}u^4-408u^3\tilde\epsilon^2+2924qu^3\tilde\epsilon^2-6036q^2u^3\tilde\epsilon^2-20300q^3u^3\tilde\epsilon^2+51656q^4u^3\tilde\epsilon^2-1424q^5u^3\tilde\epsilon^2-$
$55104q^6u^3\tilde\epsilon^2+22624q^7u^3\tilde\epsilon^2+10704q^8u^3\tilde\epsilon^2-3676q^9u^3\tilde\epsilon^2-812q^{10}u^3\tilde\epsilon^2-148q^{11}u^3\tilde\epsilon^2-1938u^2\tilde\epsilon^4+52980qu^2\tilde\epsilon^4-$
$239440q^2u^2\tilde\epsilon^4-226516q^3u^2\tilde\epsilon^4+236644q^4u^2\tilde\epsilon^4+1319404q^5u^2\tilde\epsilon^4-1192832q^6u^2\tilde\epsilon^4-235052q^7u^2\tilde\epsilon^4+214766q^8u^2\tilde\epsilon^4+$
$73760q^9u^2\tilde\epsilon^4-1776q^{10}u^2\tilde\epsilon^4-8308u\tilde\epsilon^6-119728qu\tilde\epsilon^6-5052088q^2u\tilde\epsilon^6-10787312q^3u\tilde\epsilon^6+15374592q^4u\tilde\epsilon^6+$
$15388784q^5u\tilde\epsilon^6-10250568q^6u\tilde\epsilon^6-4474640q^7u\tilde\epsilon^6-63628q^8u\tilde\epsilon^6-7104q^9u\tilde\epsilon^6-9323\tilde\epsilon^8-647824q\tilde\epsilon^8-5581380q^2\tilde\epsilon^8-$
$4380976q^3\tilde\epsilon^8+21239006q^4\tilde\epsilon^8-4380976q^5\tilde\epsilon^8-5581380q^6\tilde\epsilon^8-647824q^7\tilde\epsilon^8-9323q^8\tilde\epsilon^8$

$c_9=544u^4\tilde\epsilon+3120qu^4\tilde\epsilon-7248q^2u^4\tilde\epsilon-11424q^3u^4\tilde\epsilon+39712q^4u^4\tilde\epsilon-30352q^5u^4\tilde\epsilon-14288q^6u^4\tilde\epsilon+42480q^7u^4\tilde\epsilon-$
$25776q^8u^4\tilde\epsilon-2592q^9u^4\tilde\epsilon+7328q^{10}u^4\tilde\epsilon-1232q^{11}u^4\tilde\epsilon-272q^{12}u^4\tilde\epsilon+4464u^3\tilde\epsilon^3-81856qu^3\tilde\epsilon^3+53728q^2u^3\tilde\epsilon^3+$
$820416q^3u^3\tilde\epsilon^3-633264q^4u^3\tilde\epsilon^3-2223680q^5u^3\tilde\epsilon^3+2615424q^6u^3\tilde\epsilon^3+347584q^7u^3\tilde\epsilon^3-998320q^8u^3\tilde\epsilon^3-59904q^9u^3\tilde\epsilon^3+$
$145824q^{10}u^3\tilde\epsilon^3+8576q^{11}u^3\tilde\epsilon^3+1008q^{12}u^3\tilde\epsilon^3+22944u^2\tilde\epsilon^5-232624qu^2\tilde\epsilon^5+5638608q^2u^2\tilde\epsilon^5+8714736q^3u^2\tilde\epsilon^5-$
$13976112q^4u^2\tilde\epsilon^5-25363760q^5u^2\tilde\epsilon^5+14857008q^6u^2\tilde\epsilon^5+21165968q^7u^2\tilde\epsilon^5-6094896q^8u^2\tilde\epsilon^5-4296416q^9u^2\tilde\epsilon^5-$
$447552q^{10}u^2\tilde\epsilon^5+12096q^{11}u^2\tilde\epsilon^5+68544u\tilde\epsilon^7+1903008qu\tilde\epsilon^7+42567648q^2u\tilde\epsilon^7+240765024q^3u\tilde\epsilon^7-47883680q^4u\tilde\epsilon^7-$
$452964448q^5u\tilde\epsilon^7-40121952q^6u\tilde\epsilon^7+223471904q^7u\tilde\epsilon^7+31468384q^8u\tilde\epsilon^7+677184q^9u\tilde\epsilon^7+48384q^{10}u\tilde\epsilon^7+63504\tilde\epsilon^9+$
$4746736q\tilde\epsilon^9+60633440q^2\tilde\epsilon^9+185948064q^3\tilde\epsilon^9-251391744q^4\tilde\epsilon^9-251391744q^5\tilde\epsilon^9+185948064q^6\tilde\epsilon^9+60633440q^7\tilde\epsilon^9+$
$4746736q^8\tilde\epsilon^9+63504q^9\tilde\epsilon^9$

$c_{10}=-227u^5-982qu^5+5037q^2u^5-3532q^3u^5-8852q^4u^5+15618q^5u^5-3178q^6u^5-13952q^7u^5+14241q^8u^5-$
$698q^9u^5-6691q^{10}u^5+3660q^{11}u^5-330q^{12}u^5-114q^{13}u^5-7935u^4\tilde\epsilon^2-15410qu^4\tilde\epsilon^2-104571q^2u^4\tilde\epsilon^2+243112q^3u^4\tilde\epsilon^2+$
$193506q^4u^4\tilde\epsilon^2-474240q^5u^4\tilde\epsilon^2+294894q^6u^4\tilde\epsilon^2-573888q^7u^4\tilde\epsilon^2+425601q^8u^4\tilde\epsilon^2+319574q^9u^4\tilde\epsilon^2-334515q^{10}u^4\tilde\epsilon^2+$
$5496q^{11}u^4\tilde\epsilon^2+24924q^{12}u^4\tilde\epsilon^2+3452q^{13}u^4\tilde\epsilon^2-47660u^3\tilde\epsilon^4+969358qu^3\tilde\epsilon^4-563718q^2u^3\tilde\epsilon^4-12739128q^3u^3\tilde\epsilon^4-$
$8482126q^4u^3\tilde\epsilon^4+67525642q^5u^3\tilde\epsilon^4-20856020q^6u^3\tilde\epsilon^4-68358536q^7u^3\tilde\epsilon^4+35114656q^8u^3\tilde\epsilon^4+14697338q^9u^3\tilde\epsilon^4-$
$5081062q^{10}u^3\tilde\epsilon^4-2089280q^{11}u^3\tilde\epsilon^4-84070q^{12}u^3\tilde\epsilon^4-5394q^{13}u^3\tilde\epsilon^4-231790u^2\tilde\epsilon^6-1075792qu^2\tilde\epsilon^6-76154206q^2u^2\tilde\epsilon^6-$
$362284976q^3u^2\tilde\epsilon^6+253282108q^4u^2\tilde\epsilon^6+812743968q^5u^2\tilde\epsilon^6-10559868q^6u^2\tilde\epsilon^6-856176896q^7u^2\tilde\epsilon^6+2370826q^8u^2\tilde\epsilon^6+$
$181460208q^9u^2\tilde\epsilon^6+54946010q^{10}u^2\tilde\epsilon^6+1745136q^{11}u^2\tilde\epsilon^6-64728q^{12}u^2\tilde\epsilon^6-492565u\tilde\epsilon^8-22512472qu\tilde\epsilon^8-360209783q^2u\tilde\epsilon^8-$
$3520510256q^3u\tilde\epsilon^8-3823274802q^4u\tilde\epsilon^8+7616770240q^5u\tilde\epsilon^8+7183654450q^6u\tilde\epsilon^8-3887390064q^7u\tilde\epsilon^8-2993196809q^8u\tilde\epsilon^8-$
$186098536q^9u\tilde\epsilon^8-6480491q^{10}u\tilde\epsilon^8-258912q^{11}u\tilde\epsilon^8-339823\tilde\epsilon^{10}-29314242q\tilde\epsilon^{10}-596120771q^2\tilde\epsilon^{10}-3320927384q^3\tilde\epsilon^{10}-$
$583386318q^4\tilde\epsilon^{10}+9060177076q^5\tilde\epsilon^{10}-583386318q^6\tilde\epsilon^{10}-3320927384q^7\tilde\epsilon^{10}-596120771q^8\tilde\epsilon^{10}-29314242q^9\tilde\epsilon^{10}-$
$339823q^{10}\tilde\epsilon^{10}$

$c_{11}=6880u^5\tilde\epsilon+43512qu^5\tilde\epsilon-87016q^2u^5\tilde\epsilon-259920q^3u^5\tilde\epsilon+665392q^4u^5\tilde\epsilon-124144q^5u^5\tilde\epsilon-979488q^6u^5\tilde\epsilon+$
$1179904q^7u^5\tilde\epsilon-214816q^8u^5\tilde\epsilon-654312q^9u^5\tilde\epsilon+519704q^{10}u^5\tilde\epsilon-22064q^{11}u^5\tilde\epsilon-93808q^{12}u^5\tilde\epsilon+16736q^{13}u^5\tilde\epsilon+$
$3440q^{14}u^5\tilde\epsilon+94000u^4\tilde\epsilon^3-398160qu^4\tilde\epsilon^3+2391472q^2u^4\tilde\epsilon^3+6769808q^3u^4\tilde\epsilon^3-14925760q^4u^4\tilde\epsilon^3-21450816q^5u^4\tilde\epsilon^3+$
$32532416q^6u^4\tilde\epsilon^3+26092576q^7u^4\tilde\epsilon^3-30064976q^8u^4\tilde\epsilon^3-15830480q^9u^4\tilde\epsilon^3+12204976q^{10}u^4\tilde\epsilon^3+5204304q^{11}u^4\tilde\epsilon^3-$
$2200608q^{12}u^4\tilde\epsilon^3-387232q^{13}u^4\tilde\epsilon^3-31520q^{14}u^4\tilde\epsilon^3+503888u^3\tilde\epsilon^5-6155632qu^3\tilde\epsilon^5+16279584q^2u^3\tilde\epsilon^5+247497088q^3u^3\tilde\epsilon^5+$
$239168656q^4u^3\tilde\epsilon^5-1082651072q^5u^3\tilde\epsilon^5-906934448q^6u^3\tilde\epsilon^5+2334744544q^7u^3\tilde\epsilon^5+2252240q^8u^3\tilde\epsilon^5-1018685200q^9u^3\tilde\epsilon^5+$
$45344q^{10}u^3\tilde\epsilon^5+152266400q^{11}u^3\tilde\epsilon^5+20868432q^{12}u^3\tilde\epsilon^5+777792q^{13}u^3\tilde\epsilon^5+22384q^{14}u^3\tilde\epsilon^5+2025696u^2\tilde\epsilon^7+$
$41614400qu^2\tilde\epsilon^7+825804672q^2u^2\tilde\epsilon^7+8183829792q^3u^2\tilde\epsilon^7+3733549440q^4u^2\tilde\epsilon^7-23517612864q^5u^2\tilde\epsilon^7-13488461056q^6u^2\tilde\epsilon^7+$
$22073888064q^7u^2\tilde\epsilon^7+12170374112q^8u^2\tilde\epsilon^7-6245951808q^9u^2\tilde\epsilon^7-3242848640q^{10}u^2\tilde\epsilon^7-536036192q^{11}u^2\tilde\epsilon^7-$
$444224q^{12}u^2\tilde\epsilon^7+268608q^{13}u^2\tilde\epsilon^7+3127344u\tilde\epsilon^9+214179160qu\tilde\epsilon^9+3427104616q^2u\tilde\epsilon^9+42785299568q^3u\tilde\epsilon^9+$
$121506039168q^4u\tilde\epsilon^9-49056388784q^5u\tilde\epsilon^9-232327603536q^6u\tilde\epsilon^9-31985656448q^7u\tilde\epsilon^9+113291176592q^8u\tilde\epsilon^9+$
$31031467672q^9u\tilde\epsilon^9+1053878440q^{10}u\tilde\epsilon^9+56301776q^{11}u\tilde\epsilon^9+1074432q^{12}u\tilde\epsilon^9+1410192\tilde\epsilon^{11}+161669168q\tilde\epsilon^{11}+$
$5376925424q^2\tilde\epsilon^{11}+46049526480q^3\tilde\epsilon^{11}+83574372256q^4\tilde\epsilon^{11}-135163903520q^5\tilde\epsilon^{11}-135163903520q^6\tilde\epsilon^{11}+83574372256q^7\tilde\epsilon^{11}+$

$46049526480q^8\tilde{\epsilon}^{11} + 5376925424q^9\tilde{\epsilon}^{11} + 161669168q^{10}\tilde{\epsilon}^{11} + 1410192q^{11}\tilde{\epsilon}^{11}$

$c_{12} = -2095u^6 - 9078qu^6 + 48528q^2u^6 - 23474q^3u^6 - 129063q^4u^6 + 185244q^5u^6 + 30572q^6u^6 - 259812q^7u^6 +$
$194445q^8u^6 + 50998q^9u^6 - 145038q^{10}u^6 + 47790q^{11}u^6 + 24215q^{12}u^6 - 10680q^{13}u^6 - 4200q^{14}u^6 + 1384q^{15}u^6 +$
$306q^{16}u^6 - 36q^{17}u^6 - 6q^{18}u^6 - 122562u^5\tilde{\epsilon}^2 - 696484qu^5\tilde{\epsilon}^2 - 1585664q^2u^5\tilde{\epsilon}^2 + 8649980q^3u^5\tilde{\epsilon}^2 + 1092198q^4u^5\tilde{\epsilon}^2 -$
$25276456q^5u^5\tilde{\epsilon}^2 + 23798528q^6u^5\tilde{\epsilon}^2 + 3069072q^7u^5\tilde{\epsilon}^2 - 27264438q^8u^5\tilde{\epsilon}^2 + 29064076q^9u^5\tilde{\epsilon}^2 - 3105504q^{10}u^5\tilde{\epsilon}^2 -$
$15515444q^{11}u^5\tilde{\epsilon}^2 + 7903906q^{12}u^5\tilde{\epsilon}^2 + 743248q^{13}u^5\tilde{\epsilon}^2 - 722464q^{14}u^5\tilde{\epsilon}^2 - 37848q^{15}u^5\tilde{\epsilon}^2 + 6000q^{16}u^5\tilde{\epsilon}^2 - 144q^{17}u^5\tilde{\epsilon}^2 -$
$1011815u^4\tilde{\epsilon}^4 + 7911736qu^4\tilde{\epsilon}^4 - 12963746q^2u^4\tilde{\epsilon}^4 - 268314856q^3u^4\tilde{\epsilon}^4 - 17564777q^4u^4\tilde{\epsilon}^4 + 1314213496q^5u^4\tilde{\epsilon}^4 -$
$149488788q^6u^4\tilde{\epsilon}^4 - 2233056456q^7u^4\tilde{\epsilon}^4 + 411414663q^8u^4\tilde{\epsilon}^4 + 1737762696q^9u^4\tilde{\epsilon}^4 - 402803314q^{10}u^4\tilde{\epsilon}^4 - 522708952q^{11}u^4\tilde{\epsilon}^4 +$
$82292393q^{12}u^4\tilde{\epsilon}^4 + 51120584q^{13}u^4\tilde{\epsilon}^4 + 2901992q^{14}u^4\tilde{\epsilon}^4 + 296584q^{15}u^4\tilde{\epsilon}^4 - 1440q^{16}u^4\tilde{\epsilon}^4 - 5097748u^3\tilde{\epsilon}^6 -$
$9261520qu^3\tilde{\epsilon}^6 - 356062544q^2u^3\tilde{\epsilon}^6 - 6490402408q^3u^3\tilde{\epsilon}^6 - 6107918756q^4u^3\tilde{\epsilon}^6 + 17941186688q^5u^3\tilde{\epsilon}^6 + 41531871264q^6u^3\tilde{\epsilon}^6 -$
$43926625784q^7u^3\tilde{\epsilon}^6 - 44271863580q^8u^3\tilde{\epsilon}^6 + 38422840048q^9u^3\tilde{\epsilon}^6 + 11624715600q^{10}u^3\tilde{\epsilon}^6 - 5763518424q^{11}u^3\tilde{\epsilon}^6 -$
$2409191980q^{12}u^3\tilde{\epsilon}^6 - 174148768q^{13}u^3\tilde{\epsilon}^6 - 6452256q^{14}u^3\tilde{\epsilon}^6 - 69832q^{15}u^3\tilde{\epsilon}^6 - 15584031u^2\tilde{\epsilon}^8 - 634599106qu^2\tilde{\epsilon}^8 -$
$8881939426q^2u^2\tilde{\epsilon}^8 - 133082515706q^3u^2\tilde{\epsilon}^8 - 293238795809q^4u^2\tilde{\epsilon}^8 + 402189035004q^5u^2\tilde{\epsilon}^8 + 752791517668q^6u^2\tilde{\epsilon}^8 -$
$293875984420q^7u^2\tilde{\epsilon}^8 - 679780164641q^8u^2\tilde{\epsilon}^8 + 42160617574q^9u^2\tilde{\epsilon}^8 + 161286154718q^{10}u^2\tilde{\epsilon}^8 + 46946160414q^{11}u^2\tilde{\epsilon}^8 +$
$4203236577q^{12}u^2\tilde{\epsilon}^8 - 66369952q^{13}u^2\tilde{\epsilon}^8 - 768864q^{14}u^2\tilde{\epsilon}^8 - 17343354u\tilde{\epsilon}^{10} - 1739159756qu\tilde{\epsilon}^{10} - 35956817784q^2u\tilde{\epsilon}^{10} -$
$484521910596q^3u\tilde{\epsilon}^{10} - 2444019540258q^4u\tilde{\epsilon}^{10} - 1579662303032q^5u\tilde{\epsilon}^{10} + 4589163693824q^6u\tilde{\epsilon}^{10} + 4120456235704q^7u\tilde{\epsilon}^{10} -$
$1834619270494q^8u\tilde{\epsilon}^{10} - 2047448543292q^9u\tilde{\epsilon}^{10} - 274105094024q^{10}u\tilde{\epsilon}^{10} - 7081298868q^{11}u\tilde{\epsilon}^{10} - 445627910q^{12}u\tilde{\epsilon}^{10} -$
$3020160q^{13}u\tilde{\epsilon}^{10} - 3940145\tilde{\epsilon}^{12} - 839692852q\tilde{\epsilon}^{12} - 44941372642q^2\tilde{\epsilon}^{12} - 564907036260q^3\tilde{\epsilon}^{12} - 2096766693055q^4\tilde{\epsilon}^{12} +$
$232862848152q^5\tilde{\epsilon}^{12} + 4949191773604q^6\tilde{\epsilon}^{12} + 232862848152q^7\tilde{\epsilon}^{12} - 2096766693055q^8\tilde{\epsilon}^{12} - 564907036260q^9\tilde{\epsilon}^{12} -$
$44941372642q^{10}\tilde{\epsilon}^{12} - 839692852q^{11}\tilde{\epsilon}^{12} - 3940145q^{12}\tilde{\epsilon}^{12}$

$c_{13} = 80160u^6\tilde{\epsilon} + 543664qu^6\tilde{\epsilon} - 1008288q^2u^6\tilde{\epsilon} - 4405328q^3u^6\tilde{\epsilon} + 10358032q^4u^6\tilde{\epsilon} + 1182640q^5u^6\tilde{\epsilon} - 21282752q^6u^6\tilde{\epsilon} +$
$18173808q^7u^6\tilde{\epsilon} + 7904768q^8u^6\tilde{\epsilon} - 23259648q^9u^6\tilde{\epsilon} + 11467280q^{10}u^6\tilde{\epsilon} + 5888480q^{11}u^6\tilde{\epsilon} - 7605040q^{12}u^6\tilde{\epsilon} +$
$1719728q^{13}u^6\tilde{\epsilon} + 159520q^{14}u^6\tilde{\epsilon} + 171440q^{15}u^6\tilde{\epsilon} - 76032q^{16}u^6\tilde{\epsilon} - 15120q^{17}u^6\tilde{\epsilon} + 2352q^{18}u^6\tilde{\epsilon} + 336q^{19}u^6\tilde{\epsilon} +$
$1679120u^5\tilde{\epsilon}^3 + 6063128qu^5\tilde{\epsilon}^3 + 59581432q^2u^5\tilde{\epsilon}^3 + 15539752q^3u^5\tilde{\epsilon}^3 - 453033016q^4u^5\tilde{\epsilon}^3 + 129458032q^5u^5\tilde{\epsilon}^3 +$
$828875920q^6u^5\tilde{\epsilon}^3 - 577250480q^7u^5\tilde{\epsilon}^3 + 158350416q^8u^5\tilde{\epsilon}^3 - 362734312q^9u^5\tilde{\epsilon}^3 - 205142088q^{10}u^5\tilde{\epsilon}^3 + 556489256q^{11}u^5\tilde{\epsilon}^3 +$
$21849864q^{12}u^5\tilde{\epsilon}^3 - 222090912q^{13}u^5\tilde{\epsilon}^3 + 20779328q^{14}u^5\tilde{\epsilon}^3 + 22629216q^{15}u^5\tilde{\epsilon}^3 - 724784q^{16}u^5\tilde{\epsilon}^3 - 327936q^{17}u^5\tilde{\epsilon}^3 +$
$8064q^{18}u^5\tilde{\epsilon}^3 + 10540336u^4\tilde{\epsilon}^5 - 48758512qu^4\tilde{\epsilon}^5 - 146430464q^2u^4\tilde{\epsilon}^5 + 5927211920q^3u^4\tilde{\epsilon}^5 + 8171570528q^4u^4\tilde{\epsilon}^5 -$
$27324761648q^5u^4\tilde{\epsilon}^5 - 41111180864q^6u^4\tilde{\epsilon}^5 + 71526240336q^7u^4\tilde{\epsilon}^5 + 49660106624q^8u^4\tilde{\epsilon}^5 - 75283786576q^9u^4\tilde{\epsilon}^5 -$
$23253844736q^{10}u^4\tilde{\epsilon}^5 + 30521782576q^{11}u^4\tilde{\epsilon}^5 + 7395140768q^{12}u^4\tilde{\epsilon}^5 - 5376568464q^{13}u^4\tilde{\epsilon}^5 - 720673472q^{14}u^4\tilde{\epsilon}^5 +$
$58559728q^{15}u^4\tilde{\epsilon}^5 - 5228720q^{16}u^4\tilde{\epsilon}^5 + 80640q^{17}u^4\tilde{\epsilon}^5 + 47733936u^3\tilde{\epsilon}^7 + 926836880qu^3\tilde{\epsilon}^7 + 6859073968q^2u^3\tilde{\epsilon}^7 +$
$142796345744q^3u^3\tilde{\epsilon}^7 + 293758004144q^4u^3\tilde{\epsilon}^7 - 356712778208q^5u^3\tilde{\epsilon}^7 - 1296174868064q^6u^3\tilde{\epsilon}^7 + 240599275744q^7u^3\tilde{\epsilon}^7 +$
$2209854110656q^8u^3\tilde{\epsilon}^7 - 467844491056q^9u^3\tilde{\epsilon}^7 - 1084210989072q^{10}u^3\tilde{\epsilon}^7 + 114958749136q^{11}u^3\tilde{\epsilon}^7 + 170707905904q^{12}u^3\tilde{\epsilon}^7 +$
$22943797888q^{13}u^3\tilde{\epsilon}^7 + 1450749632q^{14}u^3\tilde{\epsilon}^7 + 40107456q^{15}u^3\tilde{\epsilon}^7 + 435312q^{16}u^3\tilde{\epsilon}^7 + 107141376u^2\tilde{\epsilon}^9 + 7320589328qu^2\tilde{\epsilon}^9 +$
$106823752176q^2u^2\tilde{\epsilon}^9 + 1804988806192q^3u^2\tilde{\epsilon}^9 + 8594883308624q^4u^2\tilde{\epsilon}^9 - 342312893024q^5u^2\tilde{\epsilon}^9 - 24749640963296q^6u^2\tilde{\epsilon}^9 -$
$7542804263776q^7u^2\tilde{\epsilon}^9 + 22582860496288q^8u^2\tilde{\epsilon}^9 + 8946197122128q^9u^2\tilde{\epsilon}^9 - 5946722572496q^{10}u^2\tilde{\epsilon}^9 - 2847205125392q^{11}u^2\tilde{\epsilon}^9 -$
$589072505840q^{12}u^2\tilde{\epsilon}^9 - 26185588480q^{13}u^2\tilde{\epsilon}^9 + 761343168q^{14}u^2\tilde{\epsilon}^9 + 1353024q^{15}u^2\tilde{\epsilon}^9 + 82129536u\tilde{\epsilon}^{11} + 12666136632qu\tilde{\epsilon}^{11} +$
$383227385048q^2u\tilde{\epsilon}^{11} + 5518178943944q^3u\tilde{\epsilon}^{11} + 40539198198504q^4u\tilde{\epsilon}^{11} + 73603296578992q^5u\tilde{\epsilon}^{11} - 46053336387728q^6u\tilde{\epsilon}^{11} -$
$148264380364912q^7u\tilde{\epsilon}^{11} - 27691173531312q^8u\tilde{\epsilon}^{11} + 70214551368216q^9u\tilde{\epsilon}^{11} + 29421863741752q^{10}u\tilde{\epsilon}^{11} + 2247502639656q^{11}u\tilde{\epsilon}^{11} +$
$65033742024q^{12}u\tilde{\epsilon}^{11} + 3287104128q^{13}u\tilde{\epsilon}^{11} + 2315520q^{14}u\tilde{\epsilon}^{11} + 1705536\tilde{\epsilon}^{13} + 4425054528q\tilde{\epsilon}^{13} + 355470697856q^2\tilde{\epsilon}^{13} +$
$6457627511168q^3\tilde{\epsilon}^{13} + 38190119195840q^4\tilde{\epsilon}^{13} + 45140353971136q^5\tilde{\epsilon}^{13} - 90147998136064q^6\tilde{\epsilon}^{13} - 90147998136064q^7\tilde{\epsilon}^{13} +$
$45140353971136q^8\tilde{\epsilon}^{13} + 38190119195840q^9\tilde{\epsilon}^{13} + 6457627511168q^{10}\tilde{\epsilon}^{13} + 355470697856q^{11}\tilde{\epsilon}^{13} + 4425054528q^{12}\tilde{\epsilon}^{13} +$
$1705536q^{13}\tilde{\epsilon}^{13}$

E.2 Hurwitz expansion for PIV

In this subsection $c_n = c_n^{PIV}$ from (8.44).
$(\alpha = g_2^{PIV}/2\ ,\ \beta = 2g_3^{PIV})$

$c_0 = 1$

$c_1 = 0$

$c_2 = -3T$

$c_3 = 4\epsilon\Lambda$

$c_4 = 15T^2 - \alpha + 8\epsilon^2$

$c_5 = 24u\epsilon - 36T\epsilon\Lambda$

$c_6 = -105T^3 + 21T\alpha - 3\beta - 168T\epsilon^2 + 106\epsilon^2\Lambda^2$

$c_7 = -672Tu\epsilon + 168T^2\epsilon\Lambda + 8\alpha\epsilon\Lambda + 176\epsilon^3\Lambda$

$c_8 = 945T^4 - 378T^2\alpha - 9\alpha^2 + 108T\beta + 3024T^2\epsilon^2 + 48\alpha\epsilon^2 + 192\epsilon^4 + 816u\epsilon^2\Lambda - 3000T\epsilon^2\Lambda^2$

$c_9 = 15120T^2u\epsilon + 336u\alpha\epsilon + 672u\epsilon^3 + 2520T^3\epsilon\Lambda - 24T\alpha\epsilon\Lambda - 216\beta\epsilon\Lambda - 7248T\epsilon^3\Lambda + 5392\epsilon^3\Lambda^3$

$c_{10} = -10395T^5 + 6930T^3\alpha + 495T\alpha^2 - 2970T^2\beta - 18\alpha\beta - 55440T^3\epsilon^2 - 4704u^2\epsilon^2 - 2640T\alpha\epsilon^2 - 432\beta\epsilon^2 - 10560T\epsilon^4 - 54288Tu\epsilon^2\Lambda + 55356T^2\epsilon^2\Lambda^2 + 3020\alpha\epsilon^2\Lambda^2 + 10592\epsilon^4\Lambda^2$

$c_{11} = -332640T^3u\epsilon - 22176Tu\alpha\epsilon + 864u\beta\epsilon - 44352Tu\epsilon^3 - 124740T^4\epsilon\Lambda - 10296T^2\alpha\epsilon\Lambda - 1404\alpha^2\epsilon\Lambda + 15120T\beta\epsilon\Lambda + 217008T^2\epsilon^3\Lambda + 20400\alpha\epsilon^3\Lambda - 12864\epsilon^5\Lambda + 8832u\epsilon^3\Lambda^2 - 347040T\epsilon^3\Lambda^3$

$c_{12} = 135135T^6 - 135135T^4\alpha - 19305T^2\alpha^2 + 69\alpha^3 + 77220T^3\beta + 1404T\alpha\beta - 54\beta^2 + 1081080T^4\epsilon^2 + 366912Tu^2\epsilon^2 + 102960T^2\alpha\epsilon^2 - 5592\alpha^2\epsilon^2 + 33696T\beta\epsilon^2 + 411840T^2\epsilon^4 + 45504\alpha\epsilon^4 - 41472\epsilon^6 + 2484144T^2u\epsilon^2\Lambda + 67632u\alpha\epsilon^2\Lambda - 119616u\epsilon^4\Lambda - 611208T^3\epsilon^2\Lambda^2 - 167928T\alpha\epsilon^2\Lambda^2 - 16776\beta\epsilon^2\Lambda^2 - 945792T\epsilon^4\Lambda^2 + 409288\epsilon^4\Lambda^4$

$c_{13} = 7567560T^4u\epsilon + 1009008T^2u\alpha\epsilon - 6600u\alpha^2\epsilon - 78624Tu\beta\epsilon + 2018016T^2u\epsilon^3 + 328416u\alpha\epsilon^3 - 874368u\epsilon^5 + 3783780T^5\epsilon\Lambda + 648648T^3\alpha\epsilon\Lambda + 121164T\alpha^2\epsilon\Lambda - 727272T^2\beta\epsilon\Lambda - 4392\alpha\beta\epsilon\Lambda - 5909904T^3\epsilon^3\Lambda - 1337856u^2\epsilon^3\Lambda - 1527984T\alpha\epsilon^3\Lambda - 87984\beta\epsilon^3\Lambda + 296256T\epsilon^5\Lambda - 3479424Tu\epsilon^3\Lambda^2 + 14050608T^2\epsilon^3\Lambda^3 + 451760\alpha\epsilon^3\Lambda^3 + 1461920\epsilon^5\Lambda^3$

$c_{14} = -2027025T^7 + 2837835T^5\alpha + 675675T^3\alpha^2 - 7245T\alpha^3 - 2027025T^4\beta - 73710T^2\alpha\beta + 513\alpha^2\beta + 5670T\beta^2 - 22702680T^5\epsilon^2 - 19262880T^2u^2\epsilon^2 - 3603600T^3\alpha\epsilon^2 + 300384u^2\alpha\epsilon^2 + 587160T\alpha^2\epsilon^2 - 1769040T^2\beta\epsilon^2 - 32112\alpha\beta\epsilon^2 - 14414400T^3\epsilon^4 - 6878592u^2\epsilon^4 - 4777920T\alpha\epsilon^4 - 52416\beta\epsilon^4 + 4354560T\epsilon^6 - 99786960T^3u\epsilon^2\Lambda - 6500592Tu\alpha\epsilon^2\Lambda + 325008u\beta\epsilon^2\Lambda - 1197504Tu\epsilon^4\Lambda - 8902530T^4\epsilon^2\Lambda^2 + 5565924T^2\alpha\epsilon^2\Lambda^2 - 213918\alpha^2\epsilon^2\Lambda^2 + 2086488T\beta\epsilon^2\Lambda^2 + 49055328T^2\epsilon^4\Lambda^2 + 5165088\alpha\epsilon^4\Lambda^2 - 5384832\epsilon^6\Lambda^2 - 6088416u\epsilon^4\Lambda^3 - 49063656T\epsilon^4\Lambda^4$

$c_{15} = -181621440T^5u\epsilon - 40360320T^3u\alpha\epsilon + 792000Tu\alpha^2\epsilon + 4717440T^2u\beta\epsilon - 74880u\alpha\beta\epsilon - 80720640T^3u\epsilon^3 - 5031936u^3\epsilon^3 - 39409920Tu\alpha\epsilon^3 + 2006784u\beta\epsilon^3 + 104924160Tu\epsilon^5 - 105945840T^6\epsilon\Lambda - 29549520T^4\alpha\epsilon\Lambda - 68738840T^2\alpha^2\epsilon\Lambda + 16752\alpha^3\epsilon\Lambda + 30663360T^3\beta\epsilon\Lambda + 452160T\alpha\beta\epsilon\Lambda - 26784\beta^2\epsilon\Lambda + 157116960T^4\epsilon^3\Lambda + 145446912Tu^2\epsilon^3\Lambda + 71974080T^2\alpha\epsilon^3\Lambda - 1699104\alpha^2\epsilon^3\Lambda + 125648864T\beta\epsilon^3\Lambda + 34686720T^2\epsilon^5\Lambda + 20543232\alpha\epsilon^5\Lambda - 29585664\epsilon^7\Lambda + 354212352T^2u\epsilon^3\Lambda^2 + 12129792u\alpha\epsilon^3\Lambda^2 - 84409344u\epsilon^5\Lambda^2 - 443953536T^3\epsilon^3\Lambda^3 - 42081408T\alpha\epsilon^3\Lambda^3 - 1292160\beta\epsilon^3\Lambda^3 - 259839744T\epsilon^5\Lambda^3 + 41785216\epsilon^5\Lambda^5$

$c_{16} = 34459425T^8 - 64324260T^6\alpha - 22972950T^4\alpha^2 + 492660T^2\alpha^3 + 321\alpha^4 + 55135080T^5\beta + 3341520T^3\alpha\beta - 69768T\alpha^2\beta - 385560T^2\beta^2 + 4968\alpha\beta^2 + 514594080T^6\epsilon^2 + 873250560T^3u^2\epsilon^2 + 122522400T^4\alpha\epsilon^2 - 40852224Tu^2\alpha\epsilon^2 - 39926880T^2\alpha^2\epsilon^2 + 22560\alpha^3\epsilon^2 + 80196480T^3\beta\epsilon^2 + 2306304u^2\beta\epsilon^2 + 4367232T\alpha\beta\epsilon^2 - 188352\beta^2\epsilon^2 + 490089600T^4\epsilon^4 + 935488512Tu^2\epsilon^4 + 324898560T^2\alpha\epsilon^4 - 3758208\alpha^2\epsilon^4 + 7128576T\beta\epsilon^4 - 296110080T^2\epsilon^6 + 34498560\alpha\epsilon^6 - 48328704\epsilon^8 + 3829381920T^4u\epsilon^2\Lambda + 4011880032T^2u\alpha\epsilon^2\Lambda - 1199136u\alpha^2\epsilon^2\Lambda - 39588480Tu\beta\epsilon^2\Lambda + 1016918784T^2u\epsilon^4\Lambda + 108258048u\alpha\epsilon^4\Lambda - 422165760u\epsilon^6\Lambda + 1008025200T^5\epsilon^2\Lambda^2 - 1185925440T^3\alpha\epsilon^2\Lambda^2 + 27893712T\alpha^2\epsilon^2\Lambda^2 - 161675424T^2\beta\epsilon^2\Lambda^2 - 661920\alpha\beta\epsilon^2\Lambda^2 - 1884868608T^3\epsilon^4\Lambda^2 - 342421248u^2\epsilon^4\Lambda^2 - 594193920T\alpha\epsilon^4\Lambda^2 - 13396992\beta\epsilon^4\Lambda^2 + 310171392T\epsilon^6\Lambda^2 + 143182080Tu\epsilon^4\Lambda^3 + 3407919648T^2\epsilon^4\Lambda^4 + 69198368\alpha\epsilon^4\Lambda^4 + 281868032\epsilon^6\Lambda^4$

$c_{17} = 4631346720T^6u\epsilon+1543782240T^4u\alpha\epsilon-60588000T^2u\alpha^2\epsilon-209952u\alpha^3\epsilon-240589440T^3u\beta\epsilon+11456640Tu\alpha\beta\epsilon-357696u\beta^2\epsilon+3087564480T^4u\epsilon^3+769886208Tu^3\epsilon^3+3014858880T^2u\alpha\epsilon^3-245952u\alpha^2\epsilon^3-307037952Tu\beta\epsilon^3-8026698240T^2u\epsilon^5+274853376u\alpha\epsilon^5-1037753856u\epsilon^7+2977294320T^7\epsilon\Lambda+1212971760T^5\alpha\epsilon\Lambda+330369840T^3\alpha^2\epsilon\Lambda-2773008T\alpha^3\epsilon\Lambda-1233020880T^4\beta\epsilon\Lambda-28861920T^2\alpha\beta\epsilon\Lambda-14256\alpha^2\beta\epsilon\Lambda+3740256T\beta^2\epsilon\Lambda-4190266080T^5\epsilon^3\Lambda-9971859456T^2u^2\epsilon^3\Lambda-2665725120T^3\alpha\epsilon^3\Lambda+82096128u^2\alpha\epsilon^3\Lambda+259716960T\alpha^2\epsilon^3\Lambda-1114731072T^2\beta\epsilon^3\Lambda-7960896\alpha\beta\epsilon^3\Lambda-4444588800T^3\epsilon^5\Lambda-3243423744u^2\epsilon^5\Lambda-2868261120T\alpha\epsilon^5\Lambda-46994688\beta\epsilon^5\Lambda+3488852736T\epsilon^7\Lambda-24712736256T^3u\epsilon^3\Lambda^2-1691665920Tu\alpha\epsilon^3\Lambda^2+100500480u\beta\epsilon^3\Lambda^2+6427782144Tu\epsilon^5\Lambda^2+10803038688T^4\epsilon^3\Lambda^3+2373394752T^2\alpha\epsilon^3\Lambda^3-41310432\alpha^2\epsilon^3\Lambda^3+298200960T\beta\epsilon^3\Lambda^3+23091631488T^2\epsilon^5\Lambda^3+1259829120\alpha\epsilon^5\Lambda^3-862029312\epsilon^7\Lambda^3-1992082176u\epsilon^5\Lambda^4-8385220224T\epsilon^5\Lambda^5$

$c_{18} = -654729075T^9+1571349780T^7\alpha+785674890T^5\alpha^2-28081620T^3\alpha^3-54891T\alpha^4-1571349780T^6\beta-142849980T^4\alpha\beta+5965164T^2\alpha^2\beta+33588\alpha^3\beta+21976920T^3\beta^2-849528T\alpha\beta^2+14904\beta^3-12570798240T^7\epsilon^2-373311461440T^4u^2\epsilon^2-4190266080T^5\alpha\epsilon^2+3492865152T^2u^2\alpha\epsilon^2+2275832160T^3\alpha^2\epsilon^2+17798976u^2\alpha^2\epsilon^2-3857760T\alpha^3\epsilon^2-3428399520T^4\beta\epsilon^2-394377984Tu^2\beta\epsilon^2-373398336T^2\alpha\beta\epsilon^2-1124064\alpha^2\beta\epsilon^2+32208192T\beta^2\epsilon^2-16761064320T^5\epsilon^4-79984267776T^2u^2\epsilon^4-18519217920T^3\alpha\epsilon^4+176721408u^2\alpha\epsilon^4+642653568T\alpha^2\epsilon^4-609493248T^2\beta\epsilon^4-25733376\alpha\beta\epsilon^4+16878274560T^3\epsilon^6-8399144448u^2\epsilon^6-5899253760T\alpha\epsilon^6+12883968\beta\epsilon^6+8264208384T\epsilon^8-145897446240T^5u\epsilon^2\Lambda-20539141056T^3u\alpha\epsilon^2\Lambda+240650208Tu\alpha^2\epsilon^2\Lambda+2990437056T^2u\beta\epsilon^2\Lambda-22496832u\alpha\beta\epsilon^2\Lambda-111287215872T^3u\epsilon^4\Lambda-2279107584u^3\epsilon^4\Lambda-18158683392Tu\alpha\epsilon^4\Lambda+1114801920u\beta\epsilon^4\Lambda+55392056064Tu\epsilon^6\Lambda-530449559240T^6\epsilon^2\Lambda^2-64954008T^4\alpha\epsilon^2\Lambda^2-2264587272T^2\alpha^2\epsilon^2\Lambda^2+8415432\alpha^3\epsilon^2\Lambda^2+10212311520T^3\beta\epsilon^2\Lambda^2+90691488T\alpha\beta\epsilon^2\Lambda^2-9951984\beta^2\epsilon^2\Lambda^2+52756329024T^4\epsilon^4\Lambda^2+51716710656Tu^2\epsilon^4\Lambda^2+41724238464T^2\alpha\epsilon^4\Lambda^2-586131264\alpha^2\epsilon^4\Lambda^2+3405687552T\beta\epsilon^4\Lambda^2+1176374016T^2\epsilon^6\Lambda^2+8935803648\alpha\epsilon^6\Lambda^2-14061376512\epsilon^8\Lambda^2+39474642816T^2u\epsilon^4\Lambda^3+2405404032u\alpha\epsilon^4\Lambda^3-34279371264u\epsilon^6\Lambda^3-181093205664T^3\epsilon^4\Lambda^4-9427516896T\alpha\epsilon^4\Lambda^4-35708256\beta\epsilon^4\Lambda^4-82478804736T\epsilon^6\Lambda^4+5457048256\epsilon^6\Lambda^6$

$c_{19} = -125707982400T^7u\epsilon-58663725120T^5u\alpha\epsilon+3837240000T^3u\alpha^2\epsilon+39890880Tu\alpha^3\epsilon+11427998400T^4u\beta\epsilon-10883380800T^2u\alpha\beta\epsilon-5434560u\alpha^2\beta\epsilon+67962240Tu\beta^2\epsilon-117327450240T^5u\epsilon^3-73139189760T^2u^3\epsilon^3-190941062400T^3u\alpha\epsilon^3-554932224u^3\alpha\epsilon^3+46730880Tu\alpha^2\epsilon^3+29168605440T^2u\beta\epsilon^3-37718784u\alpha\beta\epsilon^3+508357555200T^3u\epsilon^5-6944329728u^3\epsilon^5-52222141440Tu\alpha\epsilon^5+3166497792u\beta\epsilon^5+197173232640Tu\epsilon^7-86424237900T^8\epsilon\Lambda-48188059920T^6\alpha\epsilon\Lambda-147332574000T^4\alpha^2\epsilon\Lambda+283381200T^2\alpha^3\epsilon\Lambda-1425420\alpha^4\epsilon\Lambda+49140393120T^5\beta\epsilon\Lambda+1465128000T^3\alpha\beta\epsilon\Lambda-2725920T\alpha^2\beta\epsilon\Lambda-321343200T^2\beta^2\epsilon\Lambda+2293920\alpha\beta^2\epsilon\Lambda+1131371841160T^6\epsilon^3\Lambda+5584119091200T^3u^2\epsilon^3\Lambda+78886677600T^4\alpha\epsilon^3\Lambda-17263060992Tu^2\alpha\epsilon^3\Lambda-24649745760T^2\alpha^2\epsilon^3\Lambda+96553440\alpha^3\epsilon^3\Lambda+80322503040T^3\beta\epsilon^3\Lambda+1313998848u^2\beta\epsilon^3\Lambda+1474851456T\alpha\beta\epsilon^3\Lambda-134830656\beta^2\epsilon^3\Lambda+3382073568000T^4\epsilon^5\Lambda+595417522176Tu^2\epsilon^5\Lambda+246373735680T^2\alpha\epsilon^5\Lambda-3282111360\alpha^2\epsilon^5\Lambda+12095488512T\beta\epsilon^5\Lambda-232854393600T^2\epsilon^7\Lambda+31279299840\alpha\epsilon^7\Lambda-48489661440\epsilon^9\Lambda+1453060926720T^4u\epsilon^3\Lambda^2+143445201408T^2u\alpha\epsilon^3\Lambda^2-382435584u\alpha^2\epsilon^3\Lambda^2-16467093504Tu\beta\epsilon^3\Lambda^2-15221781504T^2u\epsilon^5\Lambda^2+34225959936u\alpha\epsilon^5\Lambda^2-228048786432u\epsilon^7\Lambda^2-119903284800T^5\epsilon^3\Lambda^3-102499933824T^3\alpha\epsilon^3\Lambda^3+7466546496T\alpha^2\epsilon^3\Lambda^3-36562637952T^2\beta\epsilon^3\Lambda^3-49680000\alpha\beta\epsilon^3\Lambda^3-1467543921408T^3\epsilon^5\Lambda^3-96641872896u^2\epsilon^5\Lambda^3-205141572864T\alpha\epsilon^5\Lambda^3-973419264\beta\epsilon^5\Lambda^3-64263217152T\epsilon^7\Lambda^3+185211867648Tu\epsilon^5\Lambda^4+889201855104T^2\epsilon^5\Lambda^5+11814500480\alpha\epsilon^5\Lambda^5+63492215552\epsilon^7\Lambda^5$

$c_{20} = 13749310575T^{10}-41247931725T^8\alpha-27498621150T^6\alpha^2+1474285050T^4\alpha^3+5763555T^2\alpha^4+160839\alpha^5+47140493400T^7\beta+5999699160T^5\alpha\beta-417561480T^3\alpha^2\beta-7053480T\alpha^3\beta-1153788300T^4\beta^2+89200440T^2\alpha\beta^2+257580\alpha^2\beta^2-3129840T\beta^3+329983453800T^8\epsilon^2+15679213804800T^5u^2\epsilon^2+146659312800T^6\alpha\epsilon^2-244500560640T^3u^2\alpha\epsilon^2-119481188400T^4\alpha^2\epsilon^2-3737784960Tu^2\alpha^2\epsilon^2+405064800T^2\alpha^3\epsilon^2-14713560\alpha^4\epsilon^2+143992779840T^5\beta\epsilon^2+414096883320T^2u^2\beta\epsilon^2+26137883520T^3\alpha\beta\epsilon^2+240848640u^2\alpha\beta\epsilon^2+236053440T\alpha^2\beta\epsilon^2-3381860160T^2\beta^2\epsilon^2+7801920\alpha\beta^2\epsilon^2+586637251200T^6\epsilon^4+5598898744320T^3u^2\epsilon^4+7100061696u^4\epsilon^4+972258940800T^4\alpha\epsilon^4-37111495680Tu^2\alpha\epsilon^4-67478624640T^2\alpha^2\epsilon^4+306453888\alpha^3\epsilon^4+42664527360T^3\beta\epsilon^4+4434269184u^2\beta\epsilon^4+5404008960T\alpha\beta\epsilon^4-419302656\beta^2\epsilon^4-$

$8861094144400T^4\epsilon^6 + 1763820334080Tu^2\epsilon^6 + 619421644800T^2\alpha\epsilon^6 - 6472535040\alpha^2\epsilon^6 - 2705633280T\beta\epsilon^6 - 8677418803 20T^2\epsilon^8 + 47779098624\alpha\epsilon^8 - 69898567680\epsilon^{10} + 5629051078560T^6u\epsilon^2\Lambda + 956054625120T^4u\alpha\epsilon^2\Lambda - 29006056800T^2u\alpha^2\epsilon^2\Lambda + 90390240u\alpha^3\epsilon^2\Lambda - 1817241135040T^3u\beta\epsilon^2\Lambda + 5206032000Tu\alpha\beta\epsilon^2\Lambda - 269075520u\beta^2\epsilon^2\Lambda + 8642028205440T^4u\epsilon^4\Lambda + 507012839424Tu^3\epsilon^4\Lambda + 1869550260480T^2u\alpha\epsilon^4\Lambda - 5092460928u\alpha^2\epsilon^4\Lambda - 225239864832Tu\beta\epsilon^4\Lambda - 4052345552640T^2u\epsilon^6\Lambda + 208936915200u\alpha\epsilon^6\Lambda - 887114769408u\epsilon^8\Lambda + 2395498931280T^7\epsilon^2\Lambda^2 + 193938993360T^5\alpha\epsilon^2\Lambda^2 + 148852423440T^3\alpha^2\epsilon^2\Lambda^2 - 1676850480T\alpha^3\epsilon^2\Lambda^2 - 581577388560T^4\beta\epsilon^2\Lambda^2 - 6919590240T^2\alpha\beta\epsilon^2\Lambda^2 - 61207920\alpha^2\beta\epsilon^2\Lambda^2 + 1820841120T\beta^2\epsilon^2\Lambda^2 - 487360177920T^5\epsilon^4\Lambda^2 - 4669735359744T^2u^2\epsilon^4\Lambda^2 - 2297513272320T^3\alpha\epsilon^4\Lambda^2 + 20166181632u^2\alpha\epsilon^4\Lambda^2 + 117995104512T\alpha^2\epsilon^4\Lambda^2 - 470217125376T^2\beta\epsilon^4\Lambda^2 - 116370432\alpha\beta\epsilon^4\Lambda^2 - 1433128032000T^3\epsilon^6\Lambda^2 - 1374321567744u^2\epsilon^6\Lambda^2 - 1667581850880T\alpha\epsilon^6\Lambda^2 - 18139292928\beta\epsilon^6\Lambda^2 + 2065774298112T\epsilon^8\Lambda^2 - 5876381903616T^3u\epsilon^4\Lambda^3 - 464802483456Tu\alpha\epsilon^4\Lambda^3 + 33048255744u\beta\epsilon^4\Lambda^3 + 4450024829952Tu\epsilon^6\Lambda^3 + 8038297821456T^4\epsilon^4\Lambda^4 + 757488032352T^2\alpha\epsilon^4\Lambda^4 - 10803767952\alpha^2\epsilon^4\Lambda^4 + 40546989504T\beta\epsilon^4\Lambda^4 + 10885286912256T^2\epsilon^6\Lambda^4 + 324650481408\alpha\epsilon^6\Lambda^4 + 5879732736\epsilon^8\Lambda^4 - 594605991168u\epsilon^6\Lambda^5 - 1740586124928T\epsilon^6\Lambda^6$

E.3 Hurwitz expansion for PIII_2 and PIII_3

In this subsection, for PIII_2 we denote $c_n = c_n^{PIII_2}$ from (8.53) and for PIII_3 we denote $c_n = c_n^{PIII_3}$ from (8.79).
(PIII_2: $\alpha = g_2^{PIII_2}/2$, $\beta = 2g_3^{PIII_2}$, $\tilde\epsilon = \epsilon/2$; PIII_3: $\alpha = g_2^{PIII_3}/2$, $\beta = 2g_3^{PIII_3}$, $\tilde\epsilon = \epsilon/2$)[111]

$c_0 = 1$

$c_1 = 0$

$c_2 = -3T - \tilde\epsilon^2$

$c_3 = 0$

$c_4 = 15T^2 - \alpha + 6T\tilde\epsilon^2 + \tilde\epsilon^4$

$c_5 = 24T^2\tilde\epsilon - 4\alpha\tilde\epsilon$

$c_6 = -105T^3 + 21T\alpha - 3\beta + 27T^2\tilde\epsilon^2 - 9\alpha\tilde\epsilon^2 - 9T\tilde\epsilon^4 - \tilde\epsilon^6$

$c_7 = -672T^3\tilde\epsilon + 240T\alpha\tilde\epsilon - 48\beta\tilde\epsilon + 96T^2\tilde\epsilon^3 - 16\alpha\tilde\epsilon^3$

$c_8 = 945T^4 - 378T^2\alpha - 9\alpha^2 + 108T\beta - 4380T^3\tilde\epsilon^2 + 1932T\alpha\tilde\epsilon^2 - 444\beta\tilde\epsilon^2 + 186T^2\tilde\epsilon^4 - 22\alpha\tilde\epsilon^4 + 12T\tilde\epsilon^6 + \tilde\epsilon^8$

$c_9 = 15120T^4\tilde\epsilon - 7944T^2\alpha\tilde\epsilon - 184\alpha^2\tilde\epsilon + 2448T\beta\tilde\epsilon - 26784T^3\tilde\epsilon^3 + 12528T\alpha\tilde\epsilon^3 - 3024\beta\tilde\epsilon^3 + 48T^2\tilde\epsilon^5 - 8\alpha\tilde\epsilon^5$

$c_{10} = -10395T^5 + 6930T^3\alpha + 495T\alpha^2 - 2970T^2\beta - 18\alpha\beta + 185211T^4\tilde\epsilon^2 - 101766T^2\alpha\tilde\epsilon^2 - 2003\alpha^2\tilde\epsilon^2 + 31068T\beta\tilde\epsilon^2 - 136026T^3\tilde\epsilon^4 + 65778T\alpha\tilde\epsilon^4 - 16182\beta\tilde\epsilon^4 - 582T^2\tilde\epsilon^6 + 82\alpha\tilde\epsilon^6 - 15T\tilde\epsilon^8 - \tilde\epsilon^{10}$

$c_{11} = -332640T^5\tilde\epsilon + 223344T^3\alpha\tilde\epsilon + 13296T\alpha^2\tilde\epsilon - 89424T^2\beta\tilde\epsilon - 576\alpha\beta\tilde\epsilon + 1749600T^4\tilde\epsilon^3 - 955824T^2\alpha\tilde\epsilon^3 - 15376\alpha^2\tilde\epsilon^3 + 283680T\beta\tilde\epsilon^3 - 549600T^3\tilde\epsilon^5 + 270672T\alpha\tilde\epsilon^5 - 67152\beta\tilde\epsilon^5 - 2208T^2\tilde\epsilon^7 + 368\alpha\tilde\epsilon^7$

$c_{12} = 135135T^6 - 135135T^4\alpha - 19305T^2\alpha^2 + 69\alpha^3 + 77220T^3\beta + 1404T\alpha\beta - 54\beta^2 - 6286878T^5\tilde\epsilon^2 + 3968340T^3\alpha\tilde\epsilon^2 + 191382T\alpha^2\tilde\epsilon^2 - 1469196T^2\beta\tilde\epsilon^2 - 9420\alpha\beta\tilde\epsilon^2 + 12987711T^4\tilde\epsilon^4 - 7002870T^2\alpha\tilde\epsilon^4 - 91015\alpha^2\tilde\epsilon^4 + 2019204T\beta\tilde\epsilon^4 - 1507596T^3\tilde\epsilon^6 + 748668T\alpha\tilde\epsilon^6 - 186492\beta\tilde\epsilon^6 - 6207T^2\tilde\epsilon^8 + 1057\alpha\tilde\epsilon^8 + 18T\tilde\epsilon^{10} + \tilde\epsilon^{12}$

$c_{13} = 7567560T^6\tilde\epsilon - 6018012T^4\alpha\tilde\epsilon - 663984T^2\alpha^2\tilde\epsilon + 2764\alpha^3\tilde\epsilon + 2948400T^3\beta\tilde\epsilon + 56736T\alpha\beta\tilde\epsilon - 3024\beta^2\tilde\epsilon - 82356768T^5\tilde\epsilon^3 + 49342800T^3\alpha\tilde\epsilon^3 + 1938000T\alpha^2\tilde\epsilon^3 - 17086608T^2\beta\tilde\epsilon^3 - 105024\alpha\beta\tilde\epsilon^3 + 76266000T^4\tilde\epsilon^5 - 40546056T^2\alpha\tilde\epsilon^5 - 419192\alpha^2\tilde\epsilon^5 + 11381328T\beta\tilde\epsilon^5 - 137952T^3\tilde\epsilon^7 + 70224T\alpha\tilde\epsilon^7 - 17712\beta\tilde\epsilon^7 - 14328T^2\tilde\epsilon^9 + 2388\alpha\tilde\epsilon^9$

$c_{14} = -2027025T^7 + 2837835T^5\alpha + 675675T^3\alpha^2 - 7245T\alpha^3 - 2027025T^4\beta - 73710T^2\alpha\beta + 513\alpha^2\beta + 5670T\beta^2 + 199971135T^6\tilde\epsilon^2 - 138607371T^4\alpha\tilde\epsilon^2 - 12259593T^2\alpha^2\tilde\epsilon^2 + 62233\alpha^3\tilde\epsilon^2 + 60239196T^3\beta\tilde\epsilon^2 + 1237284T\alpha\beta\tilde\epsilon^2 -$

[111] The coefficients of the Hurwitz expansion for PIII_2 and PIII_3 in the basis of g_2, g_3, T, ϵ are the same. What changes is the parametrization of this basis in terms of the gauge theory parameters u, Λ, m for PIII_2 ($N_f = 1$) and in terms of u, Λ for PIII_3 ($N_f = 0$).

$89982\beta^2\tilde{\epsilon}^2-8155543415T^5\tilde{\epsilon}^4+4699612747T^3\alpha\tilde{\epsilon}^4+15178059T\alpha^2\tilde{\epsilon}^4-1540776427T^2\beta\tilde{\epsilon}^4-889554\alpha\beta\tilde{\epsilon}^4+337815549T^4\tilde{\epsilon}^6-176882202T^2\alpha\tilde{\epsilon}^6-1369029\alpha^2\tilde{\epsilon}^6+48297492T\beta\tilde{\epsilon}^6+30505701T^3\tilde{\epsilon}^8-15215169T\alpha\tilde{\epsilon}^8+3799023\beta\tilde{\epsilon}^8-25827T^2\tilde{\epsilon}^{10}+4273\alpha\tilde{\epsilon}^{10}-21T\tilde{\epsilon}^{12}-\tilde{\epsilon}^{14}$

$c_{15}=-181621440T^7\tilde{\epsilon}+162882720T^5\alpha\tilde{\epsilon}+29184000T^3\alpha^2\tilde{\epsilon}-375456T\alpha^3\tilde{\epsilon}-94741920T^4\beta\tilde{\epsilon}-3738240T^2\alpha\beta\tilde{\epsilon}+25824\alpha^2\beta\tilde{\epsilon}+369792T\beta^2\tilde{\epsilon}+3412528704T^6\tilde{\epsilon}^3-2170399968T^4\alpha\tilde{\epsilon}^3-160843392T^2\alpha^2\tilde{\epsilon}^3+1016416\alpha^3\tilde{\epsilon}^3+860360832T^3\beta\tilde{\epsilon}^3+19816704T\alpha\beta\tilde{\epsilon}^3-1904256\beta^2\tilde{\epsilon}^3-6350387328T^5\tilde{\epsilon}^5+3550870848T^3\alpha\tilde{\epsilon}^5+95124288T\alpha^2\tilde{\epsilon}^5-1112400576T^2\beta\tilde{\epsilon}^5-6051072\alpha\beta\tilde{\epsilon}^5+874094976T^4\tilde{\epsilon}^7-445619904T^2\alpha\tilde{\epsilon}^7-1503808\alpha^2\tilde{\epsilon}^7+115860096T\beta\tilde{\epsilon}^7+228803136T^3\tilde{\epsilon}^9-114240864T\alpha\tilde{\epsilon}^9+28540128\beta\tilde{\epsilon}^9-30912T^2\tilde{\epsilon}^{11}+5152\alpha\tilde{\epsilon}^{11}$

$c_{16}=34459425T^8-64324260T^6\alpha-22972950T^4\alpha^2+492660T^2\alpha^3+321\alpha^4+55135080T^5\beta+3341520T^3\alpha\beta-69768T\alpha^2\beta-385560T^2\beta^2+4968\alpha\beta^2-6300435960T^7\tilde{\epsilon}^2+4642373736T^5\alpha\tilde{\epsilon}^2+679251816T^3\alpha^2\tilde{\epsilon}^2-10488216T\alpha^3\tilde{\epsilon}^2-2334296232T^4\beta\tilde{\epsilon}^2-104974704T^2\alpha\beta\tilde{\epsilon}^2+696744\alpha^2\beta\tilde{\epsilon}^2+12826512T\beta^2\tilde{\epsilon}^2+42861778596T^6\tilde{\epsilon}^4-25659822276T^4\alpha\tilde{\epsilon}^4-1686752316T^2\alpha^2\tilde{\epsilon}^4+13223724\alpha^3\tilde{\epsilon}^4+9466218288T^3\beta\tilde{\epsilon}^4+268275024T\alpha\beta\tilde{\epsilon}^4-32226408\beta^2\tilde{\epsilon}^4-38745843480T^5\tilde{\epsilon}^6+21108908304T^3\alpha\tilde{\epsilon}^6+476468472T\alpha^2\tilde{\epsilon}^6-6356572848T^2\beta\tilde{\epsilon}^6-34950960\alpha\beta\tilde{\epsilon}^6-1865304090T^4\tilde{\epsilon}^8+1046601828T^2\alpha\tilde{\epsilon}^8+18910602\alpha^2\tilde{\epsilon}^8-318442680T\beta\tilde{\epsilon}^8+1019463288T^3\tilde{\epsilon}^{10}-509252952T\alpha\tilde{\epsilon}^{10}+127253496\beta\tilde{\epsilon}^{10}+4836T^2\tilde{\epsilon}^{12}-764\alpha\tilde{\epsilon}^{12}+24T\tilde{\epsilon}^{14}+\tilde{\epsilon}^{16}$

$c_{17}=4631346720T^8\tilde{\epsilon}-4521076560T^6\alpha\tilde{\epsilon}-1226778480T^4\alpha^2\tilde{\epsilon}+31861296T^2\alpha^3\tilde{\epsilon}-4176\alpha^4\tilde{\epsilon}+3067515360T^5\beta\tilde{\epsilon}+211507200T^3\alpha\beta\tilde{\epsilon}-4816800T\alpha^2\beta\tilde{\epsilon}-29175552T^2\beta^2\tilde{\epsilon}+417312\alpha\beta^2\tilde{\epsilon}-134183604672T^7\tilde{\epsilon}^3+88037605152T^5\alpha\tilde{\epsilon}^3+11393315328T^3\alpha^2\tilde{\epsilon}^3-206444064T\alpha^3\tilde{\epsilon}^3-39765148128T^4\beta\tilde{\epsilon}^3-2206304640T^2\alpha\beta\tilde{\epsilon}^3+13223328\alpha^2\beta\tilde{\epsilon}^3+315709056T\beta^2\tilde{\epsilon}^3+420217345440T^6\tilde{\epsilon}^5-239168412720T^4\alpha\tilde{\epsilon}^5-15315721920T^2\alpha^2\tilde{\epsilon}^5+143574256\alpha^3\tilde{\epsilon}^5+83363343552T^3\beta\tilde{\epsilon}^5+3334566528T\alpha\beta\tilde{\epsilon}^5-464003136\beta^2\tilde{\epsilon}^5-170517748608T^5\tilde{\epsilon}^7+90090243264T^3\alpha\tilde{\epsilon}^7+1851307200T\alpha^2\tilde{\epsilon}^7-26117293248T^2\beta\tilde{\epsilon}^7-195771648\alpha\beta\tilde{\epsilon}^7-40879485216T^4\tilde{\epsilon}^9+21519881616T^2\alpha\tilde{\epsilon}^9+180156656\alpha^2\tilde{\epsilon}^9-5920340256T\beta\tilde{\epsilon}^9+2361830976T^3\tilde{\epsilon}^{11}-1179776352T\alpha\tilde{\epsilon}^{11}+294801696\beta\tilde{\epsilon}^{11}+172896T^2\tilde{\epsilon}^{13}-28816\alpha\tilde{\epsilon}^{13}$

$c_{18}=-654729075T^9+1571349780T^7\alpha+785674890T^5\alpha^2-28081620T^3\alpha^3-54891T\alpha^4-1571349780T^6\beta-142849980T^4\alpha\beta+5965164T^2\alpha^2\beta+33588\alpha^3\beta+21976920T^3\beta^2-849528T\alpha\beta^2+14904\beta^3+201644650935T^8\tilde{\epsilon}^2-153472402188T^6\alpha\tilde{\epsilon}^2-35572332186T^4\alpha^2\tilde{\epsilon}^2+1100992764T^2\alpha^3\tilde{\epsilon}^2-970985\alpha^4\tilde{\epsilon}^2+88837568424T^5\beta\tilde{\epsilon}^2+7497411408T^3\alpha\beta\tilde{\epsilon}^2-184446216T\alpha^2\beta\tilde{\epsilon}^2-1174529160T^2\beta^2\tilde{\epsilon}^2+18883800\alpha\beta^2\tilde{\epsilon}^2-2060347538916T^7\tilde{\epsilon}^4+1239855532236T^5\alpha\tilde{\epsilon}^4+157039543692T^3\alpha^2\tilde{\epsilon}^4-3177138132T\alpha^3\tilde{\epsilon}^4-517001103540T^4\beta\tilde{\epsilon}^4-39298810392T^2\alpha\beta\tilde{\epsilon}^4+195686388\alpha^2\beta\tilde{\epsilon}^4+6195471192T\beta^2\tilde{\epsilon}^4+3262065628212T^6\tilde{\epsilon}^6-1754323769700T^4\alpha\tilde{\epsilon}^6-130475616300T^2\alpha^2\tilde{\epsilon}^6+1334176332\alpha^3\tilde{\epsilon}^6+585720087696T^3\beta\tilde{\epsilon}^6+39701917296T\alpha\beta\tilde{\epsilon}^6-5898782952\beta^2\tilde{\epsilon}^6-311489753418T^5\tilde{\epsilon}^8+145322283996T^3\alpha\tilde{\epsilon}^8+5586917346T\alpha^2\tilde{\epsilon}^8-39363783084T^2\beta\tilde{\epsilon}^8-1372421628\alpha\beta\tilde{\epsilon}^8-298175944446T^4\tilde{\epsilon}^{10}+154807935708T^2\alpha\tilde{\epsilon}^{10}+954504302\alpha^2\tilde{\epsilon}^{10}-41564614680T\beta\tilde{\epsilon}^{10}-6370678260T^3\tilde{\epsilon}^{12}+3187499940T\alpha\tilde{\epsilon}^{12}-797145228\beta\tilde{\epsilon}^{12}+668292T^2\tilde{\epsilon}^{14}-111436\alpha\tilde{\epsilon}^{14}-27T\tilde{\epsilon}^{16}-\tilde{\epsilon}^{18}$

$c_{19}=-125707982400T^9\tilde{\epsilon}+129898248480T^7\alpha\tilde{\epsilon}+51199944480T^5\alpha^2\tilde{\epsilon}-2203382880T^3\alpha^3\tilde{\epsilon}-2789280T\alpha^4\tilde{\epsilon}-101709185760T^6\beta\tilde{\epsilon}-11218694400T^4\alpha\beta\tilde{\epsilon}+534405600T^2\alpha^2\beta\tilde{\epsilon}+3124800\alpha^3\beta\tilde{\epsilon}+1926581760T^3\beta^2\tilde{\epsilon}-86028480T\alpha\beta^2\tilde{\epsilon}+1788480\beta^3\tilde{\epsilon}+5184937293120T^8\tilde{\epsilon}^3-3392667185184T^6\alpha\tilde{\epsilon}^3-755365406688T^4\alpha^2\tilde{\epsilon}^3+27019448544T^2\alpha^3\tilde{\epsilon}^3-42949280\alpha^4\tilde{\epsilon}^3+1757337685056T^5\beta\tilde{\epsilon}^3+199084939776T^3\alpha\beta\tilde{\epsilon}^3-5206935744T\alpha^2\beta\tilde{\epsilon}^3-33530029056T^2\beta^2\tilde{\epsilon}^3+610030656\alpha\beta^2\tilde{\epsilon}^3-24434174308032T^7\tilde{\epsilon}^5+13490703299232T^5\alpha\tilde{\epsilon}^5+1943858549760T^3\alpha^2\tilde{\epsilon}^5-40343194272T\alpha^3\tilde{\epsilon}^5-5319113995296T^4\beta\tilde{\epsilon}^5-629630011008T^2\alpha\beta\tilde{\epsilon}^5+2375785056\alpha^2\beta\tilde{\epsilon}^5+103152458880T\beta^2\tilde{\epsilon}^5+19185746159808T^6\tilde{\epsilon}^7-9298051961760T^4\alpha\tilde{\epsilon}^7-1134597817728T^2\alpha^2\tilde{\epsilon}^7+10732608544\alpha^3\tilde{\epsilon}^7+3038929881984T^3\beta\tilde{\epsilon}^7+456898496256T\alpha\beta\tilde{\epsilon}^7-67794102144\beta^2\tilde{\epsilon}^7+3190569641280T^5\tilde{\epsilon}^9-1871874604320T^3\alpha\tilde{\epsilon}^9+23216772576T\alpha^2\tilde{\epsilon}^9+528275642976T^2\beta\tilde{\epsilon}^9-12995167872\alpha\beta\tilde{\epsilon}^9-1384094869056T^4\tilde{\epsilon}^{11}+711029387808T^2\alpha\tilde{\epsilon}^{11}+3168265824\alpha^2\tilde{\epsilon}^{11}-187258689216T\beta\tilde{\epsilon}^{11}-106997127744T^3\tilde{\epsilon}^{13}+53501343072T\alpha\tilde{\epsilon}^{13}-13375683168\beta\tilde{\epsilon}^{13}+1809984T^2\tilde{\epsilon}^{15}-301664\alpha\tilde{\epsilon}^{15}$

$c_{20}=13749310575T^{10}-41247931725T^8\alpha-27498621150T^6\alpha^2+1474285050T^4\alpha^3+5763555T^2\alpha^4+160839\alpha^5+$

$47140493400T^7\beta+5999699160T^5\alpha\beta-417561480T^3\alpha^2\beta-7053480T\alpha^3\beta-1153788300T^4\beta^2+89200440T^2\alpha\beta^2+$
$257580\alpha^2\beta^2-3129840T\beta^3-6638452845570T^9\tilde{\epsilon}^2+50655848266680T^7\alpha\tilde{\epsilon}^2+18315209770 20T^5\alpha^2\tilde{\epsilon}^2-92808482040T^3\alpha^3\tilde{\epsilon}^2-$
$100320690T\alpha^4\tilde{\epsilon}^2-3383275732200T^6\beta\tilde{\epsilon}^2-495222462360T^4\alpha\beta\tilde{\epsilon}^2+26590227480T^2\alpha^2\beta\tilde{\epsilon}^2+157321800\alpha^3\beta\tilde{\epsilon}^2+$
$90006169680T^3\beta^2\tilde{\epsilon}^2-4693522320T\alpha\beta^2\tilde{\epsilon}^2+114404400\beta^3\tilde{\epsilon}^2+942868841838 21T^8\tilde{\epsilon}^4-54151412884020T^6\alpha\tilde{\epsilon}^4-$
$13457360481822T^4\alpha^2\tilde{\epsilon}^4+531612223812T^2\alpha^3\tilde{\epsilon}^4-1182882771\alpha^4\tilde{\epsilon}^4+26102723565096T^5\beta\tilde{\epsilon}^4+4429354286160T^3\alpha\beta\tilde{\epsilon}^4-$
$121555350216T\alpha^2\beta\tilde{\epsilon}^4-762687063864T^2\beta^2\tilde{\epsilon}^4+15750994824\alpha\beta^2\tilde{\epsilon}^4-227746405182600T^7\tilde{\epsilon}^6+110666587239576T^5\alpha\tilde{\epsilon}^6+$
$23106793925784T^3\alpha^2\tilde{\epsilon}^6-436128912360T\alpha^3\tilde{\epsilon}^6-42619899934584T^4\beta\tilde{\epsilon}^6-9306495406800T^2\alpha\beta\tilde{\epsilon}^6+24363330936\alpha^2\beta\tilde{\epsilon}^6+$
$1513339160688T\beta^2\tilde{\epsilon}^6+65674071269550T^6\tilde{\epsilon}^8-18825895417230T^4\alpha\tilde{\epsilon}^8-10589311873890T^2\alpha^2\tilde{\epsilon}^8+74566082042\alpha^3\tilde{\epsilon}^8+$
$7022789376840T^3\beta\tilde{\epsilon}^8+5032466431992T\alpha\beta\tilde{\epsilon}^8-715199813964\beta^2\tilde{\epsilon}^8+44185477656852T^5\tilde{\epsilon}^{10}-24650424710712T^3\alpha\tilde{\epsilon}^{10}+$
$2780629806 36T\alpha^2\tilde{\epsilon}^{10}+6649057289160T^2\beta\tilde{\epsilon}^{10}-132063679800\alpha\beta\tilde{\epsilon}^{10}-3267687087630T^4\tilde{\epsilon}^{12}+1645818368460T^2\alpha\tilde{\epsilon}^{12}+$
$2009429662\alpha^2\tilde{\epsilon}^{12} - 417472660872T\beta\tilde{\epsilon}^{12} - 676344299592T^3\tilde{\epsilon}^{14} + 338171790312T\alpha\tilde{\epsilon}^{14} - 84542902440\beta\tilde{\epsilon}^{14} +$
$3897315T^2\tilde{\epsilon}^{16} - 649485\alpha\tilde{\epsilon}^{16} + 30T\tilde{\epsilon}^{18} + \tilde{\epsilon}^{20}$

E.4 Hurwitz expansion for PI

In this section $c_n = c_n^{PI}$ from (8.69).
$(\alpha = g_2^{PI}/2\ ,\ \beta = 2g_3^{PI})$

$c_0 = 1$
$c_1 = 0$
$c_2 = 0$
$c_3 = 0$
$c_4 = -\alpha$
$c_5 = 6\epsilon$
$c_6 = -3\beta$
$c_7 = 0$
$c_8 = -9\alpha^2$
$c_9 = 84\alpha\epsilon$
$c_{10} = -18\alpha\beta - 294\epsilon^2$
$c_{11} = 216\beta\epsilon$
$c_{12} = 69\alpha^3 - 54\beta^2$
$c_{13} = -1650\alpha^2\epsilon$
$c_{14} = 513\alpha^2\beta + 18774\alpha\epsilon^2$
$c_{15} = -18720\alpha\beta\epsilon - 78624\epsilon^3$
$c_{16} = 321\alpha^4 + 4968\alpha\beta^2 + 144144\beta\epsilon^2$
$c_{17} = -52488\alpha^3\epsilon - 89424\beta^2\epsilon$
$c_{18} = 33588\alpha^3\beta + 14904\beta^3 + 1112436\alpha^2\epsilon^2$
$c_{19} = -1358640\alpha^2\beta\epsilon - 8670816\alpha\epsilon^3$
$c_{20} = 160839\alpha^5 + 257580\alpha^2\beta^2 + 15053040\alpha\beta\epsilon^2 + 27734616\epsilon^4$
$c_{21} = -9642690\alpha^4\epsilon - 5786640\alpha\beta^2\epsilon - 67223520\beta\epsilon^3$
$c_{22} = 2808945\alpha^4\beta + 502200\alpha\beta^3 + 221121900\alpha^3\epsilon^2 + 47585880\beta^2\epsilon^2$
$c_{23} = -143272800\alpha^3\beta\epsilon - 12052800\beta^3\epsilon - 2725500960\alpha^2\epsilon^3$
$c_{24} = 1416951\alpha^6 + 20019960\alpha^3\beta^2 + 1506600\beta^4 + 3160803600\alpha^2\beta\epsilon^2 + 18370803024\alpha\epsilon^4$
$c_{25} = -17890596\alpha^5\epsilon - 1168920720\alpha^2\beta^2\epsilon - 33552639936\alpha\beta\epsilon^3 - 55673733024\epsilon^5$
$c_{26} = -41843142\alpha^5\beta + 162100440\alpha^2\beta^3 - 781903530\alpha^4\epsilon^2 + 22914336240\alpha\beta^2\epsilon^2 + 150406965072\beta\epsilon^4$

$c_{27} = 2956347720\alpha^4\beta\epsilon - 7272970560\alpha\beta^3\epsilon + 46552246560\alpha^3\epsilon^3 - 165826388160\beta^2\epsilon^3$

$c_{28} = -388946691\alpha^7 - 376375410\alpha^4\beta^2 + 796330440\alpha\beta^4 - 123190094160\alpha^3\beta\epsilon^2 + 92545558560\beta^3\epsilon^2 - 1059992932824\alpha^2\epsilon^4$

$c_{29} = 35919307926\alpha^6\epsilon + 52917876720\alpha^3\beta^2\epsilon - 23889913200\beta^4\epsilon + 3205542167136\alpha^2\beta\epsilon^3 + 11682746528544\alpha\epsilon^5$

$c_{30} = -6519779667\alpha^6\beta - 9465715080\alpha^3\beta^3 + 2388991320\beta^5 - 1720632974886\alpha^5\epsilon^2 - 2734614623160\alpha^2\beta^2\epsilon^2 - 45309600930384\alpha\beta\epsilon^4 - 48569368118544\epsilon^6$

$c_{31} = 978517329984\alpha^5\beta\epsilon + 1075789382400\alpha^2\beta^3\epsilon + 52109440015680\alpha^4\epsilon^3 + 61723536910848\alpha\beta^2\epsilon^3 + 239259812553216\beta\epsilon^5$

$c_{32} = 25514578881\alpha^8 - 210469286736\alpha^5\beta^2 - 144916218720\alpha^2\beta^4 - 62015233698720\alpha^4\beta\epsilon^2 - 40204399921920\alpha\beta^3\epsilon^2 - 1021069206582336\alpha^3\epsilon^4 - 453518093142144\beta^2\epsilon^4$

$c_{33} = -1666422337680\alpha^7\epsilon + 30257070700320\alpha^4\beta^2\epsilon + 11884499521920\alpha\beta^4\epsilon + 2042944257556224\alpha^3\beta\epsilon^3 + 428809408012800\beta^3\epsilon^3 + 12260416976841600\alpha^2\epsilon^5$

$c_{34} = -485174610648\alpha^7\beta - 4582619446320\alpha^4\beta^3 - 1289959784640\alpha\beta^5 - 16630361205624\alpha^6\epsilon^2 - 1654258781960640\alpha^3\beta^2\epsilon^2 - 203961254126400\beta^4\epsilon^2 - 35333389387302336\alpha^2\beta\epsilon^4 - 82954144406480256\alpha\epsilon^6$

$c_{35} = 152134674660000\alpha^6\beta\epsilon + 543425169335040\alpha^3\beta^3\epsilon + 46438552247040\beta^5\epsilon + 4829402276650944\alpha^5\epsilon^3 + 40951207980949248\alpha^2\beta^2\epsilon^3 + 315176201202046464\alpha\beta\epsilon^5 + 252377177931680256\epsilon^7$

$c_{36} = -7647989401521\alpha^9 - 41767502762088\alpha^6\beta^2 - 60648644233440\alpha^3\beta^4 - 3869879353920\beta^6 - 13963139340227424\alpha^5\beta\epsilon^2 - 21252172506998400\alpha^2\beta^3\epsilon^2 - 188196130875388464\alpha^4\epsilon^4 - 483274189528564608\alpha\beta^2\epsilon^4 - 1221327388362122496\beta\epsilon^6$

$c_{37} = 1597613431426038\alpha^8\epsilon + 7287575315043744\alpha^5\beta^2\epsilon + 4860000915779520\alpha^2\beta^4\epsilon + 5808508115632536960\alpha^4\beta\epsilon^3 + 352379520821205504\alpha\beta^3\epsilon^3 + 3629347149856321152\alpha^3\epsilon^5 + 2426497407436144896\beta^2\epsilon^5$

$c_{38} = -544306979739483\alpha^8\beta - 1028311276281264\alpha^5\beta^3 - 383302865236320\alpha^2\beta^5 - 145957471490813256\alpha^7\epsilon^2 - 450223680165654960\alpha^4\beta^2\epsilon^2 - 126206480913710400\alpha\beta^4\epsilon^2 - 13083603812270471232\alpha^3\beta\epsilon^4 - 2419559327517745536\beta^3\epsilon^4 - 39970888866183167424\alpha^2\epsilon^6$

$c_{39} = 109258810510910400\alpha^7\beta\epsilon + 125438691795788160\alpha^4\beta^3\epsilon + 20973065667632640\alpha\beta^5\epsilon + 7465233450031396416\alpha^6\epsilon^3 + 137865032820727096320\alpha^3\beta^2\epsilon^3 + 12764256434052480000\beta^4\epsilon^3 + 170610428557450446336\alpha^2\beta\epsilon^5 + 244708656405420340224\alpha\epsilon^7$

$c_{40} = -1013917176434889\alpha^{10} - 19199774752012080\alpha^7\beta^2 - 10665863758194480\alpha^4\beta^4 - 1343956990999680\alpha\beta^6 - 8993959815439913760\alpha^6\beta\epsilon^2 - 58807767682761408000\alpha^3\beta^3\epsilon^2 - 3641531490941990400\beta^5\epsilon^2 - 239802121803604192992\alpha^5\epsilon^4 - 2342301130628276832000\alpha^2\beta^2\epsilon^4 - 1230173253340180886016\alpha\beta\epsilon^6 - 676706834479537540224\epsilon^8$

$c_{41} = 257198900059618020\alpha^9\epsilon + 3280225716831278880\alpha^6\beta^2\epsilon + 10388052969859756800\alpha^3\beta^4\epsilon + 564461936219865600\beta^6\epsilon + 414215430149742286464\alpha^5\beta\epsilon^3 + 1400489309410541521920\alpha^2\beta^3\epsilon^3 + 514139402288007728352\alpha^4\epsilon^5 + 21235437032702103567360\alpha\beta^2\epsilon^5 + 4007543942148905733120\beta\epsilon^7$

$c_{42} = -8643787151905017 0\alpha^9\beta - 3496619477835237600\alpha^6\beta^3 - 6622753897906272 0\alpha^3\beta^5 - 403187097299904 0\beta^7 - 262444556907385202700\alpha^8\epsilon^2 - 2417714616812847912000\alpha^5\beta^2\epsilon^2 - 398688829974184982400\alpha^2\beta^4\epsilon^2 - 120270548805123577936320\alpha^4\beta\epsilon^4 - 16774101798663635919360\alpha\beta^3\epsilon^4 - 74935202706816131514240\alpha^3\epsilon^6 - 8614891199805709029120\beta^2\epsilon^6$

$c_{43} = 1714121636103488556 0\alpha^8\beta\epsilon + 5663455953825797568 0\alpha^5\beta^3\epsilon + 565033155117975936 0\alpha^2\beta^5\epsilon + 15282930631851255945 60\alpha^7\epsilon^3 + 102142990029361315749 12\alpha^4\beta^2\epsilon^3 + 67298014676063133849 6\alpha\beta^4\epsilon^3 + 22903541616327496286054 4\alpha^3\beta\epsilon^5 + 888379345418739420672 0\beta^3\epsilon^5 + 727102801248129087828480\alpha^2\epsilon^7$

$c_{44} = -155812911328032651\alpha^{11} - 2149281495195098670\alpha^8\beta^2 - 4849557774995228400\alpha^5\beta^4 - 342039979512624960\alpha^2\beta^6 - 1531812482581505090400\alpha^7\beta\epsilon^2 - 4022774166702416529600\alpha^4\beta^3\epsilon^2 - 1485078473391768460 80\alpha\beta^5\epsilon^2 - 58065576264612339606288\alpha^6\epsilon^4 - 268718195338965949207680\alpha^3\beta^2\epsilon^4 - 48780466191861578899 20\beta^4\epsilon^4 - 283705258144776785667302 4\alpha^2\beta\epsilon^6 - 431043680304753271353561 6\alpha\epsilon^8$

$c_{45} = 2567110805015427228 6\alpha^{10}\epsilon + 42225137934133636944 0\alpha^7\beta^2\epsilon + 7840231501245790248 00\alpha^4\beta^4\epsilon + 172253022942571200000\alpha\beta^6\epsilon + 82274221325423641453632\alpha^6\beta\epsilon^3 + 158923721697922624826880\alpha^3\beta^3\epsilon^3 + 15838652504980506700 80\beta^5\epsilon^3 + 1516158412144555288845888\alpha^5\epsilon^5 + 4452130384760643114481920\alpha^2\beta^2\epsilon^5 +$

$2110491415374883302917836\alpha\beta\epsilon^7 + 1202027155572307019553561\epsilon^9$

$c_{46} = 457002245380426137\alpha^{10}\beta - 3965154564935106648\alpha^7\beta^3 - 618335581062417271\alpha^4\beta^5 - 7217224559860665\alpha\beta^7 - 1805147237470952543490\alpha^9\epsilon^2 - 38367530463572033104080\alpha^6\beta^2\epsilon^2 - 52327000215657788644800\alpha^3\beta^4\epsilon^2 - 3110165315834241772\beta^6\epsilon^2 - 290024081554091543410550\alpha^5\beta\epsilon^4 - 372150911262287175496128\alpha^2\beta^3\epsilon^4 - 276071153435011691079681\alpha^4\epsilon^6 - 431740424909521408546368\alpha\beta^2\epsilon^6 - 728937060659609445692455\beta\epsilon^8$

$c_{47} = -48310196567979158064\alpha^9\beta\epsilon + 83562799031959655923\alpha^6\beta^3\epsilon + 91229364895781213337\alpha^3\beta^5\epsilon + 34642677887331194880\beta^7\epsilon + 669108219155586509140\alpha^8\epsilon^3 + 2028231018986670690393600\alpha^5\beta^2\epsilon^3 + 1841387437469129282933760\alpha^2\beta^4\epsilon^3 + 6879501664265158925255424\alpha^4\beta\epsilon^5 + 4887622483257001958590464\alpha\beta^3\epsilon^5 + 3445135200889785162360115\alpha^3\epsilon^7 + 1902640136342792613619507\beta^2\epsilon^7$

$c_{48} = 5858193102389670464\alpha^{12} + 1537431024070962745\alpha^9\beta^2 - 7341200232128711524\alpha^6\beta^4 - 6456985963473313497\alpha^3\beta^6 - 216516736795819968\beta^8 + 7984910171808726676632\alpha^8\beta\epsilon^2 - 7397175638883877477747\alpha^5\beta^3\epsilon^2 - 5371123219307210919859\alpha^2\beta^5\epsilon^2 - 840265898152433500383552\alpha^7\epsilon^4 - 6677087108641311784621248\alpha^4\beta^2\epsilon^4 - 3390093447118450568858880\alpha\beta^4\epsilon^4 - 10755547345724924998630656\alpha^3\beta\epsilon^6 - 2820371545090445830988083\beta^3\epsilon^6 - 2791244626252018773872926\alpha^2\epsilon^8$

$c_{49} = -164760276212418048165\alpha^{11}\epsilon - 3919443535171497272184\alpha^8\beta^2\epsilon + 1376637404766172946208\alpha^5\beta^4\epsilon + 847452877789927328640\alpha^2\beta^6\epsilon - 72071374891903697624670\alpha^7\beta\epsilon^3 + 35311919958400774248967\alpha^4\beta^3\epsilon^3 + 1461650109292067227868160\alpha\beta^5\epsilon^3 - 47383566530308128990453888\alpha^6\epsilon^5 + 134848881298208916903366144\alpha^3\beta^2\epsilon^5 + 26141148339419853007372800\beta^4\epsilon^5 + 103850291656950675722536734\alpha^2\beta\epsilon^7 + 13092369581815270041904487\alpha\epsilon^9$

$c_{50} = 3569731062346847916252\alpha^{11}\beta + 4033242497662632885960\alpha^8\beta^3 - 9637760723299526905920\alpha^5\beta^5 - 5620752638745660576000\alpha^2\beta^7 + 21372154658150761266563\alpha^{10}\epsilon^2 + 520848869331160024637472\alpha^7\beta^2\epsilon^2 - 1030085802167614771430880\alpha^4\beta^4\epsilon^2 - 379880015901869924179200\alpha\beta^6\epsilon^2 + 4394587447077989186321009\alpha^6\beta\epsilon^4 - 9230159216939607116253619\alpha^3\beta^3\epsilon^4 - 1551607892799635721085056\beta^5\epsilon^4 + 3259176861802252188541131\alpha^5\epsilon^6 - 1536300862825097506049488\alpha^2\beta^2\epsilon^6 - 5403930552493233306128545\alpha\beta\epsilon^8 - 2726576205459970094173648\epsilon^{10}$