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Bottom-up and top-down aspects of celestial CFT

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Abstract

The holographic principle suggests that quantum gravity on a manifold $\mathcal{M}$ can be described by degrees of freedom living on its boundary $\partial\mathcal{M}$. While the AdS/CFT correspondence provides a concrete realization of this idea, extending holography to asymptotically flat spacetime remains an open problem.

The celestial holography program proposes that gravity in four-dimensional flat space is dual to a two-dimensional conformal field theory, known as celestial CFT (CCFT). This thesis provides valuable insights to two key questions in this framework: the structure of the CCFT spectrum and the possibility to build a top-down construction of the dual model.

We show that the observed vanishing central charge in CCFT points beyond unitary CFTs, toward logarithmic CFTs (LCFTs). Through a bottom-up analysis of tree-level amplitudes, we demonstrate that logarithmic operators naturally arise from infrared regularization, explaining the logarithmic nature of CCFT and constraining its primary spectrum.

We also develop a top-down approach by constructing the celestial dual for MHV leaf amplitudes, extending recent proposals. This provides new insights into the role of asymptotic symmetries, stress tensor structure, and celestial OPEs, bringing us closer to a concrete flat space holographic duality.

Our results support the idea that celestial CFT is realized through a non-standard two-dimensional theory, linking asymptotic symmetries, infrared dynamics, and logarithmic structures in quantum gravity.

Declaration

I hereby declare that, except where specific reference is made to the work of others, the contents of this thesis are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university.

The discussion on the following published papers:

- A. Bissi, L. Donnay and B. Valsesia, “Logarithmic doublets in CCF”, [1]
- L. Donnay, G. Giribet and B. Valsesia, “MHV leaf amplitudes from parafermions”, [2]

and original unpublished material.

“There hasn’t been an original thought on Vesuva in centuries.”
- Emmest, Tolarian chronicler

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Contents

Introduction

Proposed initially by Gerard t'Hooft [3] and refined by Leonard Susskind [4], the holographic principle posits that in a theory of quantum gravity the description of the dynamics in a volume of spacetime can be entirely described by data encoded on its lower-dimensional boundary, which acts as a holographic screen. This idea was inspired by the development of black hole thermodynamics, in particular by the discovery of Bekenstein [5] and Hawking [6] that black hole entropy is proportional to the area of its event horizon, not its volume. This implies that the number of degrees of freedom grows with the area, suggesting that the real dynamics can be described by a model living on the black hole horizon. While groundbreaking, the original proposals of t'Hooft and Susskind remain largely conceptual in scope, serving more as foundational insights than a fully developed formulation of a complete holographic duality.

The first concrete example of such duality came with the seminal work of Maldacena [7] and Witten [8] which showed that weakly coupled type IIB supergravity in $\mathrm{AdS}_5 \times S^5$, is equivalent to the large N limit of $SU(N)$ $\mathcal{N}=4$ superconformal Yang-Mills living on the 4d boundary of 5d anti de Sitter (AdS) space. This result originally coming form string theory, soon became a fully independent line of research known today as the AdS/CFT conjecture which states that a theory of quantum gravity in asymptotically AdS spacetime can be fully described by a conformal field theory (CFT) living on the boundary of AdS.
The significance of this conjecture can hardly be overstated, not only because it allows us to give a non-perturbative definition of quantum gravity in AdS, but also because it allows us to describe strongly coupled quantum field theories using the gravitational dual. To be more specific:

- AdS/CFT provides a concrete, calculable example of how a theory with gravity can be equivalent to a quantum theory with no gravitational interactions. It offers a holographic dictionary to translate concepts and calculations between the two sides of the duality.

- AdS/CFT it is a strong-weak duality. When the gravitational theory in AdS is weakly coupled, the corresponding CFT on the boundary is strongly coupled, and vice-versa. This allows us to use relatively simple gravitational calculations to gain insights into complex, strongly interacting quantum systems [9–12].

- AdS/CFT offers a framework to address the long-standing black hole information paradox. If information is truly preserved in a quantum theory, then the evaporation of a black hole should not lead to the loss of information about what fell into it. The AdS/CFT correspondence, by relating a gravitational theory to a unitary quantum field theory, strongly suggests that information is indeed preserved, somehow encoded on the boundary [13–15].

Unfortunately, even if extremely powerful, AdS/CFT is limited to describe quantum gravity only when the boundary conditions fix the geometry of spacetime to be asymptotically anti de Sitter, and it appears to be extremely tricky to extend it to different boundary configurations such as asymptotically flat or de Sitter (dS) spacetime [16–22]. To be more specific, the three

configurations AdS, flat and dS spacetime are respectively in correspondence with a negative, vanishing or positive value of the cosmological constant Λ. A priori, to get a holographic description of gravity, for example in flat space, one could naively attempt to take the limit of vanishing cosmological constant in AdS to land on flat space. In practice this turns out to be highly non-trivial. The flat limit can be taken in different ways with no canonical candidate, leading to ambiguous results. Moreover there is not a smooth transition between the causal structure of the AdS boundary to the flat boundary, which makes difficult to identify what surface should take the role of the holographic screen.
From a phenomenological stand-point it would be extremely interesting to extend the duality beyond AdS, precisely because most recent measurements on the cosmological constant are compatible with a positive value of Λ [23], namely away from the AdS regime.

Given these difficulties to approach flat or dS holography starting from AdS/CFT, we need another strategy that allows us to determine what are the properties of the holographic duals in these different regimes. A good rule of thumb in all physical problems is to use the symmetries to guide us. Even in AdS/CFT, one can deduce that the dual theory on the boundary AdS must be a CFT because it must respect the symmetries of the bulk including the set of asymptotic isometries of spacetime, that for the case of AdS coincides with the conformal group of the boundary.

If we assume that the same principle extends to all holographic descriptions then an analysis of the asymptotic symmetries in flat space or de Sitter can be used to establish some fundamental properties of the respective holographic models.
This line of reasoning has received a huge attention in recent years, especially for the study of flat space holography which will be the main subject of this thesis. In asymptotically flat space in 4d the analysis of asymptotic symmetries was carried out by Bondi, Metzner [24] and Sachs [25] (BMS) which surprisingly found that they did not coincide with the Poincaré group, but form an infinite dimensional group which extends Poincaré to allow angle-dependent translations. These sets of transformations are now known as supertranslations and, combined in a semi-direct product with the Lorentz group, constitute the so-called BMS_4 group[1]

$$\text{BMS}_4 = SL(2,\mathbb{C}) \ltimes C^\infty(\mathbb{C},\mathbb{R}) = SL(2,\mathbb{C}) \ltimes \text{supertranslations} \tag{1}$$

In recent years there has been quite a debate if this should be considered as the group of asymptotic symmetries of flat space, as the relaxation of possibly nonphysical constraints allow extensions to larger groups, in particular the minimal extension is represented by the extended BMS group[2] [26–28]

$$\text{eBMS}_4 = \left(\text{Vir} \otimes \overline{\text{Vir}}\right) \ltimes \text{supertranslations} \tag{2}$$

which has gathered considerable interest, as it was proven to have a deep relation with the universal infrared behavior of gravity [29–31].

[1]The set of angle-dependent translations can be considered as the set of differentiable functions from the sphere S^2 to the reals, $C^\infty(\mathbb{C},\mathbb{R})$ with the group structure given by the addition in $\mathbb{R}$.

[2]With Vir we are denoting the Virasoro group

We can then assume eBMS_4 to be the actual group of asymptotic symmetries of 4d asymptotically flat space. This then suggests that, if a holographic dual exists in flat space, such dual should have eBMS_4 as a symmetry, analogously to the usual considerations drawn in AdS/CFT. Two possibilities for a holographic description then open up, and they both have been subject of intense discussion in the past years. An option falls under the name of Carrollian holography (see [32–54]), which basically suggests a holographic duality between quantum gravity (QG) in 4d flat space and a 3d conformal Carrollian theory living on the null boundary of flat space. A conformal Carrollian QFT (Carrollian CFT) in 3d is a quantum field theory leaving on a manifold endowed with a degenerate metric of rank 2 and that respects the conformal isometries of such metric. This group of conformal transformations is referred to as conformal Carroll group cCarr_3 and it is possible to prove that [55]

$$\mathrm{cCarr}_3 = \mathrm{eBMS}_4 \tag{3}$$

so that any 3d Carrollian CFT would precisely respect the asymptotic symmetries of the flat bulk.

The second option, which will be the main focus of this thesis, is the celestial holography proposal which states that the holographic dual of QG in 4d flat space must be found in a 2d CFT which lives on a 2d sphere usually known as celestial sphere. This duality can be suggested by noticing that eBMS_4 contains a double copy of the Virasoro algebra, which is precisely the full 2d conformal group.

This proposal, which is the core of celestial holography, has been largely explored (see [56–87]), especially for theories of massless particles, and today it proposes a holographic dictionary, connecting bulk scattering amplitudes to correlation function in the 2d celestial CFT (CCFT). In simple terms the dictionary trades the frequency of bulk particles with the conformal weights of operators in the CFT. Moreover it establishes a precise correspondence between the generator of eBMS_4 and currents in the CFT. In particular it turns out that the generators of the $\mathrm{Vir} \otimes \overline{\mathrm{Vir}}$ subgroup of eBMS are in correspondence with the holomorphic and antiholomorphic components of the CCFT stress tensor $T(z), \overline{T}(\bar{z})$ [88] . The connection between bulk correlation functions and operator insertions in the CFT allows us to also compute the operator product expansions (OPEs) between celestial operators using the factorization properties of collinear or small frequency limits [64, 65, 67]. This latter property also allows us to compute the central charge c of CCFT starting from the $T(z)T(0)$ OPE. Beside these successes, many questions still remain to be answered:

1. What is the complete spectrum of primaries in celestial CFT ?

2. Is it possible to obtain a fully fledged duality between a theory of QG in flat space and a specific CCFT analogous to the top down constructions in AdS/CFT ?

This thesis is organized into two main parts, each aimed at providing insights into one of these questions.

Regarding the spectrum, valuable information can be extracted by examining a notable feature of celestial CFT: the vanishing of its central charge. In fact the computations of the stress

tensor OPE mentioned above yields $c = 0$ [67,89]. If we want to treat CCFT as a wick rotated Lorentzian CFT, this curious property compels us to move beyond the framework of unitary CFTs and consider more general, possibly non-unitary structures. One possible explanation for the vanishing of the central charge is that celestial CFT (CCFT) factorizes into a direct product of two distinct conformal field theories with opposite central charges [90,91], yielding a vanishing total central charge, much like what occurs in worldsheet string theory. An alternative and intriguing possibility is that celestial CFT is a logarithmic conformal field theory (LCFT) [1,92–94]. In such theories the conformal symmetry is realized not through irreducible representations, but reducibly and indecomposably. In these models the primary fields are now organized into multiplets labeled by their spin and conformal weights. This causes logarithms to appear in the correlation functions of such CFTs, from which they take the name of logarithmic CFTs.
We will describe such models further in chapter 5, as most of the original content of this thesis examines the emergence of logarithmic CFT properties in celestial CFT [1]. Logarithmic properties of celestial CFT, have been discussed as emerging from quantum corrections in [92,94] and related to the vanishing central charge in [93]. In this thesis we will carry out an in depth bottom-up analysis of tree level CCFT and show that logarithmic operators can naturally emerge by regulating infrared divergences. The presence of a logarithmic structure can help us not only to elucidate the issue of the vanishing central charge in CCFT and also furnish information on the spectrum of primaries in such CFTs, providing valuable insight to the resolution of question 1.

Moving on to question 2, chapter 6 will be completely devoted to study the top-down approach, and will provide a novel example of top-down construction in CCFT.
Historically, celestial holography has been constructed in a bottom-up approach, with multiple attempts to infer the structure of CCFT from bulk dynamics. In recent years however different top-down models have been proposed, such as the constructions of Costello and Paquette [95,96] which furnish a CCFT dual for scalar-flat Kähler gravity on an asymptotically flat, four dimensional background known as the Burns metric. Other constructions basically propose a CCFT model for tree level MHV amplitudes both in pure Yang-Mills [2,97–99] or gravity [100,101]. These models even if not complete give us hints on some properties of CCFT, which might extend to the full holographic description.
Our discussion in this thesis will focused on the description of the celestial dual to MHV amplitudes, starting from the discussion of the results obtained in [97], and then extending to an original construction obtained in [2].

We conclude this introduction outlining the structure of the thesis.
The first chapter 1 discusses basic material, just to settle the notation. We will describe the properties of flat space, its conformal compactification and isometries. We will also focus on the description of the properties of Klein space, which is an analytic extension of Minkowski from signature $(3,1)$ to $(2,2)$ frequently used in CCFT and relevant to the discussion of the top-down approach, which will be material of chapter 6.

In the second chapter 2, we will primarily discuss results related to infrared divergences, including the soft energy expansion of amplitudes involving infrared photons and gravitons, the divergences caused by virtual gauge bosons, and previous insights into the origin and possible resolutions of these issues.

This will be used as a starting point for chapter 3 where we will finally discuss large gauge transformation, first in QED then in gravity and show how they are tightly connected to the soft behavior of amplitudes.

In chapter 4 the previous discussion will come together in the description of the holographic duality proposed by celestial holography, with the presentation of the holographic dictionary, celestial primaries and currents related to large gauge transformations and large bulk isometries. We will also comment on how to compute the central charge, which turns out to be vanishing, and use this fact to open the discussion of chapter 5.

In chapter 5, we explore logarithmic conformal field theories, beginning with an illustrative example in two dimensions and then moving on to their general properties, including how they address the issue of a vanishing central charge through the introduction of a logarithmic partner t associated with the stress tensor. Beyond this introductory discussion, most of the material in this chapter draws from the original results of [1] and additional unpublished work. Building on these, we will show, through a bottom up approach, how the key features of LCFT can be transposed to CCFT, providing explicit examples of logarithmic celestial primary fields, proposing a construction for t, and presenting a well-defined doublet of fields with finite two-point functions emerging from the soft sector of gravity. We will also relate some of the properties uncovered in CCFT to a novel construction based on the two-dimensional free scalar.

In the last chapter 6, we will come back to the top-down construction and present a novel construction showing how a celestial description for the MHV sector of $SO(N)$ pure Yang-Mills in the large N limit can be obtain by a parafermion model coupled with $SO(N)$ Kac-Moody currents.

We will conclude the thesis with a summary of the results and future perspectives regarding the development of celestial holography.

Chapter 1

The Basics

1.1 Flat space

The broad subject of this thesis is flat space holography. To lay a solid foundation, we will begin by reviewing the key properties of flat space and introducing the conventions we adopt throughout. This preliminary discussion will provide the necessary background to later extend our analysis to the appropriate spacetime symmetries in the context of asymptotically flat spacetime. This review will be based on [102, 103] for review of the properties of the Poincaré group, and [104, 105] for the asymptotic structure of flat space.

4d flat space, or Minkowski space, is defined as $\mathbb{R}^4$ equipped with a pseudo-Riemannian structure specified by the metric $\eta = \text{diag}(-1, +1, +1, +1)$ [106]. This defines the line element as:

$$ds^2 = -(dX^0)^2 + (dX^1)^2 + (dX^2)^2 + (dX^3)^2. \tag{1.1}$$

Given a Riemannian manifold it is natural to discuss its isometries. The isometry group of 4d Minkowski is the Poincaré group [106, 107] defined as the semi-direct product of translations and the Lorentz group:

$$PO(1,3) = \mathbb{R}^{1,3} \rtimes O(1,3); \quad O(1,3) = \{\Lambda \in GL(4,\mathbb{R}) \,:\, \Lambda^\intercal \eta \Lambda = \eta\}\,, \tag{1.2}$$

Because the Lorentz group $O(1,3)$ is made up by 4 disconnected components related by time-reversal and parity, we can just focus on the component connected to the identity namely the proper orthochronous Lorentz group:

$$SO(1,3)^+ = \left\{\Lambda \in O(1,3) \,:\, \det(\Lambda) = 1,\ \Lambda^0_0 \geq 1\right\}, \tag{1.3}$$

Its Lie algebra is completely spanned by

$$P_\mu = -i\partial_\mu; \quad M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu), \tag{1.4}$$

satisfying the following commutation relations:

$$[P_\mu, P_\nu] = 0, \quad [P_\lambda, M_{\mu\nu}] = i(\eta_{\mu\lambda} P_\nu - \eta_{\nu\lambda} P_\mu) \tag{1.5}$$

$$[M_{\mu\nu}, M_{\rho\lambda}] = i(\eta_{\mu\rho} M_{\nu\lambda} - \eta_{\mu\lambda} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\lambda} M_{\mu\rho}) \tag{1.6}$$

For applications in QFT we are usually interested in unitary projective representations of the Lorentz group, which by Bargmann's theorem [108] are in one-to-one correspondence with the ordinary representations of its universal cover. Focusing on

$$SO(1,3)^+ = \left\{\Lambda \in O(1,3)\ :\ \det(\Lambda) = 1,\ \Lambda^0_0 \geq 1\right\} \tag{1.7}$$

we known that its universal cover is $\mathrm{Spin}(1,3) = SL(2,\mathbb{C})$ and in particular we have the isomorphism:

$$SO(1,3)^+ \simeq SL(2,\mathbb{C})/\mathbb{Z}_2 \tag{1.8}$$

This identification is realized as follows, [103, 104] let us define:

$$\sigma^\mu = (I, \sigma^i), \quad \bar{\sigma}^\mu = (-I, \sigma^i) = \epsilon \sigma^{\mu\intercal} \epsilon, \quad \epsilon = i\sigma^2; \tag{1.9}$$

where $\sigma^i, i = 1, 2, 3$ are the Pauli matrices and I is the 2×2 identity matrix.
Then any vector $x^\mu \in \mathbb{R}^{1,3}$ can be mapped into the matrix $X = x_\mu \sigma^\mu \in \mathbb{R}^{2\times 2}$ and its "conjugate" $\bar{X} = x_\mu \bar{\sigma}^\mu$. This map is invertible in fact we can always come back to the vector using the property $\mathrm{Tr}[\bar{\sigma}^\mu \sigma^\nu] = 2\eta^{\mu\nu}$ so that:

$$x^\mu = \frac{1}{2}\,\mathrm{Tr}[\bar{\sigma}^\mu X] \tag{1.10}$$

Then it is easy to show that the following holds for any two x^μ, $y^\mu \in \mathbb{R}^{1,3}$:

$$x \cdot y = \eta_{\mu\nu} x^\mu y^\nu = \frac{1}{2}\,\mathrm{Tr}\big[X\bar{Y}\big] \tag{1.11}$$

Now we consider the adjoint representation of $SL(2,\mathbb{C})$ on X namely under $L \in SL(2, C)$, $X \to X' = L^\dagger X L$. Now using the fundamental property $\epsilon L = L^{\intercal -1}\epsilon$ it is easy to prove that $\bar{X} \to \bar{X}' = L^{-1}\bar{X}L^{\dagger -1}$ so that:

$$x' \cdot y' = \frac{1}{2}\,\mathrm{Tr}\big[X'\bar{Y}'\big] = \frac{1}{2}\,\mathrm{Tr}\big[X\bar{Y}\big] = x \cdot y \tag{1.12}$$

This means that the adjoint representation of $SL(2,\mathbb{C})$ acts as an element of $O(1,3)$. So there must exist a matrix $\Lambda \in O(1,3)$ such that:

$$\Lambda^\mu_\nu x^\nu = x'^\mu = \frac{1}{2}\,\mathrm{Tr}[\bar{\sigma}^\mu X'] = \frac{1}{2}\,\mathrm{Tr}\big[\bar{\sigma}^\mu M^\dagger X M\big] = \frac{1}{2}\,\mathrm{Tr}\big[\bar{\sigma}^\mu M^\dagger \sigma_\nu M\big] x^\mu \to \tag{1.13}$$

$$\to \Lambda^\nu_\mu(M) = \frac{1}{2}\,\mathrm{Tr}\big[\bar{\sigma}^\mu M^\dagger \sigma_\nu M\big] \tag{1.14}$$

which is the image of two matrices M and $-M$ in $SL(2,\mathbb{C})$. Using the map above we can also explicitly write the form of $\Lambda(M)$ for a generic $SL(2,\mathbb{C})$ matrix [104]:

$$M = \begin{pmatrix} a & -c \\ -b & d \end{pmatrix}, \quad ad - bc = 1 \tag{1.15}$$

$$\Lambda_M = \frac{1}{2}\begin{pmatrix} a\bar{a} + b\bar{b} + c\bar{c} + d\bar{d} & a\bar{b} + b\bar{a} + d\bar{c} + c\bar{d} & i(a\bar{b} - b\bar{a} + c\bar{d} - d\bar{c}) & b\bar{b} - a\bar{a} - c\bar{c} + d\bar{d} \\ a\bar{c} + c\bar{a} + b\bar{d} + d\bar{b} & a\bar{d} + d\bar{a} + b\bar{c} + c\bar{b} & i(a\bar{d} - d\bar{a} - b\bar{c} + c\bar{b}) & b\bar{d} + d\bar{b} - a\bar{c} - c\bar{a} \\ i(c\bar{a} - a\bar{c} - b\bar{d} + d\bar{b}) & i(d\bar{a} - a\bar{d} - b\bar{c} + c\bar{b}) & a\bar{d} + d\bar{a} - b\bar{c} - c\bar{b} & i(a\bar{c} - c\bar{a} - b\bar{d} + d\bar{b}) \\ c\bar{c} + d\bar{d} - a\bar{a} - b\bar{b} & c\bar{d} + d\bar{c} - a\bar{b} - b\bar{a} & i(b\bar{a} - a\bar{b} + c\bar{d} - d\bar{c}) & a\bar{a} - b\bar{b} - c\bar{c} + d\bar{d} \end{pmatrix}$$

By direct calculation is then easy to prove that $\det\Lambda(M) = 1,\ \Lambda^0_0 \geq 1$ so that $SL(2,\mathbb{C})$ turns out not to be a double cover of $O(1,3)$ but just of the proper orthochronous subgroup $SO(1,3)^+$. To get rid of the multiplicity $M \leftrightarrow -M$ we can take the quotient of $SL(2,\mathbb{C})$ with $\mathbb{Z}_2$ and get:

$$SO(1,3)^+ \simeq SL(2,\mathbb{C})/\mathbb{Z}_2 \tag{1.16}$$

as expected. We notice that this equality is actually a way to state the famous isomorphism between the global conformal group of $R^{1,1}$ and $SO(1,3)$. We can draw a better analogy by looking at spacetime in different coordinates, namely we parametrized flat spacetime using round retarded and advanced Bondi-Sachs coordinates [24, 61, 109] $(u, r, z, \bar{z})$ and $(v, r, z, \bar{z})$:

$$\begin{aligned} X_u &= \left(u + r, r\frac{z+\bar{z}}{1+z\bar{z}}, ir\frac{\bar{z}-z}{1+z\bar{z}}, r\frac{1-z\bar{z}}{1+z\bar{z}}\right), \\ X_v &= \left(v - r, r\frac{z+\bar{z}}{1+z\bar{z}}, ir\frac{\bar{z}-z}{1+z\bar{z}}, r\frac{1-z\bar{z}}{1+z\bar{z}}\right) \end{aligned} \tag{1.17}$$

with $r > 0$ a radial coordinates and $z, \bar{z}$ coordinate on the celestial sphere. Under these coordinates the metric takes the following form:

$$ds^2_u = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}, \quad \gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2} \tag{1.18}$$

$$ds^2_v = -dv^2 + 2dvdr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \tag{1.19}$$

The meaning of these coordinates can be better understood if we consider the following change of coordinates:

$$u = \tan U,\ \ v = \tan V, \quad U, V \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \tag{1.20}$$

$$T = U + V, \quad R = V - U \tag{1.21}$$

which take the metric to a conformally flat form:

$$ds^2 = \Omega^{-2}\left(-dT^2 + dR^2 + 2\sin^2 R\gamma_{z\bar{z}}dzd\bar{z}\right), \quad \Omega = 4\cos^2 U \sin^2 V \tag{1.22}$$

and allow us to represent all spacetime with its conformal compactification:
On the diagram we have identified different special regions:

- *Future (Past) Null Infinity* $\mathscr{I}^{+(-)}$ is reached for $r \to +\infty$ with fixed $u\,(v)$. By definition in such way we are reaching the surface where all the massless trajectories end up (begin) at $t \to +\infty(-\infty)$

- Fixed in the limit $r \to +\infty$ we can move along $\mathscr{I}^{+(-)}$ so that for $u\,(v) \to \pm\infty$ we reach respectively *the future of Future (Past) Null Infinity* $\mathscr{I}^{+(-)}_+$ and *the past of Future (Past) Null Infinity* $\mathscr{I}^{+(-)}_-$ namely the two-sphere at the boundary of $\mathscr{I}^{+(-)} \sim \mathbb{R} \times S^2$

- On the contrary if we take $u\,(v) \to +(-)\infty$, $t \to +(-)\infty$ keeping r fixed we reach a different region *Timelike Future (Past) Infinity* $i^{+(-)}$ where all massive trajectories end up (begin).

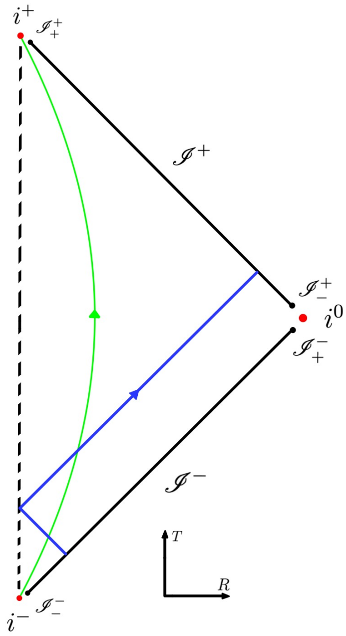

Figure 1.1: Penrose diagram of flat spacetime. Each point on this diagram represents a 2-sphere $(z, \bar{z})$. The green curve represents the trajectory of a massive particle going from past timelike infinity (i^-) to future timelike infinity (i^+). On the contrary the blue line represents the trajectory of a massless particle starting from $\mathscr{I}^-$ and ending up on $\mathscr{I}^+$

- The last non trivial limit is $u \to -\infty, r \to +\infty$ or $v \to +\infty, r \to +\infty$ for fixed t. This last region is called *Spacelike Infinity* i^0.

We notice that in T, R coordinates the surfaces $\mathscr{I}^+_+$, i^+ seem to coincide because taking the defining limit we get in both cases $(T, R) = (\pi, 0)$. However this identification is just an artifact of the coordinates of the Penrose diagram, and the two surfaces must be regarded as distinct objects. The same is valid for $\mathscr{I}^+_-, \mathscr{I}^+_-, i^0$ all in $(T, R) = (0, \pi)$ and $\mathscr{I}^-_-, i^-$ in $(T, R) = (-\pi, 0)$ which are fictitiously identified on the diagram but must be considered as distinct surfaces.
Let us now focus on the lightcone of the origin, identified choosing $u = 0$:

$$X^\mu_{u=0} = \frac{r}{1 + z\bar{z}} (1 + z\bar{z}, z + \bar{z}, i(\bar{z} - z), 1 - z\bar{z}) = \frac{r}{1 + z\bar{z}} q^\mu(z, \bar{z}) \tag{1.23}$$

If we transform this vector using the $\Lambda(M)$ defined in (1.15) we get:

$$\Lambda^\mu_\nu(M)\, X^\nu_{u=0} = (cz + d)(\bar{c}\bar{z} + \bar{d}) X^\mu_{u=0} \left(\frac{az + b}{cz + d}, \frac{\bar{a}\bar{z} + \bar{b}}{\bar{c}\bar{z} + \bar{d}} \right) \tag{1.24}$$

and we can see that, ignoring the overall "conformal weight factor", the group $SL(2, \mathbb{C})$ is acting as a 2d global conformal transformation on the $(z, \bar{z})$ coordinates. This can be considered as the action of the 2d global conformal group of an embedding of $\mathbb{R}^2$ in the lightcone.
Moreover it turns out that the whole 2d conformal group is an isometry of the origin lightcone. From the definition of conformal symmetry this is intuitively true but we can explicitly verify

this by considering the following vector [104]:

$$\xi = \frac{u}{2}\left(D_z Y^z + D_{\bar z} Y^{\bar z}\right)\partial_u - \frac{1}{2}(u+r)\left(D_z Y^z + D_{\bar z} Y^{\bar z}\right)\partial_r + \\ + \left[\left(1+\frac{u}{2r}\right)Y^z - \frac{u}{2r}D^z D_{\bar z} Y^{\bar z}\right]\partial_z + \left[\left(1+\frac{u}{2r}\right)Y^{\bar z} - \frac{u}{2r}D^{\bar z} D_z Y^z\right]\partial_{\bar z} + \\ + f\partial_u + D^z D_z f \partial_r - \frac{1}{r}D^z f\partial_z - \frac{1}{r}D^{\bar z} f\partial_{\bar z} \tag{1.25}$$

where Y^z a priori is a function of $z, \bar z$, f is a real function of $z, \bar z$ and D_A is the covariant derivative on the sphere. Taking the Lie derivative of the metric along ξ we obtain:

$$\mathcal{L}_\xi g = 2r^2\gamma_{z\bar z}\partial_z Y^{\bar z} d\bar z^2 - r\left(uD_z^3 Y^z + 2D_z^2 f\right)dz^2 - D_z(f + D^{\bar z}D_{\bar z} f)dzdu + h.c. \tag{1.26}$$

For ξ to be an isometry this must be set to zero and in particular solving the equations for $f(z,\bar z)$ and $Y^z(z)$, $Y^{\bar z}(\bar z)$ we get that the accepted functional forms are [104]:

$$f_0 = 1, \quad f_1 = \frac{z+\bar z}{1+z\bar z}, \quad f_2 = i\frac{\bar z - z}{1+z\bar z}, \qquad f_3 = \frac{1-z\bar z}{1+z\bar z} \tag{1.27}$$

$$Y^z_{12} = iz \quad Y^z_{13} = -\frac{1}{2}\left(1+z^2\right) \quad Y^z_{23} = -\frac{i}{2}\left(1-z^2\right) \tag{1.28}$$

$$Y^z_{01} = z \quad Y^z_{02} = -\frac{i}{2}\left(1+z^2\right) \quad Y^z_{03} = -\frac{1}{2}\left(1-z^2\right) \tag{1.29}$$

$$Y^{\bar z}_{ij} = (Y^z_{ij})^* \tag{1.30}$$

The f functions represent the 4 global translations while the Y are related to the 6 generators of the Lorentz group. Plugging these inside ξ we get the 10 global Killing vectors of Minkowski spacetime in retarded coordinates.
Coming back to the symmetries of the lightcone, setting $u = 0$ and $f = 0$ we can see that (1.26) is identically zero for every strictly meromorphic $Y(z)$. Namely we see that every meromorphic function gives an isometry of the lightcone, or in other words the full 2D conformal group is a lightcone isometry.

A convenient choice of coordinates - Flat Bondi

In the previous section we introduced the Bondi parametrization of flat space with round boundary representative, namely boundary geometry takes the form of a sphere with metric $\mathring{q}_{AB}dx^A dx^B = 2\gamma_{z\bar z}dzd\bar z$. However it is possible to choose a different set of coordinates such that the flat spacetime metric takes the form:

$$ds^2 = -2dudr + 2r^2 dzd\bar z, \tag{1.31}$$

so that the boundary geometry becomes that of a flat plane.
If we refer to the Bondi coordinates used in the previous section as $\{\mathring{u}, \mathring{r}, \mathring{z}, \mathring{\bar z}\}$ the new coordinates will take the form [110,111]:

$$r = \frac{\sqrt{2}}{1+\mathring{z}\mathring{\bar z}}\mathring{r} + \frac{\mathring{u}}{\sqrt{2}}, \quad u = \frac{1+\mathring{z}\mathring{\bar z}}{\sqrt{2}}\mathring{u} - \frac{\mathring{z}\mathring{\bar z}}{\sqrt{2}r}\mathring{u}^2, \quad z = \mathring{z} - \frac{\mathring{z}}{\sqrt{2}r}\mathring{u} \tag{1.32}$$

This choice of coordinates is particularly convenient because the connection on the celestial sphere becomes flat and simplifies the computations.
For example the vectors generating the Poincaré group can be written in this flat coordinates at leading order as [110, 112]:

$$\begin{aligned}\xi = &\left(\mathcal{T} + \frac{u}{2}(\partial Y^z + \bar{\partial} Y^{\bar{z}})\right)\partial_u + \left(-\frac{r}{2}(\partial Y^z + \bar{\partial} Y^{\bar{z}}) + o(r^0)\right)\partial_r \\ &+ (Y^z + o(r^{-1}))\partial_z + (Y^{\bar{z}} + o(r^{-1}))\partial_{\bar{z}}\end{aligned} \tag{1.33}$$

with parameters

$$\mathcal{T}_0 = 1, \quad \mathcal{T}_1 = z + \bar{z}, \quad \mathcal{T}_2 = i(\bar{z} - z), \qquad \mathcal{T}_3 = 1 - z\bar{z} \tag{1.34}$$

$$Y^z_{12} = iz \quad Y^z_{13} = -\frac{1}{2}\left(1 + z^2\right) \quad Y^z_{23} = -\frac{i}{2}\left(1 - z^2\right) \tag{1.35}$$

$$Y^z_{01} = z \quad Y^z_{02} = -\frac{i}{2}\left(1 + z^2\right) \quad Y^z_{03} = -\frac{1}{2}\left(1 - z^2\right) \tag{1.36}$$

$$Y^{\bar{z}}_{ij} = (Y^z_{ij})^* \tag{1.37}$$

that correspond to the 4 translations, the 3 spatial rotations and the 3 boosts.

1.2 Klein space

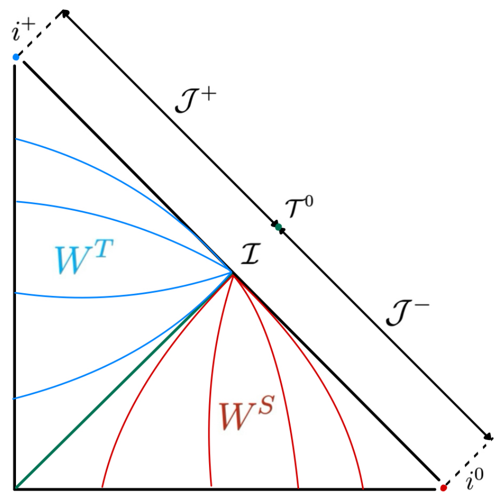

Figure 1.2: Penrose diagram of Klein space. The regions W^T and W^S represent the time-like and space-like wedges, foliated by hyperbolic slices with $\mathrm{AdS}^3/\mathbb{Z}$ geometry. The null boundary $\mathcal{I}$ has the geometry of $\mathbb{R} \times T^2$, where T^2 is a 2d torus. Time-like and space-like infinity i^+ and i^0 have both the geometry of $\mathrm{AdS}^3/\mathbb{Z}$. The full connected boundary S^3 form by the union of i^+, i^0 and $\mathcal{I}$ can be split into two caps $\mathcal{J}^+$ and $\mathcal{J}^-$ separated by a torus $\mathcal{T}^0$. These regions are relevant in the description of the scattering problem in Klein space.

Working in quantum field theory, we are well accustomed to analytic continuations. A standard example is the Wick rotation, which maps Minkowski space $\mathbb{R}^{3,1}$ to Euclidean signature. This procedure is widely used to compute Feynman diagrams in perturbation theory, to

better understand their analytic structure, to implement lattice regularization, and to access non-perturbative instantonic effects.
Unfortunately, in the context of flat space holography, this extension is not useful: it maps the extended conformal boundary of $\mathbb{R}^{3,1}$ to the boundary of $\mathbb{R}^4$, which is just a single point. As a result, it becomes much more difficult to define a boundary theory capable of capturing the rich structure of quantum gravity in $\mathbb{R}^4$.
In flat space holography a much more useful analytic extension turns Minkowski space into Klein space $\mathbb{R}^{2,2}$, namely we change from signature $(3,1)$ to $(2,2)$. Such analytic extensions is not a novelty at all and it has been used in many relevant publications, [113–120] as it provides dramatic simplifications to on-shell descriptions of scattering amplitudes. Differently from Euclidean space, Klein space has a rich boundary structure which we aim to discuss in this chapter and in recent years has found applications not only in flat space holography, and in particular celestial holography, but also in different areas of physics, like self-dual Yang-Mills, and self-dual gravity.
This section is meant to be a review of the structure of such spacetime and it will be mostly based on [121].

Starting from the basics, Klein space has the topology of $\mathbb{R}^4$ endowed with the $(2,2)$ metric [121]:

$$ds^2 = -(dX^0)^2 - (dX^1)^2 + (dX^2)^2 + (dX^3)^2 \tag{1.38}$$

As for flat space we start discussing its isometries, namely the group $\mathbb{R}^{2,2} \rtimes O(2,2)$. Neglecting the translation action $R^{2,2}$, we will focus on component connected to the identity of $O(2,2)$ namely $SO(2,2)$, which has a double cover $\text{Spin}(2,2) = SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$, in particular we will have the isomorphism:

$$SO(2,2) \simeq PSL(2,\mathbb{R})_L \times PSL(2,\mathbb{R})_R \tag{1.39}$$

The isomorphism can be explicitly realized using a construction similar to what we have seen for Minkowski space. Choosing the following coordinates:

$$z = X^2 + iX^3, \quad w = X^0 + iX^1, \tag{1.40}$$

we can express the vector fields generating the action of $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ as follows [121]:

$$L_1 = \bar{z}\partial_w + \bar{w}\partial_z, \qquad \bar{L}_1 = z\partial_w + \bar{w}\partial_{\bar{z}} \tag{1.41}$$

$$L_0 = \frac{1}{2}\left(z\partial_z + w\partial_w - \bar{z}\partial_{\bar{z}} - \bar{w}\partial_{\bar{w}}\right), \qquad \bar{L}_0 = \frac{1}{2}\left(-z\partial_z + w\partial_w + \bar{z}\partial_{\bar{z}} - \bar{w}\partial_{\bar{w}}\right) \tag{1.42}$$

$$L_{-1} = -z\partial_{\bar{w}} - w\partial_{\bar{z}}, \qquad \bar{L}_{-1} = -\bar{z}\partial_{\bar{w}} - w\partial_z \tag{1.43}$$

It is trivial to check that the metric

$$ds^2 = dzd\bar{z} - dwd\bar{w}, \tag{1.44}$$

is invariant under the action of these vector fields. Notice that the group $PSL(2,\mathbb{R})_L \times PSL(2,\mathbb{R})_R$ is also a double copy of the conformal group in 1 dimension. From an holographic perspective this reflects a change in the boundary structure. This is what we want now to investigate. To study the conformal structure of Klein space we start by choosing another set of coordinates:

$$z = re^{i\phi}, \quad w = qe^{i\varphi}, \text{ with } q, r \geq 0,\ \phi, \varphi \in [0, 2\pi), \tag{1.45}$$

$$ds^2 = -dq^2 - q^2 d\varphi^2 + dr^2 + r^2 d\phi^2 \tag{1.46}$$

In this coordinate system we see that for fixed q and r we have a Lorentzian torus parametrized by φ and ϕ. From this prospective we can then interpret φ as a compactified time direction. To conformally compactify $\mathbb{K}^{2,2}$ we can then use the change of variables

$$\tan U = q - r, \quad \tan V = q + r, \tag{1.47}$$

$$ds^2 = \frac{1}{\cos^2 U \cos^2 V}\left(-dUdV - \frac{1}{4}\sin^2(U+V)d\varphi^2 + \frac{1}{4}\sin^2(U-V)d\phi^2\right) \tag{1.48}$$

Because $q - r$ has value in $(-\infty, +\infty)$, U has domain $(-\pi/2, \pi/2)$. On the contrary $r + u$ and $2r = \tan V - \tan U$ are always positive, which restricts to domain of V to $|U| < V < \pi/2$.
The values $V = \pm U$ coincide with $r = 0$ and $q = 0$ respectively, so are the origins of the planes spanned by the two space directions or the two time directions. This is consistent with the fact that for these values the boundary is located at $V = \pi/2$ and coincide for finite $-\pi/2 < U < \pi/2$ with the null boundary $\mathcal{I}$ of Klein space. Taking away the diverging conformal factor we see that the boundary metric degenerates to

$$ds^2_{\mathcal{I}} = -d\varphi^2 + d\phi^2 \tag{1.49}$$

namely we have the Lorentzian metric on a flat torus. This implies that the null boundary is the product of a Lorentzian torus times a null direction.
This choice of variables allows us to correctly describe the null boundary of Klein space, namely the boundary where $q, r \to +\infty$ keeping $q - r$ finite. On the contrary, the points

$$\begin{aligned} &i^+ \text{ reached taking } q \to +\infty \text{ with } r \text{ finite,} \\ &i^0 \text{ reached taking } r \to +\infty \text{ with } q \text{ finite,} \end{aligned} \tag{1.50}$$

cannot be resolved within this coordinate system. This is analogous to what happens in Minkowski space where the parametrization in Bondi coordinates do not allow to resolve space-like ant timelike infinity.
To resolve these regions we chose a different set of coordinates:

$$W^T : X^\mu = \tau x_+^\mu, \text{ with } x_+^1 + ix_+^2 = e^{i\psi}\cosh\rho, \quad x_+^2 + ix_+^3 = e^{i\varphi}\sinh\rho, \quad x_+^2 = -1 \tag{1.51}$$

$$ds^2 = -d\tau^2 + \tau^2\left(-\cosh^2\rho\, d\varphi^2 + \sinh^2\rho\, d\psi^2 + d\rho^2\right), \tag{1.52}$$

$$W^S : X^\mu = \tau x_-^\mu, \text{ with } x_+^1 + ix_+^2 = e^{i\psi}\sinh\rho, \quad x_+^2 + ix_+^3 = e^{i\varphi}\cosh\rho, \quad x_-^2 = +1 \tag{1.53}$$

$$ds^2 = d\tau^2 - \tau^2\left(-\cosh^2\rho\, d\varphi^2 + \sinh^2\rho\, d\psi^2 + d\rho^2\right), \tag{1.54}$$

with $\tau > 0$, $\rho > 0$, $\varphi, \psi \in [0, 2\pi)$. We see that these two patches cover two separate regions, namely $X^2 < 0$ and $X^2 > 0$ namely the timelike W^T and spacelike W^S wedge respectively, and

intersect on the null cone $X^2 = 0$. As we can see from the metric these coordinates are clearly describing an hyperbolic foliation of Klein space into $\mathrm{AdS}_3/\mathbb{Z}$ slices. The parameter τ moves us along the slices while the coordinates ψ, ρ, φ parametrize the AdS_3 slice with compactified time ψ. This compactification is described through the identification $d\psi \sim d\psi + 2\pi$ which turns AdS_3 into $\mathrm{AdS}_3/\mathbb{Z}$. Notice that differently from Minkowski where the timelike and spacelike regions are foliated respectively by euclidean AdS_3, namely H_3 slices, and dS_3 slices, in Klein space both regions are foliated by compact time AdS_3 slices.
Then by taking τ to $+\infty$ we see that both i^+ and i^0 have the geometry of $\mathrm{AdS}^3/\mathbb{Z}$. In summary we have seen then that the boundary of Klein space decomposes into:

- A null boundary $\mathcal{I}$ which has the geometry of a null interval times a Lorentzian torus.
- A time-like and space like infinity both having the geometry of $\mathrm{AdS}_3/\mathbb{Z}$.

These pieces can be nicely put together to form the geometry of S^3. This can be seen from the following reasoning: S^3 can be decomposed into a line times a torus completed by two caps both identical to a disk times a circle. This is identical to say that a sphere S^2 can be seen as a cylinder completed by two disks at the caps. As we have shown $\mathcal{I}, i^+$ and i^0 have precisely the same geometry of the various patches building S^3 so their union reconstruct the topology of S^3 [121].
This implies that the boundary of Klein space is actually S^3 a fully connected manifold. Clearly, this makes holographic considerations in $\mathbb{K}^{2,2}$ simpler than in Minkowski space, where the boundary consists of several disconnected components.

Scattering in Klein space

Our analytic extension from Minkowski to Klein space serves as a powerful tool to simplify the treatment of scattering amplitudes in flat spacetime. This necessitates a clear understanding of how to define scattering processes within Klein space. The procedure for this definition was established in [122], which we briefly review here.

In Minkowski we usually define an S matrix, which acts as the transition matrix between the *in* and *out* states, namely for massless particles it maps states defined on the past null boundary $\mathscr{I}^-$ to states in the future null boundary $\mathscr{I}^+$. The amplitudes are computed as the inner product:

$$\mathcal{A} = \langle \mathrm{out} | \, S \, | \mathrm{in} \rangle \tag{1.55}$$

In this construction, a key point is that we have a well-defined notion of future and past represented by the separation of the past and future boundary.
In Klein space such notion is not well defined, as we have two different time directions and it is not strictly possible to have a unique notion of past and future. From a geometric standpoint, this is reflected by the fact that the boundary is fully connected and homeomorphic to S^3. States will be defined on this connected boundary, and there is no necessity for a transition matrix S.
However, we would like to connect the analytic extension of the S-matrix from Minkowski to Klein space with some quantity which is well defined in Klein space.

To do so, we introduce in Klein space a quantity called S-vector [123]. This will be a vector $|S\rangle$ on S^3 such that given a state for incoming particles to the boundary $|IN\rangle$ we have the following identification:

$$\langle S|IN\rangle = \langle \text{out}|\, S\, |\text{in}\rangle_{(3,1)\to(2,2)} \tag{1.56}$$

where the pedicle $(3,1)\to(2,2)$ denotes the analytic extension. In Klein space the $|IN\rangle$ state would contain information about the states of both *in* and *out* particles in Minkowski.
The explicit construction of $|S\rangle$ has been carried out for the free scalar in [122]. This construction relies on the decomposition of the boundary S^3 into two regions $\mathcal{J}^{\pm}$, with a single common region represented by a torus $\mathcal{T}^0$ located at $U=0$. Then we can build a Klein-Gordon inner product on these halves $\mathcal{J}^{\pm}$ by taking the boundary limit of the naturally occurring symplectic form on the hyperbolic foliation into $AdS_3/\mathbb{Z}$ slices.
The Klein Gordon inner product can be used as a basis to build the Hilbert spaces $\mathcal{H}_{\pm}$ on the two patches. $\mathcal{H}_{\pm}$ can be considered as independent Hilbert spaces of the two regions W^T and W^S. Each of these Hilbert states has a well defined vacuum states $|0\rangle_+$, $|0\rangle_-$ invariant under $SL(2,C)_L \times SL(2,C)_R$ but not under Klein space translations. This is just an accident coming from the fact that we are building the Hilbert space not of full Klein space but only for the two regions W^T, W^S, and in fact we are finding a vacuum which is invariant only under the isometries of the Lorentian hyperboloid foliating W^T and W^S.
The full Hilbert space on $\mathcal{I}$ will be obtained by taking the tensor product of $\mathcal{H}_{\pm}$:

$$\mathcal{H} = \mathcal{H}_+ \otimes \mathcal{H}_- \tag{1.57}$$

with vacuum state invariant under the full isometries of Klein space, $|S\rangle$ defined as a Bogoliubov transformation of the direct product $|0\rangle_+ \otimes |0\rangle_-$. Now suppose that we take a state $|\Psi\rangle \in \mathcal{H}_-$. Then $\langle\Psi|S\rangle \in \mathcal{H}_+$, namely the full vacuum acts as a natural map between the two regions $\mathcal{J}^{\pm}$. In [122], the authors claim that, for the free scalar theory, this vacuum state serves as the S-matrix once analytically continued back to Minkowski space.

Chapter 2

Asymptotic states and S-matrix

In flat space Quantum Field Theory (QFT), a typical problem involves computing transition probabilities between an initial and a final state, each specified, for instance, by a definite number of particles with fixed momenta.
Starting from a Lagrangian field theory this statement is usually well-posed by following a set of steps:

1. We solve the equations of motions as functions of the free data, or initial conditions, Φ, Π specified on a Cauchy surface Σ. These free data constitute a parametrization of phase space which is naturally embedded with a canonical symplectic structure. In canonical quantization, this symplectic structure Ω is fundamental to the construction of the Hilbert space, as it allows us to define the canonical commutation relations and the Klein-Gordon inner product. To be more specific we have the following relations:

$$\begin{aligned}(\Phi_1, \Phi_2)_{KG} &= -i(\Phi_1, \Phi_2)_{\text{sym}} \\ [\Phi_1, \Phi_2] &= i\{\Phi_1, \Phi_2\}\end{aligned} \tag{2.1}$$

   where $(\cdot,\cdot)_{\text{sym}}$ is the symplectic inner product defined from the symplectic current, and $\{\cdot,\cdot\}$ is the Poisson bracket constructed from the inverse symplectic form[1]. Given the commutation relations we can find the variables in phase space that will form the maximally commuting subset of Hermitian operators, whose eigenstates will fully specify the Hilbert space on the slice Σ.

2. After having specified the kinematic of the problem, we turn to the dynamics. Thanks to the geometry of Minkowski space we know that the Poincaré group must be a symmetry of our Lagrangian model, and from Noether's theorem we have a set of conserved currents associated to the generators of Poincaré. In particular given the stress-tensor $T^{\mu\nu}$ the evolution operator between Cauchy surfaces will take the form:

$$H = \int_\Sigma d\sigma_\mu T^{\mu 0} \tag{2.2}$$

[1] We recall that it is not generally possible to directly invert the symplectic form due to the presence of gauge redundancies or constraints. In this situation, the Poisson brackets get substituted by Dirac brackets [124].

where $d\sigma_\mu$ is the infinitesimal surface element of Σ proportional to the normal vector N^μ to Σ. If we consider a family of Cauchy surfaces parametrized by the parameter τ then, H will be τ independent and the evolution between Hilbert spaces will be determined as:

$$|\Psi, \tau'\rangle = e^{iH(\tau'-\tau)} |\Psi, \tau\rangle . \tag{2.3}$$

The probability of transition between states is then determined from the absolute value of:

$$\langle \Psi', \tau' | \Psi, \tau' \rangle = \langle \Psi', \tau | \, e^{-iH(\tau'-\tau)} \, |\Psi, \tau' \rangle . \tag{2.4}$$

Even if the procedure is straightforward on paper its proper realization in the case of interacting theories is an extremely difficult computational task, which forces us in particular regimes, where typically we can take a parameter to be small and solve the model perturbatively around a known solution.

This is used, for instance, to construct the Hilbert space in interacting theories by evaluating the fields on a Cauchy slice in the far past or future, where one assumes that particles can be effectively treated as free [102, 125]. Then we can treat the time evolution assuming we can consistently take strength of the interactions to be small and solve the interacting theory perturbing around the free solution.

This is in words the typical scattering set up where the states are defined at time $\tau' = t_2 = +\infty$ and $\tau = t_1 = -\infty$ as free and the evolution operator is represented by the S matrix defined in the interaction picture as [102, 125, 126]:

$$S = \lim_{t_{1,2} \to \mp\infty} S(t_1, t_2), \quad S(t_1, t_2) = \mathcal{T} \exp\left\{ i \int_{\Sigma_1}^{\Sigma_2} \mathcal{L}_{int} \right\} \tag{2.5}$$

where $\mathcal{T}$ denotes the time ordering operator.

The above discussion was aimed to be an extremely concise introduction to the basic set up of the scattering problem in QFT, which will be the subject of this chapter. To introduce our conventions we will start to consider massless free fields of arbitrary integer helicity, and massive spin 0 particles. Specializing to the cases of spin-1 and spin-2 fields, we will then review a typical problem which arises with the S-matrix of massless spinning particles which will also be a fundamental topic for the subsequent sections: infrared divergences, soft factorization, and how to handle them via Faddeev-Kulish dressing.

2.1 Free fields

In this section we will review properties of free fields, adopting the conventions and the discussion given in [110]. Let us consider a free massless bosonic spin-s Fronsdal field $\Phi_I^{(s)}$ in Minkowski spacetime ($s = 0, 1, 2, \dots$). $I = (\mu_1 \mu_2 \dots \mu_s)$ is used as a short-hand notation for a set of symmetrized indices $\mu_i \in \{0, \dots, 3\}$. The massless spin-$s$ field $\Phi_I^{(s)}$ can be conveniently put in traceless gauge [110]

$$\partial^\nu \Phi^{(s)}_{\nu\mu_2 \dots \mu_s}(X) = 0, \qquad \eta^{\mu\nu} \Phi^{(s)}_{\mu\nu\mu_3 \dots \mu_s}(X) = 0, \tag{2.6}$$

such that the equations of motion take the simple form

$$\partial^\mu \partial_\mu \Phi_I^{(s)}(X) = 0. \tag{2.7}$$

The de Donder gauge is not a complete gauge fixing but leaves some residual gauge transformations:

$$\begin{aligned} &\delta_\lambda \Phi^{(s)}_{\mu_1\ldots\mu_s}(X) = \partial_{(\mu_1}\lambda_{\mu_2\ldots\mu_s)}(X) \\ &\text{with } \partial^\nu \lambda_{\nu\mu_3\ldots\mu_s}(X) = 0 \quad \eta^{\mu\nu}\lambda_{\mu\nu\mu_4\ldots\mu_s}(X) = 0. \end{aligned} \tag{2.8}$$

The field can be quantized in the Heisenberg representation and expanded in Fourier modes. Each mode is labeled by an on-shell null 4-momentum vector

$$p^\mu(\omega, w, \bar{w}) = \omega\, q^\mu(w, \bar{w}), \quad q^\mu(w,\bar{w}) = \Big(1 + w\bar{w}, w + \bar{w}, -i(w - \bar{w}), 1 - w\bar{w}\Big) \tag{2.9}$$

parametrized by the frequency $\omega > 0$ and coordinates $(w, \bar{w})$ on the complex plane. Let $\varepsilon^{\pm}_\mu(\vec{q})$ be the polarizations,

$$\begin{aligned} \varepsilon^+_\mu(\vec{q}) &= \frac{1}{\sqrt{2}}\partial_w q_\mu = \frac{1}{\sqrt{2}}\big(-\bar{w}, 1, -i, -\bar{w}\big), \\ \varepsilon^-_\mu(\vec{q}) &= [\varepsilon^+_\mu(\vec{q})]^* = \frac{1}{\sqrt{2}}\partial_{\bar{w}} q_\mu = \frac{1}{\sqrt{2}}\big(-w, 1, i, -w\big). \end{aligned} \tag{2.10}$$

The spin-s field in de Donder gauge (2.6) can then be written in term of the creation and annihilation operators by expanding it in Fourier modes as

$$\Phi_I^{(s)}(X) = K_{(s)} \sum_{\alpha=\pm} \int \frac{d^3p}{(2\pi)^3\, 2p^0} \Big[\varepsilon_I^{*\alpha}(\vec{q})\, a_\alpha^{(s)}(\vec{p})\, e^{ip^\mu X_\mu} + \varepsilon_I^{\alpha}(\vec{q})\, a_\alpha^{(s)}(\vec{p})^\dagger\, e^{-ip^\mu X_\mu}\Big], \tag{2.11}$$

where the polarization tensor can be written in terms of the single co-vectors as

$$\varepsilon^{\pm}_{\mu_1\ldots\mu_s}(\vec{q}) = \varepsilon^{\pm}_{\mu_1}(\vec{q})\varepsilon^{\pm}_{\mu_2}(\vec{q})\ldots\varepsilon^{\pm}_{\mu_s}(\vec{q}), \tag{2.12}$$

which are fully symmetric and transverse tensors, namely $q^{\mu_i}\varepsilon^{\pm}_{\mu_1\ldots\mu_i\ldots\mu_s}(\vec{q}) = 0$ for any μ_i. The positive real constant $K_{(s)}$ is usually just a normalization that can depend on the coupling constants of the interacting model. For QED and gravity for example we have for spin 1 and 2

$$K_{(1)} = e, \quad K_{(2)} = \sqrt{32\pi G} = \kappa. \tag{2.13}$$

where e is the electric charge and G is the Newton gravitational constant.
The canonical commutation relations between the field and its momentum force the oscillators to obey the well-known commutation relations

$$\Big[a_\alpha^{(s)}(\vec{p}), a_{\alpha'}^{(s)}(\vec{p}')^\dagger\Big] = (2\pi)^3\, 2p^0\, \delta^{(3)}(\vec{p} - \vec{p}')\, \delta_{\alpha,\alpha'}. \tag{2.14}$$

Importantly, a residual gauge transformation (2.8) driven by $\lambda_{\mu_2\ldots\mu_s}(X)$ admitting a well defined Fourier transform acts on the field (2.11) by addition of a term of the form

$$\partial_{(\mu_1}\lambda_{\mu_2\ldots\mu_s)}(X) = \int \frac{d^3p}{(2\pi)^3\, 2p^0} \Big[p_{(\mu_1}\hat{\lambda}_{\mu_2\ldots\mu_s)}(\vec{p})\, e^{ip^\mu X_\mu} + p_{(\mu_1}\hat{\lambda}^*_{\mu_2\ldots\mu_s)}(\vec{p})\, e^{-ip^\mu X_\mu}\Big]. \tag{2.15}$$

This impacts the polarization tensors as

$$\delta_\lambda \varepsilon^\alpha_{\mu_1\ldots\mu_s}(\vec{q}) = q_{(\mu_1}\tilde{\lambda}^\alpha_{\mu_2\ldots\mu_s)}(\vec{p}) \tag{2.16}$$

writing $\hat{\lambda}_{\mu_2\ldots\mu_s}(\vec{p}) \equiv \frac{K_{(s)}}{\omega}\tilde{\lambda}^\alpha_{\mu_2\ldots\mu_s}(\vec{p})a^{(s)}_\alpha(\vec{p})$ and $\eta^{\mu\nu}\tilde{\lambda}^\alpha_{\mu\nu\mu_4\ldots\mu_s} = 0$, but the ladder operators $a^{(s)}_\alpha$ are left invariant by these gauge transformations[2].

Using the parametrization (2.9), we can write the field (2.11) as

$$\Phi^{(s)}_I(X) = \frac{K_{(s)}}{16\pi^3}\sum_{\alpha=\pm}\int \omega\, d\omega\, d^2w \left[a^{(s)}_\alpha(\omega,w,\bar{w})\varphi^{*\alpha,(s)}_I(\omega,w,\bar{w}|X) + a^{(s)}_\alpha(\omega,w,\bar{w})^\dagger \varphi^{\alpha,(s)}_I(\omega,w,\bar{w}|X)\right] \tag{2.17}$$

where d^2w denotes the integration measure on the complex plane with local holomorphic coordinates $(w,\bar{w})$ and

$$\varphi^{*\alpha,(s)}_I(\omega,w,\bar{w}|X) \equiv \varepsilon^{*\alpha}_I(w,\bar{w})\, e^{i\omega q^\mu X_\mu} \tag{2.18}$$

are the basis vectors of plane waves. In this parametrization, the canonical commutation relations (2.14) become

$$\left[a^{(s)}_\alpha(\omega,w,\bar{w}), a^{(s)}_{\alpha'}(\omega',w',\bar{w}')^\dagger\right] = 16\pi^3\,\omega^{-1}\delta(\omega-\omega')\,\delta^{(2)}(w-w')\,\delta_{\alpha,\alpha'}. \tag{2.19}$$

Poincaré transformations $X'^\mu = \Lambda^\mu{}_\nu X^\nu + t^\mu$ act on the gauge field as

$$\Phi^{(s)}_{\mu_1\mu_2\ldots\mu_s}(X) \mapsto \Phi'^{(s)}_{\mu_1\mu_2\ldots\mu_s}(X') = \Lambda_{\mu_1}{}^{\nu_1}\Lambda_{\mu_2}{}^{\nu_2}\ldots\Lambda_{\mu_s}{}^{\nu_s}\Phi^{(s)}_{\nu_1\nu_2\ldots\nu_s}(X). \tag{2.20}$$

Lorentz transformations induce a $SL(2,\mathbb{C})$ Möbius transformation

$$w \mapsto w'(w) = \frac{aw+b}{cw+d} \tag{2.21}$$

with $ad-bc=1$ on the complex coordinates determining the direction of the null momentum q^μ with the embedding of the Riemann sphere into the light cone (2.9).

Since p^μ is a Lorentz vector, one has

$$\omega' = \left|\frac{\partial w'}{\partial w}\right|^{-1}\omega, \qquad q^\mu(w',\bar{w}') = \left|\frac{\partial w'}{\partial w}\right|\Lambda^\mu{}_\nu q^\nu(w,\bar{w}). \tag{2.22}$$

The second equation states the fact that a Lorentz transformation on the light cone induces the corresponding Möbius transformation on the Riemann sphere via the embedding (2.9). Owing to (2.10) and (2.22), one can show that $\varepsilon^\pm_\mu(w,\bar{w})$ do not transform homogeneously under the action of (2.21) but the supplementary terms are part of the residual gauge freedom (2.16), *i.e.*

$$\varepsilon'^\pm_\mu(w',\bar{w}') = \left(\frac{\partial w'}{\partial w}\right)^{\mp\frac{1}{2}}\left(\frac{\partial \bar{w}'}{\partial \bar{w}}\right)^{\pm\frac{1}{2}}\Lambda_\mu{}^\nu\varepsilon^\pm_\nu(w,\bar{w}) + \mathscr{A}(w,\bar{w})\Lambda_\mu{}^\nu q_\nu(w,\bar{w}), \tag{2.23}$$

where $\mathscr{A}$ is a fixed function of $(w,\bar{w})$. For the expansion (2.17)–(2.18) recalling that the integration measure and the plane wave are Lorentz-invariant, we deduce from (2.23) that the transformation of ladder operators under the Poincaré group is

$$a'^{(s)}_\pm(\omega',w',\bar{w}') = \left(\frac{\partial w'}{\partial w}\right)^{-\frac{J}{2}}\left(\frac{\partial \bar{w}'}{\partial \bar{w}}\right)^{\frac{J}{2}} e^{-i\omega q^\mu(w,\bar{w})\Lambda^\nu{}_\mu t_\nu}\, a^{(s)}_\pm(\omega,w,\bar{w}). \tag{2.24}$$

[2]Notice that this is true only for proper gauge transformations. Large gauge transformations can actually impact the structure of oscillators.

In particular, one recovers that the ladder operators are eigenvectors of translations, which is expected since they are assumed to create/annihilate energy eigenstates. The infinitesimal version of (2.24) can be obtained by setting $X'^\mu = X^\mu - \epsilon\xi^\mu$, with $\xi^\mu = \varpi^\mu{}_\nu X^\nu + \tau^\mu$ ($\varpi_{\mu\nu} = \varpi_{[\mu\nu]}$) and $w'(w) = w - \epsilon\mathcal{Y}^w(w)$, and retaining only the linear terms in ϵ. One concludes that

$$\begin{aligned}&\delta_{\xi(\mathcal{T},\mathcal{Y})} a_\pm^{(s)}(\omega, w, \bar{w}) \\ &= \left[-i\omega\mathcal{T} + \mathcal{Y}^w\partial_w + \mathcal{Y}^{\bar{w}}\partial_{\bar{w}} + \frac{J}{2}\partial_w\mathcal{Y}^w - \frac{J}{2}\partial_{\bar{w}}\mathcal{Y}^{\bar{w}} - \frac{\omega}{2}(\partial_w\mathcal{Y}^w + \partial_{\bar{w}}\mathcal{Y}^{\bar{w}})\partial_\omega\right] a_\pm^{(s)}(\omega, w, \bar{w})\end{aligned} \tag{2.25}$$

where $\xi(\mathcal{T},\mathcal{Y})$ is now parametrized by the function $\mathcal{T}(w,\bar{w}) = -q^\mu(w,\bar{w})\tau_\mu$ and the vector $\mathcal{Y} = \mathcal{Y}^w(w)\partial_w + \mathcal{Y}^{\bar{w}}(\bar{w})\partial_{\bar{w}}$ on the Riemann sphere, acting on momenta parametrized by coordinates $(\omega, w, \bar{w})$.

2.1.1 Boundary fields

The free field can also be pushed at $\mathscr{I}^+$ to extract its boundary mode[3]. In retarded flat Bondi coordinates (1.32) we extract these modes restricting the field on constant r surfaces an then reach $\mathscr{I}^+$ taking $r \to +\infty$. This translates into the following expression:

$$\iota^\star\left(\Phi^{(s)}_{\mu_1\ldots\mu_s} dX^{\mu_1}\ldots dX^{\mu_n}\right) = r^{s-1}\left(\phi_Z^{(s)}(u,z,\bar{z})dz^s + \phi_{\bar{Z}}^{(s)}(u,z,\bar{z})d\bar{z}^s\right) + o(r^{s-2}), \tag{2.26}$$

where $\iota^\star$ denotes the pull back of the field on constant r surfaces, and $Z = z\ldots z$, $\bar{Z} = \bar{z}\ldots\bar{z}$. We see that the leading asymptotic behavior of the field at infinity depends on the spin, scaling as r^{s-1}. The boundary modes then are identified with the coefficients $\phi_Z^{(s)}$ and $\phi_{\bar{Z}}^{(s)}$, which may be expressed in terms of oscillators as:

$$\begin{aligned}\phi_Z^{(s)}(u,z,\bar{z}) &= -\frac{K_{(s)}}{8\pi^2 i}\int_0^{+\infty} d\omega\left[a_+^{(s)}(\omega,z,\bar{z})e^{-i\omega u} - a_-^{(s)\dagger}(\omega,z,\bar{z})e^{i\omega u}\right], \\ \phi_{\bar{Z}}^{(s)}(u,z,\bar{z}) &= -\frac{K_{(s)}}{8\pi^2 i}\int_0^{+\infty} d\omega\left[a_-^{(s)}(\omega,z,\bar{z})e^{-i\omega u} - a_+^{(s)\dagger}(\omega,z,\bar{z})e^{i\omega u}\right].\end{aligned} \tag{2.27}$$

Notice that given these boundary modes we can extract the oscillators by taking the Fourier transform:

$$\begin{aligned}a_+^{(s)}(\omega,z,\bar{z}) &= \frac{4\pi i}{K_{(s)}}\int_{-\infty}^{+\infty} du\, e^{i\omega u}\phi_Z^{(s)}(u,z,\bar{z}), \\ a_-^{(s)}(\omega,z,\bar{z}) &= \frac{4\pi i}{K_{(s)}}\int_{-\infty}^{+\infty} du\, e^{i\omega u}\phi_{\bar{Z}}^{(s)}(u,z,\bar{z}),\end{aligned} \tag{2.28}$$

which allow us also to recompose the field inside the bulk using the expression (2.17). The ladder operators $a_\alpha^{(s)}(\vec{p})$, $a_\alpha^{(s)\dagger}(\vec{p})$ can also be extracted from the field expression using the Klein-Gordon inner product, which in de Donder gauge takes the form:

$$(\Phi_1, \Phi_2)_{KG} = -i\int_\Sigma d\Sigma^\mu\left(\Phi^{(s)}_{1,I}\partial_\mu\Phi^{(s)*}_{2,J} - \Phi^{(s)*}_{2,I}\partial_\mu\Phi^{(s)}_{1,J}\right)\eta^{IJ}, \tag{2.29}$$

where $I = (\mu_1\ldots\mu_s)$, $J = (\nu_1\ldots\nu_s)$ and $\eta^{IJ} = \eta^{\mu_1\nu_1}\ldots\eta^{\mu_s\nu_s}$. The ladder operators are then derived from the contraction:

$$a_\pm(\vec{p}) = \left(\Phi^{(s)}, \epsilon^*_\pm e^{-ip\cdot X}\right)_{KG}. \tag{2.30}$$

[3]The same can be done for $\mathscr{I}^-$ by choosing advanced coordinates

At quantum level, $a_{+}^{(s)}(\omega, w, \bar{w})^{\dagger}$ (resp. $a_{-}^{(s)}(\omega, w, \bar{w})^{\dagger}$) creates a massless particle of spin s and helicity $J = +s$ (resp. $J = -s$), with energy ω and a null momentum pointing towards the direction $q^{\mu}(w, \bar{w})$. In particular if the vacuum state is set to zero by the action of annihilation operators as:

$$a_{\pm}^{(s)}(\omega, z, \bar{z}) |0\rangle = 0, \tag{2.31}$$

a state with N particles with helicity J_i and momentum $p_i = \omega q_i(z, \bar{z})$ is obtained as:

$$a_{J_1}^{(s_1)\dagger}(\omega_1, z_1, \bar{z}_1) \dots a_{J_N}^{(s_N)\dagger}(\omega_N, z_N, \bar{z}_N) |0\rangle . \tag{2.32}$$

2.1.2 Scattering

As mentioned in the introduction, the free field construction is an important step to define our scattering model, as it allows us to have control over the definition of the Hilbert space under the assumption that at early and late time interacting particles behave as free.
For massless particles, using the free field construction we can then build in and out Hilbert spaces $\mathcal{H}_{\text{in}}, \mathcal{H}_{\text{out}}$ containing states of the form:

$$\begin{aligned} &a_{J_1,\text{in}}^{(s_1)\dagger}(\omega_1, z_1, \bar{z}_1) \dots a_{J_N,\text{in}}^{(s_N)\dagger}(\omega_N, z_N, \bar{z}_N) |0\rangle_{\text{in}} \\ &a_{J_1,\text{out}}^{(s_1)\dagger}(\omega_1, z_1, \bar{z}_1) \dots a_{J_N,\text{out}}^{(s_N)\dagger}(\omega_N, z_N, \bar{z}_N) |0\rangle_{\text{out}} , \; N \in \mathbb{N}, \end{aligned} \tag{2.33}$$

where $a_{J,\text{in}}^{(s)}, a_{J,\text{out}}^{(s)}$ will be respectively the annihilation operators defined from the quantization of the free data living on the Cauchy surfaces $\mathscr{I}^{\mp}$, and generate states respectively in $\mathcal{H}_{\text{in}}, \mathcal{H}_{\text{out}}$.
The transition amplitude is then obtained thanks to the S-matrix that works as a map between the in and out Hilbert space. Using the LSZ reduction formula the amplitude can be obtained from time-ordered correlations functions G of the interacting fields as:

$$\begin{aligned} \langle p_1, J_1; \dots | \, S \, | p_1', J_1'; \dots \rangle = &\prod_{i=1}^{n} \lim_{p_i^2 \to 0} \epsilon_{N_i}^{J_i *}(p_i) p_i^2 \prod_{j=1}^{m} \lim_{p_j'^2 \to 0} \epsilon_{M_j}^{J_j}(p_j') {p_j'}^2 \times \\ &\times G^{N_1 \dots N_n M_1 \dots M_m}(p_1, \dots, p_n; p_1', \dots, p_m') \end{aligned} \tag{2.34}$$

This is the expression for the scattering of m incoming and n outgoing particles with momentum and helicity p_i', J_i' and p_i, J_i respectively. M_i and N_i represent collections of Lorentz indices depending on the spin of the interacting particles.

2.2 Infrared divergences

Now that we have discussed the general setup of the scattering problem, we want to focus on the specific issue of infrared (IR) divergences which will be the topic of this chapter. We start by discussing real infrared (IR) divergences, which appear when a physical, on-shell particle becomes soft, meaning its energy approaches zero. We also cover virtual IR divergences, which originate from soft virtual particles circulating in loops. Both of these topics are a review of the original work of Weinberg [127].
Then we will discuss the origin of this behavior, which is intimately tied to the fact that gauge interactions can never be neglect, not even on Cauchy surface infinitely far in the past or in

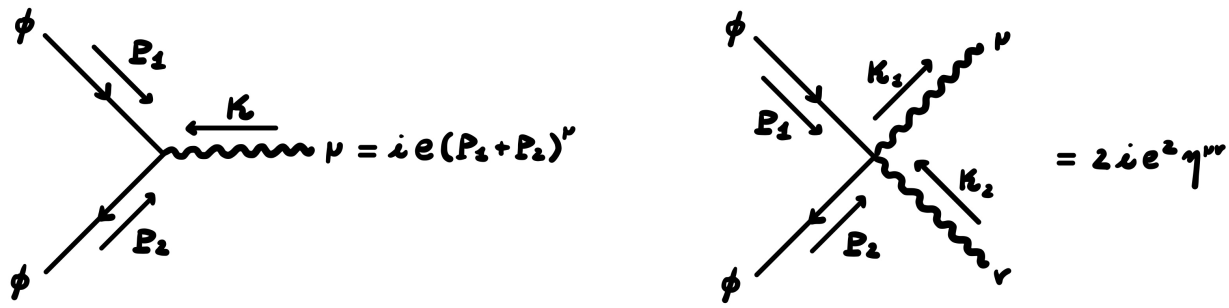

Figure 2.1: Interaction vertex of scalar QED

the future. In other words our assumption that fields behave as free on the initial and final Cauchy surfaces is invalid in gauge theories. We will also describe how the breakdown of such hypothesis was originally addressed by Chung, Faddeev and Kulish [128, 129] in the case of quantum electrodynamics.
The subtle connection between soft behavior and asymptotic symmetries will be address in the following chapters.

2.2.1 Real IR divergences

We start focusing on an extremely simple computation to define the setup. We consider massless scalar QED with Lagrangian:

$$\mathcal{L} = -\frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} + D_\mu \phi^* D^\mu \phi, \tag{2.35}$$

with $D_\mu \phi = \partial_\mu \phi - i A_\mu \phi$.
This model has two simple interaction vertices (see Fig. 2.1) which allow us to compute amplitudes perturbatively using Feynman diagrams. We are interested in computing the leading term in the case of scattering amplitude involving a photon whose frequency tends to zero, namely a soft photon. If we assume to start from an amplitude with no soft photons and a bunch of external states $\mathcal{M}$, we can turn it into a amplitude with a soft photon by attaching a photon propagator to the amplitude. If we attach the photon with momentum q to an external line with momentum p then this will add an extra propagator which carries momentum $p+q$. Combined with the vertex this produces a factor of:

$$i(2p^\mu + \eta q^\mu) \frac{-i}{(p+\eta q)^2 - i\epsilon} \overset{q \to 0}{\sim} \frac{e\eta p^\mu}{p \cdot q - i\eta\epsilon}, \tag{2.36}$$

which multiplies $\mathcal{M}$. $\eta = \pm 1$ for an outgoing or incoming particle respectively. If we add the photon propagator to internal lines it will lack the denominator $p \cdot q$ namely it will give a subleading contribution. This implies that the divergence in (2.36) is tree level exact. This computation can be repeated for any external particle with different charge eQ_i, which allows us to write a concise formula for the insertion of one soft photon of momentum k and helicity J:

$$\mathcal{M}(k, J; p_1, \ldots, p_n) = e \sum_{j=1}^{n} \frac{Q_j \eta_j \epsilon^J(k) \cdot p_j}{p_j \cdot k} \mathcal{M}(p_1, \ldots, p_n) + o(\omega^0), \tag{2.37}$$

where p_j are the momentum of the other hard external particles, and the momentum k of the photon is parametrized as in (2.9).
If we have m soft photon with momentum k_i and helicity J_i, at leading order we just have to multiply all the contributions coming from the various soft particles, namely:

$$\mathcal{M}(k_1, J_1, \dots k_m, J_m, ; p_1, \dots, p_n) = \prod_{l=1}^{m} \sum_{j=1}^{n} \frac{e\eta_j \epsilon^J(k_l) \cdot p_j}{p_j \cdot k_l} \mathcal{M}(p_1, \dots, p_n) + o(\omega_i^0), \tag{2.38}$$

and all soft contributions factorize, in front of the hard particles scattering process.
We see then that amplitudes are divergent when a photon goes soft, namely the amplitude is infrared divergent.

This is not a special property of electrodynamics. Gravitational scattering suffers from the same divergent behavior in presence of soft gravitons with a similar tree level exact factorization property namely we have:

$$\mathcal{M}(k_1, J_1, \dots k_m, J_m, ; p_1, \dots, p_n) = \prod_{l=1}^{m} \frac{\kappa}{2} \sum_{j=1}^{n} \frac{\eta_j \epsilon^J_{\mu\nu}(k_l) p_j^\mu p_j^\nu}{p_j \cdot k_l} \mathcal{M}(p_1, \dots, p_n) + o(q^0), \tag{2.39}$$

where now we have m gravitons with momentum k_i and helicity J_i.

These results (2.38),(2.39) take the name of soft photon/graviton theorem and even if they have all been presented for massless particles, they extend without any modification to the massive case, and for any external particle that interacts with photons or gravitons, independently from their spin or other quantum numbers. Moreover this leading term is also universal, namely it is left unchanged if we add additional higher order interactions to QED of general relativity, provided that any of the interaction does not spoil gauge invariance.

2.2.2 Virtual IR divergences

The factorization formula that we presented in the previous section can be used as a powerful tool to take into account the effects produced by a virtual particle propagating inside Feynman diagrams whose energy tends to zero. To be more specific, we can declare a virtual photon to be soft if it is connecting two external lines and its frequency ω lies below a certain cut-off Λ, $\omega < \Lambda$, which acts as a convenient boundary point between hard and soft modes, and is usually taken to be very small. A more physical definition of Λ can be formulated by imagining an experimental apparatus that measures the frequency of incoming photons. The parameter Λ would be the sensitivity of such instrument, as any photon with frequency below such scale would not be detected and treated as a zero frequency photon.
With this classification we can then isolate all contributions inside any Feynman diagram of photons whose frequency in loop integrals gets below the threshold Λ, namely of soft photons. Taking into account the combinatorics of all possible soft insertions and the factorization property showed in the previous section, it is possible to show that also all virtual soft contribution factorize into an exponential term as:

$$\mathcal{M}(1, \dots, n+m) = \exp\left\{ \frac{1}{2} \int_\lambda^\Lambda d^4k A(k) \right\} \mathcal{M}_0(1, \dots, n+m; \Lambda), \tag{2.40}$$

where:

$$A(k) = \frac{-ie^2}{(2\pi)^4(k^2 - i\epsilon)} \sum_{n,m} \frac{Q_n Q_m \eta_n \eta_m (p_n \cdot p_m)}{[p_n \cdot k - i\eta_n \epsilon]\,[p_m \cdot k - i\eta_m \epsilon]}, \tag{2.41}$$

and $\mathcal{M}_0$ is the IR finite part of the amplitude, depending only on the Λ cut-off.
The integration limits must be intended as the integration region over the frequencies and λ acts as the actual IR cut off which must be sent to zero, and will be the responsible for the actual IR divergences.
The expression (2.41) can be understood as the addition of a soft photon of momentum k between two external legs with momentum p_n and p_m. The photon propagator inputs the term $\frac{-i}{k^2-i\epsilon}$, while the term in the sum is precisely the product of two terms (2.36) coming from the interaction of the photon with the external lines.
The integral involving (2.41) can be performed and for the massless case it gives rise to a total contribution of the form:

$$\mathcal{M}(1,\ldots,n) = \exp\left\{-\frac{e^2}{4\pi^2}\log\lambda \sum_{n<m} Q_m Q_n \log\frac{|p_n \cdot p_m|}{2} + i\Phi_A(\lambda/\Lambda)\right\}\mathcal{M}_0(1,\ldots,n;\Lambda), \tag{2.42}$$

The quantity $\Phi(\lambda/\Lambda)$ amount to a phase factor:

$$\Phi_A(\lambda/\Lambda) = \frac{e^2}{4\pi}\log\frac{\lambda}{\Lambda}\sum_{n,m}{}' Q_n Q_m \tag{2.43}$$

where the summation $\sum_{n,m}{}'$ runs only over pairs of both incoming or outgoing particles, and is not particularly relevant for the computation of scattering probabilities as it does not bring any contribution to scattering amplitudes. However these phases are relevant to show that inclusive cross sections are IR finite (see 2.2.3).
The divergent coefficient of the real factor:

$$\alpha_{EM} = \frac{e^2}{4\pi^2}\log\lambda \tag{2.44}$$

is known as cusp anomalous dimension, as it is also related to UV divergences appearing when Wilson lines develop a cusp [130, 131].

Because soft gravitons also admit a factorization formula one can easily extend this soft exponentiation to amplitudes involving virtual soft gravitons. In particular we get the following expression:

$$\mathcal{M}(1,\ldots,n) = \exp\left\{\frac{1}{2}\int_\lambda^\Lambda d^4k\, B(k)\right\}\mathcal{M}_0(1,\ldots,n;\Lambda), \tag{2.45}$$

where:

$$B(k) = \frac{-8i\pi G}{(2\pi)^4(k^2 - i\epsilon)} \sum_{n,m} \frac{\eta_n \eta_m (p_n \cdot p_m)^2}{[p_n \cdot k - i\eta_n \epsilon]\,[p_m \cdot k - i\eta_m \epsilon]}. \tag{2.46}$$

Computing the integral we get in this situation an extra factor of $p_n \cdot p_m$ multiplying the log in (2.42) namely:

$$\mathcal{M}(1,\ldots,n) = \exp\left\{-\frac{G}{\pi}\log\lambda \sum_{n<m} p_n \cdot p_m \log\frac{|p_n \cdot p_m|}{\mu^2} + i\Phi_B(\lambda/\Lambda)\right\}\mathcal{M}_0(1,\ldots,n;\Lambda), \tag{2.47}$$

where μ is an arbitrary mass scale without any physical relevance as the conservation of momentum fixes the term to be μ independent.
In this case the phase factor is

$$\Phi_B(\lambda/\Lambda) = -G \log \frac{\lambda}{\Lambda} \sum_{n,m}{}' p_n \cdot p_m, \tag{2.48}$$

and we can also define the cusp anomaly for gravity as:

$$\alpha_{GR} = \frac{G}{\pi} \log \lambda, \tag{2.49}$$

We have shown then that virtual infrared divergences exponentiate and produce an intrinsic singularity in correlation functions, which makes extracting physical predictions a conceptually challenging problem.

2.2.3 Traditional treatments of IR divergences

Now that we know that amplitudes involving gravitons and photons are inherently divergent, we need to clarify how to deal with the ill-defined S-matrix.
One option, which is the most used in phenomenology, is in a certain sense not to deal with it at all. The cross-section, not the S-matrix, is the physically measured observable. Therefore, if we can extract a finite cross-section from a mathematically ill-defined S-matrix, there is no experimental necessity to make the S-matrix itself finite. It is possible to show that if in the computation of the cross-section one considers inclusive processes, re-summing all possible amplitudes with external soft photons, the divergences cancel and we get a well defined cross-section. The necessity of considering inclusive interactions can be physically understood thinking about an experimental apparatus with a finite sensitivity Λ. Because the apparatus cannot measure the presence of particles under the Λ cut-off, we are not able to distinguish between scattering configurations that differ by emission of soft particles, which means that we have to average over all possible soft contributions namely consider "inclusive amplitude". This has the effect of canceling all soft contributions as we have already mentioned. In QED this result is known as the Bloch-Nordsieck theorem [132], and it has been generalized in the case of the standard model by Kinoshita, Lee and Nauenberg [133, 134].

While these results successfully address infrared (IR) divergences for observable quantities, we might still want to make the S-matrix itself finite to ensure that transition probabilities are well-defined. To do so, we must get deeper into the origin of these divergences, which is precisely the objective of the next section where we will present the old analysis made by Chung, Faddeev and Kulish and their resolution of the problem.

Faddeev-Kulish dressing

To understand the origin of these divergences, we must go back to the original assumptions underlying the set-up of our scattering problem. We present this in the case of QED following the original discussion of [129].

We started from the assumptions that our particles behave as free asymptotically, namely that the contribution of the potential is vanishing if we choose our in, out Cauchy surfaces to be in the far past or in the far future. As we will now check for massless scalar QED, this assumption is not satisfied due to the presence of infrared divergences. In particular, we will show that for the specific configuration where a photon becomes soft the potential contribution cannot be considered trivially vanishing at $t = \pm\infty$.
Let us consider the potential on a fixed time slice:

$$V(t) = \int d^3x \left[-iA^\mu \left(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi\right) + |\phi|^2 A_\mu A^\mu\right], \tag{2.50}$$

which is composed by a three and four-point term. If we plug in the expression of free fields (2.11) for spin 0 and 1:

$$\begin{aligned} \phi(x) &= \int \frac{d^3p}{2p_0(2\pi)^3}\left[b\, e^{ip\cdot x} + c^\dagger\, e^{-ip\cdot x}\right] \\ A_\mu(x) &= e\sum_{\alpha=\pm}\int \frac{d^3p}{2p_0(2\pi)^3}\left[\epsilon_\mu^{\alpha *} a_\alpha e^{ip\cdot x} + \epsilon_\mu^\alpha a_\alpha^\dagger e^{-ip\cdot x}\right], \end{aligned} \tag{2.51}$$

we get a long expression written as products of creation/annihilation operators and oscillatory exponentials, integrated along the spatial directions and all particles momenta. In general the phases of the exponentials take the form $iP\cdot x$ with:

$$P = \sum_{i=1}^{f} \tilde{\eta}_i k_i + \eta_1 p_1 + \eta_2 p_2, \tag{2.52}$$

where $f = 1, 2$ for the three-point and four-point vertex respectively. P is basically the total momentum passing trough the vertex, taking into account the momenta k_i of the photons and p_i of the scalars, weighted by $\eta_i = \tilde{\eta}_i = \pm 1$ in the case of incoming or outgoing particle respectively. All momenta will be parametrized using the expression in (2.9).
The integral over the constant time slice in the potential produces a delta function in the three-momentum $\delta^3(\vec{P})$ that also removes one of the momentum integral. This leaves us only the exponential $\exp(iP_0 t)$, which will be very important to discriminate which term survives in the early/late time and soft particle photon limit.
By Riemann-Lebesgue lemma, if $P_0 \neq 0$ in the soft photon regime then the rapid oscillations of the phase for $t \to \pm\infty$ will bring a vanishing contribution. We have then to understand which configurations of momenta will give access to $P_0 \to 0$ for $k_1 \to 0$.
If we consider the conservation of three-momentum:

$$\sum_{i=1}^{f} \tilde{\eta}_i \vec{k}_i + \eta_1\vec{p}_1 + \eta_2\vec{p}_2 = 0, \tag{2.53}$$

we get that for the four-point interaction $P_0 = 0$ implies:

$$\eta_1\tilde{\eta}_1(|\vec{k}_1||\vec{p}_1| - \vec{k}_1\cdot\vec{p}_1) + \eta_1\tilde{\eta}_2(|\vec{k}_1||\vec{p}_2| - \vec{k}_1\cdot\vec{p}_2) + \eta_1\eta_2(|\vec{p}_1||\vec{p}_2| - \vec{p}_1\cdot\vec{p}_2) = 0. \tag{2.54}$$

For $k_1 \to 0$ fixes $|\vec{p}_1||\vec{p}_2| = \vec{p}_1\cdot\vec{p}_2$ which is a collinearity condition. This means that if p_1 and p_2 are collinear then $k_1 \to 0$ implies $P_0 \to 0$ in the four-point interaction. This can lead to some additional contributions, which however are not strictly related to Weinberg soft theorem and will be neglect as this is an accident of using massless charged scalars. Indeed for massive particles in fact (2.54) for $k_1 \to 0$ turns into:

$$(\vec{p}_1 \cdot \vec{p}_2)^2 - |\vec{p}_1|^2|\vec{p}_2|^2 = m^2(\eta_1\vec{p}_1 + \eta_2\vec{p}_2)^2, \tag{2.55}$$

which does not admit any solutions.

Following this reasoning we can see that the four-point interaction will be negligible at large times.

Let us turn then to the three-point interactions. In this case we have only one photon and the total energy becomes

$$P_0 = \tilde{\eta}_1|\vec{k}_1| + \eta_1|\vec{p}_1| + \eta_2|\vec{p}_2| = \tilde{\eta}_1|\vec{k}_1| + \eta_1|\vec{p}_1| + \eta_2\sqrt{|\vec{k}_1|^2 + |\vec{p}_1|^2 + 2\tilde{\eta}_1\eta_1\vec{k}_1 \cdot \vec{p}_1}, \tag{2.56}$$

which for small k_1 becomes

$$P_0 \sim (\eta_1 + \eta_2)|\vec{p}_1|^2 + \tilde{\eta}_1\eta_1\eta_2\frac{p \cdot k}{|\vec{p}_1|^2}. \tag{2.57}$$

The limit is vanishing only in the specific configuration where $\eta_1 = -\eta_2$, condition that selects what are the relevant contributions to the potential. To be more specific we have:

$$\begin{aligned} V(t) = e \int d^3x \frac{d^3k}{2k_0(2\pi)^3}\frac{d^3p_1}{2{p_1}^0(2\pi)^3}\frac{d^3p_2}{2{p_2}^0(2\pi)^3}\sum_{\alpha=\pm}\left[\epsilon^{\alpha *}_\mu a_\alpha(k)e^{ik\cdot x} + \epsilon^\alpha_\mu a^\dagger_\alpha(k)e^{-ik\cdot x}\right] \times \\ \times \Big\{ \left[b(p_1)\, e^{ip_1\cdot x} + c(p_1)^\dagger\, e^{-ip_1\cdot x}\right]\left[c(p_2)\, e^{ip_2\cdot x} - b(p_2)^\dagger\, e^{-ip_2\cdot x}\right] p_2^\mu + \\ - \left[c(p_2)\, e^{ip_2\cdot x} + b(p_2)^\dagger\, e^{-ip_2\cdot x}\right]\left[b(p_1)\, e^{ip_1\cdot x} - c(p_1)^\dagger\, e^{-ip_1\cdot x}\right] p_1^\mu \Big\} \\ \overset{t\to+\infty}{\sim} - e\int \frac{d^3k}{2k_0(2\pi)^3}\frac{d^3p}{2p_0(2\pi)^3}\sum_{\alpha=\pm}\frac{p^\mu}{p_0}\left[\epsilon^{\alpha *}_\mu a_\alpha(k)e^{i\frac{k\cdot p}{p_0}t} + \epsilon^\alpha_\mu a^\dagger_\alpha(k)e^{-i\frac{k\cdot p}{p_0}t}\right]\rho(p) = V_{as}(t), \end{aligned} \tag{2.58}$$

where $\rho(p) = b^\dagger(p)b(p) - c^\dagger(p)c(p)$ is the charge density operator. Notice that this potential would still be vanishing per se, as we can see using the approximation:

$$e^{i\frac{k\cdot p}{p_0}t} \overset{t\to+\infty}{\sim} 2\pi i\frac{k\cdot p}{p_0}\delta\left(\frac{k\cdot p}{p_0}\right). \tag{2.59}$$

The presence of the IR pole in amplitudes coming from to the insertion of a low frequency photon oscillators, gives a contribution that cancels the vanishing coefficient of the delta function, which however is not enough to balance the contribution ω_k coming from the measure, which leads to a vanishing potential. This must not come as a surprise, in fact we know the Coulomb potential to be vanishing at infinity. However this does not mean that it will automatically bring a vanishing contribution to the scattering operator.

To make this explicit we can consider the asymptotic time evolution, ruled by the free Hamiltonian H_0, to be corrected by the asymptotic potential V_{as}. The time evolution operator acting on the asymptotic states then satisfies the expression:

$$i\frac{d}{dt}U_{as}(t) = H_{as}U_{as}(t) = (H_0 + V_{as})U_{as}(t) \tag{2.60}$$

which turns the time evolution operator into:

$$U(t)_{as} = e^{-iH_0 t}\mathcal{T}\exp\left(-i\int^t V_{as}(\tau)d\tau\right) = e^{-iH_0 t}e^{R(t)}e^{i\Phi(t)}. \tag{2.61}$$

In the last step we used the commutation relations to disentangle the time-ordered exponential into two operators:

$$\begin{aligned} R(t) &= e\int \frac{d^3k}{2k_0(2\pi)^3}\frac{d^3p}{2p_0(2\pi)^3}\frac{p^\mu}{k\cdot p}\left[\epsilon^{\alpha *}_\mu a_\alpha(k)e^{i\frac{k\cdot p}{p_0}t} - \epsilon^\alpha_\mu a^\dagger_\alpha(k)e^{-i\frac{k\cdot p}{p_0}t}\right]\rho(p), \\ \Phi(t) &= -\frac{e^2}{4\pi}\int \frac{d^3p}{2p_0(2\pi)^3}\frac{d^3q}{2q_0(2\pi)^3} : \rho(p)\rho(q) : \operatorname{sign} t \log\frac{|t|}{t_0}. \end{aligned} \tag{2.62}$$

If these two operators would be vanishing in the late/early time limit the asymptotic time evolution operator would be free. However we can see that $\Phi(t)$ diverges logarithmically for $t \to \pm\infty$ and using the identity (2.59) we can see that:

$$R(t) \overset{t\to\pm\infty}{\sim} 2\pi i e\int \frac{d^3k}{2k_0(2\pi)^3}\frac{d^3p}{2p_0(2\pi)^3}\frac{p^\mu}{p_0}\left[\epsilon^{\alpha *}_\mu a_\alpha(k) + \epsilon^\alpha_\mu a^\dagger_\alpha(k)\right]\delta\left(\frac{k\cdot p}{p_0}\right)\rho(p) \tag{2.63}$$

which, taking into account the infrared behavior of the oscillators, gives a non-vanishing contribution. In the computation of a scattering amplitude we must then take into account the presence of the asymptotic time evolution operator, that changes the amplitude from

$$\lim_{t_{1,2}\to\pm\infty}\langle\text{out}|\, e^{iH_0t_1}S(t_1,t_2)e^{-iH_0t_2}\,|\text{in}\rangle \tag{2.64}$$

to

$$\lim_{t_{1,2}\to\pm\infty}\langle\text{out}|\, U^{-1}_{as}(t_1)S(t_1,t_2)U_{as}(t_2)\,|\text{in}\rangle\,. \tag{2.65}$$

Now a couple of comments are in order. The U_{as} operator is composed by two different pieces. The phase $\Phi(t)$ contains a logarithmic divergent bit in t that, if we assume t and t_0 to be the reciprocal of the energy regulator λ and Λ, turns into a factor $\log\lambda/\Lambda$. The charge density operators acting on a state extract the values of the charges and gives us:

$$-\frac{e^2}{4\pi}\log\frac{\lambda}{\Lambda}{\sum_{n,m}}' Q_nQ_m, \tag{2.66}$$

which is precisely the term that cancels Weinberg divergent phase (2.43).

The $R(t)$ term is much more interesting because it contains a net abundance of photon creation and annihilation operators. This means if it acts on a state it creates a cloud of soft photons that dresses charged particles. This cloud of photons manifestly shows that in QED charged particles can never be considered to be free, which was precisely the wrong initial assumption of the standard scattering problem. If we now scatter a bunch of charged particles, the presence of the soft photon cloud, will produce extra contributions that can be shown to precisely cancel both real and virtual infrared divergences.

An undesired result of this construction however is the fact that the Hilbert space $\mathcal{H}_{FK} = e^{R(t)}\mathcal{H}_F$ produced by the action of the operator $e^{R(t)}$ on the free Fock space $\mathcal{H}_F$ is not a Fock space as now we have also to consider configurations with an infinite number of soft particles,

which are not allowed in $\mathcal{H}_F$. Even if it is not a Fock space $\mathcal{H}_{FK}$ preserves a set of nice properties namely, is Lorentz invariant, gauge invariant and t independent.
Because the space is t independent, Faddeev and Kulish refine the notion of dressing, substituting the $R(t)$ operator with an alternative operator R_f which is now time independent but generates the same Hilbert space. This operator will still allow us to prepare the same asymptotic dressed states but with a more sensible control over the infrared regulators. For R_f to create the same Hilbert space in [129] its form was narrowed down to:

$$R_f = -e \int \frac{d^3k}{2k_0(2\pi)^3} \frac{d^3p}{2p_0(2\pi)^3} \left(f^\mu(p,k)\epsilon^{\alpha *}_\mu a_\alpha(k) - f^{\mu *}(p,k)\epsilon^\alpha_\mu a^\dagger_\alpha(k) \right) \rho(p), \tag{2.67}$$

with:

$$f_\mu(p,k) = \left[\frac{p_\mu}{p \cdot k} - c_\mu(k) \right] f(k,p), \quad c_\mu c^\mu = 0, \quad c_\mu k^\mu = 1, \tag{2.68}$$

where $f(p,k)$ is an arbitrary function that goes to 1 in a neighborhood of $k = 0$. Usually this function is taken to be vanishing above a certain energy cut of $k_0 > \lambda$, to underline that the operator R_f produces only a cloud of soft photons. This energy scale will take the same role taken by the λ parameter in (2.40).
The scattering states are then prepared by acting with e^{R_f} on free Fock states of the form:

$$|\text{in}\rangle_{FK} = e^{R_f} c^\dagger(p_1) \dots c^\dagger(p_n) b^\dagger(q_1) \dots b^\dagger(q_m) |0\rangle , \tag{2.69}$$

Commuting e^{R_f} with the creation operators, we see that each creation operator gets dressed by an exponential such that the states now take the form:

$$|\text{in}\rangle_{FK} = \hat{c}^\dagger(p_1) \dots \hat{c}^\dagger(p_n) \hat{b}^\dagger(q_1) \dots \hat{b}^\dagger(q_m) |0\rangle , \tag{2.70}$$

with

$$\hat{c}^\dagger(p) = W_+(p) c^\dagger(p), \quad \hat{b}^\dagger(p) = W_-(p) b^\dagger(p), \tag{2.71}$$

$$W_\pm(p) = \exp\left[\pm eQ \int \frac{d^3k}{2k_0(2\pi)^3} f(p,k) \sum_{\alpha=\pm} \left(\frac{p \cdot \epsilon^{\alpha *}}{p \cdot k} a_\alpha(k) - \frac{p \cdot \epsilon^\alpha}{p \cdot k} a^\dagger_\alpha(k) \right) \right]. \tag{2.72}$$

The creation operators are now dressed operators surrounded by a soft photon cloud represented by the operator $W_\pm$ depending on the charge of the particle. An explicit computation shows that states defined with these dressed operators are free of both real or virtual infrared divergences.
In this section we have explained the Faddeev-Kulish dressing for the specific case of massless scalar QED. However it is possible to show that the same results is fully general and applies in the massive case and for fermions. This is clearly due to the universality of Weinberg soft photon theorem.

The dressing can also be extended beyond QED to gravitational interaction [135]. In the case of gravity the dressing stems from the same motivations as for QED, namely the gravitational potential gives a non-vanishing contribution to the time evolution operator at infinity. However in perturbative gravity, due to the different spin and interactions of the graviton, the dressing operator R_f gets modified to:

$$R_f = \frac{\kappa}{2} \int \frac{d^3k}{2k_0(2\pi)^3} \frac{d^3p}{2p_0(2\pi)^3} \left(f^{\mu\nu}(p,k)\epsilon^{\alpha *}_{\mu\nu} a_\alpha(k) - f^{\mu\nu *}(p,k)\epsilon^\alpha_{\mu\nu} a^\dagger_\alpha(k) \right) \rho_m(p), \tag{2.73}$$

with:

$$f_{\mu\nu}(p,k) = \left[\frac{p_\mu p_\nu}{p\cdot k} - c_{\mu\nu}(k)\right] f(k,p), \quad c_{\mu\nu} = c_\mu c_\nu, \tag{2.74}$$

with c_μ defined by the properties in (2.68), and $\rho_m(p)$ the matter density operator. In the case of gravity the oscillators get dressed by the exponential operator

$$W_G(p) = \exp\left[\frac{\kappa}{2}\int \frac{d^3k}{2k_0(2\pi)^3} f(p,k) \sum_{\alpha=\pm}\left(\frac{p_\mu p_\nu \epsilon^{\alpha *}{}_{\mu\nu}}{p\cdot k} a_\alpha(k) - \frac{p_\mu p_\nu \epsilon^{\alpha}{}_{\mu\nu}}{p\cdot k} a^\dagger_\alpha(k)\right)\right]. \tag{2.75}$$

Notice that because gravitons are also self-interacting, now the W_G operators also dress the graviton operators, differently from what happens in QED as photons are not electrically charged.

We have seen how the careful treatment of the asymptotic potential allowed us to make the S-matrix finite in the limit $\lambda \to 0$. However there is still a possible drawback in this procedure, namely that even if the dressing removes the λ parameter responsible for the divergences, the cut of scale Λ, which splits the spectrum into soft and hard modes, is still present in scattering amplitudes, namely we have an extra scale in our model which is not physical but is at most parameterizing the sensitivity of the detector. Another challenging aspect of FaddeevKulish amplitudes is that their computation becomes highly non-trivial, since one must consider amplitudes involving an infinite number of photons. This difficulty is reflected in the relatively scarce literature on explicit computations in this framework. Indeed, in the fifty years since the original FK paper, only a few works [136–138] appear to have computed explicit amplitudes within this approach.

2.3 Soft expansion and subleading soft theorem

In the previous section we have seen that a scattering amplitude involving a small frequency $\omega \sim 0$, photon or graviton produces a divergent scattering amplitude with a leading behavior in $1/\omega$. However we might also be also interested in the subleading behavior, namely what are the generic contributions in a series of the form:

$$\mathcal{M}(\omega, q, J; p_1, \dots p_n) = \frac{1}{\omega}\mathcal{M}_0(q, J; p_1, \dots p_n) + \mathcal{M}_1(q; p_1, \dots p_n) + o(\omega), \tag{2.76}$$

where the momentum of the soft particle is parametrized as $k = \omega q$.

As we have already seen, from Weinberg computation:

Def 2.3.1: Leading soft factorization

$\mathcal{M}_0(q, J; p_1, \dots p_n)$ universally factorizes as

$$\mathcal{M}_0(q, J; p_1, \dots p_n) = S_0(q, J, p_i)\mathcal{M}(p_1, \dots p_n), \tag{2.77}$$

$$\textit{photon insertion: } S_0(q, J, p) = e\sum_{i=1}^n \frac{Q_i \epsilon^J(q)\cdot p}{p\cdot q}, \tag{2.78}$$

$$\textit{graviton insertion: } S_0(q, J, p) = \sum_{i=1}^n \frac{\kappa}{2}\frac{\epsilon^J_{\mu\nu}(q)p^\mu p^\nu}{p\cdot q}, \tag{2.79}$$

where the factorization is independent on the type of external particles.

The above leading soft theorem, however do not constrain in any way the form of the subleading terms, so at this level no particular properties should emerge.

However it was proven by Low [139, 140] that for QED and more recently by Strominger and Cachazo [30] for gravity, that the subleading terms also factorize. The structure of the factorization at tree level is similar to (2.77) with the S_0 substituted by terms $S_1(q, J, p)$.

Def 2.3.2: Sub-leading soft factorization

$\mathcal{M}_1(q, J; p_1, \dots p_n)$ factorizes as

$$\mathcal{M}_1(q, J; p_1, \dots p_n) = S_1(q, J, p_i)\mathcal{M}(p_1, \dots p_n), \tag{2.80}$$

$$\textit{photon insertion: } S_1(q, J, p) = -ie\sum_{i=1}^{n} Q_i \frac{q_\mu \epsilon_\nu^J(q) J^{\mu\nu}(p)}{p \cdot q} \tag{2.81}$$

$$\textit{graviton insertion: } S_1(q, J, p) = -\frac{i\kappa}{2}\sum_{i=1}^{n} \frac{\epsilon_{\mu\nu}^J(q) p^\mu}{p \cdot q} q_\rho J^{\rho\nu}(p), \tag{2.82}$$

where $J_{\mu\nu}(p)$ is the total angular momentum operator of the hard particle with momentum p. The subleading term distinguish between particles of different spin.

Notice that these are not just numerical factors that multiply the amplitude without the soft insertion but are actually operators acting on the amplitude itself. Moreover, unlike the leading terms, which are independent of the hard particles fine structure, these subleading terms distinguish between particles of different spin.

It is important here to remark an important difference between the subleading soft graviton and the subleading soft photon theorem. As for the leading soft term the subleading graviton behavior is universal, namely it will not be effected by the insertion of any additional local operators beyond GR. On the contrary in [141] it was shown that for photons turning on additional operators also modify the subleading behavior, giving proof on non-universality.

These statements can be taken beyond subleading order. In fact soft factorization still holds at higher orders, namely one would find for $n \geq 2$:

$$\mathcal{M}_n(q, J; p_1, \dots p_n) = S_n(q, J, p_i)\mathcal{M}(p_1, \dots p_n) \tag{2.83}$$

but the the terms $S_n(q, J, p)$ are now non-universal [142] namely they are sensitive to effective operators added in the action . This makes them sensitive to eventual UV corrections, which will be relevant for example in the UV completion of GR.

To summarize, in QED S_0, is the only truly universal term, while for gravity both S_0 and S_1 are universal. What we will see in the following chapter is that the universality of these terms is closely tied to the presence of gauge symmetries, in particular to the sub-group of gauge transformation that are non vanishing on the boundary of spacetime, known as asymptotic symmetries. .

Chapter 3

Asymptotic symmetries

In this chapter we will revisit the soft theorems presented in the previous chapter from the point of view of the asymptotic symmetries. In particular we will show that the universality properties of leading and subleading soft theorems are not an accident but they precisely stem from the conservation of asymptotic charges coming from gauge symmetries.

This chapter will be organized as follows: in sec. 3.1 we will review the structure of phase space and asymptotic symmetries in QED. We will show that the Ward identity of non-vanishing gauge charges, usually referred to as large gauge charges, are in one-to-one correspondence with the leading soft photon theorem.

Then in sec. 3.2 we will carry out a similar, but less detailed analysis, for the case of gravity, discussing the phase space of GR for vanishing cosmological constant, namely the phase space of asymptotically flat spacetime. In this set-up we will point out the existence of a set of non-vanishing conserved charges, all related to the set of isometries of asymptotically flat space time usually referred to as the BMS group. The Ward identity of these charges, as we will see, are in one-to-one correspondence with the leading and subleading soft graviton theorem, giving us a better explanation for the universality of such terms. In the end of the chapter we will discuss how these properties are relevant to holography in flat space, and in particular to the subject of the next chapter, celestial holography.

This discussion roughly follows the treatment given in [61], further details are given in [29, 30, 143–147]

3.1 QED and its phase space

The model of electrodynamics is specified by the following Lagrangian form [102, 148]

$$L(A,\Phi) = -\frac{1}{2e^2} F \wedge \star F + L_M(A,\Phi), \tag{3.1}$$

where $\star$ represents the Hodge dual, $F = dA$ is the field strength associated with the electromagnetic potential 1-form A and L_M is the Lagrangian of matter fields Φ coupled to the electromagnetic field. The variation of the matter Lagrangian with respect to A takes the form

$$\delta_A L_M(A,\Phi) = \delta A \wedge \star j_M, \tag{3.2}$$

where j_M is the electromagnetic 1-form current associated with matter fields. For the action to be gauge invariant under the $U(1)$ symmetry $\delta_\lambda A = d\lambda$, $\star j_M$ must be closed $d\star j_M = 0$, such that L shifts by an exact term

$$\delta_\lambda L(A, \Phi) = d\left(\lambda \star j_M\right). \tag{3.3}$$

Computing the variation of L under A we obtain the equations of motion and the electromagnetic sector of the pre-symplectic potential

$$\delta_A L = -\frac{1}{e^2}\, d\delta A \wedge \star F + \delta A \wedge \star j_M = -\frac{1}{e^2} d(\delta A \wedge \star F) + \frac{1}{e^2}\delta A \wedge \left(-d\star F + e^2 \star j_M\right), \tag{3.4}$$

such that:

$$\begin{aligned} \mathrm{EOM}_A &= d\star F - e^2 \star j_M = 0, \\ \Theta &= -\frac{1}{e^2}\delta A \wedge \star F, \end{aligned} \tag{3.5}$$

where EOM_A is the equation of motion $d-1$-form and Θ is the pre-symplectic potential, that gives us the symplectic form

$$\Omega = \frac{1}{e^2}\delta A \wedge \delta(\star F). \tag{3.6}$$

Plugging $\delta A = d\lambda$ in (3.4) and using (3.2) we can also extract the gauge current J

$$\star J = \frac{1}{e^2} d\lambda \wedge \star F + \lambda \star j_M, \tag{3.7}$$

Notice that on-shell this operator is exact, namely

$$\star J = d\left(\frac{1}{e^2}\lambda \star F\right). \tag{3.8}$$

The charge associate with the gauge transformation λ, Q_λ will then be computed by integrating $\star J$ on any co-dimension 1 surface Σ

$$Q_\lambda = \frac{1}{e^2}\int_\Sigma d(\lambda \star F) = \frac{1}{e^2}\int_{\partial\Sigma} \lambda \star F. \tag{3.9}$$

Notice also that if λ vanishes on the boundary of Σ the charge is trivially vanishing. This is a common property of gauge charges. Moreover if we take λ constant from equation (3.7), Q_λ is trivially identical to the total matter charge inside the volume Σ, while equation (3.9) shows that the charge is identical to the flux of electric field through the boundary of Σ, which is basically the statement of Gauss theorem.

Everything we have defined so far is completely independent from the target manifold, but to connect our discussion to the scattering problem, we now specialize to Minkowski space and define the phase space on a Cauchy slice in the far past and future. As we are strictly focusing on massless matter, all particles will reach, in the far future, $\mathscr{I}^+$ and are coming, in the far past, from $\mathscr{I}^-$. This means that we can specify the initial and final data only on past and future null infinity, even if they do not form a complete Cauchy surface. Then the free data can be identified through an asymptotic expansion of the fields [149]. Expanding the gauge potential A and the current j_M in Laurent series around $\mathscr{I}^\pm$ in round Bondi coordinates (1.17) we write

$$
\begin{aligned}
A(u,r,z,\bar{z}) &= \sum_{n=0}^{\infty} \frac{A^{(n)}(u,z,\bar{z})}{r^n}, \\
A(v,r,z,\bar{z}) &= \sum_{n=0}^{\infty} \frac{A^{(n)}(v,z,\bar{z})}{r^n}, \\
j_M(u,r,z,\bar{z}) &= \sum_{n=0}^{\infty} \frac{j_M^{(n)}(u,z,\bar{z})}{r^n}, \\
j_M(v,r,z,\bar{z}) &= \sum_{n=0}^{\infty} \frac{j_M^{(n)}(v,z,\bar{z})}{r^n},
\end{aligned} \tag{3.10}
$$

Solving the equations of motion order by order in r, one finds that a set of coefficients $\{A^{(0)},\dots,A^{(N)}\}$ remain unconstrained, while all higher-order terms $A^{(k)}$ with $k > N$ are fixed functions of this initial set. These unconstrained coefficients form the free data of the solution. An important point is then to determine what are the physically allowed fall-offs for the currents and the EM fields. This can be done imposing some meaningful conditions, such as that the value of the total electric charge and the energy flux at infinity remains finite. The energy flux per unit of retarded time $\partial_u E$ trough $\mathscr{I}^+$ si equal to

$$
\partial_u E = \lim_{r\to\infty} r^2 \int d^2 z\, \gamma_{z\bar{z}} T_{uu}, \quad T_{uu} = \frac{1}{\gamma_{z\bar{z}} r^2} F_{uz} F_{u\bar{z}} + O(r^{-2}). \tag{3.11}
$$

while the value of the electric charge at a $u =$ constant section of $\mathscr{I}^+$ can be expressed in terms of the field strength as

$$
Q = \frac{1}{e^2} \lim_{r\to+\infty} r^2 \int d^2 z\, \gamma_{z\bar{z}} F_{ur}. \tag{3.12}
$$

For both $\partial_u E$ and Q to be finite the gauge potential then has to fall off at $\mathscr{I}^+$ as [149]

$$
A_r \sim r^{-2}, \quad A_u \sim r^{-1}, \quad A_z \sim r^0, \quad A_{\bar{z}} \sim r^0, \tag{3.13}
$$

and similarly on $\mathscr{I}^-$. This implies that physical solutions would have $A_v^{(0)} = A_u^{(0)} = A_r^{(0)} = A_r^{(1)} = 0$ in the large r expansion (3.10).
With these fall-offs, it is possible to show, by solving the equations of motion order by order in r, that the EM field can be fully specified by the modes $A_z^{(0)}$ and $A_{\bar{z}}^{(0)}$ which represent the free data of the electromagnetic sector of QED [149]. We have indeed two independent degrees of freedom, as we should expect, as they represent the two different polarizations of the EM field.

In this description we have not been explicit about the importance of the gauge fixing procedure. It is of course crucial to specify a gauge as it makes possible to get rid of the zero modes of the Laplace operator $d \star d$ in the equation of motion and find its inverse. This is however a subtle point, as we must be careful about what can be considered as a gauge transformation and used to eliminate degrees of freedom. A gauge transformation can be considered as such only if its conserved charge, namely its charge computed over a full Cauchy slice, is vanishing as this will automatically imply a trivial action on any phase space configuration. However, if its charge is non-vanishing, it can act on the phase space and transform between inequivalent physical configurations. In massless QED, as previously mentioned, we can restrict the Cauchy surface to be null infinity ($\mathscr{I}^+$), so that the gauge charge has the form:

$$Q_\lambda = \frac{1}{e^2}\int_{\mathscr{I}^+} d(\lambda \star F) = \frac{1}{e^2}\int_{\mathscr{I}^+_+} \lambda \star F - \frac{1}{e^2}\int_{\mathscr{I}^+_-} \lambda \star F = -\frac{1}{e^2}\int_{\mathscr{I}^+_-} \lambda \star F, \tag{3.14}$$

where the first term cancels, as the assumption of no charged massive particles kills the EM field on $\mathscr{I}^+_+$. We notice then, that all charges associated to a gauge parameter $\lambda(u, r, z, \bar{z})$, such that $\lambda|_{\mathscr{I}^+_-} = \epsilon(z, \bar{z}) \neq 0$, are non-vanishing, namely they generate a set of transformations that can act non-trivially on the phase space.
Moreover, notice that once we fix the asymptotic conditions not all gauge transformations will be allowed as some can spoil the fields fall-off. The space of transformation that respect the boundary conditions will form the space of allowed gauge transformations. Among these transformations only some will have non-vanishing charges, namely they will act non-trivially on the free data of the theory. To give an explicit example, we can see that the fall-off conditions (3.13) impose that the gauge parameter must behave around $\mathscr{I}^+$ as

$$\lambda(u, r, z, \bar{z}) = \frac{1}{r}\tilde{\lambda}(u, z, \bar{z}) + \epsilon(z, \bar{z}) + o(r^{-1}). \tag{3.15}$$

It is easy to see that the charge associated to $\tilde{\lambda}$ will be vanishing, meaning that this parameter is pure gauge, while the $\epsilon(z, \bar{z})$ term will give rise to a non-vanishing charge, producing a non-trivial gauge transformations. The space of transformation that are allowed by the boundary conditions and have non-vanishing charge are usually referred to as "large gauge transformations"[1] or asymptotic symmetries [61, 104]:

Def 3.1.1: Asymptotic symmetries

$$\begin{aligned} \text{Asymptotic symmetries} &= \frac{\text{Allowed symmetries}}{\text{Trivial symmetries}} \\ &= \text{Allowed symmetry with non-vanishing charge.} \end{aligned} \tag{3.16}$$

In the case of QED the asymptotic symmetries reduce to the space gauge transformations parametrized by a function λ that, when restricted to the boundary, takes only an angle dependent functional form $\lambda|_{\mathscr{I}^+} = \epsilon(z, \bar{z}) \neq 0$.

Since these are genuine symmetries of the system, we must ensure that, when choosing a gauge condition, we do not inadvertently fix the large gauge transformations as well. Lorentz gauge $\nabla^\mu A_\mu = 0$ is well suited for this, as it only partially fixes the gauge, leaving free the set of gauge parameters satisfying $\Box\lambda = 0$. Inside this set there are still trivial gauge transformations that allow us to kill $A_u^{(1)}$, constraining the free data only to $A_z^{(0)}$ and $A_{\bar{z}}^{(0)}$ as previously mentioned. However there is also a full set of transformation satisfying $\Box\lambda = 0$ and that asymptote to some large gauge $\epsilon(z, \bar{z})$ on $\mathscr{I}^+$, which can be written as [61]:

$$\lambda(X) = \int d^2w \frac{\sqrt{\gamma_{w\bar{w}}}}{2\pi} \frac{X \cdot X}{(q(w, \bar{w}) \cdot X)^2} \epsilon(w, \bar{w}), \tag{3.17}$$

[1] In the literature the term "large gauge transformation" is also used to denote transformations that are not connected to the identity. In this context the meaning is different as the term large wants only to underline the fact that the gauge parameter is not vanishing on the boundary of spacetime.

with q specified in (2.9).

Another well suited gauge condition, that makes more easy to split between large gauge and trivial gauge transformations is radial gauge, that just fixes $A_r = 0$. This of course leaves as residual gauge all transformations of the form $\lambda(u, z, \bar{z})$. To not spoil the asymptotic conditions $\lim_{r\to+\infty} A_u = 0$ however we must also impose λ to be u independent and we are left exactly with the large gauge transformations $\lambda(z, \bar{z})$.

Now that we have pointed out this important subtlety about large gauge charges, let us analyze what are the consequences of the presence of these charges and their actions on the EM potential. The action of a large gauge transformation $\lambda(z, \bar{z})$ changes the free data at $A_z^{(0)}$, $A_{\bar{z}}^{(0)}$ to $A_z^{(0)} + \partial\lambda$, $A_{\bar{z}}^{(0)} + \bar{\partial}\lambda$. If these where actual gauge transformations, we could have used this extra freedom to cancel contributions of the form $\lim_{u\to+\infty} A_z^{(0)}$. On the contrary, as these transformations are not pure gauge, these boundary contributions will not be vanishing at all. To isolate this behavior we can define 4 modes [57, 150]:

$$
\begin{aligned}
\phi_z &= \frac{1}{2} \lim_{u\to+\infty} \left(A_z^{(0)}(u, z, \bar{z}) + A_z^{(0)}(-u, z, \bar{z}) \right), \\
\phi_{\bar{z}} &= \frac{1}{2} \lim_{u\to+\infty} \left(A_{\bar{z}}^{(0)}(u, z, \bar{z}) + A_{\bar{z}}^{(0)}(-u, z, \bar{z}) \right), \\
N_z &= \lim_{u\to+\infty} \left(A_z^{(0)}(u, z, \bar{z}) - A_z^{(0)}(-u, z, \bar{z}) \right), \\
N_{\bar{z}} &= \lim_{u\to+\infty} \left(A_{\bar{z}}^{(0)}(u, z, \bar{z}) - A_{\bar{z}}^{(0)}(-u, z, \bar{z}) \right).
\end{aligned}
\tag{3.18}
$$

Notice that in absence of magnetic charges these terms are not independent, as the magnetic field vanishes at infinity and imposes the condition [150]:

$$
\lim_{r\to+\infty} F_{z\bar{z}} = \bar{\partial} A_z^{(0)} - \partial A_{\bar{z}}^{(0)} = 0. \tag{3.19}
$$

As a consequence we must have:

$$
\partial\phi_{\bar{z}} = \bar{\partial}\phi_z, \quad \partial N_{\bar{z}} = \bar{\partial} N_z, \tag{3.20}
$$

which forces $\phi_z = \partial\phi$, $\phi_{\bar{z}} = \bar{\partial}\phi$, $N_z = e^2\partial N$, $N_{\bar{z}} = e^2\bar{\partial}N$, where the factor of e^2 is just a choice of normalization. Let us stop for a second to comment on the meaning of these modes: first of all notice that the mode N is large gauge invariant and measures the net variation of the EM field along null infinity. This mode is related to an interesting physical effect, usually referred to as the null kick memory effect [151] that predicts that after the passage of an electromagnetic wave a particle retains a small velocity precisely proportional to the $N_z, N_{\bar{z}}$ mode.
On the contrary the mode ϕ transforms under a large gauge transformation as $\delta_\lambda \phi = \lambda$, namely just with a shift. For this reason ϕ is usually referred to as the Goldstone mode for large gauge transformations.

The introduction of the modes (3.18) is helpful to fully split the phase space into two parts, related respectively to purely radiative modes and large gauge modes. More precisely, using the field ϕ we can define:

$$
\begin{aligned}
A_z^{(0)} &= \hat{A}_z^{(0)} + \partial\phi, \\
A_{\bar{z}}^{(0)} &= \hat{A}_{\bar{z}}^{(0)} + \bar{\partial}\phi.
\end{aligned}
\tag{3.21}
$$

This allows us to rewrite the symplectic form on $\mathscr{I}^+$

$$\Omega = \frac{1}{e^2}\int_{\mathscr{I}^+} dud^2z \left(\delta F^{(0)}_{uz} \wedge \delta A^{(0)}_{\bar{z}} + \delta F^{(0)}_{u\bar{z}} \wedge \delta A^{(0)}_{z}\right), \tag{3.22}$$

as

$$\begin{aligned}\Omega = \frac{2}{e^2}&\int_{\mathscr{I}^+} dud^2z\, \partial_u \delta \hat{A}^{(0)}_z \wedge \delta \hat{A}^{(0)}_{\bar{z}} \\ &+ \frac{1}{e^2}\int d^2z \left(\delta \int_{-\infty}^{+\infty} duF^{(0)}_{uz} \wedge \bar{\partial}\delta\phi + \delta \int_{-\infty}^{+\infty} duF^{(0)}_{u\bar{z}} \wedge \partial\delta\phi\right).\end{aligned} \tag{3.23}$$

By definition we can write

$$\begin{aligned}\int_{-\infty}^{+\infty} du\, F^{(0)}_{uz} &= \lim_{u\to+\infty} \left(A^{(0)}_z(u,z,\bar{z}) - A^{(0)}_z(-u,z,\bar{z})\right) = e^2\partial N,\\ \int_{-\infty}^{+\infty} du\, F^{(0)}_{u\bar{z}} &= \lim_{u\to+\infty} \left(A^{(0)}_{\bar{z}}(u,z,\bar{z}) - A^{(0)}_{\bar{z}}(-u,z,\bar{z})\right) = e^2\bar{\partial} N,\end{aligned} \tag{3.24}$$

which fixes [57, 61, 150]

$$\Omega = \frac{2}{e^2}\int_{\mathscr{I}^+} dud^2z\, \partial_u \delta \hat{A}^{(0)}_z \wedge \delta \hat{A}^{(0)}_{\bar{z}} + 2\int d^2z\, \partial\delta N \wedge \bar{\partial}\delta\phi. \tag{3.25}$$

As we see, this symplectic form distinguishes between a radiative part $\{A^{(0)}_z, A^{(0)}_{\bar{z}}\}$ and a large gauge part represented by $\{N, \phi\}$. We have already seen that the Goldstone ϕ is connected to asymptotic symmetries as the mode charged under large gauge transformations. To see how N is related to asymptotic charges, let us come back to the original definition of gauge charge:

$$Q_\lambda = \frac{1}{e^2}\int_{\mathscr{I}^+} d\lambda \wedge \star F + \int_{\mathscr{I}^+} \lambda \star j_M, \tag{3.26}$$

where here λ is a large and therefore u independent. This means that the full charge can be rewritten as:

$$Q_\lambda = -\frac{1}{e^2}\int_{\mathscr{I}^+} d^2zdu\, \left(\partial\lambda\, F_{uz} + \bar{\partial}\lambda\, F_{u\bar{z}}\right) + \int_{\mathscr{I}^+} d^2zdu\, \gamma_{z\bar{z}}\lambda j^{(2)}_u \tag{3.27}$$

where $j^{(2)}_u$ is the leading $1/r^2$ component of the current. Using again (3.24) this reduces to:

$$Q_\lambda = 2\int d^2z\, \lambda\, \partial\bar{\partial}N + \int_{\mathscr{I}^+} d^2zdu\, \gamma_{z\bar{z}}\lambda j^{(2)}_u, \tag{3.28}$$

and we see that the only contribution coming from the EM field is related to the gauge invariant N_z mode.

We conclude the section with the Poisson brackets induced by the symplectic form (3.25):

$$\begin{aligned}\left\{\partial_u A^{(0)}_z(u,z,\bar{z}), A^{(0)}_{\bar{z}}(u',z',\bar{z}')\right\} &= -\frac{e^2}{2}\delta(u-u')\delta^2(z-z'),\\ \left\{\partial\phi(z,\bar{z}), \bar{\partial}N(z',\bar{z}')\right\} &= \frac{1}{2}\delta^2(z-z').\end{aligned} \tag{3.29}$$

At a quantum level these turn into the commutation relations:

$$\begin{aligned}\left[\partial_u A^{(0)}_z(u,z,\bar{z}), A^{(0)}_{\bar{z}}(u',z',\bar{z}')\right] &= -i\frac{e^2}{2}\delta(u-u')\delta^2(z-z'),\\ \left[\partial\phi(z,\bar{z}), \bar{\partial}N(z',\bar{z}')\right] &= \frac{i}{2}\delta^2(z-z').\end{aligned} \tag{3.30}$$

As a final remark we point out the the construction of phase space and asymptotic charges, carried out around $\mathscr{I}^+$, can also be repeated around at past null infinity $\mathscr{I}^-$, yielding an identical phase space and gauge charge.

3.1.1 Charge conservation and soft photon theorems

The previous section highlighted the subtle structure of the QED phase space arising from large gauge transformations, and detailed its construction at future null infinity $\mathscr{I}^+$ together with the associated large gauge charge Q_λ. In this section, we show how these transformations are intimately connected to the occurrence of infrared divergences.

Having defined the gauge charges, we can now ask whether they are conserved; in other words, whether the charge at future null infinity, $Q^+_{\lambda^+}$, is equal to the corresponding charge at past null infinity, $Q^-_{\lambda^-}$. In formulas this is expressed as

$$Q^+_{\lambda^+} = \frac{2}{e^2}\int_{\mathscr{I}^+_-} d^2z\, \gamma_{z\bar{z}}\lambda_+(r^2F_{ur}) = \frac{2}{e^2}\int_{\mathscr{I}^-_+} d^2z\, \gamma_{z\bar{z}}\lambda_-(r^2F_{vr}) = Q^-_{\lambda^-}. \tag{3.31}$$

It is evident that without any particular matching condition for the field strength and the gauge parameters at the junction between $\mathscr{I}^-_+$ and $\mathscr{I}^+_-$, charge conservation is not guaranteed.

Focusing on F_{ur} and F_{vr} we can get an hint on the matching conditions by studying a simple example, such as the EM field produced by n charged particle moving at a fixed velocities $\vec{v}_k$. This is the Liénard-Wiechert solution which, in global coordinates, gives a F_{rt} component of the form [148]

$$F_{rt}(\vec{x},t) = \frac{e^2}{4\pi}\sum_{k=1}^{n}\frac{Q_k\gamma_k(r - t\,\hat{x}\cdot\vec{v}_k)}{|\gamma_k^2(r - t\,\hat{x}\cdot\vec{v}_k)^2 - t^2 + r^2|^{\frac{3}{2}}}, \tag{3.32}$$

where Q_k and $\gamma_k = (1 - v_k^2)^{-1/2}$ are the charge and the Lorentz factor of the k-th particle respectively, and $\vec{x} = r\hat{x}$. If we express this potential in retarded and advanced coordinates (1.18), (1.19) we obtain the following expressions for the leading $r \to +\infty$ expansion of the field strength

$$\begin{aligned} F_{ur}(u,r,z,\bar{z}) &= \frac{e^2}{4\pi r^2}\sum_{k=1}^{n}\frac{Q_k}{\gamma_k^2(1 - \hat{x}(z,\bar{z})\cdot\vec{v_k})^2} + O(r^{-2}), \\ F_{vr}(v,r,z,\bar{z}) &= \frac{e^2}{4\pi r^2}\sum_{k=1}^{n}\frac{Q_k}{\gamma_k^2(1 - \hat{x}(z,\bar{z})\cdot\vec{v_k})^2} + O(r^{-2}) \end{aligned} \tag{3.33}$$

where $\hat{x}(z,\bar{z}) = \frac{1}{1+z\bar{z}}(z+\bar{z}, i(\bar{z}-z), 1 - z\bar{z})$. This seems to imply that the two solutions are identical on the "junction point"

$$F^{(2)}_{ur}(z,\bar{z})\big|_{\mathscr{I}^+_-} = \lim_{\substack{r\to+\infty \\ u\to-\infty}} r^2F_{ur}(u,r,z,\bar{z}) = \lim_{\substack{r\to+\infty \\ v\to-\infty}} r^2F_{vr}(u,r,z,\bar{z}) = F^{(2)}_{vr}(z,\bar{z})\big|_{\mathscr{I}^-_+}. \tag{3.34}$$

This matching condition at the boundary of $\mathscr{I}^+$ and $\mathscr{I}^-$ is usually referred to as antipodal matching [61, 150]. The term antipodal in this situation is a bit counterintuitive as the field strengths are chosen to be identical at the same value of $(z,\bar{z})$, so it would seem that we are matching them at the same point on the sphere and not on opposite points. However we have to recall that from expressions (1.18), (1.19)

$$\begin{aligned} X_u &= \left(u - r, r\frac{z+\bar{z}}{1+z\bar{z}}, ir\frac{\bar{z}-z}{1+z\bar{z}}, r\frac{1-z\bar{z}}{1+z\bar{z}}\right), \\ X_v &= \left(v + r', r'\frac{z'+\bar{z}'}{1+z'\bar{z}'}, ir'\frac{\bar{z}'-z'}{1+z'\bar{z}'}, r'\frac{1-z'\bar{z}'}{1+z'\bar{z}'}\right) \end{aligned} \tag{3.35}$$

we obtain that the change of variables between the chart resolving $\mathscr{I}^+$ and $\mathscr{I}^-$ is

$$u = v, \quad r = -r', \quad z = -\frac{1}{\bar{z}'}. \tag{3.36}$$

This means that on $\mathscr{I}^-$ the point $(z, \bar{z})$ corresponds to its antipodal location $(-1/\bar{z}, -1/z)$ on $\mathscr{I}^+$, so that F_{ur} and F_{vr} are indeed antipodally matched.
Even if we have presented this matching condition for the special case of the EM field emitted by n massive particles, it has been argued that this matching condition holds basically for all reasonable solution of Maxwell equations [61, 150]. For example, adding electromagnetic waves to the n-particle solution yields a more complicated expression for the field strength, but it does not alter the fundamental Coulombic behavior of the electric field in the deep infrared near spatial infinity, thus preserving the same matching property.
Moreover this matching satisfies other important properties, namely is Lorentz and CPT invariant, and ensures that the EM field remains continuous along the null generators of the compactified boundary of Minkowski space, even when crossing space-like infinity i^0.
Antipodal matching implies then, that choosing the same gauge parameters at $\mathscr{I}^+$ and $\mathscr{I}^-$ ($\lambda_+(z, \bar{z}) = \lambda_-(z, \bar{z}) = \lambda(z, \bar{z})$), the equation $Q^+_\lambda = Q^-_\lambda$ is satisfied, and the charges are indeed conserved. Notice that as the charges depend on an arbitrary function λ we have an infinite number of conservation laws.

Let us now examine the consequences of such a conservation law in the quantum theory. From the quantum perspective Q^+_λ, Q^-_λ are promoted to operators and the the equation $Q^+_\lambda = Q^-_\lambda$ must be interpreted in a weak sense, namely only valid inside scattering amplitudes. This can be mathematically instantiated as:

$$\langle \text{out} | \, Q^+_\lambda S - S Q^-_\lambda \, | \text{in} \rangle = 0. \tag{3.37}$$

We can then compute the action of the charges on the states to be:

$$\begin{aligned}
Q^+_\lambda \, |\text{out}\rangle &= 2 \int_{\mathscr{I}^+_-} d^2 z \, \lambda \, \partial \bar{\partial} N^+ \, |\text{out}\rangle + \int_{\mathscr{I}^+} d^2 z du \, \gamma_{z\bar{z}} \lambda j^{(2)}_u \, |\text{out}\rangle \\
&= 2 \int_{\mathscr{I}^+_-} d^2 z \, \lambda \, \partial \bar{\partial} N^+ \, |\text{out}\rangle + \sum_{i=1}^{n} Q^+_i \lambda(z^+_i, \bar{z}^+_i) \, |\text{out}\rangle \\
Q^-_\lambda \, |\text{in}\rangle &= 2 \int_{\mathscr{I}^-_+} d^2 z \, \lambda \, \partial \bar{\partial} N^- \, |\text{in}\rangle + \int_{\mathscr{I}^-} d^2 z du \, \gamma_{z\bar{z}} \lambda j^{(2)}_v \, |\text{in}\rangle \\
&= 2 \int_{\mathscr{I}^-_+} d^2 z \, \lambda \, \partial \bar{\partial} N^- \, |\text{in}\rangle + \sum_{i=1}^{m} Q^-_i \lambda(z^-_i, \bar{z}^-_i) \, |\text{in}\rangle
\end{aligned} \tag{3.38}$$

where we have used the fact that the matter current extract a localized delta function on the position of the particles multiplied by the value of the charge. The $+, -$ superscripts just indicate if the operator or quantity we are considering is localized at future or past infinity respectively. Now we have to understand what is the action of N on the states. We can do so using (3.24)

$$e^2 \partial N = \int_{-\infty}^{+\infty} du \, F^{(0)}_{uz} = \int_{-\infty}^{+\infty} du \, \partial_u A^{(0)}_z, \tag{3.39}$$

and the expression of $A_z^{(0)}$ given by (2.27) specialized for $s=1$

$$A_z^{(0)} = -\frac{ie\sqrt{2}}{8\pi^2}\int_0^{+\infty} d\omega \left(a_+ e^{-i\omega u} - a_-^\dagger e^{i\omega u}\right) \tag{3.40}$$

where we have suppressed the (1) superscripts in the oscillators.
Plugging (3.40) in (3.39) we obtain:

$$\partial N = -\frac{\sqrt{2}}{8\pi e}\lim_{\omega\to 0^+}\omega\left[a_+(\omega) + a_-^\dagger(\omega)\right] \tag{3.41}$$

and the $N_z = e^2\partial N$ mode turns out to be localizing a contribution in the zero energy limit, namely it is a mode that is purely soft. This is why N_z is referred to as the leading soft photon mode. With this expression we can then come back to our charges (3.38), whose conservation gives the general expression:

$$\begin{aligned} &\lim_{\omega\to 0^+}\sqrt{2}\int\frac{d^2z}{4\pi e}\bar{\partial}\lambda\,\omega\,\langle\text{out}|\,a_+(\omega)S - Sa_-^\dagger(\omega)\,|\text{in}\rangle \\ =&\left[\sum_{i=1}^n Q_i^+\lambda(z_i^+,\bar{z}_i^+) - \sum_{j=1}^m Q_j^-\lambda(z_j^-,\bar{z}_j^-)\right]\langle\text{out}|\,S\,|\text{in}\rangle \end{aligned} \tag{3.42}$$

If we choose λ to be constant the soft part disappears and we get

$$\left(\sum_{i=1}^n Q_i^+ - \sum_{j=1}^m Q_j^-\right)\langle\text{out}|\,S\,|\text{in}\rangle = 0 \tag{3.43}$$

which is just the conservation of the electric charge.
On the other hand, if we choose

$$\lambda = \lambda_0(w,\bar{w}) = \lim_{\epsilon\to 0^+}\frac{\bar{w}-\bar{z}}{|z-w|^2+\epsilon^2} \tag{3.44}$$

we can reduce the charge conservation relation to:

$$\lim_{\omega\to 0^+}\omega\,\langle\text{out}|\,a_+(\omega)S - Sa_-^\dagger(\omega)\,|\text{in}\rangle = e\sqrt{2}\left[\sum_{i=1}^n\frac{Q_i^+}{z-z_j^+} - \sum_{j=1}^m\frac{Q_j^-}{z-z_j^-}\right]\langle\text{out}|\,S\,|\text{in}\rangle\,. \tag{3.45}$$

If we use the momentum parametrization (2.9) this relation is fully equivalent to Weinberg soft photon theorem (2.37).
We remark here that the choice (3.44) does not only allow to extract the soft theorem but also allows us to recast the soft charge (3.9) as:

$$Q_{\lambda_0} = \frac{1}{\pi}\mathcal{N}_z + \int_{\mathscr{I}^+} d^2w du\,\frac{\gamma_{w\bar{w}}}{w-z}j_u^{(2)} \tag{3.46}$$

where

$$\mathcal{N}_z(z,\bar{z}) = N_z(z,\bar{z}) + \int\frac{d^2w}{(z-w)^2}N_{\bar{z}}(w,\bar{w}) \tag{3.47}$$

is the combination of the two soft modes, which completely captures the soft part of the charge and is known as the memory mode.

We have then showed that the conservation of asymptotic charges is in strict correspondence with infrared divergences, as anticipated at the beginning of the chapter. Moreover we have seen that the mode N, which is the only contribution given by photons in the asymptotic charge, is purely a soft operator, namely it only creates infrared photons. This explains why the large gauge part of the phase space is also referred to as the soft sector of QED.
Notice that this new interpretation gives us a novel perspective on how to interpret the presence of infrared divergences: the soft $1/\omega$ pole in amplitudes involving photons is due to the fact that matter is charged under large gauge transformations. If we were able to neutralize the large gauge charge of matter, then the right hand side of (3.45) would be vanishing, making the amplitude IR finite. In the next section we will see how Faddeev-Kulish dressing is precisely neutralizing the matter operators, and how the dressing can be intimately related to the Goldstone mode ϕ.

3.1.2 Goldstone mode and Faddeev-Kulish dressing in QED

In the previous section we have discussed the relevance of the mode N in relation with infrared physics. Defining the soft phase space we have also encountered another soft mode, the symplectic partner of N referred to as the Goldstone mode ϕ, which transforms under large gauge as:

$$\delta\phi = -i[Q_\lambda, \phi] = \lambda. \tag{3.48}$$

We will now show how this mode can be used to neutralize the large gauge charge of matter operators, and how this will cancel the leading soft term. Let us define dressed massless matter operators of the form:

$$\hat{b}(\omega, z, \bar{z}) =: e^{iQ\phi(z,\bar{z})} : b(\omega, z, \bar{z}), \quad \hat{c}(\omega, z, \bar{z}) =: e^{-iQ\phi(z,\bar{z})} : c(\omega, z, \bar{z}) \tag{3.49}$$

where we have assigned a charge Q to the particle creation operator $b^\dagger$, and $-Q$ to the antiparticle annihilation operator $c^\dagger$. If we now compute the variation of these operators under the action of a large gauge transformation we get:

$$\delta_\lambda \hat{b} = -i[Q_\lambda, \hat{b}] = -i : e^{iQ\phi} : [Q_\lambda, b] - i[Q_\lambda, : e^{iQ\phi} :]b = -iQ\lambda\hat{b}(p) + iQ\lambda\hat{b}(p) = 0 \tag{3.50}$$

This implies that charge conservation between a scattering amplitude of dressed matter states:

$$|\text{in}\rangle_D = \prod_{i=1}^{n_I} \hat{b}^\dagger(z_i, \bar{z}_i) \prod_{j=1}^{m_I} \hat{c}^\dagger(z_j, \bar{z}_j) |0\rangle \tag{3.51}$$

will just produce:

$$\lim_{\omega\to 0^+} {}_D\langle\text{out}|\, \omega \left[a_+(\omega)S - Sa_-^\dagger(\omega)\right] |\text{in}\rangle_D = 0. \tag{3.52}$$

namely the scattering amplitude is now IR finite.
The kind of dressing we are using here is resembling the Faddeev-Kulish dressing, introduced in section 2.2.3, where every matter particle operator was dressed by a photon soft cloud operator $W_\pm(k)$. We will now show that it is possible to match the two dressings.

To do so we consider the explicit expression of $W_\pm(k) \equiv e^{w_\pm(k)}$ in (2.71), focusing on the exponent

$$\begin{aligned} w_\pm &= \pm Qe \int \frac{d^3k}{2k_0(2\pi)^3} f(p,k) \sum_{\alpha=\pm} \left(\frac{p \cdot \epsilon^{\alpha *}}{p \cdot k} a_\alpha(k) - \frac{p \cdot \epsilon^{\alpha}}{p \cdot k} a^\dagger_\alpha(k) \right) \\ &= \pm \frac{eQ}{\sqrt{2}(2\pi)^3} \int_0^{+\infty} d\omega \int d^2z f(p,k) \left\{ \frac{1}{\bar{z} - \bar{z}_k} \left(a_+ - a^\dagger_- \right) + \frac{1}{z - z_k} \left(a_+ - a^\dagger_- \right) \right\} \end{aligned} \tag{3.53}$$

where we have parametrized the null momenta using the convention (2.9). For this to reduce to $iQ\phi$, it must exists a function $f(p,k)$ such that,:

$$\frac{ie}{\sqrt{2}(2\pi)^3} \int_0^{+\infty} d\omega \int d^2z f(p,k) \left\{ \frac{1}{\bar{z} - \bar{z}_k} \left(a_+ - a^\dagger_- \right) + \frac{1}{z - z_k} \left(a_- - a^\dagger_+ \right) \right\} \tag{3.54}$$

is a Lorentz scalar ϕ satisfying the commutation relation $[\partial\phi, \partial N]$ (3.30). Lorentz invariance would force $f(p,k) = g(\frac{p \cdot k}{\Lambda^2})$, where Λ is a scale that must be introduced to keep f dimensionless: however ϕ by definition does not depend on any additional scale Λ which forces us to choose $f(p,k) = 1$. This choice is unusual from the point of view of Faddeev-Kulish dressing, and it is referred to as conformal dressing. This fixes [152]:

$$\phi = \frac{ie}{\sqrt{2}(2\pi)^3} \int_0^{+\infty} d\omega \int d^2w \left\{ \frac{1}{\bar{z} - \bar{w}} \left(a_+ - a^\dagger_- \right) + \frac{1}{z - w} \left(a_- - a^\dagger_+ \right) \right\} \tag{3.55}$$

which satisfies all the required properties for ϕ including its commutation relation with N. Upon this identification this shows the two dressings are equivalent.

With the above procedure we have basically back-engineered the expression for ϕ in terms of creation and annihilation operators, but it is possible to make (3.55) more rigorous [152,153] thanks to the existence of symplectically paired classical solutions to Maxwell equations A^G and A^{CS}, which can be precisely identified as the leading soft photon and the Goldstone mode. Using the Klein-Gordon inner product it is possible to prove that A^G gives expression (3.41) for the memory mode and A^{CS} gives (3.55) for the Goldstone mode.
Using (3.55) we can also compute correlation functions between Goldstone modes, for example we can focus on the two point function and obtain:

$$\langle \phi(z,\bar{z}) \phi(w,\bar{w}) \rangle = \frac{e^2}{(2\pi)^3} \int_0^{+\infty} \frac{d\omega}{\omega} \int d^2x \left[\frac{1}{x - z} \frac{1}{\bar{x} - \bar{w}} + \frac{1}{\bar{x} - \bar{z}} \frac{1}{x - w} \right] \tag{3.56}$$

which is divergent both in the UV and the IR. We can regulate this expression in the IR using dim-reg by shifting the dimension to $d = 4 + 2\epsilon$, and in the UV by putting a cut-off Λ. We will neglect the UV cut-off in the following computations as we are interested in the IR behavior of the correlation function. This turns the two-point function into

$$\langle \phi(z,\bar{z}) \phi(w,\bar{w}) \rangle = \frac{e^2}{(2\pi)^3} \int_0^{+\infty} d\omega \omega^{-1+2\epsilon} \int d^{2+2\epsilon}x \left[\frac{1}{x - z} \frac{1}{\bar{x} - \bar{w}} + \frac{1}{\bar{x} - \bar{z}} \frac{1}{x - w} \right]. \tag{3.57}$$

Using the seed-integral computed in section B.1 of [154] we reduce the two-point function to

$$\begin{aligned} \langle \phi(z,\bar{z}) \phi(w,\bar{w}) \rangle &= \frac{e^2}{(2\pi)^3} \frac{\Lambda^{2\epsilon}}{\epsilon} \frac{2^{1-2\epsilon} \pi^{\frac{5}{2}+\epsilon} |z - w|^{2\epsilon}}{\Gamma\left(\frac{1}{2} + \epsilon\right) \sin(\pi\epsilon)} = \\ &= \frac{e^2}{4\pi^2} \left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \log|z - w|^2 \right) + \dots \end{aligned} \tag{3.58}$$

where in the dots we are absorbing the UV contribution. We are finding that the two point functions of the goldstone mode, takes a logarithmic form, with the coefficient multiplying the logarithm precisely matching the cusp anomalous dimension $\alpha_{EM} = \frac{e^2}{4\pi^2}\frac{1}{\epsilon}$ defined in (2.44)[2]. This result, aside from the extra ϵ^{-2}, is compatible with the assumption made in [152]. We expect that the extra ϵ^{-2} term in our derivation is scheme dependant and can be reabsorbed in the next ϵ^{-1} term, and we will later neglect it.
We will see that the two point function (3.58) is the fundamental ingredient that allows the dressing to cancel the divergent exponential (2.42) in amplitudes and turn them IR finite. However we will leave this computation for when we will discuss IR finite celestial amplitudes.

3.2 Phase space of asymptotically flat spacetime

In this section we will now move to gravity where we will show that the rich phase space allows us to connect the symmetries of asymptotically flat spacetimes with the infrared behavior of gravity.

In this section we will focus on general relativity, whose dynamics is specified by the Einstein-Hilbert Lagrangian [155]:

$$L = \frac{1}{\kappa^2}\sqrt{g}(R - 2\Lambda + \mathcal{L}_M)d^4x \tag{3.59}$$

where Λ is the cosmological constant, R is the Ricci scalar for the metric g of the target manifold, and $\mathcal{L}_M$ is the matter Lagrangian. The equations of motions coming from this action are the well known Einstein equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi T^M_{\mu\nu} \tag{3.60}$$

with a localized matter stress-energy tensor T^M:

$$T^M_{\mu\nu} = g_{\mu\nu}\mathcal{L}_M - 2\frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}}. \tag{3.61}$$

If one assumes all matter to give a vanishing contribution on the boundary of spacetime, we see that

$$R = 4(\Lambda - 2\pi T^M) \xrightarrow{\partial M} R = 4\Lambda, \quad T^M = g^{\mu\nu}T^M_{\mu\nu}. \tag{3.62}$$

This implies that for $\Lambda \neq 0$ we cannot have vanishing Ricci tensor on the boundary.
We will be focusing on the case of asymptotically flat spacetime, that by definition require a vanishing Ricci tensor on the boundary. For that we impose $\Lambda = 0$, such that asymptotically flat spacetime becomes a set of viable solutions of Einstein equations. We want to study the phase space of those solutions, identify what are the free data and see what is the resulting set of asymptotic symmetries on the boundary of flat space. Because this set up is designed to get flat space in a neighborhood of the boundary we would expect that the group of asymptotic symmetries reduces to the isometries of flat space, namely the Poincaré group. As we will soon observe, however, this expectation will not hold true. The group we will discover indeed contains Poincaré, yet it will be far larger and built from an infinite set of generators [24, 26–28, 109].

[2] The matching between cut-off regularization and dim-reg is done by sending $\log\lambda \to \frac{1}{\epsilon}$

Without getting ahead of ourself let us build the phase space of asymptotically flat spaces. As we are interested in a model of only gravitons and massless fields we can restrict our Cauchy surface to future or past null infinity. As the result would be analogous for the two surfaces we will focus only on future null infinity in what follows.

To parametrize $\mathscr{I}^+$ we will work in retarded Bondi coordinates, such that taking $r \to +\infty$ we reach future null infinity. As gauge fixing we choose the Bondi-Sachs gauge [24,109] defined by:

$$\partial_r \det\left(\frac{g_{AB}}{r^2}\right) = 0, \quad g_{rr} = g_{rA} = 0, \tag{3.63}$$

where A, B run on the transverse indices $z, \bar{z}$.

As done for QED, we have to establish what fall-off conditions we want to impose on the metric and the matter stress tensor when we reach $\mathscr{I}^+$. Starting from the matter stress tensor, finiteness of energy, momentum and angular momentum require the following fall-offs [104,109,110]:

$$T_{uu}(u,r,z,\bar{z}) = T^{(2)}_{uu}(u,z,\bar{z})\frac{1}{r^2} + o(r^{-3}), \tag{3.64}$$

$$T_{uA}(u,r,z,\bar{z}) = T^{(2)}_{uA}(u,z,\bar{z})\frac{1}{r^2} + o(r^{-3}), \tag{3.65}$$

$$T_{ur}(u,r,z,\bar{z}) = o(r^{-4}), \quad T_{rr}(u,r,z,\bar{z}) = o(r^{-4}), \tag{3.66}$$

$$T_{rA}(u,r,z,\bar{z}) = o(r^{-3}), \quad T_{AB}(u,r,z,\bar{z}) = o(r^{-1}). \tag{3.67}$$

These are compatible with metric fall-offs of the form [25,109,112,156]:

$$g_{uu} = g^{(0)}_{uu} + o(r^{-1}), \quad g_{ur} = g^{(0)}_{ur} + o(r^{-2}), \quad g_{uz} = o(1), \tag{3.68}$$

$$g_{zz} = o(r), \quad g_{z\bar{z}} = r^2 \mathring{q}_{z\bar{z}} + o(1), \tag{3.69}$$

where the coefficients $g^{(0)}_{uu}, g^{(0)}_{ur}, \mathring{q}_{z\bar{z}}$ of the leading terms are fixed to match the flat metric at future null infinity.

The matrix $\mathring{q}_{AB}$ is the metric of the 2-sphere parametrized by the angular coordinates x^A. For the round Bondi coordinates (1.17) this will be the standard metric on the sphere $\mathring{q}_{AB}dx^A dx^B = 2\gamma_{z\bar{z}}dzd\bar{z}$, while if we consider flat Bondi coordinates (1.32) $\mathring{q}_{AB}dx^A dx^B = 2dzd\bar{z}$. To be precise we point out that in round Bondi coordinates we have $g^{(0)}_{uu} = g^{(0)}_{ur} = -1$ [25,109] while in flat Bondi $g^{(0)}_{uu} = 0$, $g^{(0)}_{ur} = -1$ [110,112]. In the next step we will focus on flat Bondi coordinates for simplicity.

Having fixed boundary and gauge conditions, one can find the generic solution of the $\Lambda = 0$ Einstein equations written as r expansion around $\mathscr{I}^+$. The generic form of this solutions was found by Bondi, Metzner [24] and Sachs [109] and takes the form:

$$\begin{aligned} ds^2 = &\left(\frac{2M}{r} + o(r^{-2})\right)du^2 - 2dudr + (r^2\mathring{q}_{AB} + rC_{AB} + o(r^0))dx^A dx^B + \\ &+ \left[\frac{1}{2}\partial^B C_{AB} + \frac{2}{3r}\left(N_A + \frac{1}{4}C_{AB}\partial_C C^{BC}\right) + o(r^{-2})\right]dudx^A + \dots \end{aligned} \tag{3.70}$$

where $C_{AB}(u,z,\bar{z})$ is a trace-less symmetric tensor, usually referred to as the gravitational shear as it measures the tidal distortion or shearing deformation of freely falling test particles as a gravitational wave passes. It is basically a measure of the incoming gravitational radiation at $\mathscr{I}^+$. $M(u,z,\bar{z})$ is the Bondi mass aspect which must not be confused with the ADM mass. The ADM mass is a single number representing the total mass-energy of an isolated system at spatial infinity and is conserved. On the contrary $M(u,z,\bar{z})$ is a function of the angular and null time coordinates and basically measures of energy per solid angle at null infinity [24,25,109]. It is usually not conserved as it takes into account the energy that gravitational waves carry energy away from the system. $N_A(u,z,\bar{z})$ is the angular momentum aspect, which is basically analogous to the Bondi mass aspect but instead of carrying mass density is retaining information about angular momentum. This is also not conserved as gravitational waves also strip away units of angular momentum.

The time evolution of both these quantities is in particular specified by the following equations [112]:

$$\partial_u M = -\frac{1}{8}N_{AB}N^{AB} + \frac{1}{4}\partial_A\partial_B N^{AB} - 4\pi G T^{(2)}_{uu}, \tag{3.71}$$

$$\partial_u N_A = \partial_A M + \frac{1}{16}\partial_A(N_{BC}C^{BC}) - \frac{1}{4}N^{BC}\partial_A C_{BC} - 8\pi G T^{(2)}_{uA} + \tag{3.72}$$

$$-\frac{1}{4}\left(C^{BC}N_{AC} - N^{BC}C_{AC}\right) - \frac{1}{4}\partial_B\partial^B\partial^C C_{AC} + \frac{1}{4}\partial_B\partial_A\partial_C C^{BC}. \tag{3.73}$$

It is clear then that all the quantities M, N_A vary as a function of $(u,z,\bar{z})$ and $C_{AB}(u,z,\bar{z})$, which means that only the shear constitutes the free data at $\mathscr{I}^+$. This means that $C_{AB}(u,z,\bar{z})$ captures the radiative data and will specify the radiative phase space, paired with its momentum $N_{AB} = \partial_u C_{AB}$ referred to as the news tensor. The symplectic structure of this radiative phase space is given by the Ashtekar-Streubel symplectic form [157]:

$$\Omega_{AS} = \frac{1}{\kappa^2}\int dud^2z\,\left[\delta N_{zz}\wedge\delta C_{\bar{z}\bar{z}} + c.c.\right]. \tag{3.74}$$

3.2.1 Asymptotic symmetries of flat spacetime

As pointed out in our analysis of QED if we want to identify the asymptotic symmetries we need to find the residual diffeomorphisms that act non trivially on the boundary data and leave the fall-offs invariant. The space of these diffeomorphisms is generated by vectors (1.25), $\xi = \xi^u\partial_u + \xi^r\partial_r + \xi^z\partial_z + \xi^{\bar{z}}\partial_{\bar{z}}$ satisfying [25,26,110]:

$$\begin{gathered}\mathcal{L}_\xi g_{uu} = o(r^{-1}), \quad \mathcal{L}_\xi g_{ur} = o(r^{-2}),\\ \mathcal{L}_\xi g_{uz} = o(1), \quad \mathcal{L}_\xi g_{zz} = o(r), \quad \mathcal{L}_\xi g_{z\bar{z}} = o(1).\end{gathered} \tag{3.75}$$

These conditions are satisfied by vectors of the form [110,112]

$$\begin{aligned}\xi^u &= \frac{u}{2}\left(\partial_z Y^z + \partial_{\bar{z}} Y^{\bar{z}}\right) + \mathcal{T}(z,\bar{z}),\\ \xi^r &= -\frac{r}{2}\left(\partial_z Y^z + \partial_{\bar{z}} Y^{\bar{z}}\right) + o(r^0),\\ \xi^z &= Y^z + o(r^{-1}),\\ \xi^{\bar{z}} &= Y^{\bar{z}} + o(r^{-1}),\end{aligned} \tag{3.76}$$

with $Y^z\partial z$ and $Y^{\bar{z}}\partial\bar{z}$ globally holomorphic vector fields in the variables z and $\bar{z}$ respectively. Notice that this has the same functional form as the vectors in (1.33), with the key difference that the parameter $\mathcal{T}(z,\bar{z})$ now belongs to a much larger class of functions, namely any differentiable function $\mathcal{T}:\mathbb{C}\to\mathbb{R}$. If we restrict the parameters $\mathcal{T}(z,\bar{z})$, Y^z, $Y^{\bar{z}}$ to the specific form given in (1.34), we recover the transformations of the Poincaré group. These, however, represent only a subset of the allowed asymptotic transformations: they form a subgroup of an infinite-dimensional group known as the Bondi $-$ Metzner $-$ Sachs (BMS) group, generated by the vectors in (3.76).

The BMS group, originally introduced by Bondi, Metzner, and Sachs in 1962 [24,25], extends the translation subgroup of the Poincaré group while leaving the Lorentz subgroup, parametrized by $Y^z, Y^{\bar{z}}$, unchanged. This restriction follows from the requirement that $Y^z\partial_z$ and $Y^{\bar{z}}\partial_{\bar{z}}$ be globally defined holomorphic vector fields. In fact, the only globally defined holomorphic sphere on $\mathbb{C}^*$ are $\{\partial_z,\ z\partial_z,\ z^2\partial_z,\ \partial_{\bar{z}},\ \bar{z}\partial_{\bar{z}},\ \bar{z}^2\partial_{\bar{z}}\}$, which correspond to the six generators of the Lorentz group.

In recent years, this condition has been relaxed to allow for singular violations of the conformal Killing equations (3.2.1) [26–28]. This leads to an extension of the BMS group, referred to as the extended BMS (eBMS) group, in which $Y^z\partial_z$ and $Y^{\bar{z}}\partial_{\bar{z}}$ are required to be only locally holomorphic. As a result, the Lorentz subgroup is enlarged to an infinite-dimensional group.

To better understand the properties of eBMS, which will be the main focus of the next sections, we can start by analyzing its algebra.
If we consider the modified Lie brackets $[\xi_1,\xi_2]_\star = [\xi_1,\xi_2] - \delta_{\xi_1}\xi_2 + \delta_{\xi_2}\xi_1$, where the last two terms cancel the contributions given by the variation of the metric at subleading order in r, then we can see that the Killing vectors satisfy the commutation relations [26–28]:

$$[\xi_1(\mathcal{T}_1,Y_1^z,Y_1^{\bar{z}}),\xi_2(\mathcal{T}_2,Y_2^z,Y_2^{\bar{z}})]_\star = \xi(\mathcal{T}_{12},Y_{12}^z,Y_{12}^{\bar{z}}) \tag{3.77}$$

where

$$\mathcal{T}_{12} = Y_1^z\partial_z\mathcal{T}_2 - \frac{1}{2}\partial_z Y_1^z\mathcal{T}_2 - Y_2^z\partial_z\mathcal{T}_1 - \frac{1}{2}\partial_z Y_2^z\mathcal{T}_1 + c.c. \tag{3.78}$$

$$Y_{12}^z = Y_1^z\partial_z Y_2^z - Y_2^z\partial_z Y_1^z, \quad Y_{12}^{\bar{z}} = Y_1^{\bar{z}}\partial_{\bar{z}} Y_2^{\bar{z}} - Y_2^{\bar{z}}\partial_{\bar{z}} Y_1^{\bar{z}} \tag{3.79}$$

Even if relations might appear a little bit obscure at first glance, we can make them much more clear just with a with a clever choice of generators. In particular if we chose:

$$T_{k,l} = \xi\left(z^{-k+1/2}\bar{z}^{-l+1/2},0,0\right), \quad Y_n = \xi\left(0,z^{-n+1},0\right), \quad \overline{Y}_n = \xi\left(0,0,\bar{z}^{-n+1}\right), \tag{3.80}$$

we obtain the following algebra [26–28]:

$$[Y_n,Y_m]_\star = (n-m)Y_{n+m}, \quad \left[\overline{Y}_n,\overline{Y}_m\right]_* = (n-m)\overline{Y}_{n+m}, \quad \left[Y_n,\overline{Y}_m\right]_\star = 0 \tag{3.81}$$

$$[Y_n,T_{k,l}]_\star = \left(\frac{n}{2}-k\right)T_{n+k,l}, \quad \left[\overline{Y}_n,T_{k,l}\right]_\star = \left(\frac{n}{2}-l\right)T_{k,n+l}, \tag{3.82}$$

$$[T_{i,j},T_{k,l}]_\star = 0. \tag{3.83}$$

We can see then that the extended BMS group is just the semi direct product of two copies of the Witt algebra, namely centerless Virasoro, with the abelian group of all real functions on the complex plane [158]

$$\text{eBMS} = \left(\text{Vir}\otimes\overline{\text{Vir}}\right)\ltimes C^\infty(\mathbb{C},\mathbb{R}). \tag{3.84}$$

The Vir $\otimes$ $\overline{\text{Vir}}$ part of the group contains the Lorentz group $SL(2,\mathbb{C})$, and can then be seen as the extensions of rotations. For this reason they are referred to as *superrotations*. The abelian part $C^\infty(\mathbb{C},\mathbb{R})$ on the other hand can be seen as an extension of translations, which shifts points by an angle dependent function. For this reason they are referred to as *supertranslations*. Extended BMS can then also be rewritten as superrotations $\ltimes$ supertranslations. Most importantly all of these transformation have non zero canonical charge given by [156, 159–167]

$$Q^+ [f, Y] = \frac{1}{8\pi G} \int_{\mathscr{I}^+_-} d^2z \left[2M\mathcal{T} + Y^z N_z + Y^{\bar{z}} N_{\bar{z}}\right]. \tag{3.85}$$

Before commenting on how these charges are related to soft theorems we would like to determine the action of eBMS group on the free data of the theory. This can be induced from the transformation of the metric under diffeomorphism which gives us [112]:

$$\begin{aligned} \delta_{f,Y} C_{zz} &= \left(Y^z\partial + Y^{\bar{z}}\bar{\partial} + \frac{3}{2}\partial Y^z - \frac{1}{2}\bar{\partial}Y^{\bar{z}}\right) C_{zz} + \\ &\quad + \left(u\frac{\partial Y^z + \bar{\partial}Y^{\bar{z}}}{2} + \mathcal{T}\right) N_{zz} - 2\partial^2\mathcal{T} - u\partial^3 Y^z \\ \delta_{\mathcal{T},Y} N_{zz} &= \left(Y^z\partial + Y^{\bar{z}}\bar{\partial} + 2\partial Y^z\right) N_{zz} + \\ &\quad + \left(u\frac{\partial Y^z + \bar{\partial}Y^{\bar{z}}}{2} + \mathcal{T}\right) \partial_u N_{zz} - \partial^3 Y^z \end{aligned} \tag{3.86}$$

These transformations can also be obtained using the Poisson brackets, induced by the Ashtekar-Streubel symplectic form, between the charges and C_{zz}, N_{zz}, as an additional proof that they really act as the generators of the symmetry.
We notice that the transformations of C_{zz} is not homogeneous under supertranslations, but there is an inhomogeneous shift of $-2\partial^2\mathcal{T}$. This implies that if we start from a fully flat solution with $M = N_{zz} = C_{zz} = 0$, after a supertranslation we would have turned on a non-zero shear $C_{zz} = -2\partial^2\mathcal{T}$. This is does not modify the geometry on the boundary as the Riemann tensor at infinity it is still identically vanishing, so we are not creating any curvature. In particular one can prove that vanishing curvature implies $C_{zz} = -2\partial^2 C$ where C is some u independent function. As for the Goldstone mode ϕ in QED, the field C is taking into account the behavior of C_{zz} at the boundary. This is why we define C such that [112]:

$$-2\partial^2 C = \frac{1}{2}\lim_{u\to+\infty} \left(C_{zz}(u) + C_{zz}(-u)\right) = 2\mathscr{C}_{zz} \tag{3.87}$$

where C transforms under a supertranslation as:

$$\delta_{\mathcal{T}} C = \mathcal{T}. \tag{3.88}$$

Analogously to ϕ, we are going to refer to C as the supertranslation Goldstone mode, and to $\mathscr{C}_{zz}$ as the supertranslation goldstone current.

3.3 Charge conservation and graviton soft theorems

As for QED, now that we have identified the asymptotic charges we want to see what are the implications of their Ward identity. We will see that the supertranslation charge gives rise to the leading soft graviton theorem while the superrotation charge is related to the subleading soft graviton theorem. As in the case of QED we have two separate charges $Q^+[\mathcal{T}^+, Y^+]$ and $Q^-[\mathcal{T}^-, Y^-]$ located at future and past null infinity, each of which could be a priori associated to different BMS generators BMS^+ and BMS^-, such that the full group becomes $\mathrm{BMS}^+ \times \mathrm{BMS}^-$. However it is not true that all these charges would be conserved as we still have to impose the appropriate matching conditions between $\mathscr{I}^+_-$ and $\mathscr{I}^-_+$. As argued for the EM case, the right condition turns out to be again the antipodal matching [168] that forces:

$$M(z,\bar{z})|_{\mathscr{I}^+_-} = M(z,\bar{z})|_{\mathscr{I}^-_+}, \quad C(z,\bar{z})|_{\mathscr{I}^+_-} = C(z,\bar{z})|_{\mathscr{I}^-_+}, \quad N_A(z,\bar{z})|_{\mathscr{I}^+_-} = N_A(z,\bar{z})|_{\mathscr{I}^-_+} \tag{3.89}$$

These matching conditions constrain the generators of BMS to be the same at future and past null infinity, which reduces the actual symmetry group from $\mathrm{BMS}^+ \times \mathrm{BMS}^-$ to the diagonal subgroup, which is just one copy of BMS. The conservation relation is then established at a quantum level as usual in the weak sense:

$$\langle \mathrm{out}| \, Q^+[\mathcal{T}, Y]S - SQ^-[\mathcal{T}, Y] \, |\mathrm{in}\rangle = 0 \tag{3.90}$$

3.3.1 Supertranslation charges

We will start by discussing supertranslations, setting the superrotation generator to zero. Using the equation for $\partial_u M$ the action of $Q^+[f, 0]$ on states can be rewritten as:

$$\begin{aligned}\frac{\kappa^2}{2} Q^+[\mathcal{T}, 0] \, |\mathrm{out}\rangle = &-\frac{1}{8}\int d^2z du \mathcal{T}(z,\bar{z}) \left(\partial_z^2 N^{zz} + \bar{\partial}^2 N^{\bar{z}\bar{z}}\right) |\mathrm{out}\rangle + \\ &+ \int d^2z du \mathcal{T}(z,\bar{z}) T_{uu} \, |\mathrm{out}\rangle\end{aligned} \tag{3.91}$$

and again we see that the action of the charges splits into two pieces, a purely gravitational term which as we will see contains only soft modes and a term usually referred to as the hard part of the charge, that purely acts on matter fields.

Notice that if $\mathcal{T}$ corresponds to a translation $\partial^2\mathcal{T} = \bar{\partial}^2\mathcal{T} = 0$ and the soft part of the charge vanishes. We recover that the charge conservation for translations corresponds to the conservation of energy and momentum.

Because $\mathcal{T}$ is u independent it can be taken outside the null time integral, such that the charge can be rewritten as:

$$\begin{aligned}\frac{\kappa^2}{8} Q^+[\mathcal{T}, 0] = &-\frac{1}{4}\int d^2z du \mathcal{T}(z,\bar{z}) \left(\partial^2 \mathcal{N}^{(0)}_{\bar{z}\bar{z}} + \bar{\partial}^2 \mathcal{N}^{(0)}_{zz}\right) |\mathrm{out}\rangle + \\ &+ \int d^2z du \mathcal{T}(z,\bar{z}) T_{uu} \, |\mathrm{out}\rangle\end{aligned} \tag{3.92}$$

where the operators [29]:

$$\begin{aligned}\mathcal{N}^{(0)}_{zz} &\equiv \int_{-\infty}^{+\infty} du N_{zz} = -\frac{\kappa}{4\pi} \lim_{\omega\to 0^+} \omega \left(a_+(\omega) + a_-^\dagger(\omega)\right) \\ \mathcal{N}^{(0)}_{\bar{z}\bar{z}} &\equiv \int_{-\infty}^{+\infty} du N_{\bar{z}\bar{z}} = -\frac{\kappa}{4\pi} \lim_{\omega\to 0^+} \omega \left(a_-(\omega) + a_+^\dagger(\omega)\right)\end{aligned} \tag{3.93}$$

are graviton soft modes analogous to $N_z, N_{\bar{z}}$ described in QED. Notice that also here we are suppressing the (2) superscript in the gravity ladder operators for brevity. The expression as a function of creation and annihilation operators has been obtained applying definition (3.93), to the asymptotic expressions (2.27) for the case $s = 2$.
Using the same argument as for QED, the conservation of supertranslation charge turns into the leading soft graviton theorem [29]:

$$\lim_{\omega \to 0^+} \omega \langle \text{out} | a_+(\omega, z, \bar{z}) S | \text{in} \rangle = \frac{\kappa}{2} \sum_k \omega_k \frac{\bar{z} - \bar{z}_k}{z - z_k} \langle \text{out} | S | \text{in} \rangle \tag{3.94}$$

here expressed with momentum parametrization (2.9), where the sum runs over all in and out particles, and ω_k is the energy of external massless particles. The right hand side is a consequence of the presence of the hard contribution, as the matter stress tensor extracts the energy value ω_k of massless particles times a localized delta function contribution. Notice that for the following choice of $\mathcal{T}$,

$$\mathcal{T} = \mathcal{T}_0(w, \bar{w}) = \frac{1}{\pi} \frac{z - w}{\bar{z} - \bar{w}}, \tag{3.95}$$

isolates the soft modes, such that the gauge charge turns into [84, 153]:

$$Q[f_0, 0] = 2\pi \mathcal{N}_{zz} + \int d^2w du \, \frac{z - w}{\bar{z} - \bar{w}} T_{uu}(u, w, \bar{w}), \tag{3.96}$$

with

$$\mathcal{N}_{zz} = \frac{1}{2\pi} \left[\mathcal{N}^{(0)}_{zz} + \int \frac{d^2w}{\pi} \frac{z - w}{(\bar{z} - \bar{w})^3} \mathcal{N}^{(0)}_{\bar{z}\bar{z}} \right], \tag{3.97}$$

which is referred to as the gravitational memory.

In this subsection we have indeed shown how supertranslation charges are in correspondence with leading soft graviton theorem. The enhancement of translation to supertranslations originally discovered by Bondi Metzner and Sachs, and that was considered originally as an undesired feature of the asymptotic symmetry analysis, turns out to be an essential ingredient to the reconstruction of the classical well known result by Weinberg, regarding the infrared behavior of gravitons.

3.3.2 Superrotation charges

Turning now to the superrotations charges, we will fix $f = 0$ and turn on only the superrotation parameter Y. In this case we can use the equation ruling the evolution of the angular momentum aspect $\partial_u N_A$ to rewrite the charge as:

$$\begin{aligned} Q^+[0, Y] |\text{out}\rangle = &-\frac{2}{\kappa^2} \int d^2z du \left(\partial^3 Y^z u N^z_{\bar{z}} + \bar{\partial}^3 Y^{\bar{z}} u N^{\bar{z}}_z \right) \\ &+ \frac{4}{\kappa^2} \int d^2z du \left(Y_{\bar{z}} T_{uz} + Y_z T_{u\bar{z}} + u \partial_z Y_{\bar{z}} T_{uu} + u \partial_{\bar{z}} Y_z T_{uu} \right). \end{aligned} \tag{3.98}$$

As $Y^z, Y^{\bar{z}}$ are also u independent we can work out the same step as done for the supertranslation charge and define the sub-leading soft modes [31]:

$$\begin{aligned} \mathcal{N}^{(1)}_{zz} &\equiv \int_{-\infty}^{+\infty} du u N_{zz} = \frac{i\kappa}{4\pi} \lim_{\omega \to 0^+} (1 + \omega \partial_\omega) \Big(a_+(\omega) - a^\dagger_-(\omega) \Big) \\ \mathcal{N}^{(1)}_{\bar{z}\bar{z}} &\equiv \int_{-\infty}^{+\infty} du u N_{\bar{z}\bar{z}} = \frac{i\kappa}{4\pi} \lim_{\omega \to 0^+} (1 + \omega \partial_\omega) \Big(a_-(\omega) - a^\dagger_+(\omega) \Big). \end{aligned} \tag{3.99}$$

These modes are still localized in the low frequency limit, however we now see that the factor multiplying the ladder operators has a different form. Instead of being just ω it acts as a differential operator $(1+\omega\partial_\omega)$ on creation and annihilation operators. It is easy to notice that in the low energy limit this projector extracts the ω^0 contribution, namely exactly the subleading soft contribution.
Notice that in this case if we take Y^z, $Y^{\bar{z}}$ to be in $SL(2,\mathbb{C})$, the vectors satisfy $\partial^3 Y^z = \bar{\partial}^3 Y^{\bar{z}} = 0$ and again we see that the soft part is vanishing and the large gauge charge reduces to the total matter angular momentum. This implies that the conservation of large gauge charges turns into the conservation of angular momentum.
The conservation of the superrotation charges gives then rise to the subleading soft theorem [31]

$$\lim_{\omega\to 0^+}(1+\omega\,\partial_\omega)\langle\text{out}|\, a_-(\omega,z,\bar{z})S\,|\text{in}\rangle = \kappa\sum_k\left(\frac{(z-z_k)}{(\bar{z}-\bar{z}_k)}\omega_k\partial_k + \frac{(z-z_k)^2}{\bar{z}_k-\bar{z}}\right)\langle\text{out}|\, S\,|\text{in}\rangle \quad (3.100)$$

Again the right hand side comes from the matter stress tensor contribution in momentum space.

We have shown that indeed the conservation of superrotation charges is in correspondence with the subleading soft graviton theorem. The fact that the presence of superrotation is independent of additional effective operators that we can add to the Einstein Hilbert action, confirms the universality of the subleading soft factorization.
Differently from the leading soft theorem, that was originally discovered by Weinberg and then only recently connected to the ward identities of supertranslations, the subleading soft theorem was originally discovered through the ward identities of superrotations. From this perspective it is an original result coming from the analysis of asymptotic symmetries.

3.3.3 Goldstone mode and Faddeev-Kulish dressing in gravity

In section 3.1.2 we have shown that the large $U(1)$ Goldstone mode allows us to dress charged states to kill infrared divergences. The same procedure can be applied in perturbative gravity, in particular if one considers the dressed operators:

$$\hat{b}(\omega,z,\bar{z}) =: e^{i\omega C(z,\bar{z})} : b(\omega,z,\bar{z}), \quad \hat{c}(\omega,z,\bar{z}) =: e^{i\omega C(z,\bar{z})} : c(\omega,z,\bar{z}) \quad (3.101)$$

they will result to not be charged under supertranslations, such that then the dressed correlators will turn out to be IR finite:

$$\begin{aligned} &\lim_{\omega\to 0^+} {}_D\langle\text{out}|\,\omega a_+(\omega)S\,|\text{in}\rangle_D = 0, \\ &\lim_{\omega\to 0^+} {}_D\langle\text{out}|\, S\omega a_-^\dagger(\omega)\,|\text{in}\rangle_D = 0. \end{aligned} \quad (3.102)$$

The Goldstone dressing can be again matched with Faddeev-Kulish dressing in the conformal set-up $f(p,k)=1$, with C identified with the combination of creation and annihilation operators [152, 153]:

$$C(z,\bar{z}) = \frac{i\kappa}{8\pi^2}\int_0^{+\infty} d\omega \int d^2w \left[\frac{z-w}{\bar{z}-\bar{w}}(a_+ - a_-^\dagger) + \frac{\bar{z}-\bar{w}}{z-w}(a_- - a_+^\dagger)\right] \quad (3.103)$$

As for QED it possible to make (3.103) more rigorous [152, 153] thanks to the existence of symplectically paired classical solutions of the free graviton equation h^G and h^{CS}, which can be precisely identified as the memory mode and the Goldstone mode. Using the Klein-Gordon inner product it is possible to prove that h^G gives expression (3.93) for the memory mode and h^{CS} gives (3.103) for the Goldstone mode.

To see how the Goldstone mode can be used to construct Faddeev-Kulish amplitudes it is useful to compute the two-point function $\langle C(z,\bar{z})C(w,\bar{w})\rangle$. In [169], the supertranslation Goldstone two-point function was originally derived assuming that a correlation function of dressing operators $\mathcal{W}_k = e^{i\eta_k\omega_k C(z_k,\bar{z}_k)}$ would precisely reproduce the soft factor from Weinberg's factorization formula [127]. As we saw in equation (2.47) soft factorization states that a scattering amplitude $\mathcal{A}$ with an IR cutoff can be expressed as a product of a soft $\mathcal{S}$-matrix (which includes all IR divergences) and a hard piece (which is IR finite),

$$\mathcal{A} = \mathcal{A}_{\text{soft}}\mathcal{A}_{\text{hard}}\,. \tag{3.104}$$

The two-point function of the Goldstone supertranslation current C is so that the correlators of vertex operators exactly account for the soft part, namely

$$\mathcal{A}_{\text{soft}} = \langle \mathcal{W}_1 \dots \mathcal{W}_n\rangle\,. \tag{3.105}$$

This procedure is basically back engineering the form of the supertranslation Goldstone mode such that conformal dressing reproduces the same results as FK dressing.

However following [1] the form of the two point function $\langle CC\rangle$ can be established directly from the expression (3.103), without any additional assumptions.
We will now present such computation following the same steps as in [1]. At first we notice that (2.19) fixes the two point function if $\delta a_\pm(z,\bar{z},\omega) = a_\pm - a_\pm^\dagger$:

$$\begin{aligned}
&\langle \delta a_\pm(z,\bar{z},\omega)\delta a_\pm(w,\bar{w},\xi)\rangle = 0\\
&\langle \delta a_\pm(z,\bar{z},\omega)\delta a_\mp(w,\bar{w},\xi)\rangle = -\frac{16\pi^3}{\omega}\delta(\omega-\xi)\delta^{(2)}(z-w)\,,
\end{aligned} \tag{3.106}$$

This implies that the $\langle C(z,\bar{z})C(w,\bar{w})\rangle$ two-point function can be put in the form

$$\begin{aligned}
&\langle C(z,\bar{z})C(w,\bar{w})\rangle \\
&= \frac{-\kappa^2}{4(2\pi)^4}\int_0^\infty d\omega\int_0^\infty d\xi\int\frac{d^2x}{2\pi}\frac{d^2y}{2\pi}\left[\frac{z-x}{\bar{z}-\bar{x}}\frac{\bar{w}-\bar{y}}{w-y}\langle\delta a_+(x)\delta a_-(y)\rangle+\frac{\bar{z}-\bar{x}}{z-x}\frac{w-y}{\bar{w}-\bar{y}}\langle\delta a_-(x)\delta a_+(y)\rangle\right]\\
&= \frac{\kappa^2}{2(2\pi)^3}\int_0^\infty\frac{d\omega}{\omega}\int d^2x\left[\frac{z-x}{\bar{z}-\bar{x}}\frac{\bar{w}-\bar{x}}{w-x}+\frac{\bar{z}-\bar{x}}{z-x}\frac{w-x}{\bar{w}-\bar{x}}\right]\,.
\end{aligned} \tag{3.107}$$

The integrals over ω and $(x,\bar{x})$ diverge so they need to be regularized. We are interested in the IR behaviour and will neglect UV divergences. We will use dimensional regularization (dim-reg) with $d = 2+2\epsilon$ to treat the IR divergences in the $(x,\bar{x})$ integral. Notice that this also cures the IR divergence in ω as in d-dimension the two-point function (3.106) becomes

$$\langle \delta a_\pm(z,\bar{z},\omega)\delta a_\mp(w,\bar{w},\xi)\rangle = -\frac{16\pi^3}{\omega^{d-1}}\delta(\omega-\xi)\delta^{(d)}(\vec{z}-\vec{w})\,. \tag{3.108}$$

We will then focus on the following regulated expression:

$$C_\epsilon(z,\bar{z}) = \frac{i\kappa}{8\pi^2}\int_0^\infty d\omega\omega^{2\epsilon}\int\frac{d^{2+2\epsilon}w}{2\pi}\left[\frac{z-w}{\bar{z}-\bar{w}}\,\delta a_+(w,\bar{w},\omega)+\frac{\bar{z}-\bar{w}}{z-w}\,\delta a_-(w,\bar{w},\omega)\right]\,. \tag{3.109}$$

Notice that in the limit $\epsilon \to 0$ we recover the expression for the Goldstone mode C.

The regulated two-point function is thus

$$\langle C_\epsilon(z,\bar{z})C_\epsilon(w,\bar{w})\rangle = \frac{\kappa^2\mu_0^{2\epsilon}}{2(2\pi)^3}\int_0^\Lambda d\omega\omega^{-1+2\epsilon}\int d^{2+2\epsilon}x\left[\frac{z-x}{\bar{z}-\bar{x}}\frac{\bar{w}-\bar{x}}{w-x}+\frac{\bar{z}-\bar{x}}{z-x}\frac{w-x}{\bar{w}-\bar{x}}\right], \quad (3.110)$$

where μ_0 is the mass scale introduced in dim-reg to preserve the mass dimension of the two-point function, and Λ is the UV cutoff. The ω integral reduces to

$$\int_0^\Lambda d\omega\omega^{-1+2\epsilon} = \frac{\Lambda^{2\epsilon}}{2\epsilon} = \frac{1}{2\epsilon}+\log\Lambda \sim \frac{1}{2\epsilon}, \quad (3.111)$$

neglecting as announced the UV divergence. Hence,

$$\langle C_\epsilon(z,\bar{z})C_\epsilon(w,\bar{w})\rangle = \frac{\kappa^2}{4(2\pi)^3}\frac{\mu_0^{2\epsilon}}{\epsilon}\int d^{2+2\epsilon}x\left[\frac{z-x}{\bar{z}-\bar{x}}\frac{\bar{w}-\bar{x}}{w-x}+\frac{\bar{z}-\bar{x}}{z-x}\frac{w-x}{\bar{w}-\bar{x}}\right]. \quad (3.112)$$

The integral

$$I_\epsilon := \mu_0^{2\epsilon}\int d^{2+2\epsilon}x\,\frac{z-x}{\bar{z}-\bar{x}}\frac{\bar{w}-\bar{x}}{w-x} \quad (3.113)$$

can then be computed using standard QFT techniques and gives the following result (see appendix A.2):

$$I_\epsilon = \frac{6\pi^{2+\epsilon}(1+\epsilon)\Gamma(2+\epsilon)^2}{\Gamma(1+\epsilon)\Gamma(4+2\epsilon)\sin(\pi\epsilon)}|z-w|^{2+2\epsilon}\mu_0^{2\epsilon}. \quad (3.114)$$

This lead to the following expression for the two-point function:

$$\begin{aligned}\langle C_\epsilon(z,\bar{z})C_\epsilon(w,\bar{w})\rangle &= \frac{\kappa^2}{16\pi^3}\frac{1}{\epsilon}I_\epsilon \\ &= \frac{3\kappa^2}{8\pi^3\epsilon}\frac{\pi^{2+\epsilon}(1+\epsilon)\Gamma(2+\epsilon)^2}{\Gamma(1+\epsilon)\Gamma(4+2\epsilon)\sin(\pi\epsilon)}|z-w|^{2+2\epsilon}\mu_0^{2\epsilon}.\end{aligned} \quad (3.115)$$

We can then expand (3.115) for small ϵ and get:

$$\begin{aligned}\langle C_\epsilon(z,\bar{z})C_\epsilon(w,\bar{w})\rangle &= \frac{\kappa^2}{16\pi^2}|z-w|^2\frac{1}{\epsilon}\left(\frac{1}{\epsilon}-\frac{2}{3}+\gamma_E+\log\pi+\log|z-w|^2\mu_0^2\right)+\dots \\ &= \frac{\kappa^2}{16\pi^2}|z-w|^2\left(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\log|z-w|^2\mu^2\right)+\dots,\end{aligned} \quad (3.116)$$

where $\mu^2 = \mu_0^2\pi e^{\gamma_E-\frac{2}{3}}$ and the dots denote finite terms in limit $\epsilon \to 0$.

A few comments are now in order: the second term in the expression coincides with the value computed in [169][3]. Notice that the prefactor $\frac{\kappa^2}{16\pi^2\epsilon} = \frac{1}{\epsilon}\frac{2G}{\pi} = 2\alpha_{GR}$ matches with the gravitational cusp anomalous dimension which appears in the level of the Goldstone current (see also [144, 170] for related works). We also see that (3.116) contains an extra leading divergent piece of the form $\frac{1}{\epsilon^2}|z-w|^2$, which is not present in the expression given in [169]. However it is important to note that this extra term will not spoil at all the soft factorization (3.104). Actually, such a divergence is also present in Weinberg's soft (virtual particle) exponential for massless states, but this term eventually drops out from the final expression due to total momentum conservation. We expect that the extra ϵ^{-2} term in our derivation is scheme dependent and can be reabsorbed in the next ϵ^{-1} term.

We will come back to the goldstone boson correlation functions in sec. 4.4, when we will discuss how to obtain IR finite celestial amplitudes.

[3]The extra μ^2 factor compared to [169] can be reabsorbed in the UV regulator.

Chapter 4

Celestial CFT

In the previous sections we have examined the relation between the infrared behavior of QED, gravity and asymptotic symmetries.
Being robust against higher-order effective terms in the Lagrangian, such as those that may appear in a UV-complete description of gravity, these IR effects allow us to capture general features expected to persist in quantum gravity. In particular, their striking connection with asymptotic symmetries provides valuable hints about the holographic nature of gravity in asymptotically flat spacetime

In the AdS/CFT correspondence, it is well established that the asymptotic symmetries constraining the holographic theory on the AdS boundary are those of a conformal field theory. Specifically, the asymptotic isometries of gravity in a d-dimensional asymptotically AdS spacetime coincide with the conformal group in $d-1$ dimensions [7, 8]. Notably, in $d = 3$ there is a remarkable enhancement: the symmetry group extends from the global conformal group in $d = 2$ to the full local conformal group, which includes the entire Virasoro algebra [171, 172].
We should then expect, that if a holographic description for quantum gravity also exists in asymptotically flat spacetime, the asymptotic symmetries of $\Lambda = 0$ general relativity should be respected by the holographic model living on the flat space boundary.
Following this reasoning we can then recast flat space holography as some flat/eBMS correspondence. Given the structure of the eBMS group:

$$\text{eBMS} = \left(\text{Vir} \otimes \overline{\text{Vir}}\right) \ltimes C^{\infty}(\mathbb{C}^*, \mathbb{R}) = \text{superrotations} \ltimes \text{supertranslations} \tag{4.1}$$

we can see that the superrotations coincide with the 2d conformal group. From this stems the idea that the actual holographic duality should not be established between a 4d and a 3d theory, but between a 4d and a 2d conformal field theory, living on the sphere parametrized by the angular coordinate $(z, \bar{z})$ also known as the celestial sphere [88, 145, 173]. This CFT is the celebrated celestial CFT (CCFT) that will be the subject of this chapter.

We will start by pointing out the basics of celestial CFT, starting from the dictionary connecting bulk S matrix and CCFT correlation functions, and the action of the Poincaré group on celestial primary fields. Then we will show how soft modes turn into celestial operators acting as the currents of the BMS group, and focus on the problem of the vanishing central charge in CCFT. This will be the starting point of our next section where we will discuss how the $c \to 0$ "catastrophe" could be solved considering logarithmic field theories.

4.1 Holographic dictionary in CCFT

We have seen that the standard scattering problem is formulated starting from free field states generated by creation and annihilation operators $a^\dagger(p), a(p)$. These ladder operators are obtained from the Klein Gordon product between a plane wave and the free field as in equation (2.30), and create or annihilate states that can be interpreted as plane waves of fixed momentum p. Then the scattering is defined by the S matrix element (2.34). As this is a fundamental quantity we would like the CCFT dictionary to establish a connection between 2d CFT correlation functions and scattering amplitudes [173]. This connection can be outlined by considering a change of basis of solutions of the free field equations from plane waves to the *conformally primary wave functions*. This basis has been described in [60, 153] for spin $0, 1, 2$ and in [174] for Dirac spinors. In this section we summarize the discussion for integer spin.

Def 4.1.1: Conformal primary wave function

A conformal primary wave functions are solutions of free EOMs transforming under $SL(2,\mathbb{C})$ as quasi-primary fields of weight Δ and helicity l.
For a transformation in $SL(2,\mathbb{C})$ represented by the matrix

$$M = \begin{pmatrix} a & -c \\ -b & d \end{pmatrix} \in SL(2,\mathbb{C}) \tag{4.2}$$

we have:

$$\Phi^l_{\pm,\Delta}\left(\Lambda^\mu_\nu X^\nu, \frac{az+b}{cz+d}, \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}\right) = (cz+d)^{\Delta+l}(\bar{c}\bar{z}+\bar{d})^{\Delta-l}U_l(\Lambda(M))\Phi^l_{\pm,\Delta}(X,z,\bar{z}) \tag{4.3}$$

where U_l denotes the helicity l representation of the Lorentz Group. The $\pm$ sign in (4.3) denotes if the conformal wave is outgoing $(+)$ or incoming $(-)$.

For the spin zero case this result can be easily achieved using the Mellin transform. In particular we can define the spin zero conformal primary wave function as the Mellin transform of the plane wave (2.18)

$$\Phi^0_{\pm,\Delta}(X,z,\bar{z}) = \int_0^\infty d\omega\omega^{\Delta-1}e^{\pm i\omega q\cdot X-\epsilon\omega} = \frac{\Gamma(\Delta)(\mp i)^\Delta}{(-q(z,\bar{z})\cdot X \mp i\epsilon)^\Delta} \tag{4.4}$$

and using:

$$q^\mu\left(\frac{az+b}{cz+d}, \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}\right) = (cz+d)^{-1}(\bar{c}\bar{z}+\bar{d})^{-1}\Lambda^\mu_\nu(M)q^\nu\,(z,\bar{z}) \tag{4.5}$$

it is easy to verify that (4.4) satisfies (4.3). Notice that the relation of the conformal wave with the plane waves trough the Mellin transform, makes manifest the transformation between the two basis.
For spin 1 and 2, fixing radial and Lorenz gauge:

$$X^\mu\Phi^{\pm1}_{\pm,\Delta\,\mu} = 0, \quad \partial^\mu\Phi^{\pm1}_{\pm,\Delta\,\mu} = 0 \tag{4.6}$$

$$X^\mu\Phi^{\pm2}_{\pm,\Delta\,\mu\nu} = 0, \quad \partial^\mu\Phi^{\pm2}_{\pm,\Delta\,\mu\nu} = 0, \quad \eta^{\mu\nu}\Phi^{\pm2}_{\pm,\Delta\,\mu\nu} = 0 \tag{4.7}$$

we can obtain for the solutions of the free EOMs the following two expressions:

$$\Phi^l_{\pm,\Delta\,\mu} = \frac{\partial_J q_\mu}{(-q\cdot X \mp i\epsilon)^\Delta} + \frac{(\partial_J q\cdot X)q_\mu}{(-q\cdot X\mp i\epsilon)^\Delta}, \quad l=\pm 1, \tag{4.8}$$

$$\Phi^l_{\pm,\Delta\,\mu\nu} = \frac{1}{2}\frac{[(-q\cdot X)\partial_J q_\mu + (\partial_J q\cdot X)q_\mu]\,[(-q\cdot X)\partial_J q_\nu + (\partial_J q\cdot X)q_\nu]}{(-q\cdot X\mp i\epsilon)^{\Delta+2}}, \quad l=\pm 2, \tag{4.9}$$

where $J = z,\, \bar{z}$ for positive or negative helicity l respectively. We remark that these solutions can still be related to the Mellin transform of (2.18) defined in (4.4) as follows

$$\Phi^l_{\pm,\Delta\,\mu} = \frac{\Delta-1}{\Delta}\frac{1}{(\mp i)^\Delta\Gamma(\Delta)}\epsilon^l_\mu\Phi^0_{\pm,\Delta}(X,z,\bar{z}) + \partial_\mu\lambda^\Delta_{\pm,l}, \quad l=\pm 1 \tag{4.10}$$

$$\Phi^l_{\pm,\Delta\,\mu\nu} = \frac{(\Delta-1)}{2(\Delta+1)}\frac{1}{(\mp i)^\Delta\Gamma(\Delta)}\epsilon^l_{\mu\nu}\Phi^0_{\pm,\Delta}(X,z,\bar{z}) + \partial_\mu\xi^\Delta_{\pm,\nu} + \partial_\nu\xi^\Delta_{\pm,\mu}, \quad l=\pm 2 \tag{4.11}$$

where:

$$\lambda^\Delta_{\pm,l} = \frac{1}{\Delta}\frac{\partial_l q\cdot X}{(\pm q\cdot X)^\Delta}, \quad \xi^\Delta_{\pm,l,\mu} = \frac{1}{2(\Delta)}\frac{\partial_J q\cdot X}{(\mp q\cdot X)^{\Delta+1}}\left[q_\mu\partial_l q\cdot X - q\cdot X\partial_l q_\mu\right]. \tag{4.12}$$

Recalling that on a photon or a graviton an infinitesimal gauge transformations act as

$$A_\mu \to A_\mu + \partial_\mu\lambda, \quad h_{\mu\nu}\to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu, \tag{4.13}$$

we can see that (4.10),(4.11), are basically the Mellin transform of the solutions obtained in the De Donder gauge (2.18):

$$\epsilon^l_I\Phi^0_{\pm,\Delta}(X,z,\bar{z}) = \int_0^{+\infty} d\omega\omega^{\Delta-1}\varphi_I^{l,(s)}(\omega,w,\bar{w}|X) \tag{4.14}$$

up to pure gauge terms.
Notice that working in the de Donder gauge these extra gauge terms will not be present, however (4.14) will fail to satisfy (4.3), but will present an extra inhomogeneous shift. In the radial gauge solutions these terms are canceled by additional contributions coming from the transformation of the gauge parts in λ and ξ, and allow (4.10),(4.11) to satisfy (4.3).

The bulk modes acting as creation and annihilation operators of the conformal waves, are then defined through the Klein-Gordon product of the on-shell fields with the conformal primary wave basis namely:

$$\mathcal{O}^l_{\pm,\Delta}(z,\bar{z}) = \frac{1}{c^l_\Delta}\left(\Phi^l_{\pm,\Delta}(X;z,\bar{z}),\Phi\right)_{KG}, \tag{4.15}$$

where Φ is the helicity l on-shell field, the KG product defined in (2.29) and c^l_Δ are just constants dependent from both helicity and conformal weight, defined as:

$$c^{\pm 1}_\Delta = \frac{\Delta-1}{\Delta}, \quad c^{\pm 2}_\Delta = \frac{\Delta-1}{2(\Delta+1)}. \tag{4.16}$$

$\mathcal{O}^l_{\pm,\Delta}$ transforms under a Lorentz transformation as:

$$\begin{gathered}{\mathcal{O}^l_{\pm,\Delta}}'(z',\bar{z}') = (\partial f)^{-h}(\bar{\partial}\bar{f})^{-\bar{h}}\mathcal{O}^l_{\pm,\Delta}(z,\bar{z}),\\ z' = f(z) = \frac{az+b}{cz+d}, \quad \bar{z}' = \bar{f}(\bar{z}) = \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}\end{gathered} \tag{4.17}$$

where $h = \frac{\Delta+l}{2}, \bar{h} = \frac{\Delta-l}{2}$ are also referred to, with a slight abuse of notation, as conformal weights. Equation (4.17) coincide precisely with the transformation of a conformal primary in a 2d CFT. We want to specify that at the moment these operators transform as primaries only under the action of the global part of the 2d conformal group, namely the bulk Lorentz group, but nothing can be said regarding the action of the superrotations, namely the action of the local 2d conformal group. We cannot naively promote $\mathcal{O}^l_{\pm,\Delta}$ to a full Virasoro primary.
The transformation property (4.17) means that the scattering element:

$$\langle 0|\, \mathcal{O}^{l_1}_{+,\Delta_1} \dots \mathcal{O}^{l_n}_{+,\Delta_n}\, S\, \mathcal{O}^{l'_1}_{-,\Delta'_1} \quad \mathcal{O}^{l'_m}_{-,\Delta'_m}\, |0\rangle \tag{4.18}$$

precisely transforms as a 2d CFT correlation function. Because the conformal primary wave functions differ from the Mellin transform of plane waves only by a gauge transformation, the conformal amplitude (4.18) satisfies:

$$\begin{aligned} &\langle 0|\, \mathcal{O}^{l_1}_{+,\Delta_1} \dots \mathcal{O}^{l_n}_{+,\Delta_n}\, S\, \mathcal{O}^{l'_1}_{-,\Delta'_1} \quad \mathcal{O}^{l'_m}_{-,\Delta'_m}\, |0\rangle = \\ &= \int_0^\infty \prod_{i=1}^n d\omega_i\, \omega_i^{\Delta_i-1} \prod_{i=1}^m d\omega'_i\, \omega_i'^{\Delta'_i-1} \langle 0|\, a^{l_1}(\omega_1) \dots a^{l_n}(\omega_n)\, S\, a^{l'_1\dagger}(\omega'_1) \quad a^{l'_m\dagger}(\omega'_m)\, |0\rangle \end{aligned} \tag{4.19}$$

namely it can be computed just by taking the Mellin transform with respect of all the frequencies of the S matrix element in momentum space.
Now given the transformation property of (4.18) it is obvious to define the celestial holography dictionary for massless particles as follows [56–60, 62, 63, 67, 88, 145, 173]

Def 4.1.2: Celestial Dictionary

Every bulk amplitude $\mathcal{A}(p_i, l_i)$ of n massless particles with momentum $p_i = \omega_i q_i(z, \bar{z})$, and helicity l_i corresponds to a celestial amplitude trough a Mellin transform of the frequencies

$$\langle \mathcal{O}^{l_1}_{1,\Delta_1}(z_1, \bar{z}_1) \dots \mathcal{O}^{l_n}_{n,\Delta_n}(z_n, \bar{z}_n) \rangle_{\text{CCFT}} = \prod_{i=1}^n \int_0^\infty d\omega_i\, \omega_i^{\Delta_i-1} \mathcal{A}(p_1, l_1; \dots; p_n, l_n) \tag{4.20}$$

where the celestial operators $\mathcal{O}^{l_i}_{i,\Delta_i}$ are all primary fields of conformal weight Δ_i and helicity l.

From a bulk representation there is then a straightforward mapping between the celestial operators and the ladder operators, of the form[1]:

$$\mathcal{O}^l_{+,\Delta}(z, \bar{z}) = \int_0^{+\infty} d\omega \omega^{\Delta-1} a_l(\omega, z, \bar{z}), \quad \mathcal{O}^l_{-,\Delta}(z, \bar{z}) = \int_0^{+\infty} d\omega \omega^{\Delta-1} a_l^\dagger(\omega, z, \bar{z}) \tag{4.21}$$

Notice that from a bulk perspective (4.20) is just an integral transform, which trades the frequency ω with the conformal weight Δ. Up to now there is no constraint to the form of the conformal weight Δ which can take any value on the complex plane $\Delta \in \mathbb{C}$. However imposing

[1]For brevity here we are adopting the following notation $a_l = a^{|l|}_{\text{sign}(l)}$

that the states created by $\mathcal{O}^l_{\pm,\Delta}$ are normalizable we can compute:

$$\begin{aligned}
\langle\mathcal{O}^{\pm}_{+,\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{\pm}_{-,\Delta_2}(z_2,\bar{z}_2)\rangle &= \int_0^{+\infty} d\omega_1 d\omega_2\, \omega_1^{\Delta_1-1}\omega_2^{\Delta_2-1}\langle a_{\pm}(\omega_1,z_1,\bar{z}_1)a^{\dagger}_{\pm}(\omega_2,z_2,\bar{z}_2)\rangle = \\
&= \int_0^{+\infty} d\omega_1 d\omega_2\, \omega_1^{\Delta_1-1}\omega_2^{\Delta_2-1}\frac{16\pi^3}{\omega_1}\delta(\omega_1-\omega_2)\delta(z_{12}) = \\
&= 16\pi^3\delta(z_{12})\int_0^{+\infty} d\omega\, \omega^{\Delta_1+\Delta_2-3}
\end{aligned} \tag{4.22}$$

where $z_{ij} = z_i - z_j$. We notice that the Mellin integral typically gives a divergent result, unless $\Delta = 1 + i\lambda, \lambda \in \mathbb{R}$ [56]. If the conformal weights belongs to this line on the complex plane, usually referred to as *principal line*, the two point function has a well defined distributional behavior

$$\langle\mathcal{O}^{\pm}_{+,\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{\pm}_{-,\Delta_2}(z_2,\bar{z}_2)\rangle = 2(2\pi)^4\delta^2(z_1-z_2)\delta(\lambda_1+\lambda_2). \tag{4.23}$$

This expression from the bulk perspective is telling us that the conformal primary waves form a normalizable unitary basis only if the conformal weights are complex and lie on the principal line[2]. From the CCFT perspective this must be interpreted as a two point function of celestial primaries. In a 2d CFT the two point point functions between operators with conformal weights $(h,\bar{h})$ is fixed by symmetry and normalized to take the form

$$\langle\Phi(z_1,\bar{z}_1)\Phi(z_2,\bar{z}_2)\rangle = z_{12}^{-2h}\bar{z}_{12}^{-2\bar{h}} \tag{4.24}$$

which does not coincide with form (4.23). However (4.23) is not ruled out by symmetry, but perfectly respects conformal invariance. In standard CFT these behavior is ruled out by unitarity constraints, which do not necessarily apply to celestial CFT.

Another important difference between celestial CFT and standard 2d CFTs, is that while usually the conformal weights are real positive numbers in celestial CFT normalizability fixes the weights to lie on the principal line, namely to be complex numbers. This mismatch has an explanation coming from representation theory. In standard 2d CFT we are looking for unitary representation of the Lorentzian conformal group $SO(2,2)$ in Euclidean signature, as we are analytically continuing the Minkowski plane to the Euclidean plane. These representations correspond to the highest-weight representation of $SL(2,\mathbb{C})$, which have real conformal weights [177]. On the other hand in celestial holography we are looking for representations of $SO(3,1)$ in Euclidean signature, which are mapped to the principal series representation of $SL(2,\mathbb{C})$, which has now weights lying on the principal line $\mathcal{P} = 1 + i\mathbb{R}$. This explains the observed discrepancy between the conformal weights of primary operators in standard and celestial CFT.
Notice that the condition that conformal weights should lie on the principal line is defining for us the spectrum of normalizable operators in CCFT. We are always free to consider non-normalizable operators and compute their correlation functions by analytically extending the values of the conformal weights outside the principal line whenever the Δ dependence of the correlation functions allows it.

[2]An alternative basis given by operators with $\Delta \in \mathbb{Z}$ has been proposed in [175,176] for wave functions living in Schwartz space

4.1.1 Action of Poincaré group on celestial primaries

In the previous section we have focused on the action of the Lorentz group on celestial primaries. However because the global part of BMS is constructed by the whole Poincaré group we also expect celestial operators to form a representation of translation. Starting from expression (2.24) we can use the dictionary to obtain:

$$\mathcal{O}^l_{\pm,\Delta}(z',\bar{z}')' = (\partial f)^{-h}(\bar{\partial}\bar{f})^{-\bar{h}} e^{-iq_\mu(z,\bar{z})\Lambda^\mu{}_\nu t^\nu \delta_\Delta}\mathcal{O}^l_{\pm,\Delta}(z,\bar{z}), \tag{4.25}$$

$$\text{with:} \quad h = \frac{\Delta \pm l}{2},\ \bar{h} = \frac{\Delta \mp l}{2}, \quad f(z) = \frac{az+b}{cz+d},\ \bar{f}(\bar{z}) = \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}, \tag{4.26}$$

where $\Lambda^\mu{}_\nu$ is the Lorentz transformations associated to the $SL(2,\mathbb{C})$ functions $f(z), \bar{f}(\bar{z})$, and δ_Δ is an operator shifting the conformal weight by 1, $\delta_\Delta \mathcal{O}_\Delta = \mathcal{O}_{\Delta+1}$. The infinitesimal transformation takes the form:

$$\delta_{\mathcal{T},\mathcal{Y}}\mathcal{O}^l_{\pm,\Delta}(z,\bar{z}) = \left(-i\mathcal{T}(z,\bar{z})\delta_\Delta + \mathcal{Y}\partial + \overline{\mathcal{Y}}\bar{\partial} + h\partial\mathcal{Y} + \bar{h}\bar{\partial}\overline{\mathcal{Y}}\right)\mathcal{O}^l_{\pm,\Delta}(z,\bar{z}) \tag{4.27}$$

where $\mathcal{T} = q_\mu(z,\bar{z})t^\mu$, is the infinitesimal generator of translations and $\mathcal{Y} = \mathcal{Y}_{-1} + \mathcal{Y}_0 z + \mathcal{Y}_1 z^2$, $\overline{\mathcal{Y}} = \overline{\mathcal{Y}}_{-1} + \overline{\mathcal{Y}}_0\bar{z} + \overline{\mathcal{Y}}_1\bar{z}^2$ the infinitesimal conformal transformations. Notice that these infinitesimal generators can be matched with the functions (1.32).

The presence of the operator δ_Δ shifts the conformal weight of the operators taking them away from the principal line, which shows that in general is necessary to consider operators outside the normalizable spectrum.

Notice that for two point functions translation invariance forces the correlation functions to be proportional to $\delta^2(z_{12})$, which gives us another explanation for the peculiar behavior of CCFT two point functions.

4.2 Celestial currents

In chapter 3 we have showed the profound relation between asymptotic symmetries and soft modes. In this section we will show how the asymptotic symmetries can be re-casted as operators on the celestial sphere.

However we will take a slightly longer route to show how it is possible to construct a set of celestial operators in one to one correspondence with the coefficients of the soft frequency expansion (2.76)

$$\mathcal{M}(\omega,q,J;p_1,\dots p_n) = \sum_{n=0}^{\infty}\omega^{n-1}S_n(q,J,p_i)\mathcal{M}(p_1,\dots p_n) \tag{4.28}$$

where $J = \pm1, \pm2$ depending if the operator is a photon or a graviton.

Starting from the ladder operators of gravitons and photons in the bulk we can define a set of soft operators as:

$$s^l_{+,1-n}(z,\bar{z}) = \lim_{\omega\to0^+}\partial_\omega{}^n\left(\omega a_l(\omega,z,\bar{z})\right), \quad s^l_{-,1-n}(z,\bar{z}) = \lim_{\omega\to0^+}\partial_\omega{}^n\left(\omega a^\dagger_l(\omega,z,\bar{z})\right). \tag{4.29}$$

If we insert these operators in a correlation function with n matter operators, they satisfy:

$$\langle\text{out}|\, s^l_{+,1-n}(z,\bar{z})S\,|\text{in}\rangle = S_n(q,l,p_i)\,\langle\text{out}|\,S\,|\text{in}\rangle \tag{4.30}$$

namely they extract the soft mode contribution.
The structure of $s^J_{+,1-n}(z,\bar{z})$ does not seem to be connected to Celestial operators, however they can actually be mapped to Celestial primaries with the following reasoning. Considering the celestial operator $\mathcal{O}^J_{+,\Delta}(z,\bar{z})$ we split the Mellin integral into two regions

$$\mathcal{O}^l_{+,\Delta}(z,\bar{z}) = \int_0^\lambda d\omega\omega^{\Delta-1}a_l(\omega,z,\bar{z}) + \int_\lambda^{+\infty} d\omega\omega^{\Delta-1}a_l(\omega,z,\bar{z}). \tag{4.31}$$

If we consider λ to be small, in the region $[0,\lambda)$ we can use the expansion of a_l in terms of the $s^l_{+,1-n}$

$$a_l(\omega,z,\bar{z}) = \sum_{n=0}^{\infty}\omega^{n-1}s^l_{+,1-n} \tag{4.32}$$

to rewrite

$$\mathcal{O}^l_{+,\Delta}(z,\bar{z}) = \sum_{n=0}^{\infty} s^J_{+,1-n}(z,\bar{z})\int_0^\lambda d\omega\omega^{\Delta+n-2} + \int_\lambda^{+\infty} d\omega\omega^{\Delta-1}a_l(\omega,z,\bar{z}) \tag{4.33}$$

under the assumption that $\Delta > 1-n$, the integral around $[0,\lambda)$ is well defined and we obtain

$$\mathcal{O}^l_{+,\Delta}(z,\bar{z}) = \sum_{n=0}^{\infty}\frac{\lambda^{\Delta+n-1}}{\Delta+n-1}s^J_{+,1-n}(z,\bar{z}) + \int_\lambda^{+\infty} d\omega\omega^{\Delta-1}a_l(\omega,z,\bar{z}). \tag{4.34}$$

We notice the presence of a set of poles for $\Delta = 1,0,-1,-2,\ldots$, namely the set $\mathbb{Z}_{\leq 1}$. Extracting the residue around $1-n$ we then obtain

$$\lim_{\epsilon\to 0}\epsilon\,\mathcal{O}^l_{+,1-n+\epsilon}(z,\bar{z}) = s^J_{+,1-n}(z,\bar{z}) + \lim_{\epsilon\to 0}\epsilon\int_\lambda^{+\infty} d\omega\omega^{\epsilon-n}a_l(\omega,z,\bar{z}) = s^J_{+,1-n}(z,\bar{z}) \tag{4.35}$$

where in the second equality we have assumed that no extra poles are coming from the UV region $[\lambda+\infty)$. We can then appreciate how soft operators are in one to one correspondence with celestial operators localized at $1-n, n\in\mathbb{N}$ [178]:

$$s^J_{+,1-n}(z,\bar{z}) = \lim_{\epsilon\to 0}\epsilon\,\mathcal{O}^l_{+,1-n+\epsilon}(z,\bar{z}) \sim \left(\frac{1-n+l}{2},\frac{1-n-l}{2}\right) \tag{4.36}$$

where we are using the notation $\phi\sim(h,\bar{h})$ to specify the conformal weights of the ϕ operator. It is easy to see then that the soft modes can all be mapped to celestial operators:

$$\begin{aligned}
N_z &= -\frac{\sqrt{2}}{8\pi e}\left(s^1_{+,1}+s^{-1}_{-,1}\right)\sim(1,0),\\
N_{\bar{z}} &= -\frac{\sqrt{2}}{8\pi e}\left(s^{-1}_{+,1}+s^{1}_{+,1}\right)\sim(0,1),\\
\mathcal{N}^{(0)}_{zz} &= -\frac{\kappa}{4\pi}\left(s^2_{+,1}+s^{-2}_{-,1}\right)\sim(3/2,-1/2),\\
\mathcal{N}^{(0)}_{\bar{z}\bar{z}} &= -\frac{\kappa}{4\pi}\left(s^{-2}_{+,1}+s^{2}_{-,1}\right)\sim(-1/2,3/2),\\
\mathcal{N}^{(1)}_{zz} &= \frac{i\kappa}{4\pi}\left(s^2_{+,0}-s^{-2}_{-,0}\right)\sim(1,-1),\\
\mathcal{N}^{(1)}_{\bar{z}\bar{z}} &= \frac{i\kappa}{4\pi}\left(s^{-2}_{+,0}-s^{2}_{-,0}\right)\sim(-1,1).
\end{aligned} \tag{4.37}$$

As the memory modes (3.47)(3.97) are made out of soft operators we can also consider them as celestial operators, and will precisely constitute the currents on the celestial sphere of the asymptotic symmetries. Notice that it is not trivial that $\mathcal{N}_z, \mathcal{N}_{zz}$ transform as primary fields because they contain the integral of a primary, which is in general not a primary. However these integrated operators:

$$\begin{aligned} \widetilde{\mathcal{N}}_z^{(0)} &= \int d^2w \frac{\mathcal{N}_{\bar{w}\bar{w}}}{(\bar{z}-\bar{w})^2}, \\ \widetilde{\mathcal{N}}_{zz}^{(0)} &= \int d^2w \frac{z-w}{(\bar{w}-\bar{z})^3} \mathcal{N}_{\bar{w}\bar{w}} \end{aligned} \tag{4.38}$$

are defined using a very specific integral transform, known as the *shadow transform* [179–182]. We will later comment more on shadow operator and here we just give a functional definition: the shadow transform acts on a primary field ϕ of weight $(h, \bar{h})$ as:

$$\tilde{\phi}(z,\bar{z}) = \int d^2w \frac{\phi(w,\bar{w})}{(z-w)^{2-2h}(\bar{z}-\bar{w})^{2-2\bar{h}}} \tag{4.39}$$

and turns it into a primary $\tilde{\phi}$ of weights $(1-h, 1-\bar{h})$. It is precisely this property that preserves the nice transformation properties of the soft modes and turns $\mathcal{N}_z, \mathcal{N}_{zz}$ into primary operators. As the bulk correlations functions of the memory modes are fixed by charge ward identities, we can immediately compute the celestial correlation functions:

$$\begin{aligned} \langle \mathcal{N}_z \mathcal{O}^{l_1}_{1,\Delta_1} \dots \mathcal{O}^{l_n}_{n,\Delta_n} \rangle &= \sum_{i=1}^{n} \frac{eQ_i}{z-z_i} \langle \mathcal{O}^{l_1}_{1,\Delta_1} \dots \mathcal{O}^{l_n}_{n,\Delta_n} \rangle, \\ \langle \mathcal{N}_{zz} \mathcal{O}^{l_1}_{1,\Delta_1} \dots \mathcal{O}^{l_n}_{n,\Delta_n} \rangle &= \sum_{i=1}^{n} \frac{\bar{z}-\bar{z}_i}{z-z_i} \langle \mathcal{O}^{l_1}_{1,\Delta_1} \dots \mathcal{O}^{l_i}_{i,\Delta_i+1} \dots \mathcal{O}^{l_n}_{n,\Delta_n} \rangle. \end{aligned} \tag{4.40}$$

The form of the correlation functions can be easily rewritten as the OPE[3] identities between the celestial currents and matter fields:

$$\begin{aligned} \mathcal{N}_z(z,\bar{z}) \mathcal{O}^l_\Delta(w,\bar{w}) &\simeq \frac{eQ}{z-w} \mathcal{O}^l_\Delta(w,\bar{w}) \\ \mathcal{N}_{zz}(z,\bar{z}) \mathcal{O}^l_\Delta(w,\bar{w}) &\simeq \frac{\bar{z}-\bar{w}}{z-w} \mathcal{O}^l_{\Delta+1}(w,\bar{w}). \end{aligned} \tag{4.41}$$

We notice that the first one is exactly the OPE of a $U(1)$ kac-moody current with a charged operator with charge Q [149], while the action of the gravitational memory shifts the conformal weight of the celestial primaries, which is compatible with the action of translation of primary fields as we should expect. The first descendant of $\mathcal{N}_{zz}$, $P_z = \bar{\partial}\mathcal{N}_{zz} \sim (3/2, 1/2)$ will also have an OPE with operators charged under supertranslation which strictly resembles that of a Kac-Moody currents [169]. Notice that the memory modes are symplectically coupled with the goldstone modes ϕ and C, which from the expressions (3.55),(3.103) can also be mapped to celestial operators:

$$\begin{aligned} \phi(z,\bar{z}) &= \frac{ie}{\sqrt{2}(2\pi)^3} \int d^2w \left[\frac{1}{\bar{z}-\bar{w}} \left(\mathcal{O}^1_{+,1} - \mathcal{O}^{-1}_{-,1} \right) + \frac{1}{z-w} \left(\mathcal{O}^{-1}_{+,1} - \mathcal{O}^1_{-,1} \right) \right] \\ C(z,\bar{z}) &= \frac{i\kappa}{8\pi^2} \int d^2w \left[\frac{z-w}{\bar{z}-\bar{w}} \left(\mathcal{O}^2_{+,1} - \mathcal{O}^{-2}_{-,1} \right) + \frac{\bar{z}-\bar{w}}{z-w} \left(\mathcal{O}^{-2}_{+,1} - \mathcal{O}^2_{-,1} \right) \right] \end{aligned} \tag{4.42}$$

[3]We will give a definition of operator product expansion (OPE) in chapter 5, when we will describe more in detail the properties of conformal field theories. For details see [177]

which have the following two points functions with soft modes:

$$\langle N_z(z,\bar z)\phi(w,\bar w)\rangle = \frac{i}{4\pi}\frac{1}{z-w}, \quad \langle \mathcal{N}^{(0)}_{zz}(z,\bar z)C(w,\bar w)\rangle = \frac{i\kappa^2}{2}\frac{\bar z-\bar w}{z-w}. \tag{4.43}$$

These imply the leading OPEs [57, 169]:

$$N_z(z,\bar z)\phi(w,\bar w) \simeq \frac{i}{4\pi}\frac{1}{z-w}, \quad \mathcal{N}^{(0)}_{zz}(z,\bar z)C(w,\bar w) \simeq \frac{i\kappa^2}{2}\frac{\bar z-\bar w}{z-w} \tag{4.44}$$

which are perfectly compatible with the OPEs of goldstone bosons with the current of their respective symmetry transformation.

4.3 Celestial stress tensor

In the previous section we have examined how the leading soft modes gets mapped to currents for large $U(1)$ and superrotations on the celestial sphere. However we did not comment on the sub-leading soft mode.
Here we define two operators [88]:

$$\begin{aligned} T(z) &= \frac{i}{8\pi G}\int d^2w \frac{1}{z-w}\partial^3\mathcal{N}^{(1)}_{\bar w\bar w} = -\frac{3i}{4\pi G}\int d^2w \frac{\mathcal{N}^{(1)}_{\bar w\bar w}}{(z-w)^4} \sim (2,0) \\ \bar T(\bar z) &= \frac{i}{8\pi G}\int d^2w \frac{1}{\bar z-\bar w}\partial^3\mathcal{N}^{(1)}_{ww} = -\frac{3i}{4\pi G}\int d^2w \frac{\mathcal{N}^{(1)}_{ww}}{(\bar z-\bar w)^4} \sim (0,2) \end{aligned} \tag{4.45}$$

which we can notice to correspond to the superrotation charges for the careful choice $Y_w = (z-w)^{-1}, Y_{\bar w} = 0$ and $Y_z = 0, Y_{\bar w} = (\bar z-\bar w)^{-1}$ respectively. Notice that these are still primary fields, as the integral acting on the sub-leading soft mode is the shadow transform which gives to T and $\bar T$ the conformal weights $(2,0)$ and $(0,2)$. Now using the sub-leading soft theorem it is possible to prove that [67]

$$\begin{aligned} \left\langle T(z)\mathcal{O}^{l_1}_{\Delta_1}\dots\mathcal{O}^{l_n}_{\Delta_n}\right\rangle &= \sum_{i=1}^{n}\left[\frac{h_i}{(z-z_i)^2}+\frac{\partial_i}{z-z_i}\right]\left\langle\mathcal{O}^{l_1}_{\Delta_1}\dots\mathcal{O}^{l_n}_{\Delta_n}\right\rangle, \\ \left\langle \bar T(\bar z)\mathcal{O}^{l_1}_{\Delta_1}\dots\mathcal{O}^{l_n}_{\Delta_n}\right\rangle &= \sum_{i=1}^{n}\left[\frac{h_i}{(\bar z-\bar z_i)^2}+\frac{\bar\partial_i}{\bar z-\bar z_i}\right]\left\langle\mathcal{O}^{l_1}_{\Delta_1}\dots\mathcal{O}^{l_n}_{\Delta_n}\right\rangle, \end{aligned} \tag{4.46}$$

which are the ward identities of a 2d CFT stress-tensor !. This implies that the superrotation charge, which is precisely related to a $\mathrm{Vir}\otimes\overline{\mathrm{Vir}}$ algebra, is mapped directly to the *celestial stress tensor*. Moreover we can deduce the OPE of the T and $\bar T$ with the celestial primaries:

$$T(z)\mathcal{O}^{l}_{\pm,\Delta}(w,\bar w) \simeq \left[\frac{h_i}{(z-w)^2}+\frac{\partial_w}{z-w}\right]\mathcal{O}^{l}_{\pm,\Delta}(w,\bar w) \tag{4.47}$$

which is exactly the OPE of the stress tensor with a Virasoro primary field. When we stated the transformation properties of celestial primaries, we where restricting ourself just to the bulk isometries namely, only the Lorentz group $SL(2,\mathbb{C})$. The action of the celestial stress tensor shows that the transformation properties of celestial primaries extend beyond the global part of the Virasoro algebra and they can be considered as full Virasoro primaries.

Now that we have the structure of the stress tensor $T(z)$ we can also try to compute the TT OPE to determine a fundamental quantity for any CFT, namely the central charge c. This was done in [67] using a double soft limit, which returns:

$$T(z)T(w) \simeq \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{4.48}$$

As we can see there is no $c/(z-w)^4$ which implies that the celestial CFT central charge is vanishing. This has interesting consequences for celestial CFT. We know that any unitary CFT with vanishing central charge is trivial namely they only contain the identity operator, so this implies that celestial CFT cannot be a unitary theory. This should not come as a surprise as this property was already suggested by the complex conformal weights of celestial operators lying on the principal line. Usually a non-trivial CFT with vanishing central charge falls into at least one of three options [183]:

1. The model can be a topological CFT. We can exclude this option as the correlators of celestial CFT depend on the insertion point of the operators.

2. The CFT can split into two decoupled sectors $\text{CFT}=\text{CFT}_1\times\text{CFT}_2$ with two separate stress tensors T_1 and T_2 with opposite central charge $c_1 = -c_2$ such that the central charge of the full model, with stress tensor $T = T_1 + T_2$, is vanishing $c = c_1 + c_2 = 0$. The total CFT is non trivial as the two models have a non empty spectrum, and is also non-unitary as one of the two CFTs will have negative central charge.

3. Another possibility is that the CFT, is not a conventional CFT but a logarithmic CFT where the problem of vanishing central charge gets solved by the introduction of an additional operator, usually referred to as the logarithmic partner of the stress tensor. We will explain this construction in the next chapter.

While the second option has been explored in [90], we will analyze the last possibility in the next chapter, showing how a logarithmic CFT structure can be built in celestial CFT due to the infrared divergences present in celestial operators.

4.4 IR finite celestial amplitudes

In sec. 2.2.3 we have shown that IR divergent amplitudes can be made finite by dressing the incoming and outgoing particles with a cloud of soft particles. This method developed by Faddeev and Kulish has the effect of erasing both real and virtual IR divergences, making scattering amplitudes well defined even in the low energy limit.

With this section we conclude the chapter showing how the divergent behavior is translated in celestial CFT and comment, as anticipated, on how Goldstone modes, both in gravity and QED, can be used to conformally dress celestial amplitudes and make them IR finite. We will discuss, following [152], both QED and gravity in two different sections, starting from QED.

4.4.1 Scalar QED

In sec. 2.2.2 we saw that virtual IR divergences in massless scalar QED factorize as (2.42)

$$\begin{aligned} \mathcal{M}(1,\dots,n) &= e^{A(p_i)}\mathcal{M}_0(1,\dots,n), \\ A(p_i) &= -\alpha_{EM}\sum_{i<j} Q_iQ_j \log\frac{|p_i\cdot p_j|}{2}, \quad \alpha_{EM}=\frac{e^2}{4\pi\epsilon}. \end{aligned} \tag{4.49}$$

In this case we are using dim-reg, so the divergent factor $\log\lambda$ in the cusp anomalous dimension is substituted by a factor $1/\epsilon$. The amplitude $\mathcal{M}$ denotes the original IR divergent amplitude, whereas $\mathcal{M}_0$ represents the IR-finite part, obtained by factoring out the virtual divergence contribution encapsulated in the overall multiplicative factor. p_n are the null momenta of the massless particles participating in the scattering.

Using the momentum parametrization (2.9) we can write $A(p_i)$ as

$$\begin{aligned} A &= \exp\left\{-\frac{e^2}{4\pi^2\epsilon}\sum_{i<j}Q_iQ_j\log\left(\omega_i\omega_j z_{ij}\bar{z}_{ij}\right)\right\} \\ &= \prod_{i<j}\left(z_{ij}\bar{z}_{ij}\right)^{-\alpha_{EM}Q_iQ_j}\left(\omega_i\omega_j\right)^{-\alpha_{EM}Q_iQ_j}, \end{aligned} \tag{4.50}$$

where $z_{ij}=z_i-z_j$. Thanks to charge conservation the frequency dependent term can also be written as:

$$\prod_{i<j}\left(\omega_i\omega_j\right)^{-\alpha_{EM}Q_iQ_j} = \prod_i \omega_i^{\alpha_{EM}Q_n^2} \tag{4.51}$$

We can then use the celestial dictionary to compute the celestial amplitude corresponding to $\mathcal{M}$, namely

$$\begin{aligned} \langle \mathcal{O}_{\Delta_1},\dots,\mathcal{O}_{\Delta_n}\rangle &= \prod_{i<j}\left(z_{ij}\bar{z}_{ij}\right)^{-\alpha_{EM}Q_iQ_j}\int\prod_i d\omega_i\omega^{\Delta_i+\alpha_{EM}Q_i^2}\mathcal{M}_0(p_1,\dots,p_n) \\ &= \prod_{i<j}\left(z_{ij}\bar{z}_{ij}\right)^{-\alpha_{EM}Q_iQ_j}\langle \mathcal{O}_{\Delta_1'},\dots,\mathcal{O}_{\Delta_n'}\rangle_0 \end{aligned} \tag{4.52}$$

and we see that the also the celestial amplitude factorizes, into a hard part:

$$\langle \mathcal{O}_{\Delta_1'},\dots,\mathcal{O}_{\Delta_n'}\rangle_0 = \int\prod_i d\omega_i\omega^{\Delta_i+\alpha_{EM}Q_i^2}\mathcal{M}_0(p_1,\dots,p_n) \tag{4.53}$$

which is basically the Mellin transform of the IR finite amplitude, with renormalized conformal dimensions $\Delta_i'=\Delta_i+\alpha_{EM}Q_i^2$, and a soft part:

$$\mathcal{M}_{soft} = \prod_{i<j}\left(z_{ij}\bar{z}_{ij}\right)^{-\alpha_{EM}Q_iQ_j}. \tag{4.54}$$

There is now a fundamental observation to make: we can use the large gauge goldstone mode ϕ to reconstruct the soft amplitude. We have computed in fact that the two point function of goldstone modes $\langle\phi\phi\rangle$ takes the form (3.58)[4]

$$\langle\phi(z,\bar{z})\phi(w,\bar{w})\rangle = \alpha_{EM}\log|z-w|^2 \tag{4.55}$$

[4]As anticipated we are neglecting the scheme dependent term $1/\epsilon^2$

which implies that:

$$\langle e^{iQ_1\phi(z_1,\bar{z}_1)}\dots e^{iQ_n\phi(z_n,\bar{z}_n)}\rangle = \exp\left\{-\alpha_{EM}\sum_{i<j}Q_iQ_j\log|z_{ij}|^2\right\} = \mathcal{M}_{soft}. \tag{4.56}$$

This result should not come as a surprise, in fact as discussed in sec. 3.1.2 a wise choice of $f(\vec{p},\vec{k})$ in the soft photon FK clouds (2.71) turns them into vertex operators of the Goldstone mode, so we should expect their correlation function to reproduce the IR divergent part. The result of the conformal dressing defined in 3.1.2, is then to dress operators in the celestial basis (2.71) with the Goldstone boson vertex operator [152]:

$$\hat{\mathcal{O}}_{\Delta'}(z,\bar{z}) =: e^{-iQ\phi}\mathcal{O}_{\Delta} : (z,\bar{z}), \tag{4.57}$$

where $:\cdot:$ denotes the normal ordering.
A correlation function of such operators yields directly to the hard part of the celestial amplitude

$$\left\langle \hat{\mathcal{O}}_{\Delta_1'}(z_1,\bar{z}_1)\dots\hat{\mathcal{O}}_{\Delta_n'}(z_n,\bar{z}_n)\right\rangle = \langle\mathcal{O}_{\Delta_1'},\dots,\mathcal{O}_{\Delta_n'}\rangle_0 \tag{4.58}$$

which is IR finite.

We have seen then the fundamental role of the large gauge Goldstone boson to obtain IR finite celestial operators in QED.

4.4.2 Gravity

In the case of gravity virtual IR divergences factor as (2.47):

$$\begin{aligned}\mathcal{M}(1,\dots,n) &= e^{B(p_i)}\mathcal{M}_0(1,\dots,n),\\ B &= -\alpha_{GR}\sum_{i<j}p_i\cdot p_j\log|p_i\cdot p_j|, \quad \alpha_{GR} = \frac{G}{\pi\epsilon}.\end{aligned} \tag{4.59}$$

As already commented for QED we are using dim-reg, so the divergent factor $\log\lambda$ in the cusp anomalous dimension is substituted by a factor $1/\epsilon$. The amplitude $\mathcal{M}$ denotes the original IR divergent amplitude, whereas $\mathcal{M}_0$ represents the IR-finite part, obtained by factoring out the virtual divergence contribution encapsulated in the overall multiplicative factor. Using the momentum parametrization (2.9) we can rewrite B as

$$B = -\alpha_{GR}\sum_{i<j}\eta_i\eta_j\omega_i\omega_j|z_{ij}|^2\log|z_{ij}|^2 \tag{4.60}$$

where $\eta_i = \pm 1$, depending on whether the particle is incoming or outgoing.

To obtain the corresponding celestial amplitude we can just Mellin transform $\mathcal{M}$:

$$\begin{aligned}\langle\mathcal{O}_{\Delta_1}\dots\mathcal{O}_{\Delta_n}\rangle &= \int_0^{+\infty}\prod_{k=1}^{n}d\omega_k\omega_k^{\Delta_k-1}\exp\left\{2\alpha_{GR}\sum_{ij}\eta_i\eta_j\omega_i\omega_j|z_{ij}|^2\log|z_{ij}|\right\}\mathcal{M}_0\\ &= \exp\left\{2\alpha_{GR}\sum_{ij}P_iP_j|z_{ij}|^2\log|z_{ij}|\right\}\langle\mathcal{O}_{\Delta_1},\dots,\mathcal{O}_{\Delta_n}\rangle_0\end{aligned} \tag{4.61}$$

where P_i act as operators on the amplitude, in particular they shift the conformal weight of the i-th particle by 1:

$$P_i\langle\mathcal{O}_{\Delta_1}\dots\mathcal{O}_{\Delta_n}\rangle=\eta_i\langle\mathcal{O}_{\Delta_1}\dots\mathcal{O}_{\Delta_i+1}\dots\mathcal{O}_{\Delta_n}\rangle. \tag{4.62}$$

We see that the celestial amplitude factorizes in a soft and a hard part in a slightly different way with respect to QED, as we see that the soft amplitude:

$$\mathcal{M}_{soft}=\exp\left\{2\alpha_{GR}\sum_{ij}P_iP_j|z_{ij}|^2\log|z_{ij}|\right\} \tag{4.63}$$

acts on the hard part

$$\langle\mathcal{O}_{\Delta_1}\dots\mathcal{O}_{\Delta_n}\rangle_0=\int_0^{+\infty}\prod_{k=1}^{n}d\omega_k\omega_k^{\Delta_k-1}\mathcal{M}_0 \tag{4.64}$$

as an operator.

As anticipated in section 3.3.3 the Goldstone mode two point functions (3.116) is intimately related to the soft amplitude, such that also for the case of gravity, vertex operators of the supertranslation Goldstone of the form $\exp\{iP_iC(z_i,\bar{z}_i)\}$ reconstruct $\mathcal{M}_{soft}$:

$$\langle e^{iP_1C(z_1,\bar{z}_1)}\dots e^{iP_nC(z_n,\bar{z}_n)}\rangle=\exp\left\{2\alpha_{GR}\sum_{ij}P_iP_j|z_{ij}|^2\log|z_{ij}|\right\}=\mathcal{M}_{soft} \tag{4.65}$$

Notice that the vertex operators $\exp\{iP_iC(z_i,\bar{z}_i)\}$ precisely coincide with the Mellin transform of the $\mathcal{W}$ operators defined in (3.105), which matched the soft amplitude in momentum space, so the result in (4.65) is again perfectly compatible with our discussion of the Faddeev-Kulish conformal dressing reported in section 3.3.3, where we saw that the supertranslation Goldstone mode can be used to dress matter operators as in (3.101). This implies that after the Mellin transform the IR finite bulk operators turn into dressed celestial operators of the form [152,169]:

$$\hat{\mathcal{O}}_\Delta=:e^{-iPC}\mathcal{O}_\Delta: \tag{4.66}$$

A correlation function of such operators yields again to the hard part of the celestial amplitude:

$$\left\langle\hat{\mathcal{O}}_{\Delta_1}(z_1,\bar{z}_1)\dots\hat{\mathcal{O}}_{\Delta_n}(z_n,\bar{z}_n)\right\rangle=\langle\mathcal{O}_{\Delta_1},\dots,\mathcal{O}_{\Delta_n}\rangle_0 \tag{4.67}$$

which is IR finite.

Chapter 5

Logarithmic CFTs and CCFT

As pointed out at the end of the previous chapter the $c = 0$ behavior of celestial CFT might find a solution if we interpret celestial CFT as a logarithmic conformal field theory. This chapter will be dedicated to an analysis of the logarithmic CFT behavior of CCFT. In recent years there have been some papers reporting succinct results on the presence of logarithmic structure in CCFT coming from loop corrections [92,94] and at the level of the stress tensor [93]. In this chapter, we present our original results, which provide a more concrete and detailed analysis of the emergence of logarithmic CFT properties in CCFT at tree level. We discuss the findings of [1] alongside additional observations drawn from unpublished material

We begin with a lightning introduction to the general properties of CFTs, followed by a discussion of the distinctive features of logarithmic CFTs and their resolution of the so-called $c \to 0$ catastrophe through the introduction of a logarithmic partner t coupled to the stress tensor T. We then examine the emergence of logarithmic primary fields in celestial CFT, highlighting the presence of the t mode and a well-defined logarithmic primary in the soft sector of gravity. The chapter concludes with additional remarks on this framework, including its connection to the behavior of the free scalar in two dimensions, as well as a novel construction embedding the free scalar into a larger free CFT exhibiting a logarithmic structure.

5.1 CFTs generalities

Let us start this chapter by giving an extremely contained introduction to properties of standard CFTs. This foundation will make clear the aspects in which logarithmic CFTs deviate from standard CFTs. For more details on the generalities of CFT see [177, 184].

Conformal field theories (CFTs) are widely studied as they play a fundamental role in the understanding of many physical and theoretical problems, spanning from critical phenomena in statistical physics to the understanding of quantum gravity in holography. The key property of this class of QFTs is that they are invariant under an extension of the usual isometries, namely the collection of coordinate transformations that change the metric as $g_{\mu\nu} \to \lambda(x) g_{\mu\nu}$. These are the so called conformal transformations whose group structure, working in $d > 2$ Minkowski spacetime, is isomorphic to $SO(d, 2)$ and contains translations, rotations, dilatations and special conformal transformations. In the case $d = 2$ things turn out to be different as every holomorphic function corresponds to a conformal transformation, and the symmetry algebra becomes infinite dimensional, isomorphic to the Virasoro algebra.

Conformal invariance has extremely important consequences: in a generic QFT given two operators $A(x)$, $B(y)$, where $x \to y$, because the operators close an algebra, it is possible to write the operator product expansion (OPE):

$$A(x)B(y) = \sum_C f_{ABC}(x,y,z)C(z). \tag{5.1}$$

This is usually considered to be an asymptotic expansion valid within a certain radius of convergence that can also turn out to be zero. However in a CFT due to the absence of scales, the OPE becomes a proper series with infinite radius of convergence.
Another consequence is the operator state correspondence which implies that every state of the Hilbert space corresponds to a field in the CFT, and vice versa. Using the symmetry we can classify fields and states using representation theory. Usually the representations are considered to be irreducible and built on highest weight states called "primary state" $|\Delta\rangle$ which are eigenstates of the dilation operator D and annihilated by the boost generator K:

$$D\,|\Delta\rangle = -i\Delta\,|\Delta\rangle\,, \quad K_\mu\,|\Delta\rangle = 0, \tag{5.2}$$

These conditions constrain the transformations properties of the corresponding operators which are refereed to as "primary operator". As an example we can write the transformation of a primary scalar field:

$$\phi'_\Delta(x') = \Omega(x)^{-\Delta}\phi_\Delta(x), \quad \Omega(x) = \det\left|\frac{dx'}{dx}\right|^{\frac{1}{d}}, \tag{5.3}$$

All the other fields in the theory are obtained from the primary fields by taking derivatives or from a state point of view acting with the translation operator on the primary states:

$$|\Delta, \mu_1 \ldots \mu_n\rangle = P_{\mu_1}\ldots P_{\mu_n}\,|\Delta\rangle\,. \tag{5.4}$$

These other fields are called descendants. The transformation properties allow us to fix the form of the coefficients of the OPE of primary fields. In the case of scalars in particular we get:

$$\phi_{\Delta_1}(x_1)\phi_{\Delta_2}(x_2) \sim \sum_{p,k} C^{(k)}_{12p}|x_1-x_2|^{2(\Delta_p-\Delta_1-\Delta_2)}(x_1-x_2)^{\mu_1}\ldots(x_1-x_2)^{\mu_k}\partial_{\mu_1}\ldots\partial_{\mu_k}\phi_{\Delta_p}(x_2) \tag{5.5}$$

where including derivatives of primaries we have also taken into account the presence of descendants fields. From the OPE is also possible to extract the form of the two point function just computing the coefficient of the identity field. By symmetries this is also fixed to be of the form:

$$\langle\phi_\Delta(0)\phi_{\Delta'}(x)\rangle \sim \frac{c_\Delta\delta_{\Delta'\Delta}}{|x|^{2\Delta}} \tag{5.6}$$

All of these results are valid for $d > 2$ and can also be applied for $d = 2$ but in a different language that allows for a better and more constrained treatment of the theories. As we already mention, in $2d$ the conformal algebra gets enhanced to a double copy of infinite dimensional

Virasoro algebra Vir $\otimes$ $\overline{\text{Vir}}$. The generators $L_n, \bar{L}_n, n \in \mathbb{Z}$ have the following commutation relation:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \tag{5.7}$$

$$[\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \tag{5.8}$$

$$[L_n, \bar{L}_m] = 0 \tag{5.9}$$

which identify the Virasoro algebra as a central extension of the Weyl algebra. The central charge c plays an extremely important role as it specifies many properties of the CFT.
On the complex coordinates $(z, \bar{z})$ the algebra acts by generating a generic holomorphic/anti-holorphic transformation:

$$z \to z - \mathcal{Y}(z) \tag{5.10}$$

$$\bar{z} \to \bar{z} - \overline{\mathcal{Y}}(\bar{z}) \tag{5.11}$$

Focusing just on the holomorphic side[1] we notice that Vir contains a centerless subalgebra generated by $L_0, L_{\pm 1}$:

$$[L_{\pm 1}, L_0] = \pm L_{\pm 1} \tag{5.12}$$

$$[L_1, L_{-1}] = 2L_0 \tag{5.13}$$

which corresponds to the algebra of the group $SL(2, \mathbb{C}) \sim SO(3,1)$. In particular acting on the coordinates: L_0 generates complex dilatation (real dilatation + rotation), L_{-1} generates translations and L_1 complex special transformations. Combining this with $\overline{\text{Vir}}$ the algebra of $SO(3,1)$ can be recovered using the following identifications

$$D = L_0 + \bar{L}_0 \to \text{Dilatation} \tag{5.14}$$

$$J_{12} = i(\bar{L}_0 - L_0) \to \text{Rotations} \tag{5.15}$$

$$P_1 = L_{-1} + \bar{L}_{-1} \to \text{Translations on the } x\text{-axis} \tag{5.16}$$

$$P_2 = i(\bar{L}_{-1} - L_{-1}) \to \text{Translations on the } y\text{-axis} \tag{5.17}$$

$$K_1 = L_1 + \bar{L}_1 \to \text{Special conformal on the } x\text{-axis} \tag{5.18}$$

$$K_2 = i(\bar{L}_1 - L_1) \to \text{Special conformal on the } y\text{-axis} \tag{5.19}$$

The finite transformations induced by $L_{0,\pm 1}$ on z take the form:

$$z \to \frac{az+b}{cz+d}, \quad ad - bc = 1 \tag{5.20}$$

which is a general mobius transformations namely the automorphisms on the Riemann sphere. We see then that this $SL(2, C)$ group corresponds to the only set of conformal transformations that actually allows for a well defined inverse. This is the group of the *global* (everywhere invertible) conformal transformations. The transformations generated by the other generators of Vir are non inverible holomorphic functions so they cannot be related to a group, and are

[1]The same follows for the anti-holomorphic part

identified with the *local* 2d conformal transformations.
The states in the CFT are now classified under representations of the Virasoro algebra, which are usually considered to be irreducible representations. The previously defined primary states are now extended to Virasoro primaries, which satisfy:

$$L_0 \left|h, \bar{h}\right\rangle = h \left|h, \bar{h}\right\rangle, \tag{5.21}$$

$$\bar{L}_0 \left|h, \bar{h}\right\rangle = \bar{h} \left|h, \bar{h}\right\rangle, \tag{5.22}$$

$$L_n \left|h, \bar{h}\right\rangle = 0, \bar{L}_n \left|h, \bar{h}\right\rangle = 0; \quad n > 0. \tag{5.23}$$

The fields corresponding to the primary states, transform under the action of conformal transformation as:

$$\phi'(z', \bar{z}') = (\partial f)^{-h} (\bar{\partial} \bar{f})^{-\bar{h}} \phi(z, \bar{z}), \quad z' = f(z), \, \bar{z}' = \bar{f}(\bar{z}). \tag{5.24}$$

The primary states are hightest weight states eigenvalues of the dilatation operators. All other states in the theory are obtained by acting on the primary fields with the negative Virasoro generators $L_{-n}, n > 0$:

$$\left|h, \bar{h}; k_1, \ldots, k_n\right\rangle = L_{-k_1} \ldots L_{-k_n} \left|h, \bar{h}\right\rangle. \tag{5.25}$$

All the descendants coming from a fixed primary are grouped inside a set called Verma module.

5.2 Logarithmic CFT

Logarithmic CFTs (Log CFTs or LCFTs) get their name from the particular behaviour of their correlation functions which contain logarithms of the spacetime coordinates. In the previous section we have specified that the representation of the Virasoro algebra are usually considered to be irreducible. However this is not mandatory, and in fact the peculiar behaviour of correlation functions for Log CFTs, is due to the fact that in logarithmic CFTs the representation of the conformal group is not irreducible but reducible and indecomposable. This means that the dilatation operator D cannot be completely diagonalized but it can be at most put into a Jordan block form such that the primary states fall into rank r multiplets ($a = 1, \ldots, r$),

$$D \left|\mathcal{O}_a\right\rangle = -i \boldsymbol{\Delta}_a{}^b \left|\mathcal{O}_b\right\rangle, \quad \boldsymbol{\Delta} = \begin{pmatrix} \Delta & 1 & 0 & \cdots & 0 \\ 0 & \Delta & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \Delta & 1 \\ 0 & 0 & 0 & 0 & \Delta \end{pmatrix}. \tag{5.26}$$

It follows then that all of these theories are non-unitary. Indeed, reflection positivity would require the two-point function $\langle \mathcal{O}_a(x) \mathcal{O}_a(-x) \rangle$ to be positive, for all x and (Hermitian) fields $\mathcal{O}_a$. However, for Log CFTs, this is not possible, unless all multiples have rank $r = 1$. This follows from the fact that the dilatation operator is not Hermitian, thus in particular time translation on the cylinder is not implemented by a unitary operator.

In what follows we will give some details on the properties of 2d logarithmic field theories, starting from the first example found by Gurarie, and then moving to the general properties of logarithmic primaries. Then we will specialize to the case $c = 0$ and show how the so called $c \to 0$ catastrophe can be solved by the introduction of a rank $r = 2$ multiplet containing the stress tensor as a primary field. For a more exhaustive treatment see [184, 185].

5.3 An explicit example of log-CFT

The explicit example of Log CFT that we will review is the first case of logarithmic CFT discovered in 1993 by Gurarie [186], namely the $c = -2$ non unitary minimal model. This model contains a $(-1/8, 0)$ chiral primary field $\mu(z)$ corresponding to the state:

$$|\mu\rangle = \mu(0)\,|0\rangle \tag{5.27}$$

and the Identity operator. This state admits at level 2 a null vector of the form:

$$|\mu; 2\rangle = \left(L_{-2} - 2L_{-1}^2\right)|\mu\rangle \tag{5.28}$$

A null vector is a particular element of the Hilbert space $|\mathcal{V}\rangle$, such that:

$$\langle\mathcal{V}|\,\mathcal{O}_1 \ldots \mathcal{O}_n\,|0\rangle = 0 \tag{5.29}$$

for any combination of operators $\mathcal{O}_1 \ldots \mathcal{O}_2$. The form of these vectors allow us usually to extract differential equations that help us fixing explicitly the correlation functions. In this case (5.28) allows us to determine the $\mu(z)$ four point function. In fact by writing the four point function in the form:

$$\langle\mu(z_1)\mu(z_2)\mu(z_3)\mu(z_4)\rangle = z_{13}^{1/4} z_{24}^{1/4} \left[x(1-x)\right]^{1/4} F(x), \tag{5.30}$$

$$x = \frac{z_{12}z_{34}}{z_{13}z_{24}}, \quad z_{ij} = z_i - z_j, \tag{5.31}$$

Gurarie showed that the structure of (5.28) leads to the following hypergeometric equation for $F(x)$

$$x(1-x)\frac{d^2F}{dx^2} + (1-2x)\frac{dF}{dx} - \frac{1}{4}F(x) = 0. \tag{5.32}$$

The solutions can be found explicitly and has the following form:

$$F(x) = c_1 G(x) + c_2 H(x) + c_2 G(x)\log x, \tag{5.33}$$

$$G(1-x) = H(x) + G(x)\log x, \tag{5.34}$$

where $H(x)$ is a regular function in $x = 0$ and $G(x)$ can be expressed as an elliptic integral:

$$G(x) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - x\sin^2\theta}}, \tag{5.35}$$

We recognize that the solution splits into a regular and a new logarithmic branch. The appearance of this new set of solutions has some drastic consequences on the structure of the OPEs of the CFT, as they now have to reconstruct the logarithmic side of the correlation function. Usually we know that 2d OPEs take the form:

$$\phi_{h_1}(z)\phi_{h_2}(0) = \sum_p \sum_{\{k\}} C_{12}^{p,\{k\}} z^{h_p - h_1 - h_2 + \sum_i k_i} \phi_{h_p}^{\{k\}}(0) = \sum_p C_{12}^p z^{h_p - h_1 - h_2} \Psi_p(z|0), \tag{5.36}$$

$$\Psi_p(z|0) = \sum_{\{k\}} \beta_{nm}^{p,\{k\}} z^{\sum_i k_i} \phi_{h_p}^{\{k\}}(0), \quad C_{12}^{p,\{k\}} = C_{12}^p \beta_{nm}^{p,\{k\}}, \tag{5.37}$$

where the p summation runs over primary fields and $\phi_{h_p}^{\{k\}}(0)$ is a short hand notation for the fields in the conformal primary of the field $\phi_{h_p}(0)$ at level $\{k\}$.
Because the indicial equation of (5.32) gives two null roots we know from [187] that the only fields that can appear inside the OPE are the one in the identity conformal family:

$$\mu(z)\mu(0) = C_{\mu\mu}^0 z^{\frac{1}{4}} \Psi_0(z|0) = z^{\frac{1}{4}} \sum_{\{k\}} C_{\mu\mu}^{0,\{k\}} z^{\sum_i k_i} \mathbb{I}^{\{k\}}(0) = z^{\frac{1}{4}} \sum_{k=0}^{\infty} z^k \mathbb{I} = z^{\frac{1}{4}} \mathbb{I}(z). \tag{5.38}$$

Even if it is manifest that this OPE can well reproduce the prefactors in front of $F(x)$ in (5.31) it is also clear that $\mathbb{I}(z)\mathbb{I}(w)$ cannot reproduce the logarithmic behavior due to his regularity in zero. Moreover it is possible to prove that this OPE breaks the crossing symmetries [186].
This implies that we need to modify the OPE extending the field content of the theory to make the theory consistent . The constraint given by the indicial equation forces us to add a new weight $(0, 0)$ field that, following the proposal in [186], changes the OPE as:

$$\mu(z)\mu(0) = z^{\frac{1}{4}} \left(\mathbb{I}(z) \log(z) + \tilde{\mathbb{I}}(z) \right) = z^{\frac{1}{4}} \sum_{k=0}^{\infty} z^k \left(\mathbb{I}_k \log(z) + \tilde{\mathbb{I}}_k \right), \tag{5.39}$$

where $\tilde{\mathbb{I}}(z)$ is a new operator with peculiar properties. The Identity comes this time multiplied by the logarithm to provide directly for the logarithmic branch of solutions in (5.31). Let us analyze the properties of this new additional field. Acting with $\mu(z)\mu(0)$ on the vacuum:

$$\mu(z)\mu(0) \left|0\right\rangle = z^{\frac{1}{4}} \sum_{k=0}^{\infty} z^k \left(\log(z) \left|\mathbb{I}, k\right\rangle + \left|\tilde{\mathbb{I}}, k\right\rangle \right), \tag{5.40}$$

we can use the Virasoro's commutation properties with the primaries to obtain:

$$L_0 \left|\mathbb{I}, k\right\rangle = k \left|\mathbb{I}, k\right\rangle, \quad L_0 \left|\tilde{\mathbb{I}}, k\right\rangle = \left|\mathbb{I}, k\right\rangle + k \left|\tilde{\mathbb{I}}, k\right\rangle, \tag{5.41}$$

$$L_k \left|\mathbb{I}, n+k\right\rangle = (n + (k-1)h_\mu) \left|\mathbb{I}, n\right\rangle, \quad L_k \left|\tilde{\mathbb{I}}, n+k\right\rangle = \left|\mathbb{I}, n\right\rangle + (n + (k-1)h_\mu) \left|\tilde{\mathbb{I}}, n\right\rangle, \tag{5.42}$$

where $k > 0$ and $h_\mu = -1/8$. Because $\left|\mathbb{I}, -n\right\rangle, \left|\tilde{\mathbb{I}}, -n\right\rangle, n > 0$ are vanishing it is immediate to extract the action of L_k on the highest weight states:

$$L_0 \left|\mathbb{I}\right\rangle = 0, \quad L_0 \left|\tilde{\mathbb{I}}\right\rangle = \left|\mathbb{I}\right\rangle, \tag{5.43}$$

$$L_k \left|\mathbb{I}\right\rangle = 0, \quad L_k \left|\tilde{\mathbb{I}}\right\rangle = 0, \tag{5.44}$$

$$\left|\mathbb{I}\right\rangle = \left|\mathbb{I}, 0\right\rangle, \quad \left|\tilde{\mathbb{I}}\right\rangle = \left|\tilde{\mathbb{I}}, 0\right\rangle, \tag{5.45}$$

so we see that the additional field behaves almost exactly as a $(0,0)$ primary field except for the anomalous action of the L_0 operator. In particular we can see that, setting $h=0$, L_0 is not diagonal anymore but it acts on the doublet as:

$$L_0 \begin{pmatrix} \left|\tilde{\mathbb{I}}\right\rangle \\ \left|\mathbb{I}\right\rangle \end{pmatrix} = \begin{pmatrix} h & 1 \\ 0 & h \end{pmatrix} \begin{pmatrix} \left|\tilde{\mathbb{I}}\right\rangle \\ \left|\mathbb{I}\right\rangle \end{pmatrix}, \tag{5.46}$$

namely it is set at most in Jordan block form. This commutation relation also allows us to compute the transformation under the conformal group $z \to f(z) = z - \mathcal{Y}(z)$ of the field associated to $\left|\tilde{\mathbb{I}}\right\rangle$, which turns out to be:

$$\tilde{\mathbb{I}}'(z') = \tilde{\mathbb{I}}(z) - \log \partial f\, \mathbb{I}, \tag{5.47}$$

and we see that the usual primary rule gets shifted by an anomalous logarithmic term. This field takes the name of *logarithmic partner* of the Identity field.

Stepping away for a moment from the $c=-2$ model, in [186] Gurarie also generalizes this kind of behavior considering a theory with three fields $A(z)$, $B(z)$, $C(z)$. In this theory we assume that the four point function $\langle A(z_1)B(z_2)A(z_3)B(z_4)\rangle$ can again be found through null vectors, leading to an hypergeometric equation with degenerate characteristic polynomial with roots $\alpha_\pm = h_C$ where h_C is the C field conformal weight. Again we have to modify the AB OPE to account for the logarithmic branch, by introducing a new field $\tilde{C}$ of dimension h_C such that:

$$A(z)B(0) = z^{h_C - h_A - h_B}\left(\tilde{C}(z) + C(z)\log(z) + \dots\right), \tag{5.48}$$

where the dots include all the descendants coming from C and $\tilde{C}$.
Using the same procedure considered to extract (5.41), (5.42) we can then deduce:

$$L_0 \left|C,k\right\rangle = (h_C + k)\left|C,k\right\rangle, \quad L_0\left|\tilde{C},k\right\rangle = \left|C,k\right\rangle + (h_C+k)\left|\tilde{C},k\right\rangle, \tag{5.49}$$

$$L_k\left|C,n+k\right\rangle = (h_C + n - h_B + kh_A)\left|C,n\right\rangle, \tag{5.50}$$

$$L_k\left|\tilde{C},n+k\right\rangle = \left|C,n\right\rangle + (h_C+n-h_B+kh_A)\left|\tilde{C},n\right\rangle, \tag{5.51}$$

where $k>0$. This implies again:

$$L_0\left|C\right\rangle = (h_C)\left|C\right\rangle, \quad L_0\left|\tilde{C}\right\rangle = \left|C\right\rangle + h_C\left|\tilde{C}\right\rangle, \tag{5.52}$$

$$L_k\left|C\right\rangle = 0, \quad L_k\left|\tilde{C}\right\rangle = 0, \tag{5.53}$$

$$\left|C\right\rangle = \left|C,0\right\rangle, \quad \left|\tilde{C}\right\rangle = \left|\tilde{C},0\right\rangle, \tag{5.54}$$

namely $\tilde{C}$ looks like behaves almost like a standard primary field with the exception of the L_0 shift.
Again this means that L_0 is in Jordan block form. The action of the Virasoro operators imply again that C and $\tilde{C}$ transform as:

$$C'(z') = (\partial f)^{-h_C}C(z) \to \delta C = (\mathcal{Y}\partial + h_C\partial\mathcal{Y})C, \tag{5.55}$$

$$\tilde{C}(z) = (\partial f)^{-h_C}\left(\tilde{C}(z) - \log\partial f C(z)\right) \to \delta\tilde{C} = (\mathcal{Y}\partial + h_C\partial\mathcal{Y})\tilde{C} + \partial\mathcal{Y}C, \tag{5.56}$$

where the $\mathcal{Y}$ are taken to be infinitesimal. We notice that for $h = 0$ this reduces to the transformation properties of the logarithmic partner of the identity. The new field $\tilde{C}$ gets as for $\tilde{\mathbb{I}}$ a logarithmic shift proportional to the original primary, and for this reason is identified as the logarithmic partner of C.

The cases that we just analyzed are just examples of the general realization in logarithmic CFTs. The key point of the examples is that to maintain the consistence of the theory we are forced to introduce a new type of field that break the diagonalizability of L_0. This has the effect of changing the representation of the conformal group, from an irreducible and indecomposable representation to a reducible but indecomposable one.

5.3.1 Log primaries and log CFT doublets

In the explicit example we have shown how in Log CFTs states are organized into logarithmic multiplets of rank $r \geq 1$.

Def 5.3.1: Logarithmic doublet

A logarithmic doublet $\mathcal{O}_a = (\Psi, \Phi)$ of weights $(h, \bar{h})$ is composed of a *primary operator* Φ and its *logarithmic partner* Ψ which transform under the global conformal group as [186]:

$$\begin{aligned} \Psi'\left(f(z), \bar{f}(\bar{z})\right) &= (\partial f)^{-h}(\bar{\partial}\bar{f})^{-\bar{h}}\left(\Psi(z,\bar{z}) - \log|\partial f|\Phi(z,\bar{z})\right), \\ \Phi'\left(f(z), \bar{f}(\bar{z})\right) &= (\partial f)^{-h}(\bar{\partial}\bar{f})^{-\bar{h}}\Phi(z,\bar{z}), \end{aligned} \tag{5.57}$$

where $f(z), \bar{f}(\bar{z})$ are elements of the $SL(2,\mathbb{C})$ group. If this transformation extends to the full Virasoro algebra, we refer to these fields as *Virasoro primary* and *log Virasoro primary*.

Using (5.57), we can also find the general equation for the infinitesimal transformations of the logarithmic partner. Writing $f(z) = z - \mathcal{Y}(z)$, $\bar{f}(\bar{z}) = \bar{z} - \bar{\mathcal{Y}}(\bar{z})$, we get

$$\begin{aligned} \Psi'(z-\mathcal{Y}, \bar{z}-\bar{\mathcal{Y}}) &= (1-\partial\mathcal{Y})^{-h}(1-\bar{\partial}\bar{\mathcal{Y}})^{-\bar{h}}\left[\Psi(z,\bar{z}) - \frac{1}{2}\log\left((1-\partial\mathcal{Y})(1-\bar{\partial}\bar{\mathcal{Y}})\right)\Phi(z,\bar{z})\right], \\ &= (1+h\partial\mathcal{Y})(1+\bar{h}\bar{\partial}\bar{\mathcal{Y}})\left[\Psi(z,\bar{z}) + \frac{\partial\mathcal{Y}+\bar{\partial}\bar{\mathcal{Y}}}{2}\Phi(z,\bar{z})\right]. \end{aligned} \tag{5.58}$$

Then if we consider $z \to z + \mathcal{Y}$, $\bar{z} \to \bar{z} + \bar{\mathcal{Y}}$ the field transformation becomes

$$\Psi'(z,\bar{z}) = (1+h\partial\mathcal{Y})(1+\bar{h}\bar{\partial}\bar{\mathcal{Y}})\left[(1+\mathcal{Y}\partial+\bar{\mathcal{Y}}\bar{\partial})\Psi + \frac{\partial\mathcal{Y}+\bar{\partial}\bar{\mathcal{Y}}}{2}\Phi\right], \tag{5.59}$$

so that the transformation properties of the log doublet can be written as

$$\begin{aligned} \delta\Psi(z,\bar{z}) &= \left(\mathcal{Y}\partial + \bar{\mathcal{Y}}\bar{\partial} + h\partial\mathcal{Y} + \bar{h}\bar{\partial}\bar{\mathcal{Y}}\right)\Psi(z,\bar{z}) + \frac{1}{2}(\partial\mathcal{Y}+\bar{\partial}\bar{\mathcal{Y}})\Phi(z,\bar{z}), \\ \delta\Phi(z,\bar{z}) &= \left(\mathcal{Y}\partial + \bar{\mathcal{Y}}\bar{\partial} + h\partial\mathcal{Y} + \bar{h}\bar{\partial}\bar{\mathcal{Y}}\right)\Phi(z,\bar{z}). \end{aligned} \tag{5.60}$$

These relations constrain the $T\Psi$ OPE with the stress tensor to be

$$T(z)\Psi(w,\bar{w}) = \frac{1}{(z-w)^2}\left(h\Psi(w,\bar{w}) + \frac{1}{2}\Phi(w,\bar{w})\right) + \frac{\partial\Psi(w,\bar{w})}{z-w} + \ldots, \tag{5.61}$$

If we identify $\Psi = \mathcal{O}_1$, $\Phi = \mathcal{O}_2$, then the two-point function of a logarithmic scalar doublet can always be written in the form [186, 188]

$$\langle \mathcal{O}_a(z_1, \bar{z}_1)\mathcal{O}_b(z_2, \bar{z}_2)\rangle = \frac{1}{z_{12}^{2h}\, \bar{z}_{12}^{2\bar{h}}} \begin{pmatrix} \tilde{k}_{\mathcal{O}} - k_{\mathcal{O}} \log |z_{12}|^2 & k_{\mathcal{O}} \\ k_{\mathcal{O}} & 0 \end{pmatrix}_{ab}, \tag{5.62}$$

with $k_{\mathcal{O}} \neq 0$ and $\tilde{k}_{\mathcal{O}}$ some constant that cannot be fixed by conformal invariance. We remark that Ψ is not uniquely specified because we may add to it any multiple of Φ, $\Psi \to \Psi + k\Phi$, without affecting its defining properties, but changing the constant $\tilde{k}_{\mathcal{O}}$ in (5.62) while $k_{\mathcal{O}}$ will remain invariant. Because of this, $\tilde{k}_{\mathcal{O}}$ may be tuned to any desired value, and it is not expected to be physical [189]. The constant $k_{\mathcal{O}}$, on the other hand, is expected to be physically meaningful. Another remarkable feature of a log CFT which can be observed from (5.62) is that the two-point function of the two primary fields vanishes, $\langle \Phi\Phi\rangle = 0$.

5.4 Scale invariance

The presence of logarithms would seem to signal that the n-point function is now scale dependent, as the quantity inside logarithms has to be dimensionless. However, this scale is physically irrelevant due to the Ward identities induced by global conformal invariance, as we review below (see [184]).

Let us start by exploring the idea in the context of standard irreducible CFTs. Suppose that we analyze the system at a fixed reference scale μ, the basis of primary operators at this scale frame will then be defined as $\mathcal{O}_{i,\mu}(z)$. Of course, the n-point function should not depend on this scale so we can fix a Callan-Symanzik equation to be such that

$$\mu\frac{d}{d\mu}\langle \mathcal{O}_{i_1,\mu}(z_1)\ldots\mathcal{O}_{i_n,\mu}(z_n)\rangle = \sum_{k=1}^{n}\langle \mathcal{O}_{i_1,\mu}(z_1)\ldots\mu\frac{d}{d\mu}\mathcal{O}_{i_k,\mu}(z)\ldots\mathcal{O}_{i_n,\mu}(z_n)\rangle = 0. \tag{5.63}$$

The total derivative with respect to the scale can be explicitly written as

$$\frac{d}{d\mu}\mathcal{O}_{i,\mu}(z) = \lim_{\delta\mu\to 0}\frac{\mu}{\delta\mu}\left[\mathcal{O}_{i,\mu+\delta\mu}\left(z + \frac{\delta\mu}{\mu}z\right) - \mathcal{O}_{i,\mu}(z)\right], \tag{5.64}$$

where we have taken into account the coordinate scale variation. The expression in the brackets is then the standard primary field variation under a rescaling and we can use its transformation properties to rewrite

$$\mu\frac{d}{d\mu}\mathcal{O}_{i,\mu}(z) = \lim_{\delta\mu\to 0}\frac{\mu}{\delta\mu}\left[z\frac{\delta\mu}{\mu}\partial\mathcal{O}_{i,\mu}(z) + h_i\frac{\delta\mu}{\mu}\mathcal{O}_{i,\mu}(z)\right] = z\partial\mathcal{O}_{i,\mu}(z) + h_i\mathcal{O}_{i,\mu}(z). \tag{5.65}$$

Thus, it follows that the Callan-Symanzik equation turns exactly into the dilatation Ward identity,

$$\mu\frac{d}{d\mu}\langle \mathcal{O}_{i_1,\mu}(z_1)\ldots\mathcal{O}_{i_n,\mu}(z_n)\rangle = \sum_{k=1}^{n}(z_k\partial_{z_k} + h_k)\langle \mathcal{O}_{i_1,\mu}(z_1)\ldots\mathcal{O}_{i_n,\mu}(z_n)\rangle = 0\,, \tag{5.66}$$

which is trivially satisfied due to conformal invariance.

Even in a standard CFT, it is of course natural to introduce a reference scale, and all the operators will then be defined at that scale. However, due to conformal invariance, the theory will not depend on the value assigned at the specific chosen scale, namely the field basis is dependent on the original choice of scale but the theory is not. The exact same thing happens in a logarithmic CFT, even if the construction is more subtle due to the presence of logarithms. In this case, the different transformation properties of log primaries lead to a Callan-Symanzik equation of the form [184]

$$\mu\frac{d}{d\mu}\langle\mathcal{O}_{i_1,l_1;\mu}(z_1)\dots\mathcal{O}_{i_n,l_n;\mu}(z_n)\rangle=\sum_{k=1}^{n}\langle\mathcal{O}_{i_1,l_1;\mu}(z_1)\dots\left(\delta^{j_k}_{l_k}z_k\partial_{z_k}+\boldsymbol{h}^{j_k}_{l_k}\right)\mathcal{O}_{j_k,\mu}(z_k)\dots\mathcal{O}_{i_n,\mu}(z_n)\rangle=0, \tag{5.67}$$

where

$$\boldsymbol{h}=\begin{pmatrix} h & 1 & 0 & \dots & 0 & 0\\ 0 & h & 1 & \dots & 0 & 0\\ 0 & 0 & h & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & h & 1\\ 0 & 0 & 0 & \dots & 0 & h \end{pmatrix}. \tag{5.68}$$

The RHS of (5.67) turns out to coincide with the dilatation Ward identity for a logarithmic primary field which must be satisfied in a logarithmic CFT, suggesting again that any n-point function will be independent on the chosen reference scale. We can see an explicit realization of this fact by considering the two-point function of logarithmic field in a doublet $(\tilde{\mathcal{O}}_{h,\mu},\mathcal{O}_{h,\mu})$ at a chosen scale μ:

$$\langle\tilde{\mathcal{O}}_{h,\mu}(z)\tilde{\mathcal{O}}_{h,\mu}(0)\rangle=-\frac{\kappa}{(\mu z)^{2h}}\log(\mu z). \tag{5.69}$$

This can be rewritten using (5.57) as

$$\langle\mu^h\left(\tilde{\mathcal{O}}_{h,\mu}+\log\mu\mathcal{O}_{h,\mu}\right)(z)\mu^h\left(\tilde{\mathcal{O}}_{h,\mu}+\log\mu\mathcal{O}_{h}\right)(0)\rangle=-\frac{\kappa}{z^{2h}}\log(z). \tag{5.70}$$

By equating (5.69) and (5.70) we can see that, while the form of the logarithmic primary field depends on the specific scale μ, we can fix μ to an arbitrary value and the theory will still be invariant under the conformal group.

5.5 Log-shadow transform

In (4.39) we have defined the shadow transform of a conformal primary Φ of weights $(h,\bar{h})$ as (see e.g. [190])

$$\widetilde{\Phi}(z,\bar{z})=K_{h,\bar{h}}\int d^2w\frac{\Phi(w,\bar{w})}{(z-w)^{2-2h}(\bar{z}-\bar{w})^{2-2\bar{h}}}, \tag{5.71}$$

With $K_{h,\bar{h}}=1$. A more suitable normalization is given by $K_{h,\bar{h}}=\frac{\Gamma(2-2\bar{h})}{\pi\Gamma(2h-1)}$ such that the shadow becomes idempotent $\widetilde{\widetilde{\Phi}}=\Phi$. The shadow operator $\widetilde{\Phi}$ is still a conformal primary but now of weights $(1-h, 1-\bar{h})$. Given a logarithmic primary Ψ of weights $(h, \bar{h})$, we define its 'logarithmic shadow transform' as

$$S_{\log}[\Psi](z,\bar{z}) = -K_{h,\bar{h}} \int d^2w \frac{\Psi(w,\bar{w}) + \log|z-w|^2 \Phi(w,\bar{w})}{(z-w)^{2-2h}(\bar{z}-\bar{w})^{2-2\bar{h}}}, \tag{5.72}$$

and $S_{\log}[\Phi](z,\bar{z})$ being the shadow defined as (5.71). We prove in appendix A.1 that $S_{\log}[\Psi](z,\bar{z})$ transforms as a logarithmic primary field under $SL(2,\mathbb{C})$ with $\widetilde{\Phi}(z,\bar{z})$ a primary field. We also check that it squares to

$$S_{\log}[S_{\log}[\Psi]](z,\bar{z}) = (-1)^{-4\bar{h}} \left[\Psi(z,\bar{z}) + \left(\frac{1}{1-2h} + \frac{1}{1-2\bar{h}} - 2\pi i\right)\Phi(z,\bar{z})\right], \tag{5.73}$$

which allows us to define the inverse log-shadow $S^{-1}_{\log}$ of a logarithmic doublet as

$$\begin{aligned} S^{-1}_{\log}[\Psi](z,\bar{z}) &= (-1)^{4\bar{h}} \left[S_{\log}[\Psi](z,\bar{z}) - \left(\frac{1}{1-2h} + \frac{1}{1-2\bar{h}} - 2\pi i\right)\widetilde{\Phi}(z,\bar{z})\right], \\ S^{-1}_{\log}[\Phi](z,\bar{z}) &= (-1)^{4\bar{h}}\widetilde{\Phi}(z,\bar{z}). \end{aligned} \tag{5.74}$$

5.6 The $c \to 0$ catastrophe

In this section we want to discuss precisely the vanishing central charge problem and its possible solutions. The following considerations where originally proposed by Cardi and Gurarie in [191, 192] and are here reported following the discussion in [193].

We start by considering an holomorphic primary field ϕ of conformal weights $(h, 0)$ with two point function:

$$\langle \phi(z_1)\phi(z_2)\rangle = \frac{K_\phi}{z_{12}^{2h}} \tag{5.75}$$

Because of the stress-tensor Ward identity the three point function $\langle T\phi\phi\rangle$ takes the form:

$$\langle T(z)\phi(z_1)\phi(z_2)\rangle = \frac{hK_\phi}{(z_- z_2)^2(z-z_2)^2(z_1-z_2)^{2h-2}} \tag{5.76}$$

If we assume the existence of a single operator of conformal weight $(2,0)$ a consequence of (5.76) and the stress-tensor two point function:

$$\langle T(z)T(0)\rangle = \frac{c}{2z^4} \tag{5.77}$$

is that the $\phi\phi$ OPE must take the form:

$$\phi(z)\phi(0) \simeq \frac{K_\phi}{z^{2h}}\left[1 + \frac{2h}{c}z^2 T(0) + \dots\right] \tag{5.78}$$

We can see then that (5.6) becomes ill-defined in the limit $c \to 0$, unless there is some relation between c and K_ϕ such that $\lim_{c\to 0} K_\phi = 0$ or $h = 0$. The latter case, as it must be true for all operators ϕ in the CFT, implies that the spectrum of the theory is trivial and we find the consistent unitary model containing only the identity.

As already stated, this singular behavior is solved if the CFT splits into two decoupled models $\mathrm{CFT} = \mathrm{CFT}_1 \times \mathrm{CFT}_2$, such that the stress tensor is the linear combination of the 2 independent stress tensors T_1 and T_2 of the two CFTs $T = T_1 + T_2$. The total central charge $c = c_1 + c_2 = 0$ can be vanishing even if $c_1 = -c_2 \neq 0$. In this case also the operator ϕ split as $\phi = \phi_1\phi_2$ where $\phi_{1,2}$ are primaries inside CFT$_{1,2}$ respectively, so that the OPE gets modified as:

$$\phi(z)\phi(0) \simeq \frac{K_\phi}{z^{2h}}\left[1 + \frac{2h_1}{c_1}z^2 T_1(0) + \frac{2h_2}{c_2}z^2 T_2(0) + \dots\right] \tag{5.79}$$

which is perfectly regular.

If the CFT does not split, we can make a generic ansatz assuming that there is an operator $\tilde{T} \sim (2 + \delta(c), 0)$ with two point function of the form:

$$\langle \tilde{T}(z)\tilde{T}(0)\rangle = \frac{K_{\tilde{T}}(c)}{z^{4+2\delta(c)}}. \tag{5.80}$$

In the limit $c \to 0$ we also assume that $\delta(c) \to 0$ such that $\tilde{T}$ has the same dimension as T. Introducing this extra operator we can modify (5.78) to:

$$\phi(z)\phi(0) \simeq \frac{K_\phi}{z^{2h}}\left[1 + \frac{2h}{c}z^2 T(0) + 2h\tilde{T}z^{2+\delta(c)} + \dots\right] \tag{5.81}$$

such that expanding in series for small c we obtain:

$$\begin{aligned}\phi(z)\phi(0) &\simeq \frac{K_\phi}{z^{2h}} + K_\phi z^{2+2\delta(c)-2h}\left[\frac{2h}{c}z^{-\delta(c)}T(0) + 2h\tilde{T} + \dots\right] = \\ &= \frac{K_\phi}{z^{2h}} + K_\phi z^{2-2h}\left[-2h\frac{\delta(c)}{c}\log z T(0) + \frac{2h}{c}T(0) + 2h\tilde{T} + \dots\right]\end{aligned} \tag{5.82}$$

If we define the parameter $b^{-1} = -\lim_{c\to 0}\delta(c)/c$, which we assume to be finite, we can rewrite this OPE as:

$$\phi(z)\phi(0) \simeq \frac{K_\phi}{z^{2h}} + K_\phi z^{2-2h}\left[\frac{2h}{b}\log z\, T(0) + \frac{2h}{b}t(0) + \dots\right] \tag{5.83}$$

with:

$$\frac{1}{b}t = \frac{1}{c}T + \tilde{T} \tag{5.84}$$

Let us examine now the properties of this new field t. At first t seems to be ill defined for $c \to 0$ due to the presence of the term T/c in its definition. However we have to recall that the stress tensor n point functions are proportional to the central charge, which can cancel the $1/c$ factor and keep the contribution of t finite in any correlation function. To see this we can start by computing the two point function $\langle Tt\rangle$ and $\langle tt\rangle$,

$$\langle T(z)t(0)\rangle = \frac{b}{c}\langle T(z)T(0)\rangle = \frac{b}{2z^4} \tag{5.85}$$

and we see that the two point function is finite.

The correlator $\langle tt\rangle$ is a bit more subtle

$$\begin{aligned}\langle t(z)t(0)\rangle &= \frac{b^2}{c^2}\langle T(z)T(0)\rangle + b^2\left\langle \tilde{T}(z)\tilde{T}(0)\right\rangle = \\ &= \frac{b^2}{2cz^4} + \frac{b^2 K_{\tilde{T}}(c)}{z^{4+2\delta(c)}} = \\ &= \frac{1}{z^4}\left[\left(\frac{b^2}{2c} + b^2 K_T(c)\right) - 2b^2 K_T(c)\delta(c)\right] + o(c).\end{aligned} \tag{5.86}$$

To keep this finite we get then a constraint on the form of $K_T(c)$

$$K_T(c) = -\frac{1}{2c} + \lambda + o(c) \tag{5.87}$$

which turns $\langle t(z)t(0)\rangle$ into:

$$\langle t(z)t(0)\rangle = \frac{1}{z^4}\left(\lambda - b\log z\right). \tag{5.88}$$

To sum up we get the following list of two point functions:

$$\begin{aligned}\langle t(z)t(0)\rangle &= \frac{1}{z^4}\left(\lambda - b\log z\right),\\ \langle T(z)t(0)\rangle &= \frac{b}{2z^4},\\ \langle T(z)T(0)\rangle &= 0\end{aligned} \tag{5.89}$$

which are precisely the two point functions of a logarithmic pair (T, t) of weight $(2, 0)$. The operator t here introduced is usually referred to as the logarithmic partner of the stress tensor. As we have seen it recouples the stress tensor to the CFT thanks to the non-vanishing b parameter. This resolves the $c \to 0$ catastrophe, as now b replaces c in the $\phi\phi$ OPE which contains the stress tensor log partner

$$\phi(z)\phi(0) \simeq \frac{K_\phi}{z^{2h}}\left[1 + \frac{2h}{b}\left(\log z\, T(0) + t(0)\right) + \dots\right] \tag{5.90}$$

The three point function (5.76) is now perfectly consistent with this OPE thanks to the correlation functions (5.91). Here we have treated the case where the starting field $\tilde{T}$ is purely holomorphic. This is not mandatory, and in fact we can take a field $\tilde{T}(z, \bar{z}) \sim (2 + \delta, \delta)$ and obtain an analogous result just with a slightly modification of the two point functions due to the introduction of the anti-holomorphic variable:

$$\begin{aligned}\langle t(z_1, \bar{z}_1)t(z_2, \bar{z}_2)\rangle &= \frac{1}{2z_{12}^4}\left(\lambda - b\log|z_{12}|^2\right),\\ \langle T(z_1)t(z_2, \bar{z}_2)\rangle &= \frac{b}{2z_{12}^4},\\ \langle T(z_1)T(z_2)\rangle &= 0.\end{aligned} \tag{5.91}$$

$t(z, \bar{z})$ transforms under the action of a conformal transformation as:

$$t'(z', \bar{z}') = (\partial f)^{-2}\left(t(z, \bar{z}) - \log|\partial f| T(z)\right) \tag{5.92}$$

which implies the following stress tensor OPE

$$T(z)t(0,0) = \frac{b}{2z^4} + \frac{1}{z^2}\left(2t(0,0) + \frac{1}{2}T(0)\right) + \frac{\partial t(0,0)}{z} + \dots. \tag{5.93}$$

Notice that the value of the b parameter it was shown to be relevant to fix the form of null vectors in the $c = 0$ logarithmic CFTs, allowing us to determine the exact form of correlation functions in such models.

5.7 Logarithmic multiplets in celestial CFT

To show how logarithmic operators can emerge in celestial CFT we will now focus on celestial operators of conformal dimension $\Delta = 1 - n, n \in \mathbb{N}$. We previously shown how these operators are ill defined as IR divergences produce simple poles for $\Delta = 1 - n$ with residue proportional to bulk soft modes.
However we want now to come back to such operators, and regulate them by slightly shifting their conformal weight by an infinitesimal ϵ, $\Delta = 1 - n \to 1 - n + \epsilon$. We can then compute the transformation properties of such operators under a conformal transformation:

$$\mathcal{O}^{l\prime}_{\pm,1-m+\epsilon} = (\partial f)^{-h-\frac{\epsilon}{2}}(\bar{\partial}\bar{f})^{-\bar{h}-\frac{\epsilon}{2}}\mathcal{O}^{l}_{\pm,1-m+\epsilon}, \quad h = \frac{1-n\pm l}{2}, \bar{h} = \frac{1-n\mp l}{2} \tag{5.94}$$

Expanding the series for small ϵ we get:

$$\mathcal{O}^{l\prime}_{\pm,1-m+\epsilon} = (\partial f)^{-h}(\bar{\partial}\bar{f})^{-\bar{h}}\left[\mathcal{O}^{l}_{\pm,1-m+\epsilon} + \log|\partial f|\epsilon\mathcal{O}^{l}_{\pm,1-m+\epsilon}\right] + o(\epsilon) \tag{5.95}$$

Now if we move away from natural n the term $\epsilon\mathcal{O}^{l}_{1-n+\epsilon,\pm}$ would be vanishing and the transformation properties of $\mathcal{O}^{l}_{1-n+\epsilon,\pm}$ would be that of a primary. On the contrary for $n \in \mathbb{N}$ we get a contribution coming from the soft poles, with $\lim_{\epsilon\to 0}\epsilon\mathcal{O}^{l}_{\pm,1-n+\epsilon} = s^{l}_{\pm,1-n}$ so that:

$$\mathcal{O}^{l\prime}_{\pm,1-m+\epsilon} = (\partial f)^{-h}(\bar{\partial}\bar{f})^{-\bar{h}}\left[\mathcal{O}^{l}_{\pm,1-m+\epsilon} - \log|\partial f|s^{l}_{\pm,1-n}\right] + o(\epsilon) \tag{5.96}$$

and we see that the singular operators $\mathcal{O}^{l}_{\pm,1-n+\epsilon}$ seem to have the structure of logarithmic primary in a doublet with the soft mode $s^{l}_{\pm,1-n}$.
Notice that due to the singularity for $\epsilon \to 0$ the operator $\mathcal{O}^{l}_{\pm,1-n+\epsilon}$ will give a divergent contribution in correlation functions. However it is possible to avoid the singularity just removing it from the operator, namely we can define:

$$\mathcal{Q}^{l}_{\pm,1-n} = \lim_{\epsilon\to 0}\left[\mathcal{O}^{l}_{\pm,1-n+\epsilon} - \frac{1}{\epsilon}s^{l}_{\pm,1-n}\right] \tag{5.97}$$

The n-point amplitudes for $n > 2$, defined using this operator will be well defined, as the IR singularity gets easily removed. For $n = 2$ it is difficult to recover a well defined two point function, however we will show how it is possible to recover a logarithmic two point function in a specific set up.
The operators $(\mathcal{Q}^{l}_{\pm,1-n}, s^{l}_{\pm,1-n})$ form then a logarithmic doublet of weight $(\frac{1-n\pm l}{2}, \frac{1-n\pm l}{2})$. We can also examine the transformation properties under translations. From equation (4.25) we get that under a translation (5.97) transforms as:

$$\begin{aligned}\mathcal{Q}^{\prime l}_{\pm,1-n} &= \lim_{\epsilon\to 0}\sum_{k=0}^{\infty}\frac{(iq\cdot t)^k}{k!}\left[\mathcal{O}^{l}_{\pm,1-n+\epsilon+k} - \frac{1}{\epsilon}s^{l}_{\pm,1-n+k}\right]\\ &= \sum_{k=0}^{n}\frac{(iq\cdot t)^k}{k!}\mathcal{Q}^{l}_{\pm,1-n+k} + \sum_{k>n}\frac{(iq\cdot t)^k}{k!}\mathcal{O}^{l}_{\pm,1-n+k}.\end{aligned} \tag{5.98}$$

This expression comes from the fact that $s^{l}_{\pm,1-n+k}$ is vanishing for $k > n$ as there is no soft operators with conformal weight bigger then 1, which just leaves for $k > n$ only regular operators.

These transformation properties have been checked just for the global part of BMS. However it is easy to prove that they extend to full BMS by computing the action of P_z and T on (5.97). From equations (4.41) (4.47) we obtain:

$$P_z(z)\mathcal{Q}^l_{\pm,1-n}(w,\bar{w}) \simeq \frac{1}{z-w}\mathcal{Q}^l_{\pm,2-n}(w,\bar{w}), \tag{5.99}$$

$$T(z)\mathcal{Q}^l_{\pm,2-n}(w,\bar{w}) \simeq \frac{1}{(z-w)^2}\left(h\mathcal{Q}^l_{\pm,2-n}(w,\bar{w}) + \frac{1}{2}s^l_{\pm,1-n}(w,\bar{w})\right) + \frac{1}{z-w}\partial\mathcal{Q}^l_{\pm,2-n}(w,\bar{w})$$

where we consider $\mathcal{Q}^l_{\pm,2} = \mathcal{O}^l_{\pm,2}$. We see then that the first equation is perfectly compatible with the shift seen in equation (5.98) but now extended to all supertranslation, while the stress tensor OPE turns (5.97) into a full Virasoro logarithmic primary field paired with its primary soft mode.

5.7.1 Alternative construction - cut off regularization

In the previous section we have chosen as a regulator for the IR divergence a shift of conformal weight to define the logarithmic operators. This is a convenient choice as it does not spoil the conformal properties of the celestial operator. However for $\mathcal{Q}^l_{+,1-n}$ to be well defined it must be independent from the choice of regularization of the IR divergences, namely we should obtain the same results working with another regulator. In particular we can consider cut off regularization and write:

$$\mathcal{Q}^l_{+,1-n} = \lim_{\lambda\to 0^+}\left[\int_\lambda^{+\infty} d\omega\omega^{-n}a_l + \log\lambda\, s^l_{+,1} + \sum_{m=0}^{n-1}\frac{\lambda^{m-n}}{m-n}s^l_{+,1-m}\right]. \tag{5.100}$$

The extra terms added to the celestial operators regulated with an IR cut-off are designed to cancel the infrared divergences in the integral.

We will now show that this mode is perfectly equivalent to that defined in (5.97). To do so we split the frequency integral of celestial operator $\mathcal{O}^l_{\pm,1-n+\epsilon}$ in (5.97) into two regions, separated by a small cut-off:

$$\mathcal{Q}^l_{+,1-n} = \lim_{\epsilon\to 0}\left[\int_0^\lambda d\omega\omega^{-n+\epsilon}a_l + \int_\lambda^{+\infty} d\omega\omega^{-n+\epsilon}a_l - \frac{1}{\epsilon}s^l_{\pm,1-n}\right], \tag{5.101}$$

Now considering λ small we can expand the oscillator a_l in the small frequency region and obtain:

$$\begin{aligned}\mathcal{Q}^l_{+,1-n} &= \lim_{\epsilon\to 0}\left[\sum_{k=0}^{\infty}\frac{\lambda^{m-n+\epsilon}}{m-n+\epsilon}s^l_{+,1-m} + \int_\lambda^{+\infty} d\omega\omega^{-n+\epsilon}a_l - \frac{1}{\epsilon}s^l_{\pm,1-n}\right] = \\ &= \sum_{\substack{k=0\\k\neq n}}^{\infty}\frac{\lambda^{m-n}}{m-n}s^l_{+,1-m} + \log\lambda\, s^l_{+,1-n} + \int_\lambda^{+\infty} d\omega\,\omega^{-n}a_l(\omega),\end{aligned} \tag{5.102}$$

where we took the limit $\epsilon\to 0$, as the expression is now finite in the limit. As this expression was λ independent we could take $\lambda\to 0$ and get back the expression (5.100), which shows that the two regularization schemes give rise to the same operator. In this case the logarithmic transformation property is made explicit by the presence of the $\log\lambda$ contribution. In fact if we now take a conformal transformation we get:

$$\begin{aligned}\mathcal{Q}'^{l}_{+,1-n} = \lim_{\lambda\to 0^+}\Bigg[&(\partial f)^{-h}(\bar\partial \bar f)^{-\bar h}\int_{|\partial f|\lambda}^{+\infty} d\omega\omega^{-n}a_l + \log\lambda\,(\partial f)^{-h}(\bar\partial\bar f)^{-\bar h}s^l_{+,1-n}+\\ &+\sum_{m=0}^{n-1}\frac{\lambda^{m-n}}{m-n}(\partial f)^{-\frac{1-m+l}{2}}(\bar\partial\bar f)^{-\frac{1-m-l}{2}}s^l_{+,1-m}\Bigg].\end{aligned} \tag{5.103}$$

Now we can redefine the limit parameter $\lambda \to |\partial f|^{-1}\lambda$ to obtain:

$$\begin{aligned}\mathcal{Q}'^{l}_{+,1-n} &= (\partial f)^{-h}(\bar\partial\bar f)^{-\bar h}\lim_{\lambda\to 0^+}\Bigg[\int_{\lambda}^{+\infty} d\omega\omega^{-n}a_l + \log|\partial f|^{-1}\lambda\, s^l_{+,1-n}+\\ &\quad+\sum_{m=0}^{n-1}\frac{\lambda^{m-n}}{m-n}s^l_{+,1-m}\Bigg] =\\ &= (\partial f)^{-h}(\bar\partial\bar f)^{-\bar h}\left[\mathcal{Q}^l_{+,1-n} - \log|\partial f| s^l_{+,1-n}\right]\end{aligned} \tag{5.104}$$

which is exactly the logarithmic primary transformation we have previously obtained.
This shows that this set of logarithmic operators is well defined independently from the regularization scheme.

5.8 Logarithmic partner of stress tensor

As a starting point we will now consider the following operator

$$t_\epsilon(z,\bar z) = -\frac{6}{\pi\kappa}\int d^2w\frac{\mathcal{O}^{-2}_{+,\epsilon}-\mathcal{O}^{+2}_{-,\epsilon}}{(z-w)^{4-\epsilon}(\bar z-\bar w)^{-\epsilon}} \tag{5.105}$$

which is the shadow transform of the difference of the two celestial operators $\mathcal{O}^{-2}_{+,\epsilon} - \mathcal{O}^{+2}_{-,\epsilon}$, and has conformal weights $(2-\epsilon/2, -\epsilon/2)$. If we now consider the limit $\lim_{\epsilon\to 0}\epsilon t_\epsilon(z,\bar z)$ this reconstructs:

$$\lim_{\epsilon\to 0}\epsilon\, t_\epsilon(z,\bar z) = -T(z) \tag{5.106}$$

namely we see that the only IR poles related to the t_ϵ operator is the stress tensor $T(z)$. This implies that we can get a finite operator, as previously shown, by subtracting the pole such that:

$$t(z,\bar z) = \lim_{\epsilon\to 0}\left[t_\epsilon(z,\bar z) + \frac{1}{\epsilon}T(z)\right] \tag{5.107}$$

is a well defined operator.
Following the same reasoning as before for the previous regulated operators, we can see that t transforms precisely as the logarithmic partner of the stress tensor

$$t'(z',\bar z') = \lim_{\epsilon\to 0}\left[(\partial f)^{-2+\epsilon/2}(\bar\partial\bar f)^{\epsilon/2}t_\epsilon(z,\bar z) + \frac{1}{\epsilon}\,(\partial f)^{-2}T(z)\right] = \tag{5.108}$$

$$= (\partial f)^{-2}\Big[t(z,\bar z) - \log|\partial f| T(z)\Big] \tag{5.109}$$

This operator can be easily related to the $\mathcal{Q}$ operator previously defined, just expanding at first order in ϵ the shadow transform, giving us

$$t(z,\bar{z}) = \lim_{\epsilon\to 0}\left\{-\frac{6}{\pi\kappa}\int d^2w \frac{\mathcal{O}^{-2}_{+,\epsilon}-\mathcal{O}^{+2}_{-,\epsilon}}{(z-w)^{4-\epsilon}(\bar{z}-\bar{w})^{-\epsilon}} + \frac{1}{\epsilon}\frac{6}{\pi\kappa}\int d^2w \frac{s^{-2}_{+,0}-s^{+2}_{-,0}}{(z-w)^4}\right\} = \tag{5.110}$$

$$= \lim_{\epsilon\to 0}\left\{-\frac{6}{\pi\kappa}\int d^2w \frac{\mathcal{O}^{-2}_{+,\epsilon}-\mathcal{O}^{+2}_{-,\epsilon}-\log|z-w|^2\epsilon\left(\mathcal{O}^{-2}_{+,\epsilon}-\mathcal{O}^{+2}_{-,\epsilon}\right)}{(z-w)^4} + \frac{1}{\epsilon}\frac{6}{\pi\kappa}\int d^2w \frac{s^{-2}_{+,0}-s^{+2}_{-,0}}{(z-w)^4}\right\}$$

$$= -\frac{6}{\pi\kappa}\int d^2w \frac{1}{(z-w)^4}\left[\mathcal{N}^{(1)}_{zz} + \log|z-w|^2\mathcal{N}^{(1)}_{zz}\right]$$

where we have defined:

$$\mathcal{N}^{(1)}_{zz} = \frac{i\kappa}{4\pi}\left(\mathcal{Q}^{-2}_{+,0} - \mathcal{Q}^{+2}_{-,0}\right) \tag{5.111}$$

We see then that t is nothing else than the logarithmic shadow transform of the logarithmic partner of the subleading soft mode which is built starting from the regulated operators ${\mathcal{Q}_{\pm,0}}^{\mp 2}$.

5.8.1 Tt OPE computation

In section 5.6 we have seen that the OPE of the stress tensor with t takes a very specific form which also contains information about the value of the b parameter. In this section we want to explore if it is possible to reconstruct such OPE thanks to the aid of soft theorems.
To do so, we can consider an expression of the form:

$$\langle T(x)t(z,\bar{z})...\rangle = \lim_{\epsilon\to 0}\left\langle T(x)\left[t_\epsilon(z,\bar{z}) + \frac{1}{\epsilon}T(z)\right]...\right\rangle \tag{5.112}$$

where the dots represent a series of primary operators of weights $(h_i,\bar{h}_i)$ and located at $(z_i,\bar{z}_i)$.
We now focus on the first term that explicitly takes the form:

$$\langle T(x)t_\epsilon(z,\bar{z})...\rangle = -\frac{6}{\pi\kappa}\int d^2w \frac{1}{(z-w)^{4-\epsilon}(\bar{z}-\bar{w})^{-\epsilon}}\langle T(x)\mathcal{O}_{\epsilon,-2}(w,\bar{w})...\rangle = \tag{5.113}$$

$$= -\frac{6}{\pi\kappa}\int d^2w \frac{(\bar{z}-\bar{w})^{\epsilon}}{(z-w)^{4-\epsilon}}\left[\frac{1}{(x-w)^2}\left(\frac{\epsilon-2}{2}\right) + \frac{\partial_w}{(x-w)}\right]\langle \mathcal{O}_{\epsilon,-2}(w,\bar{w})...\rangle$$

where:

$$\mathcal{O}_{\epsilon,-2}(w,\bar{w}) = \mathcal{O}^{-2}_{+,\epsilon}(w,\bar{w}) - \mathcal{O}^{2}_{-,\epsilon}(w,\bar{w}) \tag{5.114}$$

is as short hand for the difference of weight $\left(-1+\frac{\epsilon}{2}, 1+\frac{\epsilon}{2}\right)$ graviton primaries, and we used the fact that $T(z)$ acts on $\mathcal{O}_{\epsilon,-2}$ as the stress energy tensor.
In the standard form of the Tt OPE we expect a term of the form $\partial t/(x-z)$, namely it must appear a contribution of the form:

$$\frac{1}{x-z}\langle \partial t_\epsilon(z,\bar{z})...\rangle = -\frac{6}{\pi\kappa}\frac{1}{x-z}\partial_z\int d^2w \frac{(\bar{z}-\bar{w})^{\epsilon}}{(z-w)^{4-\epsilon}}\langle \mathcal{O}_{\epsilon,-2}(w,\bar{w})...\rangle \tag{5.115}$$

$$= -\frac{6}{\pi\kappa}\frac{1}{x-z}\int d^2w \frac{(\bar{z}-\bar{w})^{\epsilon}}{(z-w)^{4-\epsilon}}\partial_w\langle \mathcal{O}_{\epsilon,-2}(w,\bar{w})...\rangle \tag{5.116}$$

Indeed we can find an analogous term inside (5.113) following a similar procedure as in [67].
If we introduce for brevity:

$$Z = z-w, \quad X = x-w \tag{5.117}$$

we can use the following algebraic identity:

$$\frac{1}{XZ^4} = \frac{1}{x-z}\left[\frac{1}{Z^4} - \frac{1}{Z^3 X}\right], \tag{5.118}$$

to obtain:

$$\begin{aligned}\langle T(x)t_\epsilon(z,\bar{z})...\rangle = &-\frac{6}{\pi\kappa}\int d^2w|Z|^{2\epsilon}\left[\left(\frac{\epsilon-2}{2}\right)\frac{1}{Z^4X^2} - \frac{1}{Z^3X}\frac{1}{x-z}\partial_w\right]\langle \mathcal{O}_{\epsilon,-2}(w,\bar{w})...\rangle + \\ &-\frac{6}{\pi\kappa}\frac{1}{x-z}\int d^2w\frac{\bar{Z}^\epsilon}{Z^{4-\epsilon}}\partial_w\langle \mathcal{O}_{\epsilon,-2}(w,\bar{w})...\rangle\end{aligned} \tag{5.119}$$

In the second line of (5.119) we can recognize the expression given in (5.116) for the derivative and so we can rewrite:

$$\langle T(x)t_\epsilon(z,\bar{z})...\rangle = \frac{1}{x-z}\langle \partial t_\epsilon(z,\bar{z})...\rangle - \frac{6}{\pi\kappa}\int d^2w\frac{|Z|^{2\epsilon}}{Z^3X}\left[\left(\frac{\epsilon-2}{2}\right)\frac{1}{ZX} - \frac{1}{x-z}\partial_w\right]\langle \mathcal{O}_{\epsilon,-2}(w,\bar{w})...\rangle$$

We now analyze the second part. At first we integrate by part to obtain:

$$\int d^2w\frac{|Z|^{2\epsilon}}{Z^3X}\left[\left(\frac{\epsilon-2}{2}\right)\frac{1}{ZX} - \frac{1}{x-z}\partial_w\right]\langle \mathcal{O}_{\epsilon,-2}(w,\bar{w})...\rangle \tag{5.120}$$

$$= \int d^2w|Z|^{2\epsilon}\left[\left(\frac{\epsilon-2}{2}\right)\frac{1}{Z^4X^2} + \frac{1}{x-z}\left(\frac{3-\epsilon}{Z^4X} + \frac{1}{Z^3X^2}\right)\right]\langle \mathcal{O}_{\epsilon,-2}(w,\bar{w})...\rangle \tag{5.121}$$

$$= -\int d^2w|Z|^{2\epsilon}\left[\left(\frac{2-\epsilon}{2}\right)\frac{1}{Z^4X^2} - \frac{1}{x-z}\left(\frac{3-\epsilon}{Z^4X} + \frac{1}{Z^3X^2}\right)\right]\langle \mathcal{O}_{\epsilon,-2}(w,\bar{w})...\rangle \tag{5.122}$$

then we unpack the algebraic expression using (5.118) and the following relation:

$$\frac{1}{Z^4X^2} = \frac{1}{(x-z)^2}\left[\frac{1}{Z^4} - \frac{1}{Z^3X} - \frac{x-z}{Z^3X^2}\right]. \tag{5.123}$$

The expression becomes:

$$\frac{2-\epsilon}{2}\frac{1}{Z^4X^2} - \frac{1}{x-z}\left(\frac{3-\epsilon}{Z^4X} + \frac{1}{Z^3X^2}\right) \tag{5.124}$$

$$= \frac{1}{Z^4X^2} - \frac{1}{x-z}\left(\frac{3}{Z^4X} + \frac{1}{Z^3X^2}\right) + \epsilon\left(\frac{1}{x-z}\frac{1}{Z^4X} - \frac{1}{2}\frac{1}{Z^4X^2}\right) \tag{5.125}$$

$$= \frac{1}{(x-z)^2}\left[\frac{1}{Z^4} - \frac{1}{Z^3X} - \frac{x-z}{Z^3X^2}\right] - \frac{3}{(x-z)^2}\left[\frac{1}{Z^4} - \frac{1}{Z^3X}\right] - \frac{1}{x-z}\frac{1}{Z^3X^2} + \frac{\epsilon}{2}\frac{X+Z}{x-z}\frac{1}{Z^4X^2}$$

$$= -\frac{2}{(x-z)^2}\frac{1}{Z^4} + \frac{2}{(z-x)^2}\frac{1}{Z^2X^2} + \frac{\epsilon}{2}\frac{X+Z}{x-z}\frac{1}{Z^4X^2} \tag{5.126}$$

The last summand can be further simplified:

$$\frac{1}{Z^3X^2} + \frac{1}{Z^4X} = \frac{1}{x-z}\left[\frac{1}{Z^4} - \frac{1}{Z^3X}\right] + \frac{1}{(x-z)^2}\left[\frac{1}{Z^3} - \frac{1}{Z^2X} - \frac{x-z}{Z^2X^2}\right] \tag{5.127}$$

$$= \frac{1}{x-z}\frac{1}{Z^4} - \frac{1}{(x-z)^2}\frac{1}{Z^3} + \frac{1}{(x-z)^3}\left[\frac{1}{Z^2} - \frac{1}{ZX}\right] + \frac{1}{(x-z)^2}\left[\frac{1}{Z^3} - \frac{1}{Z^2X} - \frac{x-z}{Z^2X^2}\right]$$

$$= \frac{1}{x-z}\frac{1}{Z^4} + \frac{1}{(x-z)^3}\left[\frac{1}{Z^2} - \frac{1}{ZX}\right] - \frac{1}{(x-z)^2}\frac{1}{Z^2X} - \frac{1}{(x-z)^3}\left[\frac{1}{Z^2} - \frac{1}{ZX} - \frac{x-z}{ZX^2}\right]$$

$$= \frac{1}{x-z}\frac{1}{Z^4} + \frac{1}{(x-z)^2}\left[\frac{1}{ZX^2} - \frac{1}{Z^2X}\right] = \frac{1}{x-z}\frac{1}{Z^4} - \frac{1}{(x-z)}\frac{1}{Z^2X^2} \tag{5.128}$$

so that the full polynomial expression becomes:

$$\frac{2-\epsilon}{2}\frac{1}{Z^4X^2}-\frac{1}{x-z}\left(\frac{3-\epsilon}{Z^4X}+\frac{1}{Z^3X^2}\right)=-\frac{2}{(x-z)^2}\frac{1}{Z^4}+\frac{\epsilon}{2}\frac{1}{(x-z)^2}\frac{1}{Z^4}+\frac{4-\epsilon}{2}\frac{1}{(x-z)^2}\frac{1}{Z^2X^2}$$

Plugging this expression back in (5.120) we get:

$$\langle T(x)t_\epsilon(z,\bar z)...\rangle=\frac{1}{x-z}\langle\partial t_\epsilon(z,\bar z)...\rangle-\frac{2}{(x-z)^2}\frac{6}{\pi\kappa}\int d^2w\frac{|Z|^{2\epsilon}}{Z^4}\langle\mathcal{O}_{\epsilon,-2}(w,\bar w)...\rangle \tag{5.129}$$

$$+\frac{\epsilon}{2}\frac{1}{(x-z)^2}\frac{6}{\pi\kappa}\int d^2w\frac{|Z|^{2\epsilon}}{Z^4}\langle\mathcal{O}_{\epsilon,-2}(w,\bar w)...\rangle \tag{5.130}$$

$$+\frac{4-\epsilon}{2}\frac{1}{(x-z)^2}\frac{6}{\pi\kappa}\int d^2w\frac{|Z|^{2\epsilon}}{Z^2X^2}\langle\mathcal{O}_{\epsilon,-2}(w,\bar w)...\rangle= \tag{5.131}$$

$$=\left\langle\left[\frac{1}{x-z}\partial t_\epsilon(z,\bar z)+\frac{2}{(x-z)^2}t_\epsilon(z,\bar z)+\frac{1}{(x-z)^2}\frac{1}{2}T(z)\right]...\right\rangle+ \tag{5.132}$$

$$+\frac{4-\epsilon}{2}\frac{1}{(x-z)^2}\frac{6}{\pi\kappa}\int d^2w\frac{|Z|^{2\epsilon}}{Z^2X^2}\langle\mathcal{O}_{\epsilon,-2}(w,\bar w)...\rangle \tag{5.133}$$

Now we can come back to (5.112) and focus on the second term, where we can substitute to $T(x)T(z)$ the stress energy tensor OPE with itself:

$$\langle T(x)T(z)...\rangle=\left\langle\left[\frac{2}{(x-z)^2}+\frac{\partial}{x-z}\right]T(z)...\right\rangle \tag{5.134}$$

and notice that together with the first two terms of (5.132) the expression:

$$\frac{1}{\epsilon}\langle T(x)T(z)...\rangle \tag{5.135}$$

rebuilds $t(z,\bar z)$ in the $\epsilon\to 0$ limit and we get:

$$\langle T(x)t(z,\bar z)...\rangle=\left\langle\left[\frac{1}{(x-z)^2}\left(2t(z,\bar z)+\frac{1}{2}T(z)\right)+\frac{1}{x-z}\partial t(z,\bar z)\right]...\right\rangle+ \tag{5.136}$$

$$+\lim_{\epsilon\to 0}\frac{4-\epsilon}{2}\frac{1}{(x-z)^2}\frac{6}{\pi\kappa}\int d^2w\frac{|Z|^{2\epsilon}}{Z^2X^2}\langle\mathcal{O}_{\epsilon,-2}(w,\bar w)...\rangle \tag{5.137}$$

As a nice consistency check we can already see that the first addendum on the RHS is compatible with the OPE of a log-primary with its partner. In the second term

$$g=\lim_{\epsilon\to 0}g_\epsilon,\quad g_\epsilon=\frac{4-\epsilon}{2}\frac{1}{(x-z)^2}\frac{6}{\pi\kappa}\int d^2w\frac{|Z|^{2\epsilon}}{Z^2X^2}\langle\mathcal{O}_{\epsilon,-2}(w,\bar w)\dots\rangle, \tag{5.138}$$

things become more complicated. To gather some information about the $(x-z)^n$ expansion of g we will use the following well known theorem:

$$g(x)=\sum_{n=0}^{\infty}\frac{g_n}{(x-z)^n},\quad g_n=\oint_z\frac{dx}{2\pi i}(x-z)^{n-1}g(x) \tag{5.139}$$

where the integral runs over an infinitesimal circle around z. We emphasize the fact that the limit $R\to 0$ of the radius of the circle must be taken after the $\epsilon\to 0$ limit as we are trying to extract the poles of the final $\epsilon=0$ operator.
With this condition and recalling that

$$g(x)=\lim_{\epsilon\to 0}\frac{4-\epsilon}{2}\frac{1}{(x-z)^2}\frac{6}{\pi\kappa}\int d^2w\frac{|Z|^{2\epsilon}}{Z^2X^2}\langle\mathcal{O}_{\epsilon,-2}(w,\bar w)...\rangle, \tag{5.140}$$

we can compute

$$g_n = \lim_{\epsilon\to 0}\frac{4-\epsilon}{2}\frac{6}{\pi\kappa}\int d^2w\frac{|Z|^{2\epsilon}}{Z^2}\langle\mathcal{O}_{\epsilon,-2}(w,\bar{w})...\rangle\oint_z\frac{dx}{2\pi i}\frac{(x-z)^{n-3}}{(x-w)^2} \tag{5.141}$$

Then using the residue theorem we evaluate the line integral:

$$I(x,n) = \oint_z\frac{dx}{2\pi i}\frac{(x-z)^{n-3}}{(x-w)^2} = \begin{cases}(n-3)(w-z)^{n-4}\mathbb{I}_{D_+}(w) & n=1,2\\ (n-3)(w-z)^{n-4}\mathbb{I}_{D_-}(w) & n\geq 3\end{cases} \tag{5.142}$$

where $\mathbb{I}$ is the indicator function, and $D_\pm$ is the region outside or inside the infinitesimal disk respectively. We conclude that g_n are given by:

$$g_n = \lim_{\epsilon\to 0}\frac{4-\epsilon}{2}(-1)^n(n-3)\frac{6}{\pi\kappa}\int_{D_+}d^2w\frac{|Z|^{2\epsilon}}{Z^{6-n}}\langle\mathcal{O}_{\epsilon,-2}(w,\bar{w})...\rangle, \quad n=1,2\,, \tag{5.143}$$

$$g_n = \lim_{\epsilon\to 0}\frac{4-\epsilon}{2}(-1)^n(n-3)\frac{6}{\pi\kappa}\int_{D_-}d^2w\frac{|Z|^{2\epsilon}}{Z^{6-n}}\langle\mathcal{O}_{\epsilon,-2}(w,\bar{w})...\rangle, \quad n\geq 3. \tag{5.144}$$

Let us start by ruling out the first two terms. We known from (5.108) how t transforms under an $SL(2,C)$ so we can impose:

$$\begin{aligned}\delta_{\mathcal{Y}}t &= \oint_z\frac{dx}{2\pi i}\mathcal{Y}(x)T(x)t(z,\bar{z}) = \oint_z\frac{dx}{2\pi i}\mathcal{Y}(x)\left[\frac{1}{(x-z)^2}\left(2t(z,\bar{z})-\frac{1}{2}T(z)\right)+\frac{1}{x-z}\partial t(z)\right]+\\ &-\lim_{\epsilon\to 0}\frac{4-\epsilon}{2}\oint_z\frac{dx}{2\pi i}\mathcal{Y}(x)\frac{1}{(x-z)^2}\frac{6}{\pi\kappa}\int d^2w\frac{|Z|^{2\epsilon}}{Z^2X^2}\langle\mathcal{O}_{\epsilon,-2}(w,\bar{w})...\rangle = \qquad (5.145)\\ &= (\mathcal{Y}\partial+2\partial\mathcal{Y})t(z,\bar{z})+\frac{\partial\mathcal{Y}}{2}T(z) \qquad (5.146)\end{aligned}$$

The first integral cancels the last line and we are left with:

$$\lim_{\epsilon\to 0}\frac{4-\epsilon}{2}\frac{6}{\pi\kappa}\int d^2w\frac{|Z|^{2\epsilon}}{Z^2}\langle\mathcal{O}_{\epsilon,-2}(w,\bar{w})...\rangle\oint_z\frac{dx}{2\pi i}\frac{\mathcal{Y}(x)}{(x-z)^2(x-w)^2}=0 \tag{5.147}$$

where $\mathcal{Y}(x)$ is at most a quadratic polynomial in x. If we choose:

$$\mathcal{Y}(x) = \mathcal{Y}_0+\mathcal{Y}_1(x-z)+\mathcal{Y}_2(x-z)^2 \tag{5.148}$$

this imposes:

$$\lim_{\epsilon\to 0}\frac{4-\epsilon}{2}\frac{6}{\pi\kappa}\int_{D_+}d^2w\frac{|Z|^{2\epsilon}}{Z^4}\langle\mathcal{O}_{\epsilon,-2}(w,\bar{w})...\rangle=0 \tag{5.149}$$

$$\lim_{\epsilon\to 0}\frac{4-\epsilon}{2}\frac{6}{\pi\kappa}\int_{D_+}d^2w\frac{|Z|^{2\epsilon}}{Z^5}\langle\mathcal{O}_{\epsilon,-2}(w,\bar{w})...\rangle=0 \tag{5.150}$$

namely $g_{1,2}$ are vanishing. Now we can analyze the additional terms: $n=3$ is identically vanishing while for $n>3$ we can make the following change of coordinates $w=re^{i\theta}-z$, $\bar{w}=re^{-i\theta}-\bar{z}$ and rewrite the integral as:

$$g_n = \lim_{\substack{\epsilon\to 0\\ R\to 0}}\frac{4-\epsilon}{2}(-1)^n(n-3)\frac{6}{\pi\kappa}\int_0^R dr\int_0^{2\pi}r^{n-5+2\epsilon}e^{i\theta(n-6)}\langle\mathcal{O}_{\epsilon,-2}(re^{i\theta}-z,re^{-i\theta}-\bar{z})...\rangle \tag{5.151}$$

For $n > 4$ we see that the total dimension in r is also positive so that in the limit $R \to 0$ we can assume that the integral is vanishing. However for $n = 4$ the total r-weight becomes $\Delta_r = \epsilon$ so that in the limit $\epsilon \to 0$ it vanishes and the integral could be non zero. The only remaining term is then:

$$g_4 = \lim_{\substack{\epsilon\to 0\\ R\to 0}} \frac{4-\epsilon}{2}\frac{6}{\pi\kappa}\int_0^R \frac{dr}{r}\int_0^{2\pi} r^{2\epsilon}e^{-2i\theta}\langle \mathcal{O}_{\epsilon,-2}(re^{i\theta}-z, re^{-i\theta}-\bar{z})...\rangle = \tag{5.152}$$

$$= \lim_{\epsilon\to 0}\frac{4-\epsilon}{2}\frac{6}{\pi\kappa}\int_{D_-} d^2w \frac{|Z|^{2\epsilon}}{Z^2}\langle \mathcal{O}_{\epsilon,-2}(w,\bar{w})...\rangle = \langle \mathcal{I}(z,\bar{z})...\rangle \tag{5.153}$$

This last operator can include both a contribution coming from the b-parameter present in logarithmic CFTs or an extra potential (0,0) operator, as it easy to prove that $\mathcal{I}(z,\bar{z})$ has (0,0) weight.

Then we can conclude that the OPE is given at leading order by the following expression:

$$T(x)t(z,\bar{z}) = \frac{\mathcal{I}(z,\bar{z})}{(x-z)^4} + \frac{1}{(x-z)^2}\left(2t(z,\bar{z}) + \frac{1}{2}T(z)\right) + \frac{1}{x-z}\partial t(z,\bar{z}) + \dots \tag{5.154}$$

We can get some information about the content of $\mathcal{I}$ by making some assumption about the structure of the Tt_ϵ two-point function. In particular we can work with an ansatz similar to what considered in [93] namely, if we consider the expression for the stress tensor:

$$T(z) = \lim_{\epsilon\to 0} T_\epsilon, \quad T_\epsilon(z,\bar{z}) = \epsilon\, t_\epsilon(z,\bar{z}) \tag{5.155}$$

we can work for an ansatz for the regulated stress tensor of the form

$$\langle T(x)T_\epsilon(z)\rangle = -\frac{\epsilon b}{2(x-z)^4} \tag{5.156}$$

So that in the the limit $\epsilon \to 0$ we get the expected vanishing result. If we combine this ansatz with the expression for t we see that:

$$\begin{aligned}\langle T(x)t(z,\bar{z})\rangle &= \lim_{\epsilon\to 0}\langle T(x)t_\epsilon(z,\bar{z})\rangle - \lim_{\epsilon\to 0}\frac{1}{\epsilon}\langle T(x)T(z)\rangle = \\ &= -\lim_{\epsilon\to 0}\frac{1}{\epsilon}\langle T(x)T_\epsilon(z,\bar{z})\rangle = \frac{b}{2(x-z)^4}\end{aligned} \tag{5.157}$$

and we get that the subleading ϵ parameter will precisely carry the value of the b parameter, which will be present in the $\mathcal{I}$ operator defined above. This consideration is interesting as it connects our construction of the logarithmic partner of the stress tensor with the procedure carried out in [93], as we will obtain the same value of b in both constructions.

Overall we find an expression for the b parameter that might be computable:

$$b = 2\lim_{\substack{z\to 0\\ \epsilon\to 0}}\frac{z^4}{\epsilon}\langle T(z)T_\epsilon(0)\rangle \tag{5.158}$$

where T_ϵ is defined as:

$$T_\epsilon(z) = \frac{6}{\pi k}\int d^2w \frac{1}{(z-w)^4}\epsilon\mathcal{O}_{\epsilon,-2}(w,\bar{w}) \tag{5.159}$$

To determine if such a value ϵb is present in the stress tensor OPEs we could study the subleading order in ϵ of an amplitude

$$\langle T(z)T_\epsilon(w)\mathcal{O}_1 \dots \mathcal{O}_n\rangle \tag{5.160}$$

helping the computation considering a double soft limit.

As a final remark, we recall that in this construction we have treated ϵ as a mere soft regulator, in a dimensional regularization scheme. However the entire procedure can be also carried out considering a soft Λ cut-off regulator, which can be interpreted as the cosmological constant of an AdS spacetime which turns flat in the limit $\Lambda \to 0$. An interesting future exploration could be to compute expression (5.158) starting from its regulated form in AdS, as a non-vanishing value would give us a better understanding of how the holographic structure is carried from AdS to Flat space in the vanishing cosmological constant limit.

5.9 Supertranslation log doublet

Having commented on how it is possible to reconstruct a logarithmic doublet (T, t) we now turn back to the study of log CFT structures in the context of the soft sector of CCFT associated to supertranslation.

In particular we will show how the goldstone supertranslation current defined in (3.87) as the descendant of the supertranslation goldstone (3.103), has to be regulated in the IR, and as a consequence turns into a logarithmic primary field.

The primary field associated to it corresponds to a $\log u$ piece in the radiative data C_{zz}, at future null infinity $\mathscr{I}^+$. This $\log u$ behavior gives rise to the log CFT doublet associated with the IR-regulated Goldstone supertranslation current. The following sections will be mostly based on the [1], and original unpublished material.

5.9.1 Supertranslation Goldstone and its partner

Let us consider the following combinations of soft Fourier modes

$$\begin{aligned} \mathcal{B}_{zz}(z,\bar{z}) &= \frac{i\kappa}{8\pi^2}\lim_{\omega\to 0}\omega(a_+(\omega,z,\bar{z}) - a_-^\dagger(\omega,z,\bar{z})),\\ \mathcal{B}_{\bar{z}\bar{z}}(z,\bar{z}) &= \frac{i\kappa}{8\pi^2}\lim_{\omega\to 0}\omega(a_-(\omega,z,\bar{z}) - a_+^\dagger(\omega,z,\bar{z}))\,. \end{aligned} \tag{5.161}$$

As we will see below, from the gravitational phase space point of view, these modes correspond to the presence of a $\log u$ piece in the radiative data at the corners of $\mathscr{I}^+$,

$$C_{zz}(u,z,\bar{z}) \overset{u\to+\infty}{\sim} 2\mathcal{B}_{zz}(z,\bar{z})\,\log u + \dots \tag{5.162}$$

This mode and its analog for QED have been studied in [194, 195][2]. It was argued there that, while absent in the classical theory, such a $\log u$ mode turns out to be non-vanishing in the quantum theory. Most interestingly, such a term for QED was shown to be associated with the τ^{-1} decay of Coulombic modes as they approach timelike infinity, and the quantization of

[2]The QED analog of $\mathcal{B}_{zz}$ was denoted $\overset{\ln}{A}_z$ in [194].

(5.161) leads to an asymptotic charge which reproduces the quantum part of Sahoo-Sen's logarithmic[3] corrections [202] to the subleading soft photon theorem [194,195]. It was also argued in [195] that including the term (5.162) does not amount to introducing a new independent mode in the quantum system, as $\mathcal{B}_{zz}$ is fixed in terms of the classical free data.

The relation between soft operators of the form (5.161) and the $\log u$ fall-off at $u \to +\infty$ was first presented in [194]; we give below an alternative proof of this relationship (which holds for gravity but is also readily adapted to the QED case). The aim is thus to show that $\mathcal{B}_{zz}$ can be expressed in terms of creation and annihilation operators as in (5.161), under the assumption that the fall-offs of the shear C_{zz} (and thus the news tensor) are given by

$$\begin{aligned}
&C_{zz}(u,z,\bar{z}) \overset{u\to+\infty}{=} uN^{vac}_{zz}(z) + C^{+}_{zz}(z,\bar{z}) + 2\mathcal{B}_{zz}(z,\bar{z})\log u + o(1/u),\\
&N_{zz}(u,z,\bar{z}) \overset{u\to+\infty}{=} N^{vac}_{zz}(z) + \frac{2}{u}\mathcal{B}_{zz}(z,\bar{z}) + o(1/u^2),\\
&C_{zz}(u,z,\bar{z}) \overset{u\to-\infty}{=} uN^{vac}_{zz}(z) + C^{-}_{zz}(z,\bar{z}) + o(1/u),\\
&N_{zz}(u,z,\bar{z}) \overset{u\to-\infty}{=} N^{vac}_{zz}(z) + o(1/u^2)\,,
\end{aligned} \tag{5.163}$$

where the Goldstone currents can be written as $C^{\pm}_{zz} = -2\mathcal{D}_z^2 C^{\pm}$ with $\mathcal{D}_z$ being the superrotation-covariant derivative [156,203] and $C^{\pm}$ the supertranslation Goldstone boson at $\mathscr{I}^{\pm}$. Notice that we included for generality in the above expressions the 'vacuum news' N^{vac}_{zz} considered in [156,160,161,203], but what follows does not depend on whether this term is present or not. If we consider the expression of the gravitational shear:

$$C_{zz}(u,z,\bar{z}) = \frac{\kappa}{8i\pi^2}\int_0^\infty d\omega\left[a_+e^{-i\omega u} - a_-^\dagger e^{i\omega u}\right] \tag{5.164}$$

which is just equation (2.27) specialized for $s = 2$, the first step is to analyze its Fourier transform

$$\begin{aligned}
\tilde{f}(\omega,z,\bar{z}) &= \int_{-\infty}^{+\infty} du e^{i\omega u-|u|\epsilon} C_{zz}(u,z,\bar{z})\\
&= \frac{\kappa}{4i\pi}\int_0^\infty d\xi\,\left[a_+\delta(\omega-\xi) - a_-^\dagger\delta(\omega+\xi)\right] = \frac{\kappa}{4i\pi}\left[a_+(\omega)\theta(\omega) - a_-^\dagger(-\omega)\theta(-\omega)\right]\,,
\end{aligned} \tag{5.165}$$

where ϵ is a regulator meant to be small. By inspecting the relation above, we can see that showing (5.161) is equivalent to prove the following equality

$$\mathcal{B}_{zz} = -\frac{1}{2\pi}\lim_{\omega\to0^+}\omega\left(\tilde{f}(\omega)+\tilde{f}(-\omega)\right)\,. \tag{5.166}$$

In the remaining part of this section, we are going to show how to prove (5.166). We start by

[3] See also [196–201] for other works on logarithmic soft theorems.

analysing $\tilde{f}(\omega)$ which can be rewritten using integration by parts as

$$\begin{aligned}\tilde{f}(\omega) =& \int_{-\infty}^{0} du\, e^{i(\omega - i\epsilon)u} C_{zz} + \int_{0}^{+\infty} du\, e^{i(\omega + i\epsilon)u} C_{zz} \\ =& \frac{1}{i\omega - \epsilon}\int_{0}^{\infty} du\, \frac{d}{du}\left[e^{i(\omega+i\epsilon)u} C_{zz}\right] - \frac{1}{i\omega - \epsilon}\int_{0}^{\infty} du\, e^{i(\omega+i\epsilon)u} N_{zz} \\ &+ \frac{1}{i\omega + \epsilon}\int_{-\infty}^{0} du\, \frac{d}{du}\left[e^{i(\omega-i\epsilon)u} C_{zz}\right] - \frac{1}{i\omega + \epsilon}\int_{-\infty}^{0} du\, e^{i(\omega-i\epsilon)u} N_{zz} \\ =& 2\pi\delta(\omega) C_{zz}(0) + \frac{i}{\omega - i\epsilon}\int_{-\infty}^{0} du\, e^{i(\omega-i\epsilon)u} N_{zz} + \frac{i}{\omega + i\epsilon}\int_{0}^{+\infty} du\, e^{i(\omega+i\epsilon)u} N_{zz}\,,\end{aligned} \tag{5.167}$$

where we used the shear fall-off conditions to see that the total derivative will always be suppressed as u goes to infinity. We can then study the behavior of

$$\begin{aligned}\omega\left[\tilde{f}(\omega) + \tilde{f}(-\omega)\right] =& \; 4\pi\omega\delta(\omega) C_{zz}(0) \\ &+ \frac{i\omega}{\omega - i\epsilon}\int_{-\infty}^{0} du\, e^{i(\omega-i\epsilon)u} N_{zz} + \frac{i\omega}{\omega + i\epsilon}\int_{0}^{+\infty} du\, e^{i(\omega+i\epsilon)u} N_{zz} \\ &- \frac{i\omega}{\omega + i\epsilon}\int_{-\infty}^{0} du\, e^{-i(\omega+i\epsilon)u} N_{zz} - \frac{i\omega}{\omega - i\epsilon}\int_{0}^{+\infty} du\, e^{-i(\omega-i\epsilon)u} N_{zz}\end{aligned} \tag{5.168}$$

in the limit $\omega \to 0$. We would like to firstly analyse the following terms in (5.168)

$$\frac{i\omega}{\omega - i\epsilon}\int_{-\infty}^{0} du\, e^{i(\omega-i\epsilon)u} N_{zz} - \frac{i\omega}{\omega + i\epsilon}\int_{-\infty}^{0} du\, e^{-i(\omega+i\epsilon)u} N_{zz}. \tag{5.169}$$

In order to do so, we split the integrals into two regions $(-\infty, -1/\Lambda) \cup (-1/\Lambda, 0]$, where Λ is a small but positive finite number. We can use (5.163) and the asymptotic behavior of the news tensor to evaluate the integrals, thus allowing (5.169) to be written in the region $(-\infty, -1/\Lambda)$ as

$$\begin{aligned}&\frac{i\omega N_{zz}^{vac}}{\omega - i\epsilon}\int_{-\infty}^{-1/\Lambda} du\, e^{i(\omega-i\epsilon)u} - \frac{i\omega N_{zz}^{vac}}{\omega + i\epsilon}\int_{-\infty}^{-1/\Lambda} du\, e^{-i(\omega+i\epsilon)u} \\ &= N_{zz}^{vac}\omega\left[\frac{e^{-\frac{\epsilon+i\omega}{\Lambda}}}{(\omega - i\epsilon)^2} - \frac{e^{-\frac{\epsilon-i\omega}{\Lambda}}}{(\omega + i\epsilon)^2}\right].\end{aligned} \tag{5.170}$$

If we expand for small $|\omega \pm i\epsilon|$, we get

$$\begin{aligned}&N_{zz}^{vac}\omega\left[\frac{1}{(\omega - i\epsilon)^2} - \frac{1}{(\omega + i\epsilon)^2} + \frac{i\Lambda}{\omega - i\epsilon} - \frac{i\Lambda}{\omega + i\epsilon}\right] + o(\omega) \\ &= 2\pi N_{zz}^{vac}\omega\left[-i\delta'(\omega) + \Lambda\delta(\omega)\right] + o(\omega)\,.\end{aligned} \tag{5.171}$$

This is vanishing in the limit of small frequencies so we do not get contributions in the interval $(-\infty, -1/\Lambda)$. We then can focus on the region $(-1/\Lambda, 0]$. If we consider the news tensor to be regular inside $(-1/\Lambda, 0]$ then we can expand the exponential for $|\omega \pm i\epsilon| \ll \Lambda$ and get

$$\begin{aligned}&\frac{i\omega}{\omega - i\epsilon}\int_{-1/\Lambda}^{0} du\, e^{i(\omega-i\epsilon)u} N_{zz} - \frac{i\omega}{\omega + i\epsilon}\int_{-1/\Lambda}^{0} du\, e^{-i(\omega+i\epsilon)u} N_{zz} \\ &= -\left(\frac{i\omega}{\omega + i\epsilon} - \frac{i\omega}{\omega - i\epsilon}\right)\int_{-1/\Lambda}^{0} du\, N_{zz} - 2\omega\int_{-1/\Lambda}^{0} du\, u N_{zz} + o(\omega^2) \\ &= -2\pi\omega\delta(\omega)\int_{-1/\Lambda}^{0} du\, N_{zz} - 2\omega\int_{-1/\Lambda}^{0} du\, u N_{zz} + o(\omega^2)\,,\end{aligned} \tag{5.172}$$

which is also vanishing in the limit $\omega \to 0^+$. We conclude then that (5.168) receives no contribution from the integral between $(-\infty, 0]$ for small frequencies.

We now analyse the other terms in (5.168) in the same way

$$\frac{i\omega}{\omega+i\epsilon}\int_0^{+\infty} du\, e^{i(\omega+i\epsilon)u} N_{zz} - \frac{i\omega}{\omega-i\epsilon}\int_0^{+\infty} du\, e^{-i(\omega-i\epsilon)u} N_{zz}. \tag{5.173}$$

Splitting again the integration region into $[0, 1/\Lambda) \cup (1/\Lambda, +\infty)$, we can perform the same steps. The main difference is that in the regime $(1/\Lambda, +\infty)$ the news tensor has a different asymptotic behaviour, which makes

$$\begin{aligned}
&\frac{i\omega}{\omega+i\epsilon}\int_{1/\Lambda}^{+\infty} du\, e^{i(\omega+i\epsilon)u} N_{zz} - \frac{i\omega}{\omega-i\epsilon}\int_{1/\Lambda}^{+\infty} du\, e^{-i(\omega-i\epsilon)u} N_{zz} \\
&= \frac{i\omega}{\omega+i\epsilon}\int_{1/\Lambda}^{+\infty} du\, e^{i(\omega+i\epsilon)u}\left(N_{zz}^{vac} + \frac{2}{u}\mathcal{B}_{zz}\right) - \frac{i\omega}{\omega-i\epsilon}\int_{1/\Lambda}^{+\infty} du\, e^{-i(\omega-i\epsilon)u}\left(N_{zz}^{vac} + \frac{2}{u}\mathcal{B}_{zz}\right) \\
&= \frac{i\omega}{\omega+i\epsilon}\left[\frac{e^{-\frac{\epsilon-i\omega}{\Lambda}}}{\epsilon-i\omega}N_{zz}^{vac} + 2\mathcal{B}_{zz}\Gamma\left(0, \frac{\epsilon-i\omega}{\Lambda}\right)\right] - \frac{i\omega}{\omega-i\epsilon}\left[\frac{e^{-\frac{\epsilon+i\omega}{\Lambda}}}{\epsilon+i\omega}N_{zz}^{vac} + 2\mathcal{B}_{zz}\Gamma\left(0, \frac{\epsilon+i\omega}{\Lambda}\right)\right]
\end{aligned} \tag{5.174}$$

where $\Gamma(x, y)$ is the incomplete gamma function. Expanding the expression above for small $|\omega \pm i\epsilon|$ we obtain

$$\begin{aligned}
&-2\pi\omega N_{zz}^{vac}\left(i\delta'(\omega) + \Lambda\delta(\omega)\right) \\
&+ 2\omega\mathcal{B}_{zz}\left\{\frac{i}{\omega+i\epsilon}(-\gamma_E - \log\Lambda(\epsilon-i\omega)) - \frac{i}{\omega-i\epsilon}(-\gamma_E - \log\Lambda(\epsilon+i\omega))\right\} + o(\omega) \\
&= -2\pi\omega N_{zz}^{vac}\left(i\delta'(\omega) + \Lambda\delta(\omega)\right) - (\gamma_E + \log\Lambda)4\pi\mathcal{B}_{zz}\omega\delta(\omega) \\
&- 2i\omega\mathcal{B}_{zz}\left[\frac{\log(\epsilon-i\omega)}{\omega+i\epsilon} - \frac{\log(\epsilon+i\omega)}{\omega+i\epsilon}\right] + o(\omega).
\end{aligned} \tag{5.175}$$

By inspecting the regions with positive and negative frequencies, it is possible to show that

$$-i\left[\frac{\log(\epsilon-i\omega)}{\omega+i\epsilon} - \frac{\log(\epsilon+i\omega)}{\omega+i\epsilon}\right] = -\pi P\frac{1}{|\omega|} + 2\pi P\frac{1}{|\omega|}\theta(-|\omega|) - 2\pi\delta(\omega)P\log|\omega| \tag{5.176}$$

which implies that

$$\begin{aligned}
&\frac{i\omega}{\omega+i\epsilon}\int_{1/\Lambda}^{+\infty} du\, e^{i(\omega+i\epsilon)u} N_{zz} - \frac{i\omega}{\omega-i\epsilon}\int_{1/\Lambda}^{+\infty} du\, e^{-i(\omega-i\epsilon)u} N_{zz} \\
&= -2\pi\omega N_{zz}^{vac}\left(i\delta'(\omega) + \Lambda\delta(\omega)\right) - \gamma_E 4\pi\mathcal{B}_{zz}\omega\delta(\omega) \\
&- 2\mathcal{B}_{zz}\pi\omega P\frac{1}{|\omega|} + 4\pi\mathcal{B}_{zz}\omega P\frac{1}{|\omega|}\theta(-|\omega|) - 4\pi\mathcal{B}_{zz}\omega\delta(\omega)P\log|\Lambda\omega|\,.
\end{aligned} \tag{5.177}$$

The only non vanishing term in the limit $\omega \to 0^+$ is the first one in the third line[4], namely

$$\lim_{\omega\to 0}\left[\frac{i\omega}{\omega+i\epsilon}\int_{1/\Lambda}^{+\infty} du\, e^{i(\omega+i\epsilon)u} N_{zz} - \frac{i\omega}{\omega-i\epsilon}\int_{1/\Lambda}^{+\infty} du\, e^{-i(\omega-i\epsilon)u} N_{zz}\right] = -\pi\mathcal{B}_{zz}\,. \tag{5.178}$$

As in the other case, no contributions are coming from the region $[0, 1/\Lambda)$.

Combining all the pieces, we obtain that

$$\mathcal{B}_{zz} = -\frac{1}{2\pi}\lim_{\omega\to 0^+}\omega\left(\tilde{f}(\omega) + \tilde{f}(-\omega)\right) = \frac{i\kappa}{8\pi^2}\lim_{\omega\to 0}\omega\left(a_+(\omega) - a_-^\dagger(\omega)\right)\,. \tag{5.179}$$

[4] $\lim_{\omega\to 0}\theta(-|\omega|) = 0$

5.9.2 Transformation properties

Thanks to the fall-off conditions we can already prove that the goldstone mode $\mathscr{C}_{zz}$, will transform as a logarithmic primary field. To do so we recall that under an the infinitesimal conformal transformation the shear transforms as a $(\frac{3}{2},-\frac{1}{2})$ 'quasi-conformal Carrollian primary' (3.86)[5]

$$\delta C_{zz} = \left(\mathcal{Y}\partial + \bar{\mathcal{Y}}\bar{\partial} + \frac{3}{2}\partial\mathcal{Y} - \frac{1}{2}\bar{\partial}\bar{\mathcal{Y}}\right) C_{zz} + \frac{u}{2}(\partial\mathcal{Y} + \partial\bar{\mathcal{Y}})N_{zz} - u\partial^3\mathcal{Y}\,. \tag{5.180}$$

This corresponds to a finite transformation of the form

$$C'_{zz}(u',z',\bar{z}') = (\partial f)^{-\frac{3}{2}}(\bar{\partial}\bar{f})^{\frac{1}{2}}C(u,z,\bar{z}) + (\partial f)^{\frac{1}{2}}(\bar{\partial}\bar{f})^{\frac{1}{2}}\,uS(f,z)\,, \tag{5.181}$$

where $z' = f(z), \bar{z}' = \bar{f}(\bar{z}), u' = |\partial f|u$ and $S(f,z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$ denotes the Schwarzian derivative.

From this property, we can then deduce the transformations of the asymptotic components of C_{zz} in the limit $u \to +\infty$ under a superrotation. Using (5.163), we have

$$\begin{aligned} C'_{zz}(u',z',\bar{z}') &= |\partial f|uN^{vac\prime}_{zz} + C^{+\prime}_{zz} + 2\log u\,\mathcal{B}'_{zz} + 2\log|\partial f|\,\mathcal{B}'_{zz} + \widetilde{C}'_{zz}(u',z',\bar{z}') \\ C_{zz}(u,z,\bar{z}) &= uN^{vac}_{zz} + C^{+}_{zz} + 2\log u\,\mathcal{B}_{zz} + \widetilde{C}_{zz}(u,z,\bar{z})\,, \end{aligned} \tag{5.182}$$

where $\widetilde{C}_{zz}$ denotes the $o(u^{-1})$ terms. Using (5.181) we can match the two expressions:

$$\begin{aligned} &|\partial f|uN^{vac\prime}_{zz} + C^{+\prime}_{zz} + 2\log u\,\mathcal{B}'_{zz} + 2\log|\partial f|\,\mathcal{B}'_{zz} + \widetilde{C}'_{zz}(u',z',\bar{z}') \\ =&(\partial f)^{-\frac{3}{2}}(\bar{\partial}\bar{f})^{\frac{1}{2}}\left[uN^{vac}_{zz} + C^{+}_{zz} + 2\log u\,\mathcal{B}_{zz} + \widetilde{C}_{zz}(u,z,\bar{z})\right] + (\partial f)^{\frac{1}{2}}(\bar{\partial}\bar{f})^{\frac{1}{2}}S(f,z)u\,, \end{aligned} \tag{5.183}$$

implying the following finite transformation laws

$$\begin{aligned} &N^{vac\prime}_{zz} = (\partial f)^{-2}N^{vac}_{zz} + S(f,z) \\ &C^{+\prime}_{zz} = (\partial f)^{-\frac{3}{2}}(\bar{\partial}\bar{f})^{\frac{1}{2}}\left(C^{+}_{zz} - 2\log|\partial f|\mathcal{B}_{zz}\right) \\ &\mathcal{B}_{zz}{}' = (\partial f)^{-\frac{3}{2}}(\bar{\partial}\bar{f})^{\frac{1}{2}}\mathcal{B}_{zz} \\ &\widetilde{C}'_{zz}(u',z',\bar{z}') = (\partial f)^{-\frac{3}{2}}(\bar{\partial}\bar{f})^{\frac{1}{2}}\widetilde{C}_{zz}(u,z,\bar{z}). \end{aligned} \tag{5.184}$$

These can be written in infinitesimal form as

$$\begin{aligned} \delta N^{vac}_{zz} &= (\mathcal{Y}\partial + 2\partial\mathcal{Y})N^{vac}_{zz} - \partial^3\mathcal{Y} \\ \delta C^{+}_{zz} &= \left(\mathcal{Y}\partial + \bar{\mathcal{Y}}\bar{\partial} + \frac{3}{2}\partial\mathcal{Y} - \frac{1}{2}\bar{\partial}\bar{\mathcal{Y}}\right) C^{+}_{zz} + \frac{\partial\mathcal{Y} + \bar{\partial}\bar{\mathcal{Y}}}{2}2\mathcal{B}_{zz} \\ \delta\mathcal{B}_{zz} &= \left(\mathcal{Y}\partial + \bar{\mathcal{Y}}\bar{\partial} + \frac{3}{2}\partial\mathcal{Y} - \frac{1}{2}\bar{\partial}\bar{\mathcal{Y}}\right) \mathcal{B}_{zz}\,. \end{aligned} \tag{5.185}$$

Hence, comparing with the log CFT doublet transformation laws (5.60), we can see that the presence of $\mathcal{B}_{zz}$ modifies the transformation property of the Goldstone current C^{+}_{zz} to that of a logarithmic primary field of weights $\left(\frac{3}{2},-\frac{1}{2}\right)$ in a doublet with $\mathcal{B}_{zz}$. On the contrary, the transformation properties for C^{-}_{zz} remain the one of a conformal primary as in the $u \to -\infty$

[5] We use the terminology introduced in [110] for tensors living at $\mathscr{I}$; see also [164, 204, 205].

asymptotic region there is no $\log u$ contribution[6]. Defining the supertranslation Goldstone current $\mathscr{C}_{zz}$ as the linear combination

$$\mathscr{C}_{zz} = \frac{C^+_{zz} + C^-_{zz}}{2}\,, \tag{5.186}$$

we thus have

$$\mathscr{C}'_{zz} = (\partial f)^{-\frac{3}{2}}(\bar\partial \bar f)^{\frac{1}{2}}\left(\mathscr{C}_{zz} - \log|\partial f|\mathcal{B}_{zz}\right)\,. \tag{5.187}$$

5.9.3 Two-point functions

We will now argue that the logarithmic primary field (5.186) can be interpreted as an IR-regulated Goldstone current. Indeed, we will show below that the following expression

$$\begin{aligned}\mathscr{C}_{zz} =&\frac{i\kappa}{8\pi^2}\Big[\int_0^\infty d\omega\,\omega^{2\epsilon}(a_+(\omega,z,\bar z) - a^\dagger_-(\omega,z,\bar z)) \\ &+\int\frac{d^{2+2\epsilon}y}{\pi}\frac{\bar z-\bar y}{(z-y)^3}|z-y|^{-2\epsilon}\int_0^\infty d\omega\,\omega^{\epsilon}(a_-(\omega,y,\bar y) - a^\dagger_+(\omega,y,\bar y))\Big]\end{aligned} \tag{5.188}$$

transforms as a logarithmic primary field. In the above, we used dimensional regularization with $d = 2+2\epsilon$ and one can see that (5.188) consists of the sum of two primary operators, with conformal weights $\left(\frac{3}{2}+\epsilon, -\frac{1}{2}+\epsilon\right)$ and $\left(\frac{3}{2}+\frac{\epsilon}{2}, -\frac{1}{2}+\frac{\epsilon}{2}\right)$ respectively, which turn out to be the same in the limit $\epsilon \to 0$. The primary partner of $\mathscr{C}_{zz}$ can be written as

$$\begin{aligned}\mathscr{B}_{zz} &= \left[\mathcal{B}_{zz} + \int\frac{d^2y}{\pi}\frac{\bar z-\bar y}{(z-y)^3}\mathcal{B}_{\bar y\bar y}\right] \\ &= \frac{i\kappa}{8\pi^2}\lim_{\omega\to 0}\omega\left[(a_+(\omega,z,\bar z) - a^\dagger_-(\omega,z,\bar z)) + \int\frac{d^2y}{\pi}\frac{\bar z-\bar y}{(z-y)^3}\left(a_-(\omega,y,\bar y) - a^\dagger_+(\omega,y,\bar y)\right)\right]\end{aligned} \tag{5.189}$$

where we used (5.161).

Let us check that the transformation property of $\mathscr{C}_{zz}$, in the limit $\epsilon \to 0$, does correspond to the one of a logarithmic field with $\mathscr{B}_{zz}$ as primary partner. Under a conformal transformation, the field $\mathscr{C}_{zz}$ transforms as[7]

$$\begin{aligned}\mathscr{C}'_{zz} = \frac{i\kappa}{8\pi^2}(\partial f)^{-\frac{3}{2}}(\bar\partial\bar f)^{\frac{1}{2}}\Big[&|\partial f|^{-2\epsilon}\int_0^\infty d\omega\,\omega^{2\epsilon}(a_+ - a^\dagger_-) + \\ &|\partial f|^{-\epsilon}\int\frac{d^{2+2\epsilon}y}{\pi}\frac{\bar z-\bar y}{(z-y)^3}|z-y|^{-2\epsilon}\int_0^\infty d\omega\,\omega^{\epsilon}(a_- - a^\dagger_+)\Big]\,.\end{aligned} \tag{5.190}$$

Now let us focus on the first term, and expand the expression for ϵ small,

$$|\partial f|^{-2\epsilon}\int_0^\infty d\omega\,\omega^{2\epsilon}(a_+ - a^\dagger_-) = \int_0^\infty d\omega\,\omega^{2\epsilon}(a_+ - a^\dagger_-) - \log|\partial f|2\epsilon\int_0^\infty d\omega\,\omega^{2\epsilon}(a_+ - a^\dagger_-) + o(\epsilon)\,. \tag{5.191}$$

In the second term, we notice the appearance of a $\log|\partial f|$ term multiplied by the operator

$$2\epsilon\int_0^\infty d\omega\,\omega^{2\epsilon}(a_+ - a^\dagger_-). \tag{5.192}$$

[6]This can be relaxed easily to include them as well.

[7]We drop the explicit dependence of the arguments for notation simplicity.

By splitting the integral in (5.192) into two regions, using a small cut-off λ:

$$2\epsilon\int_0^\lambda d\omega\,\omega^{2\epsilon}(a_+ - a_-^\dagger) + 2\epsilon\int_\lambda^\infty d\omega\,\omega^{2\epsilon}(a_+ - a_-^\dagger)\,, \tag{5.193}$$

we can expand the first integral in a series around $\omega = 0$. Using the soft expansion

$$(a_+(\omega,z,\bar{z}) - a_-^\dagger(\omega,z,\bar{z})) \simeq \frac{1}{\omega}\lim_{\xi\to 0}\xi(a_+(\xi,z,\bar{z}) - a_-^\dagger(\xi,z,\bar{z})) + \sum_{n=0}^{\infty}\omega^n c_n(z,\bar{z}) \tag{5.194}$$

and assuming that this expression is convergent in a small radius around $\omega = 0$, we can plug it in the first term of (5.193) to get

$$\begin{aligned}&2\epsilon\int_0^\lambda d\omega\,\omega^{2\epsilon-1}\lim_{\xi\to 0}\xi(a_+(\xi,z,\bar{z}) - a_-^\dagger(\xi,z,\bar{z})) + \sum_{n=0}^{\infty}2\epsilon\int_0^\lambda d\omega\,\omega^{2\epsilon+n}c_n(z,\bar{z})\\ &= \lambda^\epsilon\lim_{\xi\to 0}\xi(a_+ - a_-^\dagger) + \sum_{n=0}^{\infty}\frac{2\epsilon}{2\epsilon+1+n}\lambda^{2\epsilon+n+1}c_n(z,\bar{z}) = \lim_{\xi\to 0}\xi(a_+ - a_-^\dagger) + o(\epsilon).\end{aligned} \tag{5.195}$$

This implies that, as no other $1/\epsilon$ poles will come from the UV region $(\lambda,+\infty)$, (5.192) reduces to

$$2\epsilon\int_0^\infty d\omega\,\omega^{2\epsilon}(a_+ - a_-^\dagger) = \lim_{\xi\to 0}\xi(a_+ - a_-^\dagger) + o(\epsilon)\,. \tag{5.196}$$

Notice that this is equivalent to writing

$$\lim_{\Delta\to 1}(\Delta-1)\int_0^\infty d\omega\,\omega^{\Delta-1}(a_+ - a_-^\dagger) = \lim_{\xi\to 0}\xi(a_+ - a_-^\dagger)\,, \tag{5.197}$$

which is the statement that soft modes can be extracted as poles of Mellin-transformed operators [67, 206]. Using this result we can then write

$$|\partial f|^{-2\epsilon}\int_0^\infty d\omega\,\omega^{2\epsilon}(a_+ - a_-^\dagger) = \int_0^\infty d\omega\,\omega^{2\epsilon}(a_+ - a_-^\dagger) - \log|\partial f|\lim_{\xi\to 0}\xi(a_+ - a_-^\dagger) + o(\epsilon)\,. \tag{5.198}$$

Plugging this back into (5.190) and using a similar argument for the second term, we then have

$$\begin{aligned}\mathscr{C}'_{zz} = (\partial f)^{-\frac{3}{2}}(\bar\partial\bar f)^{\frac{1}{2}}\Bigg[&\mathscr{C}_{zz} -\\ &\log|\partial f|\frac{i\kappa}{8\pi^2}\left(\lim_{\xi\to 0}\xi(a_+ - a_-^\dagger) + \int\frac{d^{2+2\epsilon}y}{\pi}\frac{\bar z - \bar y}{(z-y)^3}|z-y|^{-2\epsilon}\lim_{\xi\to 0}\xi(a_- - a_+^\dagger)\right) + o(\epsilon)\Bigg]\,.\end{aligned} \tag{5.199}$$

This implies that for $\epsilon\to 0$ and using (5.189), we are left with

$$\begin{aligned}\mathscr{C}'_{zz} &= (\partial f)^{-\frac{3}{2}}(\bar\partial\bar f)^{\frac{1}{2}}\left[\mathscr{C}_{zz} - \log|\partial f|\frac{i\kappa}{8\pi^2}\lim_{\xi\to 0}\xi\left((a_+ - a_-^\dagger) + \int\frac{d^2y}{\pi}\frac{\bar z - \bar y}{(z-y)^3}(a_- - a_+^\dagger)\right)\right]\\ &= (\partial f)^{-\frac{3}{2}}(\bar\partial\bar f)^{\frac{1}{2}}\left[\mathscr{C}_{zz} - \log|\partial f|\mathscr{B}_{zz}\right],\end{aligned} \tag{5.200}$$

which is the transformation property of a logarithmic primary of weights $(\frac{3}{2},-\frac{1}{2})$ partnered with $\mathscr{B}_{zz}$. Notice that this proof can be seen as an extension of of what previously proven for the operators of the form (5.97) for regulated shadow transform operators, as the origin of the logarithmic structure is basically the same. In an upcoming section we will provide an alternative definition of $\mathscr{C}_{zz}$ related to derivative operators of the form $\partial_\Delta \mathcal{O}^{(\Delta)}$, with $\mathcal{O}^{(\Delta)}$ a primary operator of conformal dimension Δ, and show how any operator of the form (5.97) be related to such derivative operators

Let us now turn to the two-point functions of the log pair. Computing first the $\langle \mathscr{B}_{zz}\mathscr{B}_{ww}\rangle$ two-point function, we have

$$\begin{aligned}\langle \mathscr{B}_{zz}\mathscr{B}_{ww}\rangle &= -\frac{\kappa^2}{64\pi^4}\Bigg\langle \lim_{\omega\to 0}\omega\left[a_+ - a_-^\dagger + \int \frac{d^2x}{\pi}\frac{\bar z-\bar x}{(z-x)^3}\left(a_- - a_+^\dagger\right)\right] \\ &\qquad\times \lim_{\xi\to 0}\xi\left[a_+ - a_-^\dagger + \int \frac{d^2y}{\pi}\frac{\bar y-\bar w}{(y-w)^3}(a_- - a_+^\dagger)\right]\Bigg\rangle \\ &= -\frac{\kappa^2}{64\pi^5}\Bigg[\int d^2y\frac{\bar w-\bar y}{(w-y)^3}\lim_{\omega,\xi\to 0}\omega\xi\langle (a_- - a_+^\dagger)(a_+ - a_-^\dagger)\rangle \\ &\qquad + \int d^2x\frac{\bar z-\bar x}{(z-x)^3}\lim_{\omega,\xi\to 0}\omega\xi\langle (a_+ - a_-^\dagger)(a_- - a_+^\dagger)\rangle\Bigg] \\ &= \frac{\kappa^2}{2\pi^2}\frac{\bar z-\bar w}{(z-w)^3}\lim_{\omega\to 0}\omega\delta(\omega) = 0\,,\end{aligned} \tag{5.201}$$

which is expected from the primary field in a logarithmic doublet. We then find

$$\begin{aligned}\langle \mathscr{C}_{zz}\mathscr{B}_{ww}\rangle &= -\frac{\kappa^2}{64\pi^4}\Bigg\langle \int_0^\infty d\omega\left[\omega^{2\epsilon}\left(a_+ - a_-^\dagger\right) + \int \frac{d^{2+2\epsilon}x}{\pi}\frac{\bar z-\bar x}{(z-x)^3}|z-x|^{-2\epsilon}\omega^\epsilon\left(a_- - a_+^\dagger\right)\right]\times \\ &\qquad\times \lim_{\xi\to 0}\xi\left[a_+ - a_-^\dagger + \int \frac{d^{2+2\epsilon}y}{\pi}\frac{\bar w-\bar y}{(w-y)^3}(a_- - a_+^\dagger)\right]\Bigg\rangle \\ &= -\frac{\kappa^2}{64\pi^5}\Bigg[\int d^{2+2\epsilon}y\frac{\bar w-\bar y}{(w-y)^3}\int_0^\infty d\omega\omega^{2\epsilon}\lim_{\xi\to 0}\xi\langle (a_- - a_+^\dagger)(a_+ - a_-^\dagger)\rangle \\ &\qquad + \int d^{2+2\epsilon}x\frac{\bar z-\bar x}{(z-x)^3}|z-x|^{-2\epsilon}\int_0^\infty d\omega\omega^\epsilon\lim_{\xi\to 0}\xi\langle (a_+ - a_-^\dagger)(a_- - a_+^\dagger)\rangle\Bigg]\end{aligned} \tag{5.202}$$

$$= \frac{\kappa^2}{4\pi^2}\left[\frac{\bar z-\bar w}{(z-w)^3} + \frac{\bar z-\bar w}{(z-w)^3}|z-w|^{-2\epsilon}\lim_{\xi\to 0}\xi^{-\epsilon}\right]. \tag{5.203}$$

Hence, using that as $\epsilon\to 0$, $\lim_{\xi\to 0}\xi^{-\epsilon}=1$ we conclude

$$\langle \mathscr{C}_{zz}\mathscr{B}_{ww}\rangle = \frac{\kappa^2}{2\pi^2}\frac{\bar z-\bar w}{(z-w)^3}\,. \tag{5.204}$$

Finally, we turn to the two-point function $\langle \mathscr{C}_{zz}\mathscr{C}_{ww}\rangle$,

$$\begin{aligned}\langle \mathscr{C}_{zz}\mathscr{C}_{ww}\rangle &= -\frac{\kappa^2}{64\pi^5}\int_0^\infty d\omega d\xi\,\omega^{2\epsilon}\xi^\epsilon\int d^2x\frac{\bar z-\bar x}{(z-x)^3}|z-x|^{-2\epsilon}\langle (a_- - a_+^\dagger)(a_+ - a_-^\dagger)\rangle \\ &\quad - \frac{\kappa^2}{64\pi^5}\int_0^\infty d\omega d\xi\,\omega^\epsilon\xi^{2\epsilon}\int d^2x\frac{\bar w-\bar y}{(w-y)^3}|w-y|^{-2\epsilon}\langle (a_+ - a_-^\dagger)(a_- - a_+^\dagger)\rangle \\ &= \frac{\kappa^2}{2\pi^2}\frac{\Lambda^\epsilon}{\epsilon}\frac{\bar z-\bar w}{(z-w)^3}|z-w|^{-2\epsilon} \\ &= \frac{\kappa^2}{2\pi^2}\frac{\bar z-\bar w}{(z-w)^3}\left(\frac{1}{\epsilon} - \log|z-w|^2\right) + o(\epsilon)\,,\end{aligned} \tag{5.205}$$

where, in the last step, we have expanded for $\epsilon \to 0$ and neglected the UV divergence. Combining this last equation with (5.201) and (5.204), we thus recognise the logarithmic doublet structure of the form (5.62) with $\tilde{k}_{\mathcal{O}} = \epsilon^{-1}$ and $k_{\mathcal{O}} = \frac{\kappa^2}{2\pi^2}$. As discussed there, we know that we should not expect the divergence in $\tilde{k}_{\mathcal{O}}$ to be physical as it can always be reabsorbed with a shift $\mathscr{C}_{zz} \to \mathscr{C}'_{zz} = \mathscr{C}_{zz} - \frac{\tilde{k}_{\mathcal{O}}}{2}\mathscr{B}_{zz}$. As a result, we arrive at the following IR-finite two-point functions of a logarithmic doublet of weights $(\frac{3}{2}, -\frac{1}{2})$, namely

$$\begin{aligned} \langle \mathscr{C}'_{zz}\mathscr{C}'_{ww}\rangle &= -\frac{\kappa^2}{2\pi^2}\frac{\bar{z}-\bar{w}}{(z-w)^3}\log|z-w|^2 \\ \langle \mathscr{C}'_{zz}\mathscr{B}_{ww}\rangle &= \frac{\kappa^2}{2\pi^2}\frac{\bar{z}-\bar{w}}{(z-w)^3} \\ \langle \mathscr{B}_{zz}\mathscr{B}_{ww}\rangle &= 0\,. \end{aligned} \tag{5.206}$$

This is a neat result, as we are obtaining finite two point functions, which exactly correspond to those of logarithmic partners (5.62) as expected.

To sum up in this section we have shown that the presence of the operator (5.161) induces a logarithmic CFT doublet structure in the soft sector of celestial CFT. While we focused on the case of gravity, our analysis carries along for the QED case as well as for the field ϕ_z. The quantization of this operator was shown in [194, 195] to source the fields at timelike infinity and to induce the quantum part of Sahoo-Sen's logarithmic soft photon theorem. It was also pointed out there that the $\log u$ quantum mode is also closely related to the asymptotic dressed operators considered in the Faddeev-Kulish construction [129]. Recently, logarithmic corrections to the subleading soft graviton theorem were derived from the Ward identity associated with superrotation charge conservation [207]. It is interesting to note that the derivation there did not require the inclusion of a $\log u$ operator; the fall-off conditions considered there for the gravitational shear at $u \to \infty$ were only relaxed so as to encompass gravitational tail effects (namely $\sim u^{-1}$). The object that played a central role in the derivation of [207] (see also [91, 167]) was rather the supertranslation Goldstone two-point function. The latter appeared because the superrotation charge involves an extra contribution accounting for the dressing of operators by the Goldstone boson, which is obviously related to the Faddeev-Kulish construction as well. It would thus be interesting to clarify the connection between different derivations of logarithmic corrections to soft theorems [91, 167, 194, 195, 207, 208] based on symmetry principles. From what we have discussed, logarithmic CFT structures can be expected to play an important role once accounting for (quantum) loop corrections. This is also in line with the analyses of [92, 94] which have unveiled the presence of logarithmic CFT operators in IR-finite gluon OPEs in CCFT at one-loop.

5.9.4 Liouville theory in CCFT

As discussed in [156, 160, 161, 203, 209] (see also eq. (5.163)), the inclusion of superrotations in the soft gravitational phase space involves the vacuum news tensor N^{vac}_{zz}, a $(2,0)$ mode which has the transformation properties of a $2d$ stress-tensor,

$$\delta_{\mathcal{Y}} N^{\text{vac}}_{zz} = (\mathcal{Y}\partial + 2\partial\mathcal{Y})\, N^{\text{vac}}_{zz} - \partial^3\mathcal{Y}\,. \tag{5.207}$$

It is related to the so-called 'superboost scalar field' Φ [209], which satisfies

$$D_A D^A \Phi(z,\bar{z}) = \mathring{R} \quad \to \quad \Phi(z,\bar{z}) = \varphi(z) + \bar{\varphi}(\bar{z}) - \log\Omega(z,\bar{z}), \tag{5.208}$$

where D_A is the covariant derivative with respect to the $2d$ metric $\mathring{q}_{AB}dx^A dx^B = 2\Omega(z,\bar{z})\,dzd\bar{z}$ of Ricci tensor $\mathring{R}$. The vacuum news tensor can be expressed in terms of the holomorphic part of Φ as

$$N^{\text{vac}}_{zz} = \frac{1}{2}(D_z\varphi)^2 - D_z^2\varphi\,. \tag{5.209}$$

The superboost field $\varphi(z)$ was used in [93] to construct a composite operator which was argued to form a logarithmic pair with the celestial stress tensor. It has also been noted in [159, 209] that N^{vac}_{zz} is precisely proportional to the stress tensor of a Euclidean Liouville theory defined by the action

$$S = \frac{\gamma^2}{4\pi}\int d^2x\sqrt{\mathring{q}}\left[\frac{1}{2}D_A\Phi D^A\Phi + \mathring{R}\Phi\right], \tag{5.210}$$

where we introduced a parameter γ. Notice that since the Liouville cosmological constant Λ is zero in (5.210), the action reduces to that of a Coulomb gas. In particular, we can make a field redefinition $\Phi = \phi/\gamma$ to bring the action in the form

$$S = \frac{1}{4\pi}\int d^2x\sqrt{\mathring{q}}\left[\frac{1}{2}D_A\phi D^A\phi + \gamma\mathring{R}\phi\right]. \tag{5.211}$$

Following the notation of [177], we see that this is a Coulomb gas with central charge

$$c = 1 + 12\gamma^2 = 1 - 24\alpha_0^2\,, \tag{5.212}$$

where we introduced α_0 such that $\gamma = i\sqrt{2}\alpha_0$ to make contact with standard notation.

We want to analyze the properties of the operator content of this theory, which is usually represented by vertex operators

$$V_\alpha =: e^{i\sqrt{2}\alpha\phi} : \tag{5.213}$$

of conformal weight

$$h_\alpha = \alpha(\alpha - 2\alpha_0) = \alpha(\alpha + i\sqrt{2}\gamma)\,. \tag{5.214}$$

For the moment, we will be agnostic about the value of γ but we remark that, if γ is purely imaginary, then vertex operators with real conformal weights will be those with $\alpha \in \mathbb{R}$, while for $\gamma \in \mathbb{R}$ we will have to consider purely imaginary α. In this case, vertex operators turn into those usually considered in Liouville theory, $V_\lambda(z) =: e^{\lambda\phi} : (z)$ with $\lambda \in \mathbb{R}$.

We can now discuss the appearance of logarithmic operators in Coulomb gas models, as studied in [210]. We first recall that, in the Coulomb gas model, only correlators which satisfy the neutrality condition $\sum \alpha_i = 2\alpha_0$ are non-vanishing; in particular, the only non-zero two-point functions will be those of the form $\langle V_\alpha V_{2\alpha_0-\alpha}\rangle$ [177]. From (5.214), we see that the operators V_α and $V_{2\alpha_0-\alpha}$ have the same conformal dimension and when $\alpha = \alpha_0$, the two operators degenerate to the same one. It turns out that, for this value of α, there exists a second operator,

$$V_P =: \phi\, e^{i\sqrt{2}\alpha_0\phi} := \frac{1}{i\sqrt{2}}\frac{dV_\alpha}{d\alpha}\bigg|_{\alpha=\alpha_0}, \tag{5.215}$$

which is also a primary field of the same dimension than V_{α_0}, namely $h_{\alpha_0} = -\alpha_0^2$. In Liouville theory, where α_0 is imaginary, V_P is known as the 'puncture operator' [211]. Notice that $\frac{dV_\alpha}{d\alpha}$ will be a primary field only if $\alpha = \alpha_0$, while for generic values of α it will turn into a logarithmic primary. This is easy to see from the relation between h_α and α as we can rewrite

$$\frac{dV_\alpha}{d\alpha} = 2(\alpha - \alpha_0)\frac{dV_\alpha}{dh} = 4(\alpha - \alpha_0)\frac{dV_\alpha}{d\Delta}\,, \tag{5.216}$$

which shows that for $\alpha \neq \alpha_0$, $\frac{dV_\alpha}{d\alpha}$ is related to operators of the form $\partial_\Delta \mathcal{O}^{(\Delta)}$ (with $\mathcal{O}^{(\Delta)}$ a primary) which are known to be logarithmic[8] [184, 185, 212]. While the puncture operator (5.215) is an ordinary primary field, its inclusion in the spectrum will give rise to logarithmic correlation functions [210, 213, 214]. For this reason, it is sometimes referred to as a 'pre-logarithmic' operator, namely a primary operator whose OPEs with other primaries will contain logarithmic operators. An interesting case in this context is the one of a gravitationally dressed CFT where $\alpha_0^2 < 1$ and the Coulomb gas is used to couple another a CFT with $2d$ gravity [215]. If we denote by Φ_h the operators of the CFT coupled to gravity, the dressed operators will take the form

$$\mathcal{O}(z) =: V_\alpha \Phi_h : \tag{5.217}$$

with conformal weight equal to $h_\alpha + h = 1$. This means that, if $h = 1 + \alpha_0^2$, then $h_\alpha = -\alpha_0^2$ and one thus needs to also include the puncture operator to dress the fields, which will give rise to a logarithmic structure [210, 215]. If the superrotation Liouville field can be interpreted as some gravitational dressing (see [84, 176, 216] for subleading conformally soft dressings), then the above discussion would allow to understand the appearance of logarithmic operators in CCFT as a result of the inclusion in the spectrum of the Liouville puncture operator. It would also be interesting to explore possible connections with the recent works highlighting the use of Liouville theory in celestial holography [97–99, 217].

5.10 Relation between $\mathscr{C}_{zz}$ and $\partial_\Delta \mathcal{O}^{(\Delta)}$

In section 5.9.2, we have proved that $\mathscr{C}_{zz}$ behaves like a logarithmic primary field. The aim of this section is to relate the latter to logarithmic fields of the form $\partial_\Delta \mathcal{O}^{(\Delta)}$. Such derivative operators were already considered in celestial CFT in the case of loop corrected gluon OPEs [92, 94]. The presence of the $\mathscr{B}_{zz}$ operator may give rise to their presence also for graviton operators.

To connect with the $\mathscr{C}_{zz}$ operator we define:

$$A_\pm = \frac{i\kappa}{8\pi^2}\lim_{\Delta\to 1}\partial_\Delta\left[(\Delta-1)\left(\mathcal{O}^{\pm 2}_{+,\Delta} - \mathcal{O}^{\mp 2}_{-,\Delta}\right)\right] = \frac{i\kappa}{8\pi^2}\lim_{\delta\to 0}\partial_\delta\left[\delta\left(\mathcal{O}^{\pm 2}_{+,1+\delta} - \mathcal{O}^{\mp 2}_{-,1+\delta}\right)\right] \tag{5.218}$$

[8]This is easy to check from the transformation properties of primary fields

$$\begin{aligned}\partial_\Delta \mathcal{O}^{(\Delta)\prime}(z', \bar{z}') &= \left(\frac{\partial f}{\bar{\partial}\bar{f}}\right)^{-J}\partial_\Delta\left(|\partial f|^{-\Delta}\mathcal{O}^{(\Delta)}(z,\bar{z})\right)\\ &= (\partial f)^{-h}(\bar{\partial}\bar{f})^{-\bar{h}}\left[\partial_\Delta \mathcal{O}^{(\Delta)} + \log|\partial f|\mathcal{O}^{(\Delta)}\right].\end{aligned}$$

. Notice that from equation (5.197) we know that $(\Delta-1)\left(\mathcal{O}^{\pm 2}_{+,\Delta}-\mathcal{O}^{\mp 2}_{-,\Delta}\right)$ is regular in the limit $\Delta\to 1$ so that A_+ is finite. We are interested at first in the transformation properties of $A_\pm$ under conformal transformations. Using the transformation properties of the Mellin ladder operators we can write

$$
\begin{aligned}
A'(z',\bar z')_\pm &= \frac{i\kappa}{8\pi^2}(\partial f)^{-\frac{3}{2}}(\bar\partial\bar f)^{\frac{1}{2}}\lim_{\delta\to 1}\partial_\delta\left[|\partial f|^{-\delta}\delta\left(\mathcal{O}^{\pm 2}_{+,1+\delta}-\mathcal{O}^{\mp 2}_{-,1+\delta}\right)\right]\\
&= \frac{i\kappa}{8\pi^2}(\partial f)^{-\frac{3}{2}}(\bar\partial\bar f)^{\frac{1}{2}}\lim_{\delta\to 1}\left[|\partial f|^{-\delta}\partial_\delta\left(\delta\left(\mathcal{O}^{\pm 2}_{+,1+\delta}-\mathcal{O}^{\mp 2}_{-,1+\delta}\right)\right)-\log|\partial f|\delta\left(\mathcal{O}^{\pm 2}_{+,1+\delta}-\mathcal{O}^{\mp 2}_{-,1+\delta}\right)\right]\\
&= (\partial f)^{-\frac{3}{2}}(\bar\partial\bar f)^{\frac{1}{2}}\left[A_\pm-\log|\partial f|\frac{i\kappa}{8\pi^2}\lim_{\delta\to 0}\delta\left(\mathcal{O}^{\pm 2}_{+,1+\delta}-\mathcal{O}^{\mp 2}_{-,1+\delta}\right)\right]\\
&= (\partial f)^{-\frac{3}{2}}(\bar\partial\bar f)^{\frac{1}{2}}\left[A_\pm-\log|\partial f|B_\pm\right]\,,
\end{aligned}
$$

where as A_+ is finite in the $\delta\to 0$ limit we get no other contribution from $|\partial f|^{-\delta}$. Then we see that A_+ is a logarithmic field in a doublet with the primary

$$
B_+=\frac{i\kappa}{8\pi^2}\lim_{\delta\to 0}\delta\left(\mathcal{O}^{+2}_{+,1+\delta}-\mathcal{O}^{-2}_{-,1+\delta}\right) \tag{5.219}
$$

However from (5.197) we also know that

$$
B_+=\frac{i\kappa}{8\pi^2}\lim_{\delta\to 0}\delta\left(\mathcal{O}^{+2}_{+,1+\delta}-\mathcal{O}^{-2}_{-,1+\delta}\right)=\frac{i\kappa}{8\pi^2}\lim_{\omega\to 0}\omega(a_+-a_-^\dagger)=\mathcal{B}_{zz}\,, \tag{5.220}
$$

which proves that the derivative operator A_+ is in a logarithmic doublet with $\mathcal{B}_{zz}$. We can then relate A_+ with C^+_{zz}. Then taking into consideration also A_- and the log-shadow transform we can also suggest an alternative definition for $\mathscr{C}_{zz}$:

$$
\mathscr{C}_{zz}\sim\left[A_+-\int\frac{d^2y}{\pi}\frac{\bar z-\bar y}{(z-y)^3}\left(A_-+\log|z-y|^2\mathcal{B}_{\bar y\bar y}\right)\right]\,. \tag{5.221}
$$

Using the properties of the log shadow it is easy to see that this operator will also be logarithmic and in a doublet with $\mathscr{B}_{zz}$. Notice that if $\mathscr{B}_{zz}=0$ the operator defined in (5.221) is also vanishing, which underlines the fact that the logarithmic structure is present only if $\mathscr{B}_{zz}$ is non vanishing.

This discussion allowed us to bridge $\partial_\Delta\mathcal{O}^{(\Delta)}$ with $\mathscr{C}_{zz}$. We now want to make some additional considerations on the A_+ operator. To do so we will at first massage its expression by explicitly writing the definition of the derivative:

$$
A_+=\lim_{\delta\to 0}\lim_{\epsilon\to 0}\frac{(\delta+\epsilon)\left(\mathcal{O}^{+2}_{+,1+\delta+\epsilon}-\mathcal{O}^{-2}_{-,1+\delta+\epsilon}\right)-\delta\left(\mathcal{O}^{+2}_{+,1+\delta}-\mathcal{O}^{-2}_{-,1+\delta}\right)}{\epsilon}. \tag{5.222}
$$

Under the assumption that the limits are well defined, we can exchange their ordering to find

$$
A_+=\frac{i\kappa}{8\pi^2}\lim_{\epsilon\to 0}\left[\left(\mathcal{O}^{+2}_{+,1+\epsilon}-\mathcal{O}^{-2}_{-,1+\epsilon}\right)+\frac{1}{\epsilon}\lim_{\delta\to 0}\delta\left(\mathcal{O}^{+2}_{+,1+\delta+\epsilon}-\mathcal{O}^{-2}_{-,1+\delta+\epsilon}\right)-\frac{1}{\epsilon}\lim_{\delta\to 0}\delta\left(\mathcal{O}^{+2}_{+,1+\delta}-\mathcal{O}^{-2}_{-,1+\delta}\right)\right]\,.
$$

For ϵ finite, the second term vanishes for $\delta\to 0$, while the last term is identical to $\mathcal{B}_{zz}$ and we are left with

$$
\begin{aligned}
A_+&=\lim_{\epsilon\to 0}\left[\frac{i\kappa}{8\pi^2}\left(\mathcal{O}^{+2}_{+,1+\epsilon}-\mathcal{O}^{-2}_{-,1+\epsilon}\right)-\frac{1}{\epsilon}\mathcal{B}_{zz}\right]\\
&=\lim_{\epsilon\to 0}\frac{i\kappa}{8\pi^2}\left[\int_0^\infty\omega^\epsilon(a_+-a_-^\dagger)-\frac{1}{\epsilon}\lim_{\omega\to 0}\omega(a_+-a_-^\dagger)\right]\\
&=\frac{i\kappa}{8\pi^2}\left[\mathcal{Q}^{+2}_{+,1}-\mathcal{Q}^{-2}_{-,1}\right]
\end{aligned} \tag{5.223}
$$

This expression shows that A_+ is basically a linear combinantion of the regulated operators defined in (5.97). Notice that regulated operators of the form (5.223) are precisely those used to write the dim-reg expression for $\mathscr{C}_{zz}$ (5.188), which tightens the relation between $\mathscr{C}_{zz}$ and A_+.
Notice that computation that we have carried out around $\Delta = 1$ in (5.223) it can be done for any conformal weight $\Delta = 1-n$, and relates $\mathcal{Q}$ to $\partial_\Delta \mathcal{O}_\Delta$ as

$$\mathcal{Q}^l_{\pm,1-n}(z,\bar{z}) = \lim_{\Delta\to 1-n} \partial_\Delta \left[(\Delta + n - 1)\, \mathcal{Q}^l_{\pm,\Delta}(z,\bar{z}) \right] \tag{5.224}$$

5.11 Remark - free scalar constructions

We will now discuss an analogy between the logarithmic structure found in the doublet $(\mathscr{C}_{zz}, \mathscr{B}_{zz})$ with the case of a free scalar in $2d$ CFT when considering the divergent part of the correlator associated to the scalar zero-mode.
Then we will discuss how a logarithmic structure can also be made precise for the free scalar, by considering a gauging of the shift symmetry. We will find that the free scalar sits in a logarithmic doublet with a constant mode, related to the scalar zero mode.

5.11.1 The example of the free scalar field

We begin this section with a recap of the treatment of the free scalar field in dimensional regularization. The action for the free scalar field ϕ in d-dimension is given by

$$S = \frac{1}{2}\int d^d x\, \partial_\mu \phi(x) \partial^\mu \phi(x). \tag{5.225}$$

The two-point correlation functions of the scalar field takes then the following form

$$\langle \phi(x)\phi(y)\rangle = \pi^{\frac{d}{2}-1}\Gamma\left(\frac{d-2}{2}\right)\frac{1}{|x-y|^{d-2}}. \tag{5.226}$$

In dimensional regularization near $2d$, we take $d = 2+2\epsilon$ and expanding (5.226) for $\epsilon \to 0$ we get

$$\langle \phi(z,\bar{z})\phi(w,\bar{w})\rangle = \frac{1}{\epsilon} - \log|z-w|^2\mu^2 + o(\epsilon) \tag{5.227}$$

where μ is the mass scale which enforces the argument to be dimensionless. Notice that, in this setup, there is a divergent part followed by the usual logarithmic two-point function. In standard expressions (see [177]), the divergent term is typically left inside the logarithm or not written explicitly. In order to extract the divergent part of the correlator, we can define the field $\varphi = \epsilon\phi$ such that the two-point correlator becomes

$$\begin{aligned} \langle \phi(z,\bar{z})\varphi(w,\bar{w})\rangle &= 1 - \epsilon \log|z-w|^2\mu^2 \xrightarrow{\epsilon\to 0} 1 \\ \langle \varphi(z,\bar{z})\varphi(w,\bar{w})\rangle &= \epsilon - \epsilon^2 \log|z-w|^2\mu^2 \xrightarrow{\epsilon\to 0} 0. \end{aligned} \tag{5.228}$$

From the expressions above, it is apparent that φ extracts the divergent part induced by the presence of the zero mode of ϕ. To make it even more explicit, it is possible to expand ϕ in modes (see e.g. [177, 218])

$$\phi(z,\bar{z}) = \chi - i\phi_0 \log|z|^2 + i\sum_{n\neq 0}\left(\frac{1}{n}\phi_n z^{-n} + \frac{1}{n}\bar{\phi}_n \bar{z}^{-n}\right), \tag{5.229}$$

with the commutation rules $[\chi, \phi_0] = i$ and $[\phi_n, \phi_m] = [\bar{\phi}_n, \bar{\phi}_m] = n\delta_{n+m,0}$, $[\phi_n, \bar{\phi}_m] = 0$. To enforce (5.227) the zero mode χ has to satisfy

$$\langle\chi\chi\rangle = \frac{1}{\epsilon}, \tag{5.230}$$

while all the other modes give a finite contribution acting on the vacuum. We can conclude then that in the limit $\epsilon \to 0$ the operator φ extracts the contribution of the zero mode as

$$\varphi(z,\bar{z}) = \epsilon\phi(z,\bar{z}) = \epsilon\chi + \epsilon\left(-i\phi_0\log|z|^2 + i\sum_{n\neq 0}\left(\frac{1}{n}\phi_n z^{-n} + \frac{1}{n}\bar{\phi}_n\bar{z}^{-n}\right)\right) \xrightarrow{\epsilon\to 0} \epsilon\chi, \tag{5.231}$$

and all the other operators evaluate to zero in this limit.

The example of the free scalar field is suggestive of the behavior of the Goldstone mode. To make it manifest, we define the normalised mode

$$\widehat{C}_\epsilon = \sqrt{\epsilon}C_\epsilon \tag{5.232}$$

such that at leading order in the ϵ the correlator is

$$\langle\widehat{C}_\epsilon(z,\bar{z})\widehat{C}_\epsilon(w,\bar{w})\rangle = \frac{\kappa^2}{16\pi^2}|z-w|^2\left(\frac{1}{\epsilon} + \log\left(|z-w|^2\mu^2\right)\right). \tag{5.233}$$

This structure resembles the two-point function of the free scalar. Following the same reasoning as in (5.231), we can then relate the following operator $\widehat{B}_\epsilon = \epsilon\widehat{C}_\epsilon$ with the $\widehat{C}_\epsilon$ zero mode. Moreover, we can use (5.197) to rewrite $\widehat{B}_\epsilon$ in the limit $\epsilon \to 0$ as follows

$$\widehat{B}_\epsilon(z,\bar{z}) = \frac{\sqrt{\epsilon}}{2}\int\frac{d^{2+2\epsilon}w}{2\pi}\left[\frac{z-w}{\bar{z}-\bar{w}}\mathcal{B}_{ww} + \frac{z-w}{\bar{z}-\bar{w}}\mathcal{B}_{ww}\right], \tag{5.234}$$

then making it clear that the mode $\mathcal{B}_{zz}$ is needed in the construction of the zero mode $\widehat{B}_\epsilon$ in this regularization scheme.

5.11.2 Gauge fixing the free scalar

We will now consider the free massless scalar but in the situation where we gauge the global shift symmetry $\delta_\xi\phi = \xi$, such that the partition function is defined as [177]:

$$Z = \int\frac{\mathcal{D}\phi}{\mathrm{Vol}(\mathbb{R})}e^{-S[\phi]}, \quad S = \frac{1}{8\pi}\int d^2x\sqrt{g}\partial_\mu\phi\partial^\mu\phi \tag{5.235}$$

We will consider the case where we are on a differentiable compact $2d$ manifold M, with coordinates such that the metric is fixed to be euclidean conformally flat, namely:

$$ds^2 = 2e^{2\sigma}dzd\bar{z}, \quad R = -4e^{-2\sigma}\partial\bar{\partial}\sigma \tag{5.236}$$

where R is the Ricci scalar. Moreover we will assume, at least in first place that we are not on a surface with vanishing Euler characteristic:

$$\chi(M) = \frac{1}{4\pi}\int d^2x\sqrt{g}R = -\frac{1}{\pi}\int d^2x\,\partial\bar{\partial}\sigma(x,\bar{x}) \tag{5.237}$$

We want to gauge fix the action using the Faddev-Popov formalism which fixes the path integral to be:

$$Z = \int \mathcal{D}\phi\mathcal{D}\varphi\mathcal{D}\Psi\mathcal{D}\overline{\Psi}e^{-S_{FP}[\phi,\varphi,\overline{\Psi},\Psi]}, \quad S_{FP} = S[\phi] + i\varphi F[\phi] + \overline{\Psi}\frac{\delta F[\phi^{\xi}]}{\delta\xi}\Psi \tag{5.238}$$

where φ is the Lagrange multiplier for the gauge fixing $F[\phi]=0$, $\overline{\Psi},\Psi$ are the ghost fields and $\frac{\delta F[\phi^{\xi}]}{\delta\xi}$ is the variation of the gauge fixing with respect to the gauge group parameter. In our case we have to gauge fix a global shift symmetry, so that the ghosts and the Lagrange multiplier do not need to be fields and the path integral over those variables will reduce to ordinary integrals. As gauge fixing function we consider the following:

$$F[\phi] = \frac{1}{4\pi}\int d^2x\sqrt{g}R\phi \tag{5.239}$$

that, under the hypothesis of non vanishing $\chi(M)$, transforms non trivially under the shift and can be considered as a valid fixing:

$$\delta_{\xi}F[\phi] = \xi\chi(M) \neq 0. \tag{5.240}$$

The gauged fixed action then becomes:

$$S_{FP} = \int d^2z\left[\frac{1}{4\pi}\partial\phi\bar{\partial}\phi + \frac{i}{4\pi}\sqrt{g}\varphi\phi R + \frac{1}{4\pi}\sqrt{g}\bar{\Psi}\Psi R\right] \tag{5.241}$$

The ghost sector is completely decoupled from the scalar field so we can just integrate it out and get an extra overall $\chi(M)$ factor:

$$Z = \chi(M)\int_{-\infty}^{+\infty}\frac{d\varphi}{2\pi}\int\mathcal{D}\phi\,\exp\left\{-\frac{1}{4\pi}\int d^2z\left[\partial\phi\bar{\partial}\phi + i\varphi\sqrt{g}R\phi\right]\right\} \tag{5.242}$$

We notice that this gauged fixed action:

$$S_{CG} = \frac{1}{4\pi}\int d^2z\left[\partial\phi\bar{\partial}\phi + i\varphi\sqrt{g}R\phi\right] \tag{5.243}$$

turns the free scalar into an averaging of coulomb gasses with $\gamma = i\varphi$. Notice that this action is still conformal invariant if we change the transformation property of the scalar field to:

$$\phi'(z') = \phi - 2i\varphi\Omega(z) + \frac{i\varphi}{\pi\chi(M)}\int d^2z\left[\partial\Omega\bar{\partial}\Omega + \frac{1}{2}\sqrt{g}R\Omega\right] \tag{5.244}$$

where $\Omega = \log|\partial f|$.

The first piece two pieces of this expression precisely resemble the transformation property of a logarithmic primary with φ as its $(0,0)$ primary partner. The last bit is a coordinate independent shift that, as we will show in the next section can be eliminated with a field redefinition. Before discussing this technical detail we better explore the connection with log CFTs compute the partition function and the two point functions $\langle\phi\varphi\rangle$, $\langle\varphi\varphi\rangle$. Notice that after the gauge fixing considering operators like ϕ would be prohibited as it is not a gauge invariant operator. However in what follows we will extend the spectrum to also include these operators, and consider the model as this averaging of coulomb gasses.

We now start by computing the partition function, expanding the free scalar in a basis of eigenvectors of the Laplacian:

$$D_\mu D^\mu \varphi_n = e^{-2\sigma}\partial\bar{\partial}\varphi_n = -\lambda_n\varphi_n, \quad \phi = \sum_n \varphi_n c_n \tag{5.245}$$

where D_μ is the covariant derivative related to the $2d$ metric $g_{\mu\nu}$. We assume that they satisfy the orthogonality condition:

$$(\varphi_n, \varphi_m) = \int_M d^2z\sqrt{g}\,\varphi_n(z,\bar{z})\varphi_m(z,\bar{z}) = \delta_{n,m} \tag{5.246}$$

The zero mode $c_0 = c$ is the component related to the eigenvector with zero eigenvalue, that we normalize on the volume of the manifold as:

$$\varphi_0 = \frac{1}{\sqrt{\mathrm{Vol}(M)}}; \tag{5.247}$$

such that $(\varphi_0, \varphi_0) = 1$. With this choice we rewrite the partition function as:

$$Z = \chi(M)\int\prod_{n\neq 0} dc_n \int_{-\infty}^{+\infty}\frac{d\varphi dc}{2\pi}\exp\left\{-\sum_{n\neq 0}\left[\frac{\lambda_n}{4\pi}c_n^2 + i\varphi c_n\chi_n(M)\right] - i\frac{\chi(M)}{\sqrt{\mathrm{Vol}(M)}}\varphi c\right\} \tag{5.248}$$

where:

$$\chi_n(M) = \int\frac{d^2z}{4\pi}\sqrt{g}R\varphi_n(z,\bar{z}) \tag{5.249}$$

Integrating over c we immediately get:

$$\begin{aligned} Z &= \sqrt{\mathrm{Vol}(M)}\int\prod_{n\neq 0} dc_n d\varphi\delta(\varphi)\exp\left\{-\sum_{n\neq 0}\left[\frac{1}{4\pi}\lambda_n c_n^2 + i\varphi c_n\chi_n(M)\right]\right\} \\ &= \sqrt{\mathrm{Vol}(M)}\prod_n\sqrt{\frac{4\pi^2}{\lambda_n}} \end{aligned} \tag{5.250}$$

which, taking into account the different normalization of the action, it precisely matches the result of [177] with the volume of the zero mode gauged away. Notice that we get the same result by introducing in the partition function a fictitious mass term for the zero mode and taking it to zero:

$$\begin{aligned} Z = \chi(M)\lim_{\epsilon\to 0}&\int\prod_{n\neq 0} dc_n\int_{-\infty}^{+\infty}\frac{d\varphi dc}{2\pi}\times \\ &\times\exp\left\{-\sum_{n\neq 0}\left[\frac{1}{4\pi}\lambda_n c_n^2 + i\varphi c_n\chi_n(M)\right] - i\frac{\chi(M)}{\sqrt{\mathrm{Vol}(M)}}\varphi c - \epsilon c^2\right\} \end{aligned} \tag{5.251}$$

We will use this expression to compute now $\langle\phi(z,\bar{z})\varphi\rangle$:

$$
\begin{aligned}
\langle\phi(z,\bar{z})\varphi\rangle = \chi(M)\lim_{\epsilon\to 0}\int\prod_{n\neq 0}dc_n\int_{-\infty}^{+\infty}\frac{d\varphi dc}{2\pi}\phi(z)\varphi\times \\
\times\exp\left\{-\sum_{n\neq 0}\left[\frac{1}{4\pi}\lambda_n c_n^2+i\varphi c_n\chi_n(M)\right]-i\frac{\chi(M)}{\sqrt{\mathrm{Vol}(M)}}\varphi c-\epsilon c^2\right\} \\
= i\sqrt{\mathrm{Vol}(M)}\lim_{\epsilon\to 0}\int\prod_{n\neq 0}dc_n\int_{-\infty}^{+\infty}\frac{d\varphi dc}{2\pi}\phi(z)\left(\frac{d}{dc}+2\epsilon c\right)\times \\
\times\exp\left\{-\sum_{n\neq 0}\left[\frac{1}{4\pi}\lambda_n c_n^2+i\varphi c_n\chi_n(M)\right]-i\frac{\chi(M)}{\sqrt{\mathrm{Vol}(M)}}\varphi c-\epsilon c^2\right\}
\end{aligned}
\tag{5.252}
$$

We can integrate by part the term containing the c derivative and use the fact that $d\phi(z)/dc = 1/\sqrt{\mathrm{Vol}(M)}$ to obtain:

$$
\begin{aligned}
\langle\phi(z,\bar{z})\varphi\rangle = i\sqrt{\mathrm{Vol}(M)}\lim_{\epsilon\to 0}\int\prod_{n\neq 0}dc_n\int_{-\infty}^{+\infty}\frac{d\varphi dc}{2\pi}\left(-\frac{1}{\sqrt{\mathrm{Vol}(M)}}+\phi(z)2\epsilon c\right)\times \\
\times\exp\left\{-\sum_{n\neq 0}\left[\frac{1}{4\pi}\lambda_n c_n^2+i\varphi c_n\chi_n(M)\right]-i\frac{\chi(M)}{\sqrt{\mathrm{Vol}(M)}}\varphi c-\epsilon c^2\right\} = \\
= -\frac{iZ}{\chi(M)}+i\sqrt{\mathrm{Vol}(M)}\int\prod_{n\neq 0}dc_n\int_{-\infty}^{+\infty}d\varphi\phi(z)\times \\
\exp\left\{-\sum_{n\neq 0}\left[\frac{1}{4\pi}\lambda_n c_n^2+i\varphi c_n\chi_n(M)\right]\right\}\lim_{\epsilon\to 0}\int_{-\infty}^{+\infty}\frac{dc}{2\pi}2\epsilon c\, e^{-i\frac{\chi(M)}{\sqrt{\mathrm{Vol}(M)}}\varphi c-\epsilon c^2}
\end{aligned}
\tag{5.253}
$$

The c integral gives us:

$$
\lim_{\epsilon\to 0}\int_{-\infty}^{+\infty}\frac{dc}{2\pi}2\epsilon c\, e^{-i\frac{\chi(M)}{\sqrt{\mathrm{Vol}(M)}}\varphi c-\epsilon c^2} = -i\frac{\chi(M)}{\sqrt{\mathrm{Vol}(M)}}\varphi\lim_{\epsilon\to 0}\frac{1}{2\sqrt{\pi\epsilon}}e^{-\frac{\chi(M)^2}{4\epsilon\mathrm{Vol}(M)}\varphi^2} = -i\varphi\delta(\varphi) \tag{5.254}
$$

so that the integral over φ kills the second term, and we obtain:

$$
\langle\phi(z,\bar{z})\varphi\rangle = -\frac{iZ}{\chi(M)} \tag{5.255}
$$

With this result we can then move to $\langle\varphi\varphi\rangle$. To see that this is vanishing is fairly trivial in the sense that being this expression ϕ independent we can integrate over c. This will produce a term of the form $\varphi^2\delta(\varphi)$ that makes the integral vanish so:

$$
\langle\varphi\varphi\rangle = 0. \tag{5.256}
$$

Notice that (5.255), (5.256) are precisely compatible with the two point function two fields $(\phi, i\chi(M)\varphi)$ in a logarithmic doublet. For the computation of the $\phi\phi$ two point function we refer to the exact computation of the generating functional, which will also give us an hint of the form of the field redefinition highlighting the logarithmic structure of the model.

5.11.3 n-point function generating functional

We will now compute the generating functional of the free scalar explicitly:

$$Z[J,j] = \left\langle e^{\int d^2x\sqrt{g}J(x)\phi(x)+ij\varphi} \right\rangle \tag{5.257}$$

If we define:

$$J_n = \int d^2x\sqrt{g}J(x)\varphi_n(x) \tag{5.258}$$

we have:

$$\begin{aligned}
Z[J,j] =& \chi(M)\int\prod_{n\neq 0} dc_n \int_{-\infty}^{\infty}\frac{dcd\varphi}{2\pi}\times\\
&\times\exp\left\{-\sum_{n\neq 0}\frac{\lambda_n}{4\pi}c_n^2 - i\varphi\left[\sum_{n\neq 0} c_n\chi_n(M)+\frac{\chi(M)}{\sqrt{\mathrm{Vol}(M)}}c-j\right]+\sum_{n\neq 0}c_nJ_n + cJ_0\right\}\\
=&\chi(M)\int\prod_{n\neq 0} dc_n \int_{-\infty}^{\infty}\frac{dcd\varphi}{2\pi}\delta\left(\sum_{n\neq 0} c_n\chi_n(M)+\frac{\chi(M)}{\sqrt{\mathrm{Vol}(M)}}c-j\right)\times\\
&\times\exp\left\{-\sum_{n\neq 0}\frac{\lambda_n}{4\pi}c_n^2+\sum_{n\neq 0}c_nJ_n + cJ_0\right\}=\\
=&\sqrt{\mathrm{Vol}(M)}\int\prod_{n\neq 0} dc_n\exp\left\{-\sum_{n\neq 0}\frac{\lambda_n}{4\pi}c_n^2+\sum_{n\neq 0}c_n\left[J_n - J_0\frac{\sqrt{\mathrm{Vol}(M)}}{\chi(M)}\chi_n(M)\right]\right\}\times\\
&\times e^{jJ_0\frac{\sqrt{\mathrm{Vol}(M)}}{\chi(M)}}=\\
=&Z\exp\left\{\sum_{n\neq 0}\frac{\pi}{\lambda_n}\left(J_n - J_0\frac{\sqrt{\mathrm{Vol}(M)}}{\chi(M)}\chi_n(M)\right)^2 + jJ_0\frac{\sqrt{\mathrm{Vol}(M)}}{\chi(M)}\right\}
\end{aligned} \tag{5.259}$$

Defining the propagator:

$$G(x,y) = -\sum_{n\neq 0}\frac{1}{\lambda_n}\varphi_n(x)\varphi_n^*(y) \tag{5.260}$$

we can then rewrite the generating functional as:

$$\begin{aligned}
Z[J,j] = Z\exp\Bigg\{&-\pi\int d^2x\sqrt{g}\int d^2y\sqrt{g}\left(J(x)-\frac{R(x)}{4\pi\chi(M)}\int d^2z\sqrt{g}J(z)\right)\times\\
&\times G(x,y)\left(J(y)-\frac{R(y)}{4\pi\chi(M)}\int d^2w\sqrt{g}J(w)\right)+\frac{j}{\chi(M)}\int d^2x\sqrt{g}J(x)\Bigg\}
\end{aligned} \tag{5.261}$$

This partition function allows us to compute the scalars two point function:

$$\begin{aligned}
\langle\phi(z)\phi(w)\rangle =& -2\pi G(z,w) - \frac{1}{2\chi(M)}\int d^2x\sqrt{g}R\left[G(z,x)+G(w,x)\right]+\\
&-\frac{1}{8\pi\chi^2(M)}\int d^2x\sqrt{g}\int d^2y\sqrt{g}\,R(x)G(x,y)R(y)
\end{aligned} \tag{5.262}$$

and confirms the form of the previous two point functions.
Notice now that instead of ϕ, φ we can consider the following two fields [9]:

$$\begin{aligned}\Phi(z) &= \phi(z) - \frac{i\varphi}{2}\int d^2x\sqrt{g}R(x)G_0(z,x) + \frac{i\varphi}{16\pi\chi}\int d^2x\sqrt{g}\int d^2y\sqrt{g}\,R(x)G_0(x,y)R(y) \\ \Psi &= i\chi(M)\varphi\end{aligned} \tag{5.263}$$

Under a conformal transformations these fields change as:

$$\begin{aligned}\Phi'(z') &= \Phi(z) - \Psi\log|\partial f| \\ \Psi' &= \Psi\end{aligned} \tag{5.264}$$

which is precisely the transformation of a logarithmic doublet of weight $(0,0)$. Then if we compute the partition function related to these fields:

$$\begin{aligned}\mathscr{Z}[J,j] &= Z\bigg[J, ij\chi(M) - \frac{1}{2}\int d^2x\sqrt{g}\int d^2x\sqrt{g}R(x)G_0(z,x)J(x) + \\ &+ \frac{1}{16\pi\chi}\int d^2z\sqrt{g}J(z)\int d^2x\sqrt{g}\int d^2y\sqrt{g}\,R(x)G_0(x,y)R(y)\bigg] \\ &= Z\exp\bigg\{-\pi\int d^2x\sqrt{g}\int d^2y\sqrt{g}J(x)G_0(x,y)J(y) + ij\int d^2x\sqrt{g}J(x)\bigg\}\end{aligned} \tag{5.265}$$

we can list the two point functions:

$$\begin{aligned}\langle\Phi(z)\Phi(w)\rangle &= -2\pi G_0(z,w) = -\log|z-w|^2 \\ \langle\Phi(z)\Psi\rangle &= 1 \\ \langle\Psi\Psi\rangle &= 0\end{aligned} \tag{5.266}$$

which coincide with the two point functions of a logarithmic doublet. Notice that nothing depends on the metric of the original manifold. This means that if we take a sphere of radius R we can consider a large radius limit and go back to the plane preserving the logarithmic structure.
We can now study a bit more in depth the operator algebra of the model. We start by showing that the logarithmic primary behavior of Φ can also be found from its OPEs with the stress tensor. We will specialize to a scalar living on a sphere of radius R, with:

$$e^{2\sigma} = \frac{2R^4}{(z\bar{z}+R^2)^2}, \quad R(z,\bar{z}) = \frac{2}{R^2}. \tag{5.267}$$

With this choice the expression of the scalar fields Φ, Ψ becomes:

$$\begin{aligned}\Phi &= \phi - 2i\varphi\log\big(|z|^2+R^2\big) + i\varphi(1+\log R^2) \\ \Psi &= 2i\varphi\end{aligned} \tag{5.268}$$

Now the stress tensor can be directly computed from the action and takes the form:

$$T(z) = -\frac{1}{2}\partial\phi\partial\phi + i\varphi\nabla_z^2\phi = -\frac{1}{2}\partial\Phi\partial\Phi + \frac{\Psi}{2}\partial^2\Phi \tag{5.269}$$

[9]The function G_0 is defined in the appendix A.3

where we have used (5.268) to get the second expression. We can then use the OPE:

$$\Phi(z)\Phi(0) \simeq -\log|z-w|^2, \quad \Phi(z)\Psi \simeq 1 \tag{5.270}$$

to prove:

$$T(z)\Phi(0) \simeq \frac{\partial\Phi(0)}{z^2} + \frac{\Psi}{2z} \tag{5.271}$$

which is exactly the OPE of a primary field with its logarithmic partner. Because Ψ is a constant field we expect $T(z)\Psi \simeq 0$ which is exactly the result we obtain computing the $T(z)\Psi$ OPE explicitly.
Using the OPEs (5.270) we can also compute TT OPE

$$T(z)T(0) \simeq \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z} + \frac{1}{2z^4}\left(1+3\Psi^2\right) \tag{5.272}$$

which is quite peculiar as the central charge gets shifted by the $(0,0)$ Ψ constant field. This is consistent with conformal transformations, and does not modify the TT two point function of the stress tensor because $\langle\Psi^2\rangle = 0$. This property implies that the conformal anomaly it is still dictated by the old value of the central charge $c=1$, with no modification coming from Ψ^2.
In this set up we can also study vertex operators $\mathcal{V}_\alpha =: e^{i\alpha\Phi}:$. Classically the transformation property of these fields will be of the form:

$$\mathcal{V}'_\alpha(z',\bar{z}') =: |\partial f|^{-i\alpha\Psi}\mathcal{V}_\alpha(z,\bar{z}): \tag{5.273}$$

Notice that in the usual set up [177] Φ is classically invariant under conformal transformation, so the function $\mathcal{V}_\alpha =: e^{i\alpha\Phi}:$ would also be invariant. However the action of the stress tensor would show that this is a $(\alpha^2/2, \alpha^2/2)$ primary field.
However in this set up we have

$$T(z)\mathcal{V}_\alpha(0,0) \simeq \frac{\partial\mathcal{V}_\alpha(0)}{z} + \frac{i\alpha}{2}\Psi\frac{\mathcal{V}_\alpha(0)}{z^2} \tag{5.274}$$

which shows that the transformation matches the classical expression (5.273).
Notice that $\mathcal{V}_\alpha$ transforms as a primary field but with an operator valued conformal weight $h_\alpha = \frac{i\alpha}{2}$. This allows the correlation functions to remain conformal invariant. In fact from the generating functional we can compute that in general:

$$\langle\mathcal{V}_\alpha(z,\bar{z})\mathcal{V}_\beta(w,\bar{w})\rangle = |z-w|^{2\alpha\beta} \tag{5.275}$$

which under a conformal transformation changes as:

$$|z'-w'|^{2\alpha\beta} \to |z-w|^{2\alpha\beta}|\partial f(z)|^{\alpha\beta}|\partial f(w)|^{\alpha\beta} \tag{5.276}$$

If we check the transformation from the properties of the vertex operators we get:

$$\begin{aligned}\left\langle\mathcal{V}'{}_\alpha(z',\bar{z}')\mathcal{V}'_\beta(w',\bar{w}')\right\rangle &= \left\langle|\partial f|^{-i\alpha\Psi}|\partial f|^{-i\beta\Psi}\mathcal{V}_\alpha(z,\bar{z})\mathcal{V}_\beta(w,\bar{w})\right\rangle = \\ &= |\partial f(z)|^{\alpha\beta}|\partial f(w)|^{\alpha\beta}\left\langle\mathcal{V}_\alpha(z,\bar{z})\mathcal{V}_\beta(w,\bar{w})\right\rangle\end{aligned} \tag{5.277}$$

which is exactly what we expected from (5.276). Notice that if we focus on correlation functions of operators that are shift invariant, which is necessary if we want to restrict to operators which are allowed in the gauged free scalar, we have to impose $\alpha = -\beta$ and we get that the only allowed correlation functions are of the form:

$$\langle \mathcal{V}_{\alpha}(z,\bar{z})\mathcal{V}_{-\alpha}(w,\bar{w})\rangle = |z-w|^{-2\alpha^2} \tag{5.278}$$

and in correlation functions the operator seem to transform exactly as primaries of weight $h_\alpha = \alpha^2/2$.
In general we notice that to recover the gauged free scalar we have to consider a current J such that

$$\int d^2x\sqrt{g}J(x) = 0 \tag{5.279}$$

as this condition is equivalent to decouple the zero mode, so that no operator charged under the shift symmetry can appear in correlation functions. Notice that with this condition Ψ decouples and we end up in a sub-sector of the model which behaves exactly like the gauged free scalar.

In summary, what we have shown in this section is that it is possible to recast the free scalar into a logarithmic CFT, where the scalar Φ takes the role of a logarithmic primary field in a doublet with a constant $(0,0)$ primary field Ψ. The mode Ψ recouples the zero mode, making the theory finite.

Chapter 6

Top-Down Celestial CFT

In the previous chapter we have analyzed how a logarithmic CFT behavior can emerge from celestial CFT and argued how its feature can be used to address some properties of CCFT. Most of the considerations we have pointed out come exclusively from the properties of the bulk theory, such as the universality of the infrared behavior of gravity or gauge theories, which are used to infer the properties of celestial CFT in a bottom-up approach.
A powerful alternative approach to explore the general properties of the 2d holographic dual is to provide a fully explicit example of the duality between a candidate CCFT and quantum gravity in flat space. This mirrors the AdS/CFT case, where well-established exact dualities not only clarify the general features of the specific dual CFTs but also reveal general properties of other boundary theories. Finding such an example would allow us to address precisely how CCFT remains non-trivial for $c \to 0$, and clarify more precisely how the logarithmic CFT structure emerges in CCFT.

Recently many directions have been explored to propose a top-down CCFT, such has the constructions coming from twisted holography [95], tensionless strings and self-dual models. In this chapter we will focus on a recent construction which focuses on maximally helicity violating (MHV) amplitudes in pure Yang Mills and their celestial counterpart in Klein space. This construction is based on the fact that celestial MHV amplitudes can be obtained as a translation invariant combination of the so-called leaf amplitudes, which are amplitudes associated with each 3D hyperbolic (H_3^+) foliation of 4D Minkowski spacetime in a Milne expanding patch [219]; see also [56, 58, 100, 220–226] for related works. By avoiding the singularities that celestial amplitudes can exhibit, leaf amplitudes provide a framework that facilitates a dual 2D CFT description of celestial correlators. Translation invariance in the bulk indeed typically forces celestial amplitudes to take on a distributional form, which contrasts with the analytic structure one might naturally expect from a standard 2D CFT perspective. This has led to consider the decomposition of celestial amplitudes into contributions from individual leaves of an H_3^+ foliation of the bulk spacetime, which are only constrained by conformal invariance. Because these contributions are related to H_3^+ leaves, they are usually referred to as celestial leaf amplitudes.
For the case of tree MHV celestial amplitudes, the Kleinian three-point function was shown to be encoded in the residue of a pole in the leaf gluon amplitudes [219]. This example illustrates how

the full translation-invariant celestial amplitudes can be reconstituted from non-distributional leaf amplitudes.

In the recent work [97], Melton, Sharma, Strominger and Wang gave a holographic dual description of tree-level 4D MHV scattering amplitudes in terms of a 2D CFT[1]. The dual theory, referred to as a 'dressed Liouville theory', is a non-compact, non-unitary theory that involves a Liouville[2] field together with N free fermions and an additional chiral fermion η of negative weight. The leaf amplitudes framework plays a key role in the derivation of a duality between a sector of this theory and MHV celestial amplitudes. It was indeed shown that, in the large N and large central charge c limit, n-point correlation functions in the dressed Liouville CFT turn out to be in correspondence with the n-point MHV leaf amplitudes on a single H_3 slice of Minkowski spacetime. The proof of this statement relies on the fact that, for large c, Liouville theory correlation functions can be represented by contact Witten diagrams in Euclidean AdS_3 space [97]. The holographic dictionary is then established in three steps: first, it involves going from the dressed Liouville observables to the so-called Euclidean leaves. Second, going from the Euclidean leaves to the Lorentzian leaves, which involves a continuation from Minkowski to Klein space $\mathbb{R}^{2,2}$, whose boundary is foliated by celestial tori. Finally, going from the Lorentzian leaves to the full MHV celestial amplitudes, which amounts to single out the pole at $\beta = \sum_{i=1}^{n}(\Delta_i - 1) = 0$, where Δ_i denotes the conformal dimension of the gluons [97].

As already mentioned above, the celestial dual of [97] consists of a specific non-unitary 2D CFT involving Liouville field theory in direct product with N fermions of conformal weights $(h, \bar{h}) = (\frac{1}{2}, 0)$ and an extra fermion η of weights $(-\frac{3}{2}, 0)$. A combination of the real (or complex) fermions gives a rank-N, level-1 affine Kac-Moody $SO(N)$ (resp. $SU(N)$) current J^a of weights $(1, 0)$ which enters in the CFT operator dual to a positive helicity gluon. This current is then dressed with the additional η field to construct a $(-1, 0)$ current, denoted $\bar{J}^a$. The latter turns out to be needed in the dual description of bulk fields of negative helicity.

In this chapter we will present an alternative 2D CFT realization of 4D MHV leaf amplitudes in terms of a perhaps more familiar theory, namely the Wess-Zumino-Witten (WZW) model. This is achieved by realizing that the dressed Liouville correlators of [97] can be expressed as specific correlators in the $SL(2,\mathbb{R})/U(1)$ coset theory, which can be realized in terms of parafermions [228, 229]. Connection between celestial amplitudes and $SL(2,\mathbb{R})$ WZW correlators has recently been explored in the literature [101, 230–232], likely motivated by the fact that, from the leaf amplitudes perspective, it is natural to expect an $H_3^+ = SL(2,\mathbb{C})/SU(2)$ structure related to the hyperbolic foliation. However, as we will discuss, a crucial ingredient needed to establish a connection between leaf amplitudes and such WZW correlators is to consider spectrally flowed representations.
The material presented in the following sections is mostly based on the results we obtained in [2]

[1]See also [74] for the description of the MHV sector of gluon scattering in terms of a 2D celestial CFT using twistor string theory.

[2]Connections between celestial amplitudes and Liouville theory were also investigated in [98, 99, 217, 227].

6.1 Celestial MHV amplitudes

For the entire chapter we will focus on pure $SO(N)$ Yang-Mills. In this model the only quantity we compute are correlation functions of gluons which can be split into a dynamical part and a color part, which is only dependent on the group structure. In particular at tree level any n-point gluon amplitude $\mathcal{A}^g$ can be written as [233]:

$$\mathcal{A}_n^g\left(1^{a_1,j_1},2^{a_2,j_2},\ldots,n^{a_n,j_n}\right)=\sum_{\sigma\in S_n/\mathbb{Z}_n}\mathrm{Tr}\{T^{a_{\sigma(1)}}\ldots T^{a_{\sigma(n)}}\}\mathcal{A}_n\left(\sigma(1^{j_1}),\sigma(2^{j_2}),\ldots,\sigma(n^{j_n})\right) \tag{6.1}$$

where with i^{a_i,j_i} we denote a gluon with momentum p_j helicity j_i and color factor a_i. As we see the total amplitude split into a sum of non-cyclic permutation of a pure color factor identical to the trace of the generators of the algebra times a dynamical piece $\mathcal{A}$ which is usually referred to as the color ordered partial amplitude.

The partial amplitude in general is a complicated function of the momenta of the external gluons, but it simplifies drastically in specific cases. One of such cases is the situation where 2 gluons have helicity ±1 and $n-2$ gluons have helicity ∓1. These are called maximally helicity violating amplitudes (MHV) because they are the non-vanishing amplitude with maximum value of total helicity. The expression of the color stripped amplitude is given by the Parke-Taylor formula [234, 235]

$$\begin{aligned}\mathcal{A}\left(1^-,2^-,3^+,\ldots,n^+\right)&=\epsilon_1\ldots\epsilon_n\frac{\omega_1^2\omega_2^2}{\omega_1\ldots\omega_n}\frac{z_{12}^4}{z_{12}z_{23}\ldots z_{n1}}\delta(P),\\ \mathcal{A}\left(1^+,2^+,3^-,\ldots,n^-\right)&=\epsilon_1\ldots\epsilon_n\frac{\omega_1^2\omega_2^2}{\omega_1\ldots\omega_n}\frac{\bar{z}_{12}^4}{\bar{z}_{12}\bar{z}_{23}\ldots\bar{z}_{n1}}\delta(P)\end{aligned} \tag{6.2}$$

where we have chosen the usual momentum parametrization (2.9), $P=p_1+\cdots+p_n$ and $\epsilon_i=\pm1$ signals if the particle is incoming or outgoing. Now that we have the momentum parametrization of the MHV amplitude is easy to obtain the expression for the corresponding celestial amplitude, as we just have to Mellin transform (6.2) such that

$$\begin{aligned}\langle\mathcal{O}^-_{\Delta_1}\mathcal{O}^-_{\Delta_2}\mathcal{O}^+_{\Delta_3}\ldots\mathcal{O}^+_{\Delta_n}\rangle&=\epsilon\frac{z_{12}^4}{z_{12}z_{23}\ldots z_{n1}}\int_0^{+\infty}\prod_{j=1}^n d\omega_j\,\omega_j^{2\bar{h}_j-1}\delta(P),\\ \langle\mathcal{O}^+_{\Delta_1}\mathcal{O}^+_{\Delta_2}\mathcal{O}^-_{\Delta_3}\ldots\mathcal{O}^-_{\Delta_n}\rangle&=\epsilon\frac{\bar{z}_{12}^4}{\bar{z}_{12}\bar{z}_{23}\ldots\bar{z}_{n1}}\int_0^{+\infty}\prod_{j=1}^n d\omega_j\,\omega_j^{2\bar{h}_j-1}\delta(P),\\ h_{1,2}&=\frac{\Delta_{1,2}-1}{2},\quad\bar{h}_{1,2}=\frac{\Delta_{1,2}+1}{2}\\ h_i&=\frac{\Delta_i+1}{2},\quad\bar{h}_i=\frac{\Delta_{1,2}-1}{2},\quad i=3,\ldots,n\end{aligned} \tag{6.3}$$

where $\epsilon=\epsilon_1\ldots\epsilon_n$.

In general it is complicated to compute the Mellin transform, however results up to 6-point functions can be found in the literature, and present a distributional form inherited by translation invariance [69].

In the upcoming section we will show a procedure that allows to splitting the celestial MHV amplitude into the sum of two pieces that independently do not respect translation invariance but take a form much more akin to that of common 2d CFTs. These partial amplitudes are usually referred to a celestial leaf amplitudes.

6.2 Leaf amplitudes

We now review the construction of leaf amplitudes for the gluon MHV case. We will carry out the construction in Klein space following [219] and then comment on the extension to Minkowski space.

In Klein space the celestial MHV amplitudes takes the form (6.3) but with momentum parametrized by

$$p_i = \epsilon_i \omega_i q_i = \omega_i \hat{p}_i, \quad q_i = (1 - z_i\bar{z}_i, z_i + \bar{z}_i, 1 + z_i\bar{z}_i, z_i - \bar{z}_i) \tag{6.4}$$

where now $z, \bar{z}$ are independent real variables.

The procedure to obtain the leaf amplitudes starts from a trivial rewriting of the δ function:

$$\delta(P) = \int \frac{d^4X}{(2\pi)^4} e^{iP\cdot X} \tag{6.5}$$

which allows us to obtain:

$$\begin{aligned} \langle \mathcal{O}^-_{\Delta_1}\mathcal{O}^-_{\Delta_2}\mathcal{O}^+_{\Delta_3}\dots\mathcal{O}^+_{\Delta_n}\rangle &= \epsilon \frac{z_{12}^4}{z_{12}z_{23}\dots z_{n1}} \int \frac{d^4X}{(2\pi)^4} \prod_{j=1}^n \int_0^{+\infty} d\omega_j\, \omega_j^{2\bar{h}_j - 1} e^{i\omega_j \hat{p}_j\cdot X} = \\ &= \epsilon \frac{z_{12}^4}{z_{12}z_{23}\dots z_{n1}} \int \frac{d^4X}{(2\pi)^4} \prod_{j=1}^n \Phi^0_{\epsilon_i, 2\bar{h}_i}(X, z_i, \bar{z}_i) \end{aligned} \tag{6.6}$$

where in the last step we have recognized that the integral reconstructs the scalar conformal primary wave function (4.4).

Now we can use the property of Klein space to foliate into $\mathrm{AdS}^3/\mathbb{Z}$ slices and rewrite:

$$\begin{aligned} \langle \mathcal{O}^-_{\Delta_1}\mathcal{O}^-_{\Delta_2}\mathcal{O}^+_{\Delta_3}\dots\mathcal{O}^+_{\Delta_n}\rangle = \epsilon \frac{z_{12}^4}{z_{12}z_{23}\dots z_{n1}} \int_0^{+\infty} \frac{d\tau}{2\pi}\tau^3 \Bigg[\int_{x_+^2=-1} \frac{d^3x_+}{(2\pi)^3} \prod_{j=1}^n \Phi^0_{\epsilon_i, 2\bar{h}_i}(\tau x_+, z_i, \bar{z}_i) + \\ + \int_{x_-^2=+1} \frac{d^3x_-}{(2\pi)^3} \prod_{j=1}^n \Phi^0_{\epsilon_i, 2\bar{h}_i}(\tau x_-, z_i, \bar{z}_i) \Bigg]. \end{aligned} \tag{6.7}$$

In this expression we have basically split the integral over full Klein space into the two regions W^T, W^S using the parametrization $X = \tau x_\pm$ described in (1.51).

Now using the conformal properties of the primary wave function:

$$\Phi^0_{\epsilon_i, 2\bar{h}_i}(\tau x_\pm, z_i, \bar{z}_i) = \tau^{-2h_i} \Phi^0_{\epsilon_i, 2\bar{h}_i}(x_\pm, z_i, \bar{z}_i) \tag{6.8}$$

we can extract completely the τ dependence which reduces to:

$$\int \frac{d\tau}{2\pi} \tau^{-\beta-1} = \delta(-i\beta) \tag{6.9}$$

with:

$$\beta = \sum_{i=1}^n (\Delta_i - 1) = i \sum_{i=1}^n \lambda_i \tag{6.10}$$

The $\delta(-i\beta)$ can be considered as a well posed distribution for celestial operators sitting on the principal line, where β is purely imaginary. The only surviving integrals are computed on the

$\mathrm{AdS}_3/\mathbb{Z}$ slices, and are equivalent to the integral of bulk to boundary propagator in AdS_3. We notice that the integrals over W^T, W^S are not independent but can be related to each other just sending $\bar{z}_i \to -\bar{z}_i$ in one of the two integrals.
The full celestial amplitude then can be rewritten as:

$$\langle \mathcal{O}^-_{\Delta_1} \mathcal{O}^-_{\Delta_2} \mathcal{O}^+_{\Delta_3} \dots \mathcal{O}^+_{\Delta_n} \rangle = \frac{\delta(-i\beta)}{8\pi^3} \left(\mathcal{L}(z_i, \bar{z}_i) + \mathcal{L}(z_i, -\bar{z}_i) \right) \tag{6.11}$$

where:

$$\mathcal{L}(z_i, \bar{z}_i) = \epsilon \frac{z_{12}^4}{z_{12} z_{23} \dots z_{n1}} \int_{\mathrm{AdS}_3/\mathbb{Z}} \frac{d^3x}{(2\pi)^3} \prod_{j=1}^{n} \Phi^0_{\epsilon_i, 2\bar{h}_i}(x, z_i, \bar{z}_i) \tag{6.12}$$

These expression are the celestial MHV leaf amplitudes, which take their name from the fact of being localized just on a single $\mathrm{AdS}_3/\mathbb{Z}$ slice. The leaf amplitude alone does not respect translation invariance, however the combination $\mathcal{L}(z_i, \bar{z}_i) + \mathcal{L}(z_i, -\bar{z}_i)$ restores the invariance. In the next section we will show that this amplitudes can be reproduced by a simple 2d CFT.

6.3 A CFT for MHV leaf amplitudes

This section is a brief review of the $2d$ CFT realization of MHV leaf amplitudes [219, 226] given in [97]. It involves a Liouville field theory dressed with extra fields, N free fermions ψ^i, and a free chiral fermion η. Important elements are Liouville primary operators of momentum α

$$V_\alpha(z, \bar{z}) = e^{2\alpha\varphi(z,\bar{z})}\,, \tag{6.13}$$

which have conformal dimension

$$h_{V_\alpha} = \bar{h}_{V_\alpha} = \alpha(Q - \alpha)\,. \tag{6.14}$$

The background charge $Q = b + b^{-1}$ is related to the Liouville central charge via $c_L = 1 + 6Q^2$. The dressed operators in (6.15) below will involve 'light' Liouville operators, for which the momentum α scales with b in the classical limit $b \to 0$.

The dictionary between bulk positive and negative helicity gluon primaries and dressed Liouville operators is [97]

$$\begin{aligned} \mathcal{O}^{+a,\epsilon}_\Delta(z, \bar{z}) &= e^{-i\epsilon\frac{\pi}{2}(\Delta-1)} \lim_{b\to 0} N^+_\Delta\, J^a(z)\, V_{\frac{b}{2}(\Delta-1)}(z, \bar{z})\,, \\ \mathcal{O}^{-a,\epsilon}_\Delta(z, \bar{z}) &= e^{-i\epsilon\frac{\pi}{2}(\Delta+1)} \lim_{b\to 0} N^-_\Delta\, \bar{J}^a(z)\, V_{\frac{b}{2}(\Delta+1)}(z, \bar{z})\,, \end{aligned} \tag{6.15}$$

where the conformal dimension Δ and the spin S are read from the weights $(h, \bar{h})$ using $h = \frac{1}{2}(\Delta + S)$, $\bar{h} = \frac{1}{2}(\Delta - S)$. The index $\epsilon = \pm 1$ labels whether the gluon is outgoing or incoming. The normalization in (6.15) is

$$\begin{aligned} N^+_\Delta &= \mu_{\text{cl}}^{(\Delta-1)/2} \Gamma(\Delta - 1) e^{-i\pi(\Delta-1)/2}\,, \\ N^-_\Delta &= \mu_{\text{cl}}^{\Delta/2 - 1/2b^2} e^{\gamma_E - 1/b^2} \sqrt{\pi b^3 \sin(\pi/b^2)}\, \Gamma(\Delta + 1) e^{-i\pi(\Delta+1)/2}\,. \end{aligned} \tag{6.16}$$

The classical Liouville cosmological constant $\mu_{\text{cl}} = \pi\mu b^2$ is held fixed in the classical limit. Importantly, the definition (6.15) also includes currents $J^a(z)$ and $\bar{J}^a(z)$, for positive helicity and negative gluons, respectively. $J^a(z)$ is built out of the real fermions $\psi^i(z)$ $(i = 1, \dots, N)$ via

$$J^a(z) = \frac{1}{2} T^a_{ij} : \psi^i \psi^j : (z) \tag{6.17}$$

where T^a are the generators of $so(N)$. From the fermions leading OPE

$$\psi^i(z)\psi^j(w) = \frac{\delta^{ij}}{z-w} + : \psi^i \psi^j : (w) + \mathcal{O}(z-w)\,, \tag{6.18}$$

one sees that $J^a(z)$ is a Kac-Moody current of weights $(1,0)$ and level $k = 1$,

$$J^a(z)J^b(w) \sim \frac{\delta^{ab}}{(z-w)^2} + \frac{if^{ab}{}_c J^c(w)}{z-w}\,. \tag{6.19}$$

The current $\bar{J}^a(z)$ in (6.15) is realized by the inclusion of an extra fermionic field η of weight $(-\frac{3}{2}, 0)$ as

$$\bar{J}^a(z) = \eta \partial \eta J^a(z)\,, \tag{6.20}$$

and is thus of weights $(-1, 0)$. The η field has a regular OPE,

$$\eta(z)\eta(w) = (z-w)\eta\partial\eta(w) + \mathcal{O}((z-w)^2)\,. \tag{6.21}$$

This operator has four zero modes on the sphere, so to have a non-vanishing correlation function, one has to insert four η (see also [236]) such that

$$\langle \eta\partial\eta(z)\eta\partial\eta(w)\rangle = (z-w)^4\,. \tag{6.22}$$

From the currents and Liouville weights (6.14), one can check that the conformal dimension and spin of dressed operators in (6.15) are given by

$$\begin{aligned} \Delta_{\mathcal{O}^+} &= \frac{\Delta - 1}{2}(2 + b^2(\Delta - 3)) + 1\,, \qquad S_{\mathcal{O}^+} = +1\,, \\ \Delta_{\mathcal{O}^-} &= \frac{\Delta + 1}{2}(2 + b^2(\Delta - 1)) - 1\,, \qquad S_{\mathcal{O}^-} = -1\,, \end{aligned} \tag{6.23}$$

which indeed gives, in the classical limit, $\lim_{b\to 0} \Delta_{\mathcal{O}^\pm} = \Delta$.

To check that correlators of the dressed Liouville CFT operators given in (6.15) reproduce the MHV leaf amplitudes [219], one only needs the following non-vanishing propagators

$$\begin{aligned} \Big\langle \bar{J}^{a_1}(z_1)\bar{J}^{a_2}(z_2) \prod_{\ell=3}^{n} J^{a_\ell}(z_\ell)\Big\rangle &= \mathrm{Tr}(T^{a_1}\dots T^{a_n})\,\frac{z_{12}^4}{z_{12}z_{23}\dots z_{n1}} + \dots \\ \Big\langle \prod_{\ell=1}^{n} V_{b\sigma_\ell}(z_\ell, \bar{z}_\ell)\Big\rangle &= \frac{e^{-2\gamma_E + \frac{2}{b^2}}\mu_{\mathrm{cl}}^{\frac{1}{b^2}-1-\frac{1}{2}\beta}}{\pi b^3} \csc[\pi(b^{-2} - \beta/2)]\, \mathcal{C}_{2\sigma_1\dots 2\sigma_n}\,, \end{aligned} \tag{6.24}$$

where the dots represent multi-trace terms, $z_{ij} \equiv z_i - z_j$, γ_E is the Euler-Mascheroni constant, and

$$\beta = 2\sum_j \sigma_j - 4\,. \tag{6.25}$$

For finite b, $\mathcal{C}_{2\sigma_1\dots 2\sigma_n}$ is a complicated expression, but in the classical limit $b \to 0$ it can be expressed in terms of contact Witten diagrams,

$$\mathcal{C}_{2\sigma_1\dots 2\sigma_n} = \int_{H_3^+} D^3x \prod_{j=1}^{n} G_{2\sigma_j}(z_j, \bar{z}_j; x) = \int_{H_3^+} \frac{dy\, d\bar{y}\, d\rho}{\rho^3} \prod_{j=1}^{n} \left(\frac{\rho}{\rho^2 + |y - z_j|^2}\right)^{2\sigma_j}\,, \tag{6.26}$$

where D^3x is the measure on the unit hyperboloid ($x = \{\rho, y, \bar{y}\}$) and $G_{2\sigma_j}(z_j, \bar{z}_j; x)$ is the scalar bulk-to-boundary propagator of weight $2\sigma_j$, namely

$$G_h(z, \bar{z}; x) = \left(\frac{\rho}{\rho^2 + |y - z|^2}\right)^h . \tag{6.27}$$

Notice that this is precisely the integral over the AdS$_3$ slices encountered in (6.12). For $N \to \infty$ we can also simplify the first correlator in (6.24) since, in the large N limit, only the first term survives. Suppressing the color indices we can then only focus on color-ordered correlators,

$$\langle \bar{J}(z_1)\bar{J}(z_2)\prod_{j=3}^{n} J(z_j)\rangle = \frac{z_{12}^4}{z_{12}z_{23}\ldots z_{n1}}. \tag{6.28}$$

Notice that the expression above contains two types of contribution: the term z_{12}^4 is given by the η bilinear, while the denominator is completely constructed from the dimension-$(1,0)$ currents. Also notice that it has a cyclic structure $z_{12}z_{23}z_{34}z_{45}\ldots z_{n1}$ coming from the single trace over $so(N)$ generators. Using (6.28) and (6.24), one thus reproduces [97], for $b \to 0$ and $N \to \infty$, the MHV leaf amplitude [219]

$$\langle \mathcal{O}^{-,\epsilon_1}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{-,\epsilon_2}_{\Delta_2}(z_2,\bar{z}_2)\prod_{j=3}^{n}\mathcal{O}^{+,\epsilon_j}_{\Delta_j}(z_j,\bar{z}_j)\rangle = \prod_{j=1}^{n} e^{-i\pi\epsilon_j\bar{h}_j}\,\Gamma(2\bar{h}_j)\,\frac{z_{12}^4}{z_{12}z_{23}\ldots z_{n1}}\mathcal{C}_{2\bar{h}_1,2\bar{h}_2,\ldots,2\bar{h}_n} . \tag{6.29}$$

This supports the duality between MHV amplitudes and dressed Liouville CFT correlation functions in the large N limit. In the upcoming section we will provide a different option for a CFT that reconstructs precisely the same behavior.

6.4 Leaf amplitudes from parafermions

In this section, we provide an alternative $2d$ CFT description of $4d$ MHV leaf amplitudes in terms of non-compact parafermions [228, 229]. These parafermions are those that are involved in the realization of the level-k gauged WZW theory on the coset $SL(2,\mathbb{R})/U(1)$. Usually denoted $\Psi^j_{m,\bar{m}}(z,\bar{z})$, these parafermions are organized in j-representations of $SL(2,\mathbb{R})$ and are closely related to the parafermions considered in the context of $\mathcal{N} = 2$ 2D superconformal algebra [228]. The two-point functions of such fields have the form

$$\langle \Psi^j_{m,\bar{m}}(z_1,\bar{z}_1)\Psi^{-1-j}_{-m,-\bar{m}}(z_2,\bar{z}_2)\rangle = (z_1 - z_2)^{\frac{2}{k-2}j(j+1)-\frac{2}{k}m^2}(\bar{z}_1 - \bar{z}_2)^{\frac{2}{k-2}j(j+1)-\frac{2}{k}\bar{m}^2} . \tag{6.30}$$

This gives the conformal weights

$$h_{\Psi^j_{m,\bar{m}}} = -\frac{j(j+1)}{k-2} + \frac{m^2}{k}, \qquad \bar{h}_{\Psi^j_{m,\bar{m}}} = -\frac{j(j+1)}{k-2} + \frac{\bar{m}^2}{k} , \tag{6.31}$$

which realizes the spectrum of the WZW model on the coset $SL(2,\mathbb{R})/U(1)$, see [237–240].

This coset model can also be realized in terms of the σ-model on $SL(2,\mathbb{R})$ in the Wakimoto free field representation [241] and supplemented by an extra scalar field X and a b-c ghost system that realizes the BRST charge to mode out the $U(1)$ factor. X is charged under the $U(1)$, with the charges being the labels $m, \bar{m}$; see [238–240] for details.

Parafermionic n-point correlation functions $\langle \Psi^{j_1}_{m_1,\bar m_1}(z_1,\bar z_1)\dots\Psi^{j_n}_{m_n,\bar m_n}(z_n,\bar z_n)\rangle$ that obey the specific condition $\sum_{\ell=1}^n(m_\ell+\bar m_\ell)=k(n-2)$ turn out to be proportional to n-point correlation functions in Liouville theory; see (6.36) below. This is a particular case of a much more general relation that exists between correlation functions in the $SL(2,\mathbb{R})$ WZW model and in Liouville theory, namely [242]

$$\langle\prod_{\ell=1}^n \Psi^{j_\ell}_{m_\ell,\bar m_\ell}(z_\ell,\bar z_\ell)\rangle = \frac{2\pi^3\, b^{1+\frac{r}{2}}}{\Gamma(n-1-r)}\prod_{\ell=1}^n\frac{\Gamma(-j_\ell-m_\ell)}{\pi^2\Gamma(j_\ell+1+\bar m_\ell)}\prod_{\ell<\ell'}^n z_{\ell\ell'}^{\beta_{\ell\ell'}}\,\bar z_{\ell\ell'}^{\bar\beta_{\ell\ell'}}\int\prod_{a=1}^{n-2-r}d^2y_a\prod_{a<a'}|y_{aa'}|^k$$
$$\times\prod_{\ell,a}(z_\ell-y_a)^{\frac{k}{2}-m_\ell}\,(\bar z_\ell-\bar y_a)^{\frac{k}{2}-\bar m_\ell}\times\langle\prod_{\ell=1}^n V_{\alpha_\ell}(z_\ell,\bar z_\ell)\prod_{a=1}^{n-2-r}V_{-\frac{1}{2b}}(y_a)\rangle\,, \tag{6.32}$$

where

$$\beta_{\ell\ell'}=\frac{2}{k}\left(m_\ell-\frac{k}{2}\right)\left(m_{\ell'}-\frac{k}{2}\right),\qquad \bar\beta_{\ell\ell'}=\frac{2}{k}\left(\bar m_\ell-\frac{k}{2}\right)\left(\bar m_{\ell'}-\frac{k}{2}\right) \tag{6.33}$$

with the restrictions

$$\sum_{\ell=1}^n\bar m_\ell=\sum_{\ell=1}^n m_\ell=\frac{k}{2}r\,,\qquad r\in\mathbb{Z}_{\geq 0}\,. \tag{6.34}$$

In an unpublished work, Fateev, Zamolodchikov and Zamolodchikov have shown that the bound $r\leq n-2$ also holds, cf. [243, 244]. We see from (6.32) that, for a violation of the spectral flow condition by r units, the correspondence between WZW and Liouville correlators involves the insertion of $n-2-r$ 'degenerate' Liouville operators $V_{-\frac{1}{2b}}$.

While the expectation value on the left hand side of (6.32) corresponds to the $SL(2,\mathbb{R})/U(1)$ coset theory with central charge $c_{SL(2)/U(1)}=2+\frac{6}{k-2}$, the one on the right hand side corresponds to Liouville theory with central charge $c_L=1+6Q^2$ with $Q=b+\frac{1}{b}$. The dictionary between momenta and parameters in both theories is given by

$$b^2=\frac{1}{k-2}\,,\qquad \alpha_\ell=\frac{1}{\sqrt{k-2}}\left(j_\ell+\frac{k}{2}\right)\,. \tag{6.35}$$

When extended to the full $SL(2,\mathbb{R})$, relation (6.32) agrees with the analytic extension of the so-called H_3^+ WZW-Liouville correspondence[3]. This correspondence was first proven by Ribault and Teschner [245] in the spectral flow-preserving case $r=0$, based on an observed relationship between the Knizhnik-Zamolodchikov equations for WZW models and the Belavin-Polyakov-Zamolodchikov equations for Liouville correlators [246]. A generalization of the H_3^+ WZW-Liouville correspondence for Riemann surfaces to higher genus $g\geq 0$ in the case $r=-2g$ was given in [247] using the path integral approach. Expression (6.32) for $g=0$ has been proven in [242] for the case $0\leq r<n-2$. The case $r=n-2$ was later proven in [248] using the Coulomb gas approach. Here we will be involved with the latter. In that case, we simply have

$$\prod_{\ell=1}^n\hat N^{j_\ell}_{m_\ell,\bar m_\ell}\,\langle\prod_{\ell=1}^n\Psi^{j_\ell}_{m_\ell,\bar m_\ell}(z_\ell,\bar z_\ell)\rangle = 2\pi^3\prod_{\ell<\ell'}^n z_{\ell\ell'}^{\beta_{\ell\ell'}}\,\bar z_{\ell\ell'}^{\bar\beta_{\ell\ell'}}\;\langle\prod_{\ell=1}^n V_{\alpha_\ell}(z_\ell,\bar z_\ell)\rangle\,, \tag{6.36}$$

with the normalization

$$\hat N^{j_\ell}_{m_\ell,\bar m_\ell}=\frac{\pi^2\Gamma(j_\ell+1+\bar m_\ell)}{b^{1/2}\,\Gamma(-j_\ell-m_\ell)}. \tag{6.37}$$

[3]See appendix A.4 for the relation between parafermions and WZW spectral flowed primaries.

Motivated by the above correspondence between Liouville theory and the spectrally-flowed H_3^+ WZW correlators, we identify the CFT operators dual to gluon primaries as,

$$\begin{aligned}\chi^{+a,\epsilon}_{2j+k+1}(z,\bar{z}) \;&=\; e^{-i\pi\epsilon(j+\frac{k}{2})}\, N^{+}_{2j+k+1}\hat{N}^{j}_{+\frac{k}{2},+\frac{k}{2}}\; J^a(z)\, \Psi^{j}_{+\frac{k}{2},+\frac{k}{2}}(z,\bar{z})\,,\\ \chi^{-a,\epsilon}_{2j+k-1,\pm}(z,\bar{z}) &= e^{-i\pi\epsilon(j+\frac{k}{2})}\, N^{-}_{2j+k-1}\hat{N}^{j}_{\pm\frac{k'}{2},\mp\frac{k}{2}}\; J^a(z)\, \Psi^{j}_{\pm\frac{k'}{2},\mp\frac{k}{2}}(z,\bar{z})\,,\end{aligned} \tag{6.38}$$

where $N^{\pm}_{\Delta}$ is given in (6.16) and $k'^2 = k(k-8)$. As we will argue, in the $k\to\infty$ limit, the positive helicity gluon operator $\mathcal{O}^{+a,\epsilon}_{\Delta}$ in the semiclassical limit corresponds to the parafermion operator $\chi^{+a,\epsilon}$ with $j=-\frac{k}{2}+\frac{1}{2}(\Delta-1)$. The negative helicity gluon $\mathcal{O}^{-a,\epsilon}_{\Delta}$ corresponds to the linear combination $\chi^{-a}=\frac{1}{2}(\chi^{-a,\epsilon}_{+}+\chi^{-a,\epsilon}_{-})$ with $j=-\frac{k}{2}+\frac{1}{2}(\Delta+1)$.

We can already note that amplitudes with $n-1$ operators $\mathcal{O}^{+a,\epsilon}_{\Delta}$ and a single operator $\mathcal{O}^{-a,\epsilon}_{\Delta}$ vanish in virtue of (6.34). Let us also mention that the prescription we are providing here is for the mostly plus amplitudes. The expression for the mostly minus amplitudes follow the same line as a $+\leftrightarrow-$ symmetric version of formula (6.32) exists; this is obtained by changing the $m_\ell\to-m_\ell$, $\bar{m}_\ell\to-\bar{m}_\ell$ and $r\to-r$ in (6.32)[4]. It is worthwhile pointing out that, in contrast with (6.15), the dressing in the right-hand side of (6.38) involves the same chiral current $J^a(z)$ for both positive and negative helicity, and hence does not require the introduction of an extra η field.

The conformal weights of the parafermion operators in (6.38) can be read from (6.31) and the values $(h,\bar{h})=(1,0)$ for the current J^a. We have

$$\begin{aligned} h_{\chi^\pm} &= -\frac{j(j+1)}{k-2}+\frac{k}{4}\pm 1\,,\\ \bar{h}_{\chi^\pm} &= -\frac{j(j+1)}{k-2}+\frac{k}{4}\,,\end{aligned} \tag{6.39}$$

so that the spin and the scaling dimension, in the large k limit, are given by

$$S_{\chi^\pm}=h_{\chi^\pm}-\bar{h}_{\chi^\pm}=\pm1\,,\qquad \lim_{k\to\infty}\Delta_{\chi^\pm}=\lim_{k\to\infty}(h_{\chi^\pm}+\bar{h}_{\chi^\pm})=\Delta\,. \tag{6.40}$$

Let us now show that correlators of parafermion operators (6.38) reconstruct MHV leaf amplitudes. The n-point function

$$\mathcal{A}_{j_1,j_2,\ldots,j_n} \;=\; \frac{1}{2\pi^3}\,\langle \chi^{-,\epsilon_1}_{2j_1+k-1}(z_1,\bar{z}_1)\chi^{-,\epsilon_2}_{2j_2+k-1}(z_2,\bar{z}_2)\prod_{\ell=3}^{n}\chi^{+,\epsilon_\ell}_{2j_\ell+k+1}(z_\ell,\bar{z}_\ell)\rangle\,, \tag{6.41}$$

is computed using (6.36). One can easily see that, since

$$\begin{aligned} m_1 &= -m_2=\pm\frac{k'}{2},\quad & m_{\ell>2}&=\frac{k}{2},\\ \bar{m}_1 &= -\bar{m}_2=\frac{k}{2}, & \bar{m}_{\ell>2}&=\frac{k}{2},\end{aligned} \tag{6.42}$$

with $k'^2=k(k-8)$ and using (6.33), the $\bar{z}_\ell$-dependent factor disappears from (6.36), while the z_ℓ-dependent factor reduces to z_{12}^4. This gives precisely the contribution obtained from the

[4]This also amounts to change $\omega_\ell\to-\omega_\ell$ in the formulae of [242], and the change $+\leftrightarrow-$ makes the z_{ij}- and $\bar{z}_{ij}$-dependence to switch.

η bilinear dressing of [97]. The n-point correlator for the $J(z)$ currents, in the large N limit, reproduces the denominator of (6.28) in the same way while the Liouville correlation function reads,

$$\langle\prod_{\ell=1}^{n} V_{\alpha_\ell}(z_\ell,\bar{z}_\ell)\rangle = \langle\prod_{\ell=1}^{n} V_{b\bar{h}_\ell}(z_\ell,\bar{z}_\ell)\rangle\,, \tag{6.43}$$

where we used (6.35) and the dictionary below (6.38). Putting everything together, we conclude that, in the large k and large N limit, the n-point correlation functions of parafermion operators (6.38) gives

$$\mathcal{A}_{j_1,j_2,\ldots,j_n} = \frac{z_{12}^4}{z_{12}z_{23}\ldots z_{n1}}\prod_{\ell=1}^{n} e^{i\pi\epsilon_\ell\bar{h}_\ell}\Gamma(2\bar{h}_\ell)\mathcal{C}_{2\bar{h}_1\ldots 2\bar{h}_n}\,, \tag{6.44}$$

reproducing the large-N limit of the MHV leaf amplitudes.

We emphasize that the realization of MHV amplitudes in terms of parafermions is totally equivalent to the CFT realization presented in [97]. The precise relation between both descriptions follows from the generalized version of the H_3^+ WZW-Liouville correspondence [242, 245]. The relations between the parameters in the dressed Liouville and parafermion theories are given by $\alpha_\ell = b\sigma_\ell$ with $\sigma_\ell = j_\ell + \frac{k}{2}$ and $k = 2 + \frac{1}{b^2}$, and $\sigma_\ell = \frac{1}{2}(\Delta_\ell \mp 1)$ for operators of helicity ± 1. Nevertheless, it is worthwhile emphasizing that the dual description in terms of parafermions only involves a single Kac-Moody current and hence does not involve the addition of an extra $(-\frac{3}{2}, 0)$-weight free fermion in the description of bulk conformal primary gluons of negative helicity. In particular, one can see that certain correlators that involve more than two negative helicity parafermion operators (6.38) can be non-vanishing. For example, correlators involving four operators χ^{-a} and $n-4$ operators χ^{+a} would correspond, in the Liouville theory, to $(n+2)$-point functions involving two degenerate fields $V_{-\frac{1}{2b}}$ and additional field dressing. While we do not provide here a bulk interpretation of the correlators involving such operators, it would be very interesting to understand the role played by the insertion of a Liouville degenerate operator from the gauge theory perspective. Similarly, understanding the role of the so-called spectral flow operator in the $SL(2,\mathbb{R})/U(1)$ theory from the gauge theory perspective is an interesting question to explore. Here above, we showed the dual CFT realization in terms of operators (6.38), understood as a correspondence between the correlators (6.41) and MHV amplitudes. In Appendix A.4, we discuss whether the parafermionic realization for the coset $SL(2,\mathbb{R})/U(1)$ theory can be uplifted to the full level-k $SL(2,\mathbb{R})$ WZW theory.

6.5 Operator product expansions

OPE of leaf amplitudes can be obtained from the collinear expansion of higher-point leaf amplitudes [226]; the leading z_{12} pole is given by

$$\begin{aligned}
\mathcal{O}^+_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^+_{\Delta_2}(z_2,\bar{z}_2) &\sim \frac{1}{z_{12}}B(\Delta_1-1,\Delta_2-1)\mathcal{O}^+_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2)+\ldots\\
\mathcal{O}^+_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^-_{\Delta_2}(z_2,\bar{z}_2) &\sim \frac{1}{z_{12}}B(\Delta_1+1,\Delta_2-1)\mathcal{O}^-_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2)+\ldots
\end{aligned} \tag{6.45}$$

where $B(x,y)$ is the Euler beta function and the dots stand for descendants. It is instructive to compute the corresponding OPE from the dual CFT description (6.15) in terms of Liouville theory dressed with affine currents (see also [227]). To see this explicitly, consider first the Liouville OPE

$$V_{\alpha_1}(z_1,\bar z_1)V_{\alpha_2}(z_2,\bar z_2) = \frac{1}{\pi}\int_{\alpha\in\frac{Q}{2}+i\mathbb{R}} d\alpha\, C(\alpha_1,\alpha_2,Q-\alpha)|z_{12}|^{2(h_\alpha-h_{\alpha_1}-h_{\alpha_2})}V_\alpha(z_2,\bar z_2)+\dots \tag{6.46}$$

where $C(\alpha_1,\alpha_2,\alpha_3)$ are the DOZZ Liouville structure constants [249, 250]

$$C(\alpha_1,\alpha_2,\alpha_3) = \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{(Q-\alpha_{123})/b}\frac{\Upsilon'(0)\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\alpha_{123}-Q)\Upsilon(\alpha_{12}^3)\Upsilon(\alpha_{13}^2)\Upsilon(\alpha_{23}^1)}\,, \tag{6.47}$$

where we have introduced the notation $\alpha_{ijk}=\alpha_i+\alpha_j+\alpha_k,\ \alpha_{ij}^k=\alpha_i+\alpha_j-\alpha_k$ and

$$\gamma(x)=\frac{\Gamma(x)}{\Gamma(1-x)}\,. \tag{6.48}$$

We refer the reader to [249, 250] for notations and definitions of the Υ-function. These structure constants obey the reflection property

$$C(\alpha_1,\alpha_2,Q-\alpha_3) = C(\alpha_1,\alpha_2,\alpha_3)\,S\left(\alpha_3-\frac{Q}{2}\right), \tag{6.49}$$

with the reflection coefficient being

$$S(P) = -\left(\pi\mu\gamma(b^2)\right)^{2P/b}\frac{\Gamma(1-2P/b)\Gamma(1-2Pb)}{\Gamma(1+2P/b)\Gamma(1+2Pb)}\,. \tag{6.50}$$

We now compute $c(\sigma_1,\sigma_2,\sigma_3)\equiv b\,C(b\sigma_1,b\sigma_2,Q-b\sigma_3)$ in the semiclassical limit ($b\to 0$), which yields

$$c(\sigma_1,\sigma_2,\sigma_3) = \frac{1}{2}\mu_{\rm cl}^{-\sigma_{12}^3}\frac{\sin\pi(2\sigma_3-1/b^2)}{\sin\pi(\sigma_{123}-1/b^2)}\frac{\Gamma(\sigma_{12}^3)\Gamma(\sigma_{13}^2)\Gamma(\sigma_{23}^1)\Gamma(\sigma_{123}-1)}{\Gamma(2\sigma_1)\Gamma(2\sigma_2)\Gamma(2\sigma_3-1)} \tag{6.51}$$

so that the OPE can be written as

$$V_{b\sigma_1}(z_1,\bar z_1)V_{b\sigma_2}(z_2,\bar z_2) = \frac{1}{\pi}\int d\sigma\, c(\sigma_1,\sigma_2,\sigma)|z_{12}|^{2(\sigma-\sigma_1-\sigma_2)}V_{b\sigma}(z_2,\bar z_2)+\dots \tag{6.52}$$

where we consider the summation over light operators, as heavy operators decouple when $b\to 0$. The ellipsis stands for higher-order contributions.

Using (6.52), the currents OPE (6.19) and the dictionary (6.15) of [97], we find

$$\begin{aligned}\mathcal{O}^{a_1,+}_{\Delta_1}(z_1,\bar z_1)\mathcal{O}^{a_2,+}_{\Delta_2}(z_2,\bar z_2) = &\int_{i\mathbb{R}_+} d\sigma\, c\left(\frac{1}{2}(\Delta_1-1),\frac{1}{2}(\Delta_2-1),\sigma\right)e^{-i\pi(\Delta_1+\Delta_2-2-2\sigma)}\mu^{\frac{1}{2}(\Delta_1+\Delta_2-2-2\sigma)}\\ &\times|z_{12}|^{2\sigma-\Delta_1-\Delta_2+2}\,\frac{\Gamma(\Delta_1-1)\Gamma(\Delta_2-1)}{2\pi\,\Gamma(2\sigma)}\\ &\times\left(\frac{\delta^{a_1a_2}}{z_{12}^2}\psi^+_{2\sigma+1}(z_2,\bar z_2)+\frac{if^{a_1a_2}{}_a}{z_{12}}\mathcal{O}^{a,+}_{2\sigma+1}(z_2,\bar z_2)\right)+\dots\end{aligned} \tag{6.53}$$

where

$$\psi^+_\Delta(z,\bar z) = e^{-i\frac{\pi}{2}(\Delta-1)}N^+_\Delta\,V_{\frac{b}{2}(\Delta-1)}(z,\bar z) \tag{6.54}$$

is a normalized Liouville field. The contribution to the OPE (6.53) corresponds to $2\sigma=\Delta_1+\Delta_2-2$. In that case one recovers the OPE in [227]; see (4.7) therein. Notice that, for the celestial operators to remain on the principal line, we have to constrain the integral over σ to pure imaginary values, $\sigma \in i\mathbb{R}_+$. Also, notice that, if $\Delta_{1,2}$ lie on the principal line, the exponent of $|z_{12}|$ in the OPE is purely imaginary. However we notice that for $\sigma = \frac{1}{2}\Delta_1 + \frac{1}{2}\Delta_2 - 1$ this exponent is vanishing and the Liouville structure constant develops a simple pole on the imaginary axis; namely

$$c\left(\frac{1}{2}(\Delta_1 - 1), \frac{1}{2}(\Delta_2 - 1), \sigma\right) \sim \frac{1}{\Delta_1 + \Delta_2 - 2 - 2\sigma} \tag{6.55}$$

for $\sigma \simeq \frac{1}{2}\Delta_1 + \frac{1}{2}\Delta_2 - 1$. By deforming the integration path we pick up the residue and we see that the OPE receives a contribution of the form

$$\begin{aligned}\mathcal{O}^{a_1,+}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{a_2,+}_{\Delta_2}(z_2,\bar{z}_2) \simeq\ & B(\Delta_1 - 1, \Delta_2 - 1)\times \\ & \times\left(\frac{\delta^{a_1 a_2}}{z_{12}^2}\psi^{+}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2) + \frac{i f^{a_1 a_2}{}_a}{z_{12}}\mathcal{O}^{a,+}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2)\right).\end{aligned} \tag{6.56}$$

The z_{12}^{-1} term precisely coincides with the leaf OPE, cf. [226]. In addition, we have the extra term

$$\frac{\delta^{a_1 a_2}}{z_{12}^2}\psi^{+}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2)\,, \tag{6.57}$$

which is a scalar contribution that contains a Liouville operator. The latter comes from the central term in the current algebra. One might also consider a realization of the affine current algebra at level κ, e.g. coming from a $SO(N)$ WZW factor. This would result in a current OPE

$$J^{a_1}(z_1)J^{a_2}(z_2) = \frac{\kappa\,\delta^{a_1 a_2}}{z_{12}^2} + \frac{i f^{a_1 a_2}{}_{a_3}}{z_{12}} J^{a_3}(z_2) + \dots \tag{6.58}$$

It has been observed in [227] that a similar scalar contribution can be interpreted from a bulk perspective as an interaction between gluons and an extra bulk scalar field.

Let us now move to the OPE produced from the parafermion representation (6.38). In the case of positive helicity operators, we obtain

$$\Psi^{j_1}_{\frac{k}{2},\frac{k}{2}}(z_1,\bar{z}_1)\Psi^{j_2}_{\frac{k}{2},\frac{k}{2}}(z_2,\bar{z}_2) \sim \frac{1}{\hat{N}^{j_1}_{\frac{k}{2},\frac{k}{2}}\hat{N}^{j_2}_{\frac{k}{2},\frac{k}{2}}}\int dj\,|z_{12}|^{\Delta-\Delta_1-\Delta_2+1}\,\tilde{C}_{\mathrm{WZW}}(j_1,j_2,j^*)\,\hat{N}^{j}_{\frac{k}{2},\frac{k}{2}}\,\Psi^{j}_{\frac{k}{2},\frac{k}{2}}(z_2,\bar{z}_2) \tag{6.59}$$

where $j^* \equiv -j-1$ and where

$$\tilde{C}_{\mathrm{WZW}}(j_1,j_2,j_3) = c_k\,C(\alpha_1,\alpha_2,\alpha_3) \tag{6.60}$$

are $SL(2,\mathbb{R})$ WZW structure constants in the spectral flow sector with $\omega_1+\omega_2-\omega_3 = -1$, with c_k being a k-dependent factor. The explicit expression of $\tilde{C}_{\mathrm{WZW}}(j_1,j_2,j_3)$ has been computed in [242–244, 248]; see those references for details. Considering (6.35), it can easily be shown that $\tilde{C}_{\mathrm{WZW}}(j_1,j_2,j^*)$ agree with the Liouville structure constants $C(\alpha_1,\alpha_2,Q-\alpha)$, and so the OPE. The conformal dimension Δ is related to j as in (6.39)-(6.40). We still have to prescribe how to integrate in the j-plane, which is done by requiring the conformal weights of the celestial theory lying on the principal line. Putting all this together, we obtain

$$\begin{aligned}\mathcal{O}^{a_1,+}_{2j_1+k+1}(z_1,\bar{z}_1)\mathcal{O}^{a_2,+}_{2j_2+k+1}(z_2,\bar{z}_2) = &\int dj\, \tilde{C}_{\mathrm{WZW}}\left(j_1,j_2,j^*\right)|z_{12}|^{\Delta-\Delta_1-\Delta_2+1} \\ &\times e^{-i\pi(\Delta_1+\Delta_2-\Delta-1)}\mu^{\frac{1}{2}(\Delta_1+\Delta_2-\Delta-1)}\frac{\Gamma(\Delta_1-1)\Gamma(\Delta_2-1)}{4b^{1/2}\,\Gamma(\Delta-1)} \\ &\times\left(\frac{\delta^{a_1a_2}}{z_{12}^2}\tilde{\psi}^+_{2j+k+1}(z_2,\bar{z}_2)+\frac{if^{a_1a_2}{}_a}{z_{12}}\mathcal{O}^{a,+}_{2j+k+1}(z_2,\bar{z}_2)\right)+\ldots\end{aligned} \tag{6.61}$$

where the first term contains an analogous scalar contribution

$$\tilde{\psi}^+_{2j+k+1}(z_2,\bar{z}_2) = e^{-i\pi(j+\frac{k}{2})}\,N^+_{2j+k+1}\hat{N}^j_{\frac{k}{2},\frac{k}{2}}\,\Psi^j_{\frac{k}{2},\frac{k}{2}}(z,\bar{z})\,. \tag{6.62}$$

In the semiclassical limit, for Δ to lie on the principal line, we can consider $j=-\frac{k}{2}+\frac{i}{2}\lambda$, $\lambda\in\mathbb{R}$. Identifying $\sigma=\frac{i}{2}\lambda$, we see that (6.61) reproduces the same structure as in (6.53); in the semiclassical limit, we recover the OPE (6.56).

We conclude the chapter with a brief summary. Specifically, we have demonstrated that celestial MHV leaf amplitudes can be reconstructed from parafermion amplitudes dressed with Kac-Moody currents. This provides an alternative realization to the dual model proposed in [97].
Moreover, our analysis of the OPEs in both the dressed Liouville and parafermion models reveals that the leaf OPEs [226] can be successfully recovered, albeit with an additional contribution that can be interpreted as arising from the presence of an extra bulk scalar coupled to gluons. Interestingly the OPE analysis provides additional information on the features of the bulk dual. Possible future development regard the extension of the proposed dualities beyond the MHV sector. In particular for the parafermion construction a preliminary analysis shows that next to next MHV amplitudes will be non-vanishing. It would be interesting to investigate if these leaf correlation functions can be related to some known bulk model.

Conclusions and outlook

Flat space holography is a broad and rapidly evolving field, with origins tracing back to discussions almost as old as the holographic principle itself [16, 251]. Only in recent years, thanks to the discovery of the deep connection between soft theorems and asymptotic symmetries, have concrete proposals emerged to relate bulk dynamics to boundary observables. Among these, Carrollian and celestial holography have become the most extensively studied approaches. With our work, we focused specifically on celestial holography, aiming to shed light on key features of CCFT, particularly the structure of its spectrum and the issue of its vanishing central charge, through both bottom-up and top-down perspectives.

Starting from the bottom up approach we have shown, using our understanding of the infrared properties of the bulk, how we can construct a set of logarithmic operators in CCFT. We have analyzed in great detail in particular two logarithmic multiplets: the stress tensor doublet (see sec. 5.8) and the supertranslation current doublet (see sec. 5.9) [1]. To prove that the introduction of the logarithmic partner of the stress tensor definitely solves the problem of the vanishing central charge, it is important to compute the value of parameter b (see sec. 5.6), which we consider an important future goal. This would allow us to further solidify the relevance of logarithmic CFT in the construction of celestial holography.
In section 5.8.1 we have already suggested a possible pathway to compute the value of b by looking at the order ϵ corrections to the subleading soft graviton theorem, where ϵ is the infrared regulator. Another way that we have not previously mentioned is to related t to the mode N^{vac}_{zz} discussed in section 5.9.1 which has already been proven to have non-vanishing correlation functions with the stress tensor T [156, 160, 161, 203].

If the value of b turns out to be finite this would open for interesting developments as, aside from consolidating CCFT as an LCFT, it will also allow us to draw some conclusion on the structure of its null vectors, which could then be used to fix four points CCFT amplitudes and in turn produce non perturbative results about the structure of bulk quantum gravity.
If on the contrary b turns out to be zero, it could still be possible to make sense of the vanishing central charge thanks to logarithmic CFT, turning the logarithmic doublet of the stress tensor into a logarithmic triplet.
In the case $b = 0$ however, it would be particularly interesting to further investigate the alternative solutions discussed in section 4.3 for addressing the vanishing central charge problem in CCFT. In [90], it was argued that the full stress tensor in CCFT can be realized as a linear combination of the stress tensors $T_{1,2}$ arising from the two CFT_2 descriptions associated with the time-like AdS_3 and space-like dS_3 slicing of Minkowski space. The corresponding central charges were computed to be $c_1 = -c_2 = i\infty$, providing a concrete explanation for how CCFT can be viewed as a non-trivial CFT with vanishing central charge, that effectively splits into two independent CFTs with opposite central charges.
Furthermore, recent studies [**?**, 252–256] have uncovered compelling connections between CCFT and the worldsheet theory of tensionless strings. Notably, the worldsheet model naturally exhibits a vanishing central charge as it splits into a matter CFT and a ghost CFT with opposite central charges. It would be valuable to examine these constructions in greater detail to provide

a bulk interpretation of the splitting between the two CFTs.

Beyond the issue of the vanishing central charge, we have identified another instance of logarithmic CFT behavior by analyzing the pair of operators $(\mathscr{C}_{zz}, \mathscr{B}_{zz})$ in (5.206) [1]. In particular, we have shown that since $\mathscr{B}_{zz}$ remains non-vanishing in standard perturbation theory, $\mathscr{C}_{zz}$ acquires the structure of a logarithmic field. A related phenomenon appears in QED, where the non-trivial action of an operator analogous to $\mathscr{B}_{zz}$ has been argued to play a key role in reconstructing the loop-corrected subleading soft photon theorem as a Ward identity of asymptotic charges [194]. This suggests that the behavior of $\mathscr{B}_{zz}$ deserves a more detailed investigation, particularly regarding its possible connection with the charges associated to the loop-corrected subleading soft graviton theorem.
Other works [92, 94] emphasize that logarithmic CFT features become manifest once loop corrections are taken into account, which is consistent with the interpretation of $\mathscr{B}_{zz}$ as being related to loop-corrected soft theorems.

An intriguing possibility is that an appropriate redefinition of the vacuum state could render the operator $\mathscr{B}_{zz}$ trivial, thereby turning $\mathscr{C}_{zz}$ into an ordinary primary field. If $\mathscr{B}_{zz}$ is indeed related to the loop-corrected subleading soft graviton theorem, such a modification of the vacuum would also affect the infrared structure of gravity, potentially leading to novel and interesting implications.

In Sec. 5.10, we have also shown that the logarithmic doublet $(\mathscr{C}_{zz}, \mathscr{B}_{zz})$ may be related to derivative operators of the form $\partial_\Delta \mathcal{O}_\Delta$. Such operators frequently appear in gravitationally dressed CFTs as a consequence of including the puncture operator in the Liouville spectrum. As discussed in the context of top-down constructions, Liouville theory appears to play a central role in several CCFT frameworks [97–99]. It would therefore be interesting to clarify the role of the puncture operator within CCFT and explore its potential connection to the logarithmic primary fields we have identified.

More broadly we hope that our analysis of the logarithmic properties of celestial CFT helps to shed some light on the nature of the spectrum of CCFT. As CCFT is also related to Carrollian holography [110], it would be interesting to study how the logarithmic operators gets transposed in the Carrollian set-up.

While analyzing the Log CFT properties of CCFT we mostly fed information from the bulk to the holographic dual. In this thesis we also worked in the opposite direction, in the context of the top-down approach where we showed that it is possible to recast the CFT_2 construction of the large N MHV leaf amplitudes proposed in [97] in terms of non-compact parafermions in direct product with $(1,0)$ affine Kac-Moody currents. This formulation does not involve the introduction of additional fields of negative dimension in the dressing of negative helicity gluon operators and follows from the H_3^+ WZW-Liouville correspondence [242, 245, 246, 257], applied to the spectrally flowed sector and restricted to the $SL(2,\mathbb{R})/U(1)$ coset theory. We have used this construction to obtain from the dual formulation the celestial OPEs computed in [64].
We hope that the 2D CFT realization of leaf amplitudes in terms of the H_3^+ WZW model presented here will help derive new insights for celestial CFTs inspired by the $\mathrm{AdS}_3/\mathrm{CFT}_2$

correspondence. Possible future development regard the extension of the proposed dualities beyond the MHV sector. In the case of the parafermion model it is possible to obtain non-vanishing amplitudes of the form $(- - - - + \cdots +)$, namely next to next to MHV. It would be interesting to compute their explicit form to investigate whether these CCFT correlation functions can be matched to those of a bulk gauge theory. While we believe that such a match is unlikely without further modifications to the model, the analysis may still provide valuable insights into the corrections required for the CCFT to correspond to a known bulk gauge theory.

Thanks to the dictionary between celestial and Carrollian operators [110], another possible exploration related to the top-down holographic construction would be to connect the leaf operators (6.15) and (6.38) to Carrollian primary operators. The structure of such Carrollian fields could provide some hint on the nature of the Carrollian dual and one of the first examples of a top-down Carrollian analysis.

Overall, we hope that the results presented here contribute to a better understanding of the algebraic and dynamical structures underlying celestial holography and motivate further work at the intersection of logarithmic CFTs, asymptotic symmetries, and top-down string-inspired constructions. In particular, clarifying the nature of celestial CFT, whether as a Logarithmic CFT or as a theory with a non-trivial decomposition into sectors, remains an exciting challenge with deep implications for the holographic description of flat space quantum gravity.

Appendix A

Technical computations

A.1 More on the log-shadow

In this appendix we would like to prove that $\widetilde{\Psi}$ transforms as a logarithmic primary of weights $(1-h, 1-\bar{h})$. Under a generic $SL(2,\mathbb{C})$ transformation:

$$\begin{aligned} z' &= f(z) = \frac{az+b}{cz+d}, \quad \bar{z}' = \bar{f}(\bar{z}) = \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}} \\ \partial f(z) &= \frac{1}{(cz+d)^2}, \quad \bar{\partial}\bar{f}(\bar{z}) = \frac{1}{(\bar{c}\bar{z}+\bar{d})^2} \end{aligned} \tag{A.1}$$

we have

$$\widetilde{\Psi}'(z', \bar{z}') = -K_{h,\bar{h}} \int d^2w \frac{\Psi'(w,\bar{w}) + \log|z'-w|^2 \Phi(w,\bar{w})}{(z'-w)^{2-2h}(\bar{z}'-\bar{w})^{2-2\bar{h}}} . \tag{A.2}$$

Changing the integration variable to $x' = w = f(x), \bar{x}' = \bar{w} = \bar{f}(\bar{x})$, we can use the following property:

$$\left|\det \frac{\partial w}{\partial x}\right| = \frac{1}{(cx+d)^2(\bar{c}\bar{x}+\bar{d})^2}, \quad z' - x' = f(z) - f(x) = \frac{z-x}{(cz+d)(cx+d)} \tag{A.3}$$

to rewrite (A.2) as:

$$\widetilde{\Psi}(z', \bar{z}') = -K_{h,\bar{h}} \int d^2x \frac{(cx+d)^{-2h}(\bar{c}\bar{x}+\bar{d})^{-2\bar{h}}}{(z-x)^{2-2h}(\bar{z}-\bar{x})^{2-2\bar{h}}} \left[\Psi'(x',\bar{x}') + \log \frac{|z-x|^2}{|cz+d|^2|cx+d|^2} \Phi'(x',\bar{x}')\right] \tag{A.4}$$

We can then substitute the transformed fields with the identities (5.57) and rewrite:

$$\begin{aligned} \widetilde{\Psi}'(z', \bar{z}') =& - K_{h,\bar{h}} (\partial f)^{h-1} (\bar{\partial}\bar{f})^{\bar{h}-1} \int d^2x \frac{1}{(z-x)^{2-2h}(\bar{z}-\bar{x})^{2-2\bar{h}}} \\ & \times \left[\Psi(x,\bar{x}) - \log|\partial f(x)|\Phi(x,\bar{x}) + \log\left(|z-x|^2|\partial f(z)||\partial f(x)|\right)\Phi(x,\bar{x})\right] \\ =& - K_{h,\bar{h}} (\partial f)^{h-1} (\bar{\partial}\bar{f})^{\bar{h}-1} \int d^2x \frac{\Psi(x,\bar{x}) + \log\left(|z-x|^2|\partial f(z)|\right)\Phi(x,\bar{x})}{(z-x)^{2-2h}(\bar{z}-\bar{x})^{2-2\bar{h}}} \\ =& (\partial f)^{h-1} (\bar{\partial}\bar{f})^{\bar{h}-1} \left[\widetilde{\Psi}(z,\bar{z}) - \log|\partial f(z)| \widetilde{\Phi}(z,\bar{z})\right] , \end{aligned} \tag{A.5}$$

which is exactly the transformation property of a $(1-h, 1-\bar{h})$ log primary.
The shadow logarithmic doublet $\widetilde{\mathcal{O}}_a = (\widetilde{\Psi}, \widetilde{\Phi})$ thus transforms as a log CFT doublet of weights

$(1-h, 1-\bar{h})$.

We can also be interested into computing the square of the log-shadow to verify if it also squares to 1. We need to compute:

$$\begin{aligned}\tilde{\tilde{\Psi}}(z,\bar{z}) = K_{1-h,1-\bar{h}}K_{h,\bar{h}}\int d^2w d^2x \frac{\Psi(x,\bar{x})}{(z-w)^{2h}(\bar{z}-\bar{w})^{2\bar{h}}(w-x)^{2-2h}(\bar{w}-\bar{x})^{2-2\bar{h}}} + \\ + K_{1-h,1-\bar{h}}K_{h,\bar{h}}\int d^2w d^2x \frac{\Phi(x,\bar{x})}{(z-w)^{2h}(\bar{z}-\bar{w})^{2\bar{h}}(w-x)^{2-2h}(\bar{w}-\bar{x})^{2-2\bar{h}}}\log\frac{|w-x|^2}{|z-w|^2}.\end{aligned} \tag{A.6}$$

Because $J = h - \bar{h} \in \mathbb{Z}/2$ the integral in (A.6) can be computed using the following formula found in [258]:

$$I_1 = \int d^2w \frac{1}{(z-w)^{2h}(\bar{z}-\bar{w})^{2\bar{h}}(w-x)^{2-2h}(\bar{w}-\bar{x})^{2-2\bar{h}}} = (-1)^{-4\bar{h}}\pi^2\frac{\Gamma(1-2h)\Gamma(2h-1)}{\Gamma(2\bar{h})\Gamma(2-2\bar{h})}\delta^2(z-x) \tag{A.7}$$

so that using the definition of $K_{h,\bar{h}}$ (A.6) turns out be equal to $(-1)^{-4\bar{h}}\Psi$. If we assume to work only with semi-integers conformal weights then it follows that the usual shadow squares to 1. To compute (A.6) we notice that:

$$\frac{1}{2}(\partial_h + \partial_{\bar{h}})I_1 = \frac{\partial I_1}{\partial \Delta} = \int d^2w \frac{1}{(z-w)^{2h}(\bar{z}-\bar{w})^{2\bar{h}}(w-x)^{2-2h}(\bar{w}-\bar{x})^{2-2\bar{h}}}\log\frac{|w-x|^2}{|z-w|^2} = I_2 \tag{A.8}$$

where $\Delta = h + \bar{h}$. This implies:

$$\begin{aligned}I_2 = \pi^2\delta^2(z-x)(-1)^{-4\bar{h}}\frac{\Gamma(1-2h)\Gamma(2h-1)}{\Gamma(2\bar{h})\Gamma(2-2\bar{h})}\times \\ \times\left(\frac{1}{1-2h} + \frac{1}{1-2\bar{h}} - 2\pi i - \pi\cot 2\pi h + \pi\cot 2\pi\bar{h}\right).\end{aligned} \tag{A.9}$$

Because h and $\bar{h}$ always differ by a semi-integer the cotangent part of this expression can be dropped and we get:

$$I_2 = \pi^2\delta^2(z-x)(-1)^{-4\bar{h}}\frac{\Gamma(1-2h)\Gamma(2h-1)}{\Gamma(2\bar{h})\Gamma(2-2\bar{h})}\left(\frac{1}{1-2h} + \frac{1}{1-2\bar{h}} - 2\pi i\right) \tag{A.10}$$

making the expression for the squared log-shadow:

$$\tilde{\tilde{\Psi}}(z,\bar{z}) = (-1)^{-4\bar{h}}\left[\Psi(z,\bar{z}) + \left(\frac{1}{1-2h} + \frac{1}{1-2\bar{h}} - 2\pi i\right)\Phi(z,\bar{z})\right]. \tag{A.11}$$

We can clearly see that it does not square to the identity in the general case, but it shifts the log primary with the partner primary. However we can see that the inverse shadow for the doublet is well defined, as:

$$S^{-1}_{\log}[\Psi](z,\bar{z}) = (-1)^{4\bar{h}}\left[\tilde{\Psi}(z,\bar{z}) - \left(\frac{1}{1-2h} + \frac{1}{1-2\bar{h}} - 2\pi i\right)\tilde{\Phi}(z,\bar{z})\right] \tag{A.12}$$

$$S^{-1}_{\log}[\Phi](z,\bar{z}) = (-1)^{4\bar{h}}\tilde{\Phi}(z,\bar{z}). \tag{A.13}$$

A.2 Regulated integral computation

In this appendix, we compute explicitly the integral (3.112) in dimensional regularization $d = 2 + 2\epsilon$. At first, let us rewrite the integral using Feynman parameters as

$$I_\epsilon = \mu_0^{2\epsilon} \int d^d z \frac{(z-z_1)^2(\bar{z}-\bar{z}_2)^2}{|z-z_1|^2|z-z_2|^2} = \mu_0^{2\epsilon} \int_0^1 du \int d^d z \frac{(z-z_1)^2(\bar{z}-\bar{z}_2)^2}{(u|z-z_1|^2+(1-u)|z-z_2|^2)^2} \,. \tag{A.14}$$

Focusing on the z integral, we re-parameterize it with the following change of variables:

$$z = x + uz_1 + (1-u)z_2, \quad \bar{z} = \bar{x} + u\bar{z}_1 + (1-u)\bar{z}_2 \quad z_{12} = z_1 - z_2 \,. \tag{A.15}$$

This leaves us with the expression

$$\int d^{2+2\epsilon}x \, \frac{|x|^4 + u^2(1-u)^2|z_{12}|^2 - 4u(1-u)|x|^2|z_{12}|^2}{(|x|^2+u(1-u)|z_{12}|^2)^2} \,, \tag{A.16}$$

where terms linear in $x, \bar{x}$ have been dropped due to the parity properties of the integral. The expression depends on $|x|^2$ so we can factorize the angular component and get:

$$J_\epsilon = \frac{2\pi^{1+\epsilon}}{\Gamma(1+\epsilon)} \int_0^\infty dr \, r^{1+2\epsilon} \frac{r^4 + R^4 - 4r^2R^2}{(r^2+R^2)^2} \,, \tag{A.17}$$

where we have defined $R^2 = u(1-u)|z_{12}|^2$.
J_ϵ is linearly divergent at infinity, as we can highlight by splitting it as:

$$J_\epsilon = \frac{2\pi^{1+\epsilon}}{\Gamma(1+\epsilon)} \left[\int_0^\infty dr \, r^{1+2\epsilon} - 6 \int_0^\infty dr \, \frac{r^{3+2\epsilon}R^2}{(r^2+R^2)^2} \right] , \tag{A.18}$$

where the second term is finite for $\epsilon < 0$. As the first term is a divergent dimensionful term that does not contain any scale, in dim-reg it can be directly set to zero. Any possible finite ambiguity will be taken into account by changing μ_0.
This leaves us with:

$$J_\epsilon = -\frac{12\pi^{1+\epsilon}}{\Gamma(1+\epsilon)} \int_0^\infty dr \, \frac{r^{3+2\epsilon}R^2}{(r^2+R^2)^2} = \frac{6\pi^{2+\epsilon}(1+\epsilon)}{\Gamma(1+\epsilon)\sin \pi\epsilon} R^{2+2\epsilon} \,. \tag{A.19}$$

I_ϵ can then be easily obtained by integrating over u, which gives

$$I_\epsilon = \frac{6\pi^{2+\epsilon}(1+\epsilon)\Gamma(2+\epsilon)^2}{\Gamma(1+\epsilon)\Gamma(4+2\epsilon)\sin \pi\epsilon} |z_{12}|^{2+2\epsilon} \mu_0^{2\epsilon} \,. \tag{A.20}$$

Notice that this expression as a function of ϵ can be analytically continued also in the region $\epsilon > 0$. If we now consider the expansion in small ϵ we get:

$$\begin{aligned} I_\epsilon &= \pi|z_{12}|^2 \left(\frac{1}{\epsilon} - \frac{2}{3} + \gamma_E + \log \pi + \log\big(|z_{12}|^2 \mu_0^2\big) \right) + \ldots \\ &= \pi|z_{12}|^2 \left(\frac{1}{\epsilon} + \log\big(|z_{12}|^2\mu^2\big) \right) + \ldots \end{aligned} \tag{A.21}$$

with $\mu^2 = \pi e^{\gamma_E - \frac{2}{3}} \mu_0^2$.

A.3 Propagator computation

We will here compute the free scalar propagator defined as:

$$G(z,w) = -\sum_{n\neq 0} \frac{1}{\lambda_n}\varphi_n(z,\bar z)\varphi_n(w,\bar w) \tag{A.22}$$

specializing in the case of a sphere of radius R, $\mathcal{M} = S^2$. In this case:

$$e^{2\sigma} = \sqrt{g} = \frac{2R^4}{(z\bar z + R^2)^2} \tag{A.23}$$

so that $\varphi_n(z,\bar z)$ must satisfy :

$$\partial\bar\partial\varphi_n = \frac{2R^4\lambda_n}{(z\bar z + R^2)^2}\varphi_n \tag{A.24}$$

With the redefinition:

$$z = R\frac{\sin\theta}{1-\cos\theta}e^{im\varphi}, \quad \Lambda_n = \frac{\lambda_n R^2}{2} \tag{A.25}$$

the differential equation turns into:

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi_n}{\partial\theta}\right) + \frac{1}{\sin\theta^2}\frac{\partial^2\varphi_n}{\partial\varphi^2} = -\Lambda_n\varphi_n \tag{A.26}$$

namely φ_n are precisely the spherical harmonics normalized such that:

$$\int d^2z\sqrt{g}\varphi_n(z,\bar z)\varphi_m^*(z,\bar z) = \delta_{nm} \tag{A.27}$$

namely:

$$\varphi_n(z,\bar z) = \frac{2}{R}Y_l^m\left(\arccos\left(\frac{|z|^2-R^2}{|z|^2+R^2}\right), \frac{i}{2}\log\frac{\bar z}{z}\right) \tag{A.28}$$

with $n = l, m$ and $\lambda_n = \frac{2}{R^2}l(l+1)$. Then we have to compute:

$$\begin{aligned} G(z_1,z_2) &= -\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\frac{2}{l(l+1)}Y_l^m(\theta_1,\varphi_1)Y_l^m(\theta_2,\varphi_2)^* \\ &= -\frac{1}{2\pi}\sum_{l=1}^{\infty}\frac{2l+1}{l(l+1)}\sum_{m=-l}^{l}\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta_1)P_l^m(\cos\theta_2)e^{im(\varphi_1-\varphi_2)}. \end{aligned} \tag{A.29}$$

We can use then the identity:

$$\begin{aligned} \sum_{m=-l}^{l}\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta_1)P_l^m(\cos\theta_2)e^{im(\varphi_1-\varphi_2)} &= P_l(\cos\theta_1\cos\theta_2 + \cos(\varphi_1-\varphi_2)\sin\theta_1\sin\theta_2) \\ &= P_l\left(1 - \frac{2R^2|z_1-z_2|^2}{(|z_1|^2+R^2)(|z_2|^2+R^2)}\right) \\ \sum_{l=1}^{\infty}\frac{2l+1}{l(l+1)}P_l(x) &= -1 - \log\frac{1-x}{2} \end{aligned} \tag{A.30}$$

to obtain:

$$G(z_1,z_2) = \frac{1}{2\pi}\left[1 + \log\left(\frac{R^2|z_1-z_2|^2}{(|z_1|^2+R^2)(|z_2|^2+R^2)}\right)\right] \tag{A.31}$$

Notice that using:

$$\frac{1}{2\pi}\partial\bar{\partial}\log|z-w|^2 = \delta^2(z-w) \tag{A.32}$$

we can prove that (A.31) satisfies:

$$\begin{aligned}\partial\bar{\partial}G(z,w) &= \frac{1}{2\pi}\partial\bar{\partial}\log R^2|z-w|^2 - \frac{1}{2\pi}\partial\bar{\partial}\log\left(|z|^2+R^2\right)\\ &= \delta^2(z-w) - \frac{1}{4\pi R^2}\frac{2R^4}{(|z|^2+R^2)}\\ &= \delta^2(z-w) - \frac{\sqrt{g}(z)}{\mathrm{Vol}(M)}\end{aligned} \tag{A.33}$$

This is expected to be a general feature of the propagator defined as in (A.22) on any manifold. In fact from (5.246) we have the completion relation:

$$\sqrt{g}\sum_n \varphi_n(z,\bar{z})\varphi_n(w,\bar{w}) = \delta^2(z-w) \tag{A.34}$$

which induces property (A.33) on the propagator:

$$\begin{aligned}\partial_z\bar{\partial}_z G(z,w) = e^{2\sigma}\sum_{n\neq 0}\varphi_n(z,\bar{z})\varphi_n(w,\bar{w}) = e^{2\sigma}(z,\bar{z})\left[\sum_n \varphi_n(z,\bar{z})\varphi_n(w,\bar{w}) - \frac{1}{\mathrm{Vol}(M)}\right] =\\ = \delta^2(z-w) - \frac{\sqrt{g}}{\mathrm{Vol}(M)}.\end{aligned} \tag{A.35}$$

We will now comment on a more general expression for the propagator. We will start by trying to obtain an expression for the logarithmic kernel:

$$G_0(z,w) = \frac{1}{2\pi}\log|z-w|^2 \tag{A.36}$$

in terms of the Laplacian basis $\varphi_n(z)$. We start from the ansatz:

$$G_0(z,w) = \sum_{n\neq 0} a_n\varphi_n^*(w)\varphi_n(z) + a_0(z)\varphi_0^2 + a_0(w)\varphi_0^2 \tag{A.37}$$

and use the completion relation to obtain:

$$\begin{aligned}\int d^2w\sqrt{g}\varphi_n(w)G_0(z,w) &= a_n\varphi_n(z) + \varphi_0^2\int d^2w\sqrt{g}\varphi_n(w)a_0(w), \text{ for } n\neq 0\\ \int d^2w\sqrt{g}G_0(z,w) &= a_0(z) + \varphi_0^2\int d^2w\sqrt{g}a_0(w)\end{aligned} \tag{A.38}$$

Taking the Laplacian of the first line we can use (A.32) to rewrite:

$$\sqrt{g}\varphi_n(z) = a_n\partial\bar{\partial}\varphi_n(z) = -a_n\lambda_n\sqrt{g}\varphi_n(z) \tag{A.39}$$

which fixes $a_n = -1/\lambda_n$. On the other hand we can assume:

$$a_0(z) = \int d^2w\sqrt{g}G_0(z,w) + k \tag{A.40}$$

and using the second line in (A.38) obtain an expression for k:

$$k = -\frac{1}{2}\varphi_0^2 \int d^2z\sqrt{g} \int d^2w\sqrt{g}G_0(z,w) \tag{A.41}$$

This gives the following expression:

$$\begin{aligned} G_0(z,w) = G(z,w) + \frac{1}{\mathrm{Vol}(M)} \int d^2x\sqrt{g}\,[G_0(z,x) + G_0(w,x)] + \\ -\frac{1}{\mathrm{Vol}^2(M)} \int d^2x\sqrt{g} \int d^2y\sqrt{g}G_0(x,y) \end{aligned} \tag{A.42}$$

which allow us to write a general expression for the propagator:

$$\begin{aligned} G(z,w) &= \frac{1}{\mathrm{Vol}(M)} \int \frac{d^2x}{2\pi}\sqrt{g}\log\frac{|z-w|^2}{|z-x|^2|w-x|^2} + \frac{1}{\mathrm{Vol}^2(M)} \int d^2x\sqrt{g} \int d^2y\sqrt{g}G_0(x,y) = \\ &= \frac{1}{2\pi}\frac{1}{\mathrm{Vol}^2(M)} \int d^2x\sqrt{g} \int d^2y\sqrt{g}\log\frac{|z-w|^2|x-y|^2}{|z-x|^2|w-y|^2} \end{aligned} \tag{A.43}$$

We can see that this expression is dilatation invariant but not trivially invariant under special conformal transformations. Using this we can see easily obtain A.31.

A.4 H_3^+ representation

Primary fields of the $SL(2,\mathbb{R})$ WZW theory, including those in the so-called spectrally flowed representation, can be expressed in terms of parafermions $\Psi^j_{m,\bar{m}}$ for the coset model $SL(2,\mathbb{R})/U(1)$ and an extra $U(1)$ field. Denoting by $\Phi^{j,w}_{m,\bar{m}}(z,\bar{z})$ fields of isospin j and spectral flow number $w \in \mathbb{Z}$, the relation is given by [243, 244, 259]

$$\Phi^{j,w}_{m,\bar{m}}(z,\bar{z}) \;=\; e^{i\sqrt{\frac{2}{k}}\left(m+w\frac{k}{2}\right)\phi(z)+i\sqrt{\frac{2}{k}}\left(\bar{m}+w\frac{k}{2}\right)\bar{\phi}(\bar{z})}\,\Psi^j_{m,\bar{m}}(z,\bar{z})\,, \tag{A.44}$$

with the timelike free boson $\phi(z,\bar{z}) = \phi(z) + \bar{\phi}(\bar{z})$. The role of this field is to restore the $U(1)$ factor of the coset theory, which is realized by the current $J^3(z) = -i\sqrt{\frac{k}{2}}\partial\phi(z)$ obeying the operator product expansion

$$J^3(z)\,\Phi^{j,w}_{m,\bar{m}}(w,\bar{w}) \;\sim\; \left(m + \frac{k}{2}w\right)\frac{\Phi^{j,w}_{m,\bar{m}}(w,\bar{w})}{(z-w)}\,. \tag{A.45}$$

The conformal weights of the spectral flowed fields (A.44) are

$$\begin{aligned} h_{\Phi^{j,w}_{m\bar{m}}} &= -j(j+1)b^2 - wm - \frac{2+b^{-2}}{4}w^2\,, \\ \bar{h}_{\Phi^{j,w}_{m\bar{m}}} &= -j(j+1)b^2 - w\bar{m} - \frac{2+b^{-2}}{4}w^2\,, \end{aligned} \tag{A.46}$$

with $b^{-2} = k-2$.

H_3^+ WZW-Liouville correspondence [242] leads to write the correlators $\langle\prod_{\ell=1}^n \Phi^{j_\ell,w_\ell}_{m_\ell,\bar{m}_\ell}(z_\ell,\bar{z}_\ell)\rangle$ in terms of Liouville correlation functions $\langle\prod_{\ell=1}^n V_{\alpha_\ell}(z_\ell,\bar{z}_\ell)\prod_{a=1}^{n-2-r} V_{-\frac{1}{2b}}(y_a)\rangle$ through a formula that looks like (6.32), but replacing

$$\beta_{\ell\ell'} \to \beta_{\ell\ell'} - \frac{2}{k}\left(m_\ell + \frac{k}{2}w_\ell\right)\left(m_{\ell'} + \frac{k}{2}w_{\ell'}\right)$$
$$\bar{\beta}_{\ell\ell'} \to \bar{\beta}_{\ell\ell'} - \frac{2}{k}\left(\bar{m}_\ell + \frac{k}{2}w_\ell\right)\left(\bar{m}_{\ell'} + \frac{k}{2}w_{\ell'}\right), \tag{A.47}$$

together with the restrictions

$$\sum_{\ell=1}^{n} m_\ell = \sum_{\ell=1}^{n} \bar{m}_\ell = \frac{k}{2}r\,, \qquad \sum_{\ell=1}^{n} w_\ell = -r\,. \tag{A.48}$$

The above conditions ensure the conservation of the charge associated to the $U(1)$ current, namely

$$\sum_{\ell=1}^{n}\left(m_\ell + \frac{k}{2}w_\ell\right) = \sum_{\ell=1}^{n}\left(\bar{m}_\ell + \frac{k}{2}w_\ell\right) = 0\,. \tag{A.49}$$

Using the above relation between Liouville theory and the WZW model, it would be natural to directly propose a map between gluon operators and H_3 WZW fields of the form

$$\mathcal{O}_\Delta^{\pm\, a,\,\epsilon}(z,\bar{z}) \,\propto J^a(z)\,\Phi^{j,w_\pm}_{m,\bar{m}}(z,\bar{z})\,, \tag{A.50}$$

for some specific value of $j, w_\pm, m$ and $\bar{m}$. To see if this is possible, let us have a closer look at what terms in (6.44) can be reproduced from an n-point function of operators of the form (A.50). From (6.44), it is clear that the cyclic $1/(z_{12}z_{23}\ldots z_{n1})$ factor will come again from the current n-point function. This means that the $SL(2,\mathbb{R})$ WZW amplitude should be able to reproduce both the Liouville and the z_{12}^4 contribution. However, it turns out that, due to the constraints specified in (A.48) and the values of the $\beta_{\ell\ell'}$ exponents given in (A.47), one can show that no combination of $j, w_\pm, m, \bar{m}$ can result in a consistent dictionary that reproduces the factor z_{12}^4. This indicates that constructing gluon operators out of a $SL(2,\mathbb{R})$ primary and a Kac-Moody current would necessarily require the introduction of additional fields.

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