Bounds on the epsilon expansion

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EPFL

Classifying CFTs

Eric P. has hopefully talked about the **conformal bootstrap**.
Grandiose ambition: (infinite) list of all relativistic, unitary CFTs in $d \geqslant 2$.
Pretty difficult (seriously!).

Bootstrap great for systems with "simple" low-energy spectrum/OPEs.
Classic examples: 3d Ising/$O(N)$ CFTs.
Have few "fundamental" DoFs or big symmetry groups.

Not great when low-energy spectrum is a mess:
too many eqns for numerics & analytics [at this point in time].
Yet expect that many (most?) CFTs are of this form.

Today: look at class of systems where we can make progress analytically.
At least get a feeling for complexity!

Multiscalar CFTs

Will look at system of N real scalars ϕ^i w/ quartic coupling in $2 \leqslant d < 4$ dimensions:

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{16\pi^2\Lambda^{4-d}}{4!}\lambda_{ijkl}\,\phi^i\phi^j\phi^k\phi^l + \text{counterterms}.$$

In MS in $d = 4 - \varepsilon$, $\varepsilon \ll 1$, beta function reads

$$\beta(\lambda)_{ijkl} = -\varepsilon\lambda_{ijkl} + \lambda_{ijmn}\lambda_{klmn} + \lambda_{ikmn}\lambda_{jlmn} + \lambda_{ilmn}\lambda_{jkmn} + \text{higher loops}.$$

There exist perturbative solutions to $\beta(\lambda) \equiv 0$, of form

$$\lambda_{ijkl} = \varepsilon\lambda^{(1)}_{ijkl} + O(\varepsilon^2).$$

Will look at one-loop term $\lambda^{(1)}_{ijkl}$ for rest of talk.
No need to discuss scheme dependencies, mass terms.

Multiscalar CFTs (2)

Well-explored since birth of RG [Wilson 1972] + many, many others.
Very relevant for stat mech, after computing higher loops and setting $\varepsilon \to 1, 2$.

Today: other POV. Many obvious open questions:

- Can we classify, or at least count, all solutions for a given N?
- Geometry of set of solutions? Are there conformal manifolds?
- What does a typical solution look like? Global symmetry?
- What can we say about observables (e.g. critical exponents)?

Although focus on multiscalar theories in $d = 4 - \varepsilon$,
lot of reasoning applies to *any* beta function of the form

$$\beta(\lambda)_I = -\varepsilon \lambda_I + C_I{}^{JK} \lambda_J \lambda_K + \dots$$

e.g. could add Yukawas, ϕ^n-type interactions at diferent critical d.
Small parameter can be something else ($1/k$ in $\mathcal{N} = 2$ Chern-Simons + matter).

State of the art

What has been done?

- Construct (families of) solutions with large global symmetry (few couplings)
- Prove structural theorems about cases with few couplings (stability etc.)
- Classify all **isotropic** CFTs for low $N \lesssim 6$
- Extensive study of group-theoretical properties.

Without imposing symmetry, classification?

- # of couplings grows as $\sim \frac{1}{24} N^4$.
  $N = 1$: textbook.
  $N = 2$: solved in by [Osborn-Stergiou 2017], after 43 years!
  $N = 3$: 15 eqns, too many for Gröbner basis.
  $N \geqslant 4$: ?
- Different spirit: scan through theory space numerically?
  Difficult in its own right. [WIP with Z. Fisher]

Let X be the space of solutions inside the set of all real couplings. We can actually bound the total number of solutions from above, using some tricks from algebraic geometry. If we define

$$B(X) = \sum_i b_i(X)$$

then

$$\ln B(X) \leqslant (\ln 3)(D(N) - 1) + \ln 2 \underset{N \gg 1}{\sim} 0.0458\, N^4$$

where

$$D(N) = \binom{N+3}{4}$$

is the dimension of the total space [Milnor-Oleinik-Thom].

Discussion of (an)isotropy

Historically, often imposed **isotropy**:
global symmetry group G has unique quadratic invariant δ_{ij}.
Many consequences in theory & practice:

- fields ϕ^i form irrep of G,

$$\langle \phi_i(x)\phi_j(0)\rangle = \delta_{ij}/|x|^{2\Delta_\phi}$$

  can talk about critical exponent η
- unique mass operator $\mathcal{O}_2 = \delta_{ij}\phi^i\phi^j$, critical exponent ν
- typically $O(\text{few})$ number of quartic couplings I_4 allowed, solvable
- tensor λ_{ijkl} obeys various simplifying identities

Without isotropy, RG picture is murky. $\sim \frac{1}{2}N^2$ mass terms, $\sim \frac{1}{24}N^4$ quartic couplings.

Famous solutions

Trivial theory $\lambda_{ijkl} = 0$. Next: $O(N)$ = N-vector = Heisenberg model

$$\lambda_{ijkl} = \frac{1}{N+8}\,\delta_{ij}\delta_{kl} + \text{symm.}$$

which for $N = 1, 2$ is known as Ising/Wilson-Fisher resp. XY model.
Zoo of other known solutions:

- cubic: $O(N)$ deformed by $\sum_i \phi_i^4$, discrete symmetry group $|G| = 2^N \cdot N!$
- biconical-type solutions with symmetry $O(N_1) \times O(N_2) \times \ldots$.
- "tensor" models built out of matrix fields Φ_{ab} [SYK literature]
- ...

Note: solutions are additive.

This talk

Complete classification every for $N = O(\text{few})$ seems too challenging.
What will be done today:

- rule out parts of high-dimensional ($\sim \frac{1}{24}N^4$) theory space
- comments about (non)existence of conformal manifolds
- bounds on one-loop anomalous dimensions

as well as generalization to N gauged complex scalars.

First hard bounds found in [Brézin–Le Guillou–Zinn-Justin 1973]
for simple = isotropic systems.
Recent revival of this strategy by [Osborn-Stergiou 2017] and [Rychkov-Stergiou 2018].
Will build on this work.

Orbits

Take sum of two Ising models

$$\lambda_{ijkl}\phi^i\phi^j\phi^k\phi^l = \frac{1}{3}(\phi_1^4 + \phi_2^4).$$

Rotate $\phi_{1,2}$: still solution — this is a 1d family of equivalent theories, $S^1 \subset \mathbb{R}^5$.

This is generic. Such sets of theories are called **orbits**: what you get when you act on λ_{ijkl} with $O(N)$.
In general

$$\dim(G) + \dim(\text{orbit}) = \dim O(N) = \tfrac{1}{2}N(N-1)$$

so the $O(N)$ fixed point is a single point, but less symmetric theories are manifolds in the landscape of couplings. Easily observed in numerics!

Consequence: instead of talking about coordinates in theory space, should discuss invariants λ_{ijkl}^2, λ_{iijj}, ..., instead (cst. on orbits).

Known results

Most basic invariant: norm

$$\|\lambda\|^2 = \lambda_{ijkl}^2 \geqslant 0.$$

R-S recently showed that

$$\beta(\lambda) = 0 \;\Rightarrow\; \|\lambda\|^2 \leqslant \begin{cases} \frac{1}{36}(3+4\sqrt{2}) \approx 0.240468 & N=2 \\ \frac{1}{12}(1+2\sqrt{3}) \approx 0.372008 & N=3 \\ \frac{1}{8}N & N \geqslant 4 \end{cases}.$$

Interpretation: fixed points can't live in the whole space of dimension $\sim N^4$, they live inside a sphere of radius $\sim \sqrt{N}$.

Proof: bound individual elements of λ_{ijkl} using $\beta = 0$.

For all but finite # of N, upper bound saturated. $N = 3$ mysterious.

New: lower bound

Proceed in same spirit. Argue that any fixed point must satisfy:

$$\lambda_{ijkl} = 0 \quad \text{or} \quad \|\lambda\| \geqslant \frac{1}{3}.$$

Interpretation: fixed points don't live inside a disk, but inside an annulus.
Or: can't have arbitrarily weak CFTs.

Proof: fixed point obeys

$$\lambda_{ijkl} = \lambda_{ijmn}\lambda_{klmn} + 2 \text{ terms.}$$

Now bound first term on RHS using Cauchy-Schwartz:

$$\sum_{mn}(\lambda_{ijmn}\lambda_{klmn})^2 \leqslant \sum_{mn}\lambda_{ijmn}^2 \sum_{pq}\lambda_{klpq}^2 \;\Rightarrow\; \|\text{RHS}\| \leqslant 3\|\lambda\|^2.$$

But then

$$\|\lambda\| \leqslant 3\|\lambda\|^2 \;\Rightarrow\; 3\|\lambda\|(\|\lambda\| - 1/3) \geqslant 0. \quad \square$$

Lower bound (2)

C-S argument gives info about limiting cases.
Here: learn that the bound is saturated if there exist matrices R, S such that

$$\lambda_{ijkl} = R_{ij} S_{kl}.$$

By permutation symmetry of λ_{ijkl} can argue that $R = S$ and

$$(S^2)_{ij} = \mathrm{tr}(S) S_{ij}.$$

But then every eigenvalue ν of S must obey

$$\nu = 0 \quad \text{or} \quad \nu = \mathrm{tr}(S).$$

Only possible if $\leqslant 1$ non-zero eigenvalue. So $\exists\, u_i$ s.t. $S_{ij} \propto u_i u_j$ which implies

$$\text{bound saturated} \quad \Leftrightarrow \quad \|\lambda\| = 1/3 \quad \Leftrightarrow \quad \lambda_{ijkl} = \tfrac{1}{3} u_i u_j u_k u_l.$$

Conclusion: Ising model is the most weakly-coupled CFT, for any N!

More bounds

To proceed, need to introduce more refined invariants, like

$$\lambda_{iijj} =: a_0 \in \mathbb{R}.$$

Use that space of rank-4 tensors splits in irreps as

$$\text{vector space of all couplings} = \text{spin-0} \oplus \text{spin-2} \oplus \text{spin-4}$$

with associated projection operators. Gives rise to invariants:

$$a_2 = \frac{6}{N+4}\lambda_{ijkk}^2 - \frac{2(N+2)}{N+4}\,a_0^2$$
$$a_4 = \|\lambda\|^2 - \frac{3}{N(N+2)}\,a_0^2 - a_2.$$

Normalization of $a_{2,4}$ given by projection operators.
Naive bound

$$0 \leqslant \frac{3}{N(N+2)}\,a_0^2 + a_2 + a_4 = \|\lambda\|^2 \leqslant \tfrac{1}{8}N$$

as follows from R-S.

More bounds (2)

Start with simplest invariant, a_0. Will argue that it lives inside a strip

$$a_0 \in [a_-, a_+]$$

instead of $\mathbb{R}$.
Proof: apply $O(N)$ projectors to the beta function equation, e.g.

$$\lambda_{iijj} = \lambda_{iimn}\lambda_{jjmn} + 2 \text{ terms}$$

and decompose this into invariants. Messy but doable:

$$\frac{1}{2N}\, a_0(N - a_0) = \|\lambda\|^2 + \frac{N+4}{12}\, a_2.$$

After tedious manipulation:

$$\frac{N}{2}\left[1 - \sqrt{1 - \frac{8}{9N}}\,\right] < a_0 \leqslant \frac{N(N+2)}{N+8}.$$

Upper bound is the $O(N)$ model. Any theories close to the lower bound

$$a_0 > \frac{2}{9} + O(1/N) = ?$$

More bounds (3)

Next invariant a_2 is measure of anisotropy. Appears in refinement of R-S:

$$\frac{1}{9} < \|\lambda\|^2 + \frac{1}{12}(N+4)\, a_2 \leqslant \begin{cases} \frac{3N(N+2)}{(N+8)^2} & N = 2, 3 \\ \frac{1}{8}N & N \geqslant 4 \end{cases}$$

(notice that by construction $a_2 \geqslant 0$).
For **isotropic** theories $a_2 = 0$, so for those theories it reduces to R-S for $N \geqslant 4$.

For $N = 2$ there is a complete classification, skip this case.

For $N = 3$ it shows that the $O(3)$ model is the most strongly coupled CFT. Refinement only tiny:

$$0.371901 < 0.372008$$

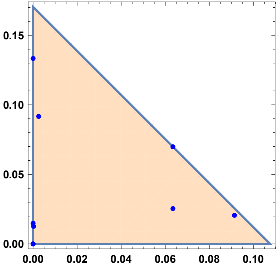
despite the fact that proof is very different.

More bounds (4)

Finally can obtain

$$\frac{1}{12}(N+16)a_2 + a_4 \leqslant \frac{N(N+2)}{8(N+8)}.$$

For $N = 3$ this is **almost** saturated (Ising + Ising + trivial CFT):



In particular can show that the anisotropy a_2 is at most of order unity:

$$a_2 \leqslant 3/2 + O(1/N)$$

even though *a priori* it could be of order N.

Bounds on anomalous dimensions

Composite operators $\mathcal{O}_a$ have an anomalous dimension at one loop:

$$\Delta[\mathcal{O}_a] = \Delta_{\text{classical}}[\mathcal{O}_a] + \gamma_a \varepsilon + O(\varepsilon^2)$$

which can be determined through eigenvalue problem (operator mixing!):

$$\mathcal{V}(x)\mathcal{O}_a(0) = \sum_b \frac{1}{|x|^4} C_a{}^b(\lambda)\mathcal{O}_b(0), \quad \mathcal{V} := \lambda_{ijkl}\phi^i\phi^j\phi^k\phi^l$$

Focus on operators of form $\mathcal{O} \sim \phi^r$:

$$\mathcal{O}_a = T_{a|i_1\cdots i_r}\,\phi^{i_1}\cdots\phi^{i_r}$$

for some tensor T_a of rank $r \geqslant 2$. There are

$$D(N,r) \sim \frac{1}{r!}N^r$$

tensors: huge mixing matrix that needs to be diagonalized.

Anomalous dimensions (3)

First, get bound: for operators with r copies of ϕ, we find

$$\mathcal{O}_a \sim \phi^r : \quad |\gamma_a| \leqslant \frac{r(r-1)}{2} \|\lambda\|.$$

Second:

bound saturated $\leftrightarrow$ $\lambda =$ Ising model and $\mathcal{O} = (u \cdot \phi)^4$.

Third, get **sum rules** which are of the form:

$$\frac{1}{D(N,r)} \sum_a \gamma_a = \frac{r(r-1)}{N(N+1)} a_0,$$

$$\frac{1}{D(N,r)} \sum_a \gamma_a^2 = p_0 a_0^2 + p_1 a_2 + p_2 \|\lambda\|^2$$

for some rational functions $p_i(N,r)$. [covariance, r=0,1,3,4]

Note: for $r = 2$ + isotropic, bound on ν appeared in [Brézin et al. 1973].

Anomalous dimensions (3)

In particular, find that

$$\text{“typical anom. dim.”} := \sqrt{\frac{1}{D(N,r)} \sum_a \gamma_a^2} \underset{r \gg 1}{\leqslant} \frac{r^2}{3N} + O(1/N).$$

This has the same scaling as a bound previously found by [Kehrein-Wegner-Pismak], valid for the $O(N)$ model:

$$O(N) \text{ model, any operator with } r \text{ fields}: \quad 0 \leqslant \gamma \underset{r \gg 1}{\leqslant} \frac{3r^2}{2(N+8)}$$

(Evanescents.) Lower bound in general case?
Can we repeat this for other operators? For example

$$\text{double-trace operators} \qquad \phi^i \partial^\ell (\partial^2)^n \phi^j + (i \leftrightarrow j)$$
$$\text{currents} \qquad \phi^i \partial^\ell \phi^j - (i \leftrightarrow j)$$

...in complete generality?

Anomalous dimensions of ϕ^i at NLO

The anomalous dimensions of operators ϕ^i are generated at **two** loops:

$$\Delta[\Phi_a] = \tfrac{1}{2}(d-2) + \gamma_a \varepsilon^2 + O(\varepsilon^3), \quad \Phi_a =: T_{a|i}\phi^i$$

so slightly different framework. Now get

$$\tfrac{1}{12}\lambda_{ipqr}\lambda_{jpqr}T_{a|j} = \gamma_a T_{a|i}.$$

The sum rule/bound reads

$$\frac{1}{N}\sum_a \gamma_a = \frac{1}{12N}\|\lambda\|^2, \quad 0 \leqslant \gamma_a \leqslant \frac{1}{12}\|\lambda\|^2.$$

Again it can be shown that

$$\gamma = \frac{1}{12}\|\lambda\|^2 \ \leftrightarrow\ \lambda = \text{Ising and } \Phi = u\cdot\phi$$

and if there are k zero eigenvalues $\Rightarrow$ interaction involves at most $N-k$ fields.

Note: isotropic case already in [Brézin et al. 1973].

Complex/gauged case

Can instead look at N complex scalars, imposing overall $U(1) + \mathcal{C}$ — want to gauge later:

$$\mathscr{L} = |\partial_\mu \phi^i|^2 + \frac{24\pi^2}{6} g_{ijkl} \phi^i \phi^j (\phi^*)^k (\phi^*)^l.$$

At fixed points, reality condition $g_{ijkl} \in \mathbb{R}$ sufficient for unitarity.
Can be embedded in action with $2N$ real fields. No new interesting bounds (surprisingly?), except Ising is replaced by $O(2) =$ XY model.

More interesting: gauge the $U(1)$ that rotates ϕ^i = (multi)scalar QED:

$$\mathscr{L}' = \tfrac{1}{4} F_{\mu\nu}^2 + |D_\mu \phi^i|^2 + \text{quartic interaction}.$$

Beta function for coupling e reads

$$\beta(e) = -\frac{\varepsilon}{2} e + \frac{N}{48\pi^2} e^3 + \ldots$$

so at any fixed point

$$e_* = 0 \quad \text{or} \quad e_*^2 \sim \frac{\varepsilon}{N}.$$

Known cases

Well-known solution: $PSU(N) = SU(N)/\mathbb{Z}_N$ global symmetry, having an interaction

$$V(\phi) \propto (\sum_i |\phi^i|^2)^2$$

which exists only for $N \geqslant 183$ (famous result!).

For even N, can split fields into two groups and get other fixed points with big global symmetries $G \sim SU(N/2)^2 \times$ Abelian factors..
These two only exist for $N \geqslant 198$ — see [Benvenuti-Khachatryan 2019] $\times$ 2.

The 3d physics of scalar QED (= Abelian Higgs) at $N = O(\text{few})$ is of serious experimental interest. Not in this talk...

Gauged beta functions

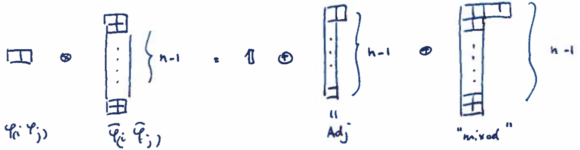
The beta functions (omitting tensor structures and $O(1)$ coefficients) are of the form

$$\beta(g)_{ijkl} = -\varepsilon g + g^2 - e^2 g + e^4 \mathbb{1} + 2 \text{ loops}$$

so right scaling with ε.

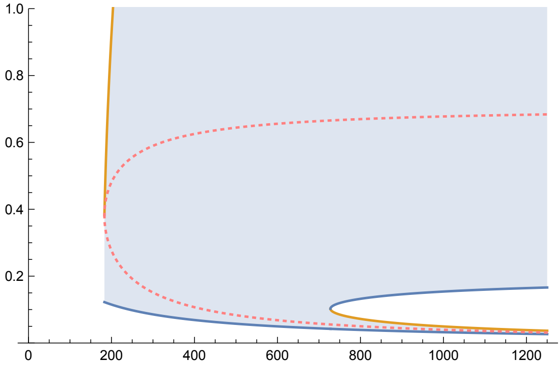
Since $e_*^2 \sim 1/N$, the beta functions differ from the *ungauged* case only by terms of order $1/N$ and $1/N^2$.
Phenomenology should not be too different from the real case for $N \geqslant O(\text{few})$.

Introduce invariants as before, only $PSU(N)$ instead of $O(N)$.



Bounds on gauged case

Allowed region in theory space ($PSU(N)$ dotted):



No solutions at all for

$$N < N_* := 90 + 24\sqrt{15} \approx 182.9516.$$

Bound for $PSU(N)$ theory applies to *all possible fixed points*!

Discussion

- Like to think that Wilson-Fisher/Ising is a completely generic CFT.
  But in many ways, it's a special point in theory space.
  To a lesser extent $O(N)$.
- Useful to consider higher invariants?
- $N = 3$ case special. Can we show that there's nothing new?
  Lemma about discrete symmetries.
  Representation as sum of quadratic forms (Hilbert)?
- For $4 \leqslant N \leqslant O(10)$, brute-force scan through landscape and compare with known results. Can we compare # of solutions to algebraic geometry bound?
- Explore bosonic QED more. Impossible (?) to explore numerically.
- Other theories.