Scattering Amplitudes and Extra Dimensions in AdS/CFT

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Caltech, Simons Collaboration on Nonperturbative Bootstrap

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One of the physical world's most fascinating features is its dependence on scale.

In quantum field theory, this dependence is encoded in the renormalization group.

A **conformal field theory** (CFT) is a renormalization group fixed point, and hence essential to the study of quantum field theory.
We are living in a **golden age of CFT**.

There has been a proliferation of new ideas about what, fundamentally, a CFT *is*.

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Space of possible consistent CFTs
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**Conformal bootstrap**: the program of classifying conformal field theories using symmetries and other abstract constraints.

- What is the range of **possible quantum critical behaviors**?
- What **hidden structures** govern CFTs?

Space of possible consistent CFTs
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The bootstrap paradigm is especially powerful in the context of the AdS/CFT Correspondence.

"Stringy" (?)
At first, AdS/CFT was mostly used as a tool for determining strongly coupled field theory dynamics from simple, semiclassical calculations in gravity.

\[ \text{AdS} \rightarrow \text{CFT} \]

More recently,

\[ \text{AdS} \leftarrow \text{CFT} \]

We are learning about quantum gravity from insights and precision computations in CFT.
The conformal bootstrap typically constrains CFT correlation functions.

AdS scattering amplitudes $\longleftrightarrow$ CFT correlation functions

Loop expansion in AdS $\longleftrightarrow$ $1/N$ expansion in CFT

\[
\langle O_1 O_2 O_3 O_4 \rangle = \text{Planar (1/N}^2\text{)} + \text{Non-planar (1/N}^4 + \ldots \text{)}
\]

Today’s talk will focus on AdS loop amplitudes: their computation, using bootstrap-inspired techniques, and their utility in answering questions about string theory.
The conformal bootstrap typically constrains CFT correlation functions.

AdS scattering amplitudes $\leftrightarrow$ CFT correlation functions
Loop expansion in AdS $\leftrightarrow$ $1/N$ expansion in CFT

The talk has 3 components.
I. Loops in AdS

Why loops?

1. **Curved space** amplitude-ology
2. The only known approach to generic **non-planar CFT** data at strong coupling
3. **Fundamental** objects in AdS quantum gravity
I. Loops in AdS

Why loops?

1. *Curved space* amplitude-ology
2. The only known approach to generic non-planar CFT data at strong coupling
3. *Fundamental* objects in AdS quantum gravity

Before 2016, what was known?

New idea: *AdS Unitarity Method*
II. Application: String amplitudes from N=4 SYM

String perturbation theory is stuck in the genus expansion.
State-of-the-art for graviton 4-pt amplitude in Minkowski space:

[D'Hoker, Phong '05: “Two-loop superstrings VI: Non-renormalization theorems and the 4-point function”]

[Gomez, Mafra '13]
II. Application: String amplitudes from N=4 SYM

String perturbation theory is stuck in the genus expansion.

State-of-the-art for graviton 4-pt amplitude in Minkowski space:

\[ \forall \alpha' \]

\[ \sim \alpha'^6 f(p_i) + \ldots \]

N=4 SYM has a type IIB string dual on AdS\(_5 \times S^5\).

Its non-planar correlators encode bulk string loop amplitudes...

\( \rightarrow \) **Compute string amplitudes** holographically.

\[ L_{\text{AdS}} \to \infty \]
III. The String Landscape and Extra Dimensions in AdS/CFT

What is the landscape of AdS vacua in string/M-theory?

One simpler (but still hard!) question is whether there exist fully rigorous AdS x M vacua with parametrically small extra dimensions (i.e. hierarchy/scale-separation).

Define D as the total number of large (AdS sized) bulk dimensions. The question is whether $D = d+1$ is possible. (There are no fully controlled examples.)
Consider the uniqueness question for N=4 SYM. Why AdS$_5 \times$ S$^5$ instead of “pure” AdS$_5$?
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Hard question: what is the bulk dual of QCD? Of a “typical” SCFT?

Easier question: Why does our universe appear 3+1-dimensional?

Could it have been otherwise? What symmetry principles govern this?
Today we will address the following modest question about the AdS landscape:

Take $D =$ number of “large” (= AdS-sized) bulk dimensions.

Given the planar OPE data of a large $N$, strongly coupled CFT, what is $D$?
Today we will address the following modest question about the AdS landscape:

Take $D = \text{number of \ "large" \ (= \text{AdS-sized}) \ \text{bulk dimensions}}$.

Given the planar OPE data of a large $N$, strongly coupled CFT, what is $D$?
Outline

1. Bootstrap basics and large N CFT

2. Loops in AdS

3. Application: String amplitudes from N=4 super-Yang-Mills

4. The String Landscape and Extra Dimensions in AdS/CFT

Based on:

- 1612.03891, with O. Aharony, F. Alday, A. Bissi
- 1808.00612, with J. Liu, V. Rosenhaus, D. Simmons-Duffin
- 1809.10670, with F. Alday, A. Bissi
- 1906.01477, with F. Alday
- To appear, with D. Meltzer, A. Sivaramakrishnan
What are Conformal Field Theories (made of)?

I. Local operators: $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \ldots$

These carry a conformal dimension ($\Delta$), Lorentz spins, and maybe other charges.

II. Their interactions: $\mathcal{O}_i(x)\mathcal{O}_j(0) \sim \sum_k C_{ijk}\mathcal{O}_k(0)x^{\Delta_k-\Delta_i-\Delta_j}$

This is the operator product expansion (OPE).

"OPE data" $\{\Delta_i, C_{ijk}\}$ completely determine local operator dynamics of a CFT.
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This is the operator product expansion (OPE).

“OPE data” \( \{\Delta_i, C_{ijk}\} \) completely determine local operator dynamics of a CFT.

Charting theory space = Constraining the sets \( \{\Delta_i, C_{ijk}\} \)

Note: No reference to Lagrangians!
What are Conformal Field Theories (made of)?

We can glue these vertices to make higher-point correlation functions.

These obey dynamical laws which constrain the underlying data \( \{\Delta_i, C_{ijk}\} \).
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- **Unitarity:** \( \Delta_i \geq \Delta_* \geq 0 \) and \( C_{ijk}^2 \geq 0 \)

- **Associativity:**

\[
\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 = \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3
\]

The latter implies crossing symmetry of four-point functions.
The conformal bootstrap program has three main threads:

1. The **space** of CFTs
2. The **properties** of *all* CFTs
3. The **properties** of *specific* (universality classes of) CFTs

Originally, these investigations were numerical. Now, **analytics** are exploding.

*How* the bootstrap works – i.e. what symmetries and abstract constraints are used – is time-dependent, as we discover new facts about field theory.
Some classic bootstrap questions:

Is there an upper bound on the dimension of the lightest operator in any CFT? In a given OPE?

Are there bounds on OPE coefficients – for example, central charges or anomaly coefficients?

Assuming certain features, is there a CFT at all? If so, can we determine the precise value of its critical exponents, etc?

How special are the CFTs we already know about?

In a given CFT, what hidden structures relate apparently independent OPE data?
Bootstrap 2.0: Analytics

Some landmark results:

- Every CFT has an infinite number of primaries.
- Every 2d CFT has a lightest primary below a universal upper bound.
- CFTs with higher spin currents are free.
- Central charges – measures of anomalies and/or degrees of freedom – are bounded.
- Many classes of superconformal theories have soluble subsectors that are completely determined by 2d chiral algebras.

[Komargodski, Zhiboedov; Fitzpatrick, Kaplan, Poland, Simmons-Duffin; Hellerman; Maldacena, Zhiboedov; Hofman, Maldacena; Beem, Rastelli, van Rees; Afkhami-Jeddi, Hartman, Kundu, Jain; Caron-Huot]
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Some of these proven using new approaches, not just crossing symmetry!

- Causality and analyticity
- Regge physics/quantum chaos
- Energy conditions (e.g. ANEC)
- In 2d, modular invariance

[Komargodski, Zhiboedov; Fitzpatrick, Kaplan, Poland, Simmons-Duffin; Hellerman; Maldacena, Zhiboedov; Hofman, Maldacena; Beem, Rastelli, van Rees; Afkhami-Jeddi, Hartman, Kundu, Jain; Caron-Huot]
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- $\phi_i$: Elementary fields
- $g_{\mu\nu}$: Graviton
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Strongly-coupled quark-gluon plasma

\[ \frac{\eta}{s} = \frac{1}{4\pi} \]

Huge landscape of non-Lagrangian CFTs

Area law entanglement

\[ S_{EE} = \frac{A_{RT}}{4G_N} \]

Strongly coupled anomalous dimensions

\[ \Delta \sim M_{\text{string}} \sim \lambda^\# > 0 \]
AdS $\leftrightarrow$ CFT
Outline

1. Bootstrap basics and large N CFT

2. Loops in AdS

3. Application: String amplitudes from N=4 super-Yang-Mills

4. The String Landscape and Extra Dimensions in AdS/CFT
(A quick word on notation:)

\[
\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{1}{x_{12}^{2\Delta\phi} x_{34}^{2\Delta\phi}} A(z, \bar{z})
\]
CFT decomposition of bulk amplitude $<\phi\phi\phi\phi>$.

$$\sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} [\phi\phi]_{n,\ell}$$

Double-trace composites:

$$[\phi\phi]_{n,\ell} \simeq \phi \Box^n \partial_{\mu_1} \ldots \partial_{\mu_\ell} \phi$$

$$\Delta_{n,\ell} = 2\Delta_\phi + 2n + \ell + \gamma_{n,\ell}$$
CFT decomposition of bulk amplitude $<\phi\phi\phi\phi>$.

$$= \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \left[\phi\phi\right]_{n,\ell}$$

Double-trace composites:

$$[\phi\phi]_{n,\ell} \sim \phi^{n} \partial_{\mu_{1}} \ldots \partial_{\mu_{\ell}} \phi$$

$$\Delta_{n,\ell} = 2\Delta_{\phi} + 2n + \ell + \gamma_{n,\ell} = 0 \text{ in MFT}$$
CFT decomposition of bulk amplitude $\langle \phi \phi \phi \phi \rangle$.

$$
\sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} [\phi \phi]_{n,\ell} = \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} a_{n,\ell}^{(0)} G_{n,\ell}^{(s)}
$$

Squared OPE coefficients of MFT

S-channel conformal blocks

Double-trace composites:

$$
[\phi \phi]_{n,\ell} \simeq \phi \Box_n \partial_{\mu_1} \ldots \partial_{\mu_\ell} \phi
$$

$\Delta_{n,\ell} = 2\Delta_\phi + 2n + \ell + \gamma_{n,\ell}$

=0 in MFT
CFT decomposition of bulk amplitude $<\phi\phi\phi\phi>$.

\[
\begin{align*}
= & \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} [\phi\phi]_{n,\ell} \\
= & \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} a_{n,\ell}^{(0)} G_{n,\ell} \\
\end{align*}
\]

Double-trace composites:

\[
[\phi\phi]_{n,\ell} \simeq \phi \square^n \partial_{\mu_1} \ldots \partial_{\mu_{\ell}} \phi
\]

\[
\Delta_{n,\ell} = 2\Delta_\phi + 2n + \ell + \gamma_{n,\ell}
\]

$[\phi\phi]$ anomalous dimension:

\[
\gamma_{n,\ell} = \frac{\gamma_{n,\ell}^{(1)}}{c} + \frac{\gamma_{n,\ell}^{(2)}}{c^2} + \ldots
\]

Tree-level

Fixed by single-trace data
CFT decomposition of bulk amplitude $\langle \phi \phi \phi \phi \rangle$.

\[
\begin{align*}
\langle \phi \phi \rangle &= \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} [\phi \phi]_{n,\ell} = \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} a_{n,\ell}^{(0)} G_{n,\ell} \\
\mathcal{O} &= \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} [\phi \phi]_{n,\ell}
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Double-trace composites:

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[\phi \phi]_{n,\ell} \simeq \phi \square^n \partial_{\mu_1} \ldots \partial_{\mu_\ell} \phi
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Tree-level Fixed by single-trace data

1-loop Fixed by tree-level data... how?
In the world of amplitudes, the dominant paradigm is that of “unitarity methods”.

Recall the optical theorem for an $S$-matrix:

$$S = 1 + iT \quad \xrightarrow{\text{Disc}(T) = T^\dagger T} \quad \text{Unitarity of } S$$

This buys you one order in perturbation theory. e.g. at 1-loop,

Unitarity methods = constructing loop-level amplitudes from low-order ones by cutting.

(Basic idea: A Lagrangian defines the set of tree-level amplitudes, so from these, one must be able to construct the $S$-matrix to all orders in perturbation theory.)
In AdS, no asymptotic states: instead, dual CFT operators…

“AdS Unitarity Method”: a prescription for constructing loop-level AdS amplitudes from OPE data of lower-order ones.

Like ordinary unitarity methods, but reconstructed from operations in the CFT.

[Aharony, Alday, Bissi, EP]
AdS Unitarity Method

Nicely phrased using CFT dispersion relation (Lorentzian inversion).

Schematically:

\[ A(z, \bar{z}) \approx \int K(z, \bar{z}; z', \bar{z}') \, d\text{Disc}(A(z', \bar{z}')) \]

(“dDisc constructibility”)

where

\[ d\text{Disc}_t(A(z, \bar{z})) = \frac{1}{2} \text{Disc}^{\circ}_{z=1}(A(z, \bar{z})) + \frac{1}{2} \text{Disc}^{\circ}_{\bar{z}=1}(A(z, \bar{z})) \]

For identical external scalar operators, dDisc acts on conformal blocks G as follows:

\[ d\text{Disc}_t(G^{(t)}_{\Delta, \ell}) = 2 \sin^2 \left( \frac{\pi}{2} (\Delta - \ell - 2\Delta_\phi) \right) G^{(t)}_{\Delta, \ell} \]

→ Annihilates double-trace operators with \( \gamma = 0 \).

→ In the 1/c expansion,

\[ d\text{Disc}_t(A^{1-\text{loop}}) \supset \frac{\pi^2}{2} \sum_{n, \ell} a^{(0)}_{n, \ell} (c^{(1)}_{n, \ell})^2 G^{(t)}_{n, \ell} \]

1-loop anomalous dimension does not appear = Fixed by tree-level!

[Caron-Huot]

[Aharony, Alday, Bissi, EP]
AdS Unitarity Method

“Glue” CFT data at leading order in $1/N$ (AdS tree) to compute higher orders (AdS loops).

Diagrammatic suggestion:
AdS Unitarity Method

This picture can be made precise.

• We can glue two tree amplitudes together (CFT shadow techniques).
• Because the internal propagator is off-shell, the correct procedure requires an infinite sum of such pairs of glued trees, with specific dimensions.

(Reverse: “split representation” of bulk-bulk propagator)
This picture can be made precise.

- We can glue two tree amplitudes together (CFT shadow techniques).
- Because the internal propagator is off-shell, the correct procedure requires an infinite sum of such pairs of glued trees, with specific dimensions.
  (Reverse: “split representation” of bulk-bulk propagator)
- Choosing to glue s- or t-channel trees gives the different 1-loop diagrams. e.g.
Q: What corresponds to a cut?

A in bulk: Taking the internal legs on-shell.

A in CFT: Isolating part of the conformal block expansion from double-trace operators whose constituents are dual to the internal lines.
Q: What corresponds to a cut?

A in bulk: Taking the internal legs on-shell.

A in CFT: Isolating part of the conformal block expansion from double-trace operators whose constituents are dual to the internal lines.
Q: What corresponds to a cut?

**A in bulk:** Taking the internal legs on-shell.

**A in CFT:** Isolating part of the conformal block expansion from double-trace operators whose constituents are dual to the internal lines.

\[
\text{dDisc}_t(\text{box})
\]

\text{dDisc is the cut operator!}
Aside: A nice mathematical connection

When gluing diagrams, \textit{6j symbols} for the conformal group appear.

\[ \sum \int (6j)^2 \]

\textit{6j symbol} \sim \text{AdS ladder kernel}

[Liu, EP, Rosenhaus, Simmons-Duffin]
We are now able to compute all of these amplitudes (and various others) using AdS unitarity:
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What are they?

In M-theory,

\[ S_{\text{M-theory}} \sim \int d^{11}x \sqrt{g}(R + R^4 + D^6 R^4 + \ldots) \]
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\[ S_{\text{M-theory}} \sim \int d^{11}x \sqrt{g} (R + R^4 + D^6 R^4 + \ldots) \]

Known (fixed by SUSY) \quad Unknown

In type II string theory,

\[ S_{\text{type-II strings}} \sim \int d^{10}x \sqrt{g} (R + f(g_s)R^4 + g(g_s)D^4 R^4 + h(g_s)D^6 R^4 + \ldots) \]

Known (fixed by SUSY) \quad Unknown

(Non-holomorphic SL(2,Z) modular forms)

Missing terms reflect a paucity of results in scattering amplitudes (as noted earlier).
Taking stock of string perturbation theory:

Most recent work on string perturbation theory has focused on issues at low-genus.

i) Shoring up issues of principle: unitarity, renormalization

ii) Mathematical structure of moduli space integrands = “Modular graph functions”

iii) Transcendentality and double-copy

[De Lacroix, Erbin, Pius, Rudra, Sen, Witten]

[Basu, D'Hoker, Duke, Green, Gurdogan, Kaidi, Miller, Pioline, Vanhove]

[D'Hoker, Green, Mafra, Schlotterer]
Taking stock of string perturbation theory:

A couple of big questions:

1. **$D^8 R^4$ coefficient is unknown.** Has been conjectured to obey **perturbative non-renormalization beyond four loops**.

   \[
   S_{\text{type-II strings}} \sim \int d^{10}x \sqrt{g} (R + \tilde{f}(g_s) R^4 + g(g_s) D^4 R^4 + h(g_s) D^6 R^4 + \ldots )
   \]

   Perturbative truncation at $g=1,2,3$, respectively...

2. The coefficients appearing in the low-energy expansion appear to obey a form of **maximal transcendentality**. No one knows why.
A new approach to string perturbation theory

1) Holographically compute the one-loop amplitude for strings in AdS, as a nonplanar correlator in a dual CFT.

2) Take a “flat space limit”

[Albay, Caron-Huot; Alday, Bissi, EP]
A new approach to string perturbation theory

1) Holographically compute the one-loop amplitude for strings in AdS, as a nonplanar correlator in a dual CFT.

2) Take a “flat space limit”

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String scattering amplitudes from 1/N expansion of local operator data in CFT.

[Albay, Caron-Huot; Alday, Bissi, EP]
A new approach to string perturbation theory

The prototypical CFT with a string dual is 4d maximally supersymmetric Yang-Mills:

We compute the leading non-planar correction to a four-point function.
In practice, we take a low-energy limit = CFT strong coupling expansion.
Genus-one strings in \( \text{AdS}_5 \times S^5 \)

Strong coupling (\( \lambda \to \infty \)) single-trace spectrum:
\[ \mathcal{O}_p \in [0p0] \text{ of } SU(4)_R \]
\[ \Delta_p = p \text{ , where } p = 2, 3, \ldots \]

Only \( \frac{1}{2} \)-BPS multiplets.

Study <2222>, where 2 = Stress-tensor multiplet.

\[
\text{dDisc}_t(A^{1-\text{loop}}_{2222}) = \frac{\pi^2}{2} \sum_{n,\ell} \langle \gamma_{n,\ell}^{(1)} \rangle^2 a_{n,\ell}^{(0)} G_{n,\ell}^{(t)}
\]
Indicates mixing:

\[ [\mathcal{O}_2 \mathcal{O}_2]_{n,\ell} , [\mathcal{O}_3 \mathcal{O}_3]_{n-1,\ell} , \ldots , [\mathcal{O}_{n+2} \mathcal{O}_{n+2}]_{0,\ell} \]

Evaluate in strong coupling (1/\( \lambda \)) expansion:

\[ \gamma^{(1)} = \gamma^{(1)}|_{\text{sugra}} + \frac{1}{\lambda^{3/2}} \gamma^{(1)}|_{R^4} + \frac{1}{\lambda^{5/2}} \gamma^{(1)}|_{D^4R^4} + \ldots \]

\( \rightarrow \) Non-planar <2222> in 1/\( \lambda \) expansion \( \iff \) \( A^{1-\text{loop}}_{2222} \) in \( \text{AdS}_5 \times S^5 \) in \( \alpha' \) expansion.
Genus-one strings in $R^{10}$

We can then take **flat space limit**.

$$L_{\text{AdS}} \rightarrow \infty$$

AdS amplitude $\rightarrow$ Graviton amplitude in $R^{10}$, with momenta in a five-plane.

At each order in $1/c$ and $1/\lambda$ expansion, leading power of $s$ must match string result.

First few orders in $1/\lambda$:

$$\text{e.g.} \quad \frac{d\text{Disc}(A_{2222}^{1-\text{loop}})R^4|R^4}_{\text{flat}} = \frac{\text{Disc}(A_{IIB}^{g=1})R^4|R^4}{\text{Disc}(A_{IIB}^{g=1})_{\text{sugra}}|R^4} = \frac{3\zeta(3)}{14} \left(\frac{\alpha' s}{4}\right)^3$$

[Okuda, Penedones; Penedones; Maldacena, Simmons-Duffin, Zhiboedov]

[Green, Schwarz]
Compute these diagrams via the strong-coupling expansion of the CFT.
Flat space limit $\rightarrow$ Low-energy expansion of the genus-one string amplitude in $10d$ flat space.

This matches the first several terms in genus-one string perturbation theory.
Outline

1. Bootstrap basics and large N CFT

2. Loops in AdS

3. Application: String amplitudes from N=4 super-Yang-Mills

4. The String Landscape and Extra Dimensions in AdS/CFT
We know necessary CFT conditions for bulk locality...

Large $N$ + higher-spin gap ($s>2$) $\Rightarrow$ Local AdS bulk

...but in what dimension is it local? (CFT: how sparse is the low-spin spectrum?)

All fully-controlled examples of the AdS/CFT Correspondence involve bulk solutions which contain manifolds of parametrically large positive curvature: $D > d+1$

$$\text{AdS}_d \times M_{D-d-1}$$

$$\text{AdS}_5 \times S^5/T^{1,1}/Y^{p,q}/L^{p,q,r}, \quad \text{AdS}_{4/7} \times S^{7/4}, \quad \text{AdS}_3 \times S^3 \times T^4, \quad \text{AdS}_{3/2} \times S^{2/3} \times CY_3, ...$$

Large transverse manifolds means light KK towers, dual to CFT local operators. No pure gravity, or even close!? [Heemskerk, Penedones, Polchinski, Sully; ...]
There are attempts at constructing AdS x Small solutions in string/M-theory. e.g.:

1. **Large Volume Scenario** (non-SUSY AdS$_4$, IIB)  
2. **KKLT** (SUSY AdS$_4$, IIB)  
3. **DGKT** (SUSY AdS$_4$ IIA)  
4. **Polchinski-Silverstein** (SUSY AdS$_4$, AdS$_5$ from F-theory)

These all involve assumptions or arguments based on effective field theory, perturbative/non-perturbative effects in $\alpha'$ and/or $g_s$, and backreaction of sources.

What we want is to make fully rigorous, quantitative statements from the bootstrap.

Today: set up a dictionary.
Today we will address the following modest question about the AdS landscape:

Take $D =$ number of “large” (= AdS-sized) bulk dimensions.

Given the planar OPE data of a large $N$, strongly coupled CFT, what is $D$?
Segue

Q: In the N=4 calculation, why did we get a $D=10$ string amplitude?

A: The bulk dual is $\text{AdS}_5 \times S^5$ where $L_{S^5} = L_{\text{AdS}_5}$...

How exactly does the CFT correlator “know” about the extra five dimensions?
Segue

To match to flat space, either:

1. Match amplitudes
2. Match partial wave coefficients

\[ d\text{Disc}_t(A_{2222}^{1-\text{loop}}) = \frac{\pi^2}{2} \sum_{n,\ell} (\gamma^{(1)}_{n,\ell})^2 a^{(0)}_{n,\ell} G^{(t)}_{n,\ell} \]

The dictionary between OPE data and flat space momentum:

\[ L\sqrt{s} \sim n \]
Segue

To match to flat space, either:
1. Match amplitudes
2. Match partial wave coefficients

\[ d\text{Disc}_t(A_{2222}^{1-\text{loop}}) = \frac{\pi^2}{2} \sum_{n,\ell} \langle \gamma_{n,\ell}^{(1)} \rangle^2 a_{n,\ell}^{(0)} G_{n,\ell}^{(t)} \]

The dictionary between OPE data and flat space momentum:

\[ L \sqrt{s} \sim n \]

In N=4 SYM at \( \lambda = \infty \), \( \gamma_{n,\ell}^{(1)} \sim n^3 \) but \( \langle \gamma_{n,\ell}^{(1)} \rangle^2 \sim n^{11} = n^{6+5} \)

[Alday, Caron-Huot]
A 1-loop sum rule for $D$

Consider a $D$-dimensional two-derivative theory of gravity + spin $\leq 2$ matter.

$$A_D(s, t) = G_N A_D^{\text{tree}}(s, t) + G_N^2 A_D^{1\text{-loop}}(s, t) + \mathcal{O}(G_N^3)$$

Suppose there exists an $\text{AdS}_{d+1} \times \mathcal{M}_{D-d-1}$ vacuum.

Define

$$A_{d+1}(s, t) \equiv \frac{A_D(s, t)}{\text{Vol}(\mathcal{M})} = \frac{L^{d-1}}{c} A_{d+1}^{\text{tree}}(s, t) + \frac{L^{D+d-3}}{c^2} A_{d+1}^{1\text{-loop}}(s, t) + \mathcal{O}(c^{-3})$$

where the CFT central charge $c \sim 1/G_N$.

At high-energy $s, t \gg 1$ and fixed-angle $\cos \theta = 1 + \frac{2t}{s}$,

$$A_{d+1}(s \gg 1, \theta) = \frac{(L \sqrt{s})^{d-1}}{c} f_{d+1}^{\text{tree}}(\theta) + \frac{(L \sqrt{s})^{D+d-3}}{c^2} f_{d+1}^{1\text{-loop}}(\theta) + \mathcal{O}(c^{-3})$$

Order-by-order in $1/c$, flat space limit of a CFT correlator must reproduce this.
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Order-by-order in $1/c$, flat space limit of a CFT correlator must reproduce this.
A 1-loop sum rule for $D$

An arbitrary 1-loop correlator has a (t-channel) dDisc of the following form:

$$dDisc_t(A^{1-\text{loop}}(z, \bar{z})) = \sum_{n, \ell} \beta^{1-\text{loop}}_{n, \ell} a^{(0)}_{n, \ell} G^{(t)}_{n, \ell}(z, \bar{z})$$

In flat space limit, matching yields a 1-loop sum rule for $D$:

$$L\sqrt{s} \sim n$$

where

$$\beta^{1-\text{loop}}_{n \gg 1, \ell} \sim n^{D+d-3} f^{1-\text{loop}}(\ell)$$

$$\beta^{1-\text{loop}}_{n, \ell} = 2 \sum_{\mathcal{O}} \rho_{\text{ST}}(\Delta_{\mathcal{O}}) \left( \frac{\pi^2}{4} \langle \gamma^{(1)}_{n, \ell}(\mathcal{O}) \rangle^2 + \sin^2(\pi(\tau_{\mathcal{O}} - \Delta_{\phi})) ||C^{2}_{\phi\phi[\mathcal{O}\mathcal{O}]_{n, \ell}}|| \right)$$

Single-trace density of states

Degenerate operators

Non-degenerate operators

$$\Delta_{\mathcal{O}} - \Delta_{\phi} \in \mathbb{Z}$$

$$\Delta_{\mathcal{O}} - \Delta_{\phi} \notin \mathbb{Z}$$
A 1-loop sum rule for D

\[ \beta_{n,\ell}^{1-\text{loop}} \sim n^{D+d-3} f^{1-\text{loop}}(\ell) \]

Comments:

1. **Positive-definite**, term-by-term \( \rightarrow \) Lower bound D

\[ \beta_{n,\ell}^{1-\text{loop}} \equiv 2 \sum_{O} \rho_{\text{ST}}(\Delta_{O}) \left( \frac{\pi^2}{4} \langle \gamma^{(1)}_{n,\ell}(O) \rangle^2 + \sin^2(\pi(\tau_{O} - \Delta_{O})) \| C_{\phi\phi[O\sigma]}^{2}[n,\ell] \| \right) \] N.B. dDisc crucial!

2. Trees are insensitive to D, as they must be: consistent truncations exist.

\[ \beta_{n,\ell}^{\text{tree}} \sim n^{d-1} : \text{Einstein scaling} \] [Cornalba, Costa, Penedones]

3. D+d-3 follows from two-derivative approx. (= large HS gap in CFT)

Let us explore some consequences of this sum rule for extra dimensions.
A 1-loop sum rule for $D$

1. Suppose we have a power law density of non-degenerate single-trace operators:

$$\rho_{\text{ST}}(\Delta_0 \gg 1) \sim \Delta_0^{x-1}$$

Sum dominated by large double-trace dimensions, $1 \ll n \sim \Delta < \Delta_{\text{gap}}$

$$\beta_{n \gg 1, \ell}^{1\text{-loop}} \sim n^{2d+x-1} \quad \Rightarrow \quad D = d + 1 + x$$

$\Rightarrow x$ large extra dimensions.

Converse of a holographic fact: Weyl's law growth of eigenvalues $\lambda$ on compact manifold $\mathcal{M}$ with smooth boundary.

Parameterizing $\lambda \sim \Delta^2$,

$$\int_{\Delta_* \gg 1} d\Delta \rho_\mathcal{M}(\Delta) \sim \frac{\text{vol}(\mathcal{M})}{(4\pi)^{\frac{\dim(\mathcal{M})}{2}} \Gamma\left(\frac{\dim(\mathcal{M})}{2} + 1\right)} \Delta_*^{\dim(\mathcal{M})}$$
A 1-loop sum rule for $D$

2. Suppose there is a tower of degenerate operators

$$\Delta_p = \Delta_\phi + p - 2 \ , \ \text{where} \ p = 2, 3, \ldots$$

Assuming a cubic coupling $\phi pp$.

Result:

$$\left. \langle \gamma^{(1)}_{[\phi\phi]_{n,\ell}} (p) \rangle \right|_{n \sim p} \sim n^{d-3} \cdot C_{\phi pp}^2 \bigg|_{p \gg 1}$$

Depends on asymptotic of $C$! If $C_{\phi pp} \bigg|_{p \gg 1} \sim \frac{p^{1+\frac{\alpha}{4}}}{\sqrt{c}}$ then $D = d + 2 + \alpha$
Stringy OPE universality

In familiar cases like $\phi = T_{\mu\nu}, \mathcal{L}$, this OPE coefficient is linear ($\alpha = 0$).

**Conjecture (OPE universality):** for any light operator $\phi$ and heavy operator $\phi_p$ with

$$\Delta_\phi \ll \Delta_{\text{gap}} \quad \text{and} \quad \Delta_\phi \ll \Delta_p \ll c#^0$$

the normalized planar OPE coefficient $\phi_{pp}$ has linear asymptotics, $C_{\phi_{pp}} \bigg|_{\Delta_p \gg 1} \sim \frac{p}{\sqrt{c}}$

$p$ can be KK mode or massive string mode.

Copious evidence from literature (N=4 SYM semiclassical and KK correlators, ABJM, D1-D5)

(N.B. This is *NOT* the same “heavy-heavy-light” as in ETH, 2d CFT, or large charge.)
Bounding holographic spectra

Now turn logic around.
Assume string/M-theory dual with $D \leq 10$ or $11$.
What does this imply about single-trace spectrum of planar CFT?

1. Density of states:
   $$\rho_{\text{ST}}(\Delta \gg 1) \lesssim \Delta^{8-d} \quad \text{(string)}$$
   $$\rho_{\text{ST}}(\Delta \gg 1) \lesssim \Delta^{9-d} \quad \text{(M)}$$

2. If $\phi_p$ furnish sequence of irreps $R_p$ of global symmetry, with asymptotics
   $$\dim(R_p\gg 1) \sim p^{r_p}$$
then since $D = d + 2 + r_p$, the above inequalities bound $r_p$. 
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2. If $\phi_p$ furnish sequence of irreps $R_p$ of global symmetry, with asymptotics
   \[ \dim(R_p) \gg 1 \sim p^{r_p} \]
then since $D = d + 2 + r_p$, the above inequalities bound $r_p$.

Why, from CFT, are these things true?
A final speculation

So, then: what is the landscape of AdS vacua?

A possible Holographic Hierarchy Conjecture:

**Large Higher-Spin Gap + No Global Symmetries → Local AdS dual with D = d+1**

This generalizes arguments of [Polchinski, Silverstein]

[Lust, Palti, Vafa] make the much stronger claim that D = d+1 is not possible...?
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So, then: what is the landscape of AdS vacua?

A possible Holographic Hierarchy Conjecture:

Large Higher-Spin Gap + No Global Symmetries $\rightarrow$ Local AdS dual with $D = d+1$

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[Lust, Palti, Vafa] make the much stronger claim that $D = d+1$ is not possible...?

Let the bootstrapping begin.
Summary

New techniques for AdS loop amplitudes using ideas from the bootstrap

A novel holographic approach to string perturbation theory

A dictionary for finding large extra bulk dimensions from CFT data
Future directions

1. AdS loops:
   - Complete analysis. e.g. a 1-loop basis?
   - What is the L-loop function space/transcendentality properties?

2. Holographic string amplitudes
   - Higher-genus, non-SUSY data

3. Extra dimensions
   - We have a dictionary. Can we bootstrap the landscape?