Classifying and Constraining 4 Graviton S matrices

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Based on

- ArXiv:1819.????? S. Duttachowdhury, A. Gadde, I. Halder, L. Janagal and S. M.

Introduction

- The study of string theory suggests a surprising rigidity in the structure of quantum theories of gravity. For instance there are only 5 known Lorentz Invariant theories of gravity in flat 10 dimensional space.
- It is possible that these 5 are the only 10 dimensional stable Lorentz Invariant quantum theories of gravity. But how could we hope to establish the non existence of a putative sixth theory?
- Atleast with our current state of understanding of quantum gravity the only practical way of tackling such a question is to employ simple general low energy consistency considerations. This is the strategy we will employ in this talk.

Intro: Example of a conjecture

- Indulging in a slight flight of fantasy, lets list a result we might hope eventually to establish (or falsify). Consider all consistent Lorentz invariant d dimensional theories that admit a classical limit.
- Conjecture: The classical gravitational S matrix in every such theory is necessarily one of either the Einstein S matrix, the or the type II S matrix on $R^d \times M$ or the Heterotic gravitational S matrix on $R^d \times M$ where M is any 'compact space'.
- Note the S matrices above are independent of M. The conjecure of the last paragraph asserts that the gravitational part of the classical limit of any consistent theory of flat space gravity admits a consistent truncation to one of the three universal theories described above. Perhaps low energy consistency is enough to establish this result?

Intro: A sub conjecture

- While I find the conjecture of the previous slide completely fascinating, I think we are as yet quite far from being able to meaningfully study it.
- We can, however, focuss on a simpler sub problem as follows. Recall that the type II and heterotic S matrices have intermediate massive poles corresponding to the exchange of higher spin massive particles.
- Consequently, the conjecture of the previous slide - if true - implies a simpler result as a special case. Namely that Einstein gravity is the only consistent local (i.e. finite number of derivatives) classical theory of gravity interacting that admits a consistent truncation involving no other fields.
- This 'special case' is simple enough that one can meaningfully begin to investigate it. Infact there is already one interesting result about this question in the literature that we now pause to review.

Review: 3 graviton scattering

- We wish to investigatte whether the most general classical gravitational S matrix of the sort described in the previous slide (i.e. local and interacting with no other particles) is the Einstein S matrix.
- We would like to check whether this is true for the scattering of n gravitons, for all $n = 3, 4, 5 \ldots$. The case $n = 3$ is especially simple.
- This simplicity has its root in the fact that 3 graviton S matrices are highly kinematically constrained. The most general 3 graviton S matrix is kinematically forced to be a linear combination of three structures.

$$
\begin{aligned}
T_1 &= (\epsilon_1.\epsilon_2\epsilon_3.p_1 + \text{perm})^2 && \text{2 der : Einstein} \\
T_2 &= (\epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3 \wedge p_1 \wedge p_2)^2 && \text{4 der : GaussBonnet} \\
T_3 &= (\epsilon_1.p_2\epsilon_2.p_3\epsilon_3.p_1)^2 && \text{6 der : Reimann}^3
\end{aligned}
$$

Review: CEMZ result

- The most general 3 graviton S matrix takes the form
$$aT_1 + bT_2 + cT_3$$
where a, b and c are pure numbers.
- CEMZ demonstrated that any theory in which either b or c is nonzero is necessarily acausal unless it couples to higher spin particles of arbitrarily high spin. In particular in a causal gravitational theory with a local S matrix, $b = c = 0$. Using the principle of causality, in other words, CEMZ have already established our conjecture for 3 graviton scattering.
- This is very encouraging. However note that 3 graviton scattering is special as it is parameterized by finite data. We encounter qualitatively greater complexity when scattering 4 (or more) gravitons. I turn, in the rest of the talk, to the study of 4 graviton S matrices. We first parameterize S matrices and then try to constrain them.

4 particle S matrices: identical scalars

- Warm up: consider the scattering of 4 identical scalars. The most general S matrix is a permutation invariant function $\mathcal{S}$ of s, t and u with $s+t+u=0$.
- If we restrict attention to local S matrices then $\mathcal{S}$ is a polynomial. Let the number of such polynomials at degree m be $d_{sym}(m)$. Define the partition function $Z(x) = \sum_{m=0}^{\infty} d_{sym}(m) x^m$. Turns out

$$Z_{sym} = \frac{1}{(1-x^2)(1-x^3)}$$

$$d_{sym}(m) \sim \frac{m+1}{6} \qquad \text{asymptotically}$$

- $d_{sym}(m)$ also counts the number of field redefinition inequivalent m derivative 4 ϕ terms one can add to the free boson Lagrangian. Three is a simple 2 way map from and S matrix to its corresponding Lagrangian structure.

Indices: S_4, S_3 and $Z_2 \times Z_2$

- We will now turn to a study of S matrices of particles with indices. Such S matrices are labelled by polarization tensors in addition to s, t and u.
- The full S matrix has to be S_4 invariant. Now it is easy to check that the $Z_2 \times Z_2$ subgroup of S_4 consisting of I, $P_{12}P_{34}$, $P_{13}P_{24}$ and $P_{14}P_{23}$ leaves s, t and u unchanged. S_4 invariance thus requires that index structure that appears in the S matrix is $Z_2 \times Z_2$ invariant.
- The conditions above just on index structure ensure the S matrix is invariant under $Z_2 \times Z_2$ permutations. To ensure invariance under all of S_4 we must now also ensure invariance of the S matrix under $S_4/(Z_2 \times Z_2) = S_3$.
- Consider an index structure that happens to be invariant under a subgroup G of S_3. The coefficient function of s, t and u that multiplies this structure must also be invariant under this subgroup - which can vary from nothing to all of S_3.

Polynomials of s, t, u and S_3

- We decompose polynomials of s, t and u into the 3 irreps of S_3, namely the 1 dim completely sym rep, the one dim completely antisym rep and the 2 dim irrep (in which it turns out that every permutation operator ($e.g. P_{12}$) has eigenvalues ± 1. We find
- $$Z_{no-sym} = \frac{1}{(1-x)^2} = \sum_{m=0}^{\infty} (m+1) x^m$$
$$Z_{sym} = \frac{1}{(1-x^2)(1-x^3)} = 1 + x^2 + x^3 + x^4 + x^5 + 2x^6 + x^7 + 2x^8 + 2$$
$$Z_{as} = \frac{x^3}{(1-x^2)(1-x^3)} = x^3 \left(1 + x^2 + x^3 + x^4 + x^5 + 2x^6 + x^7 + 2x^8 +\right.$$
$$Z_{mixed} = \frac{2x}{(1-x)(1-x^3)}$$
$$Z_{Z_2-sym} == \frac{1+x}{(1-x^2)^2} = \sum_{m=0}^{\infty} \left(\left[\frac{m}{2} \right] + 1 \right) x^m$$

Counting Data

- At large m we have

$$
\begin{aligned}
d_{no-sym}(m) &= m+1 \\
d_{sym}(m) &\sim \frac{m+1}{6} \\
d_{as}(m) &=\sim \frac{m+1}{6} \\
d_{mixed}(m) &= \frac{2(m+1)}{3} \\
d_{Z_2-sym}(m) &== \frac{m+1}{2} =
\end{aligned}
\tag{2}
$$

- The following rough characterization is sometimes useful. A function of s t and u is said to have p degrees of freedom if the number of coefficients at degree m in this function grows like $\frac{p(m+1)}{6}$ at large m. Completely symmetric functions have one degree of freedom, Z_2 inv functions has 3 degrees of freedom, and $no - sym$ functions have 6 degrees of freedom.

S matrices for 4 identical photons

- I now present our results for the most general local parity invariant S matrix for 4 photons.
- For $d \geq 5$ this function is parameterized by 2 Z_2 invariant functions (i.e. functions that are symmetric under u goes to t interchange) $A^{0,1}(t,u)$ and a single S_3 invariant function $A^{2,1}(s,t,u)$; a total of 7 degrees of freedom.
- We say a Lagrangian structure A is a descendent of a structure B if first A has more derivatives than B, but all the extra derivatives that are in A but not in B have indices that contract with each other. Second, if we remove all these contracted derivatives A reduces to B.
- $A^{0,1}$ and $A^{0,2}$ parameterize descendents of the four derivative structures $(TrF^2)^2$ and $Tr(F^4)$ respectively while $A^{1,2}$ parameterizes descendents of the six derivative term

$$F_{ab}\mathrm{Tr}(\partial_a F \partial_b F F)$$

Explicit parameterization of 4 photon S matrices

- Explicitly the most general 4 photon S matrix is given by the sum of
- $$\begin{aligned} &A^{0,1}(t,u)\,(p^1_\mu\epsilon^1_\nu - p^1_\nu\epsilon^1_\mu)\,(p^2_\mu\epsilon^2_\nu - p^2_\nu\epsilon^2_\mu)\,(p^3_\alpha\epsilon^3_\beta - p^3_\beta\epsilon^3_\alpha)\,(p^4_\alpha\epsilon^4_\beta - p^4_\beta\epsilon^4_\alpha)\\ +&A^{0,1}(s,u)\,(p^1_\mu\epsilon^1_\nu - p^1_\nu\epsilon^1_\mu)\,(p^3_\mu\epsilon^3_\nu - p^3_\nu\epsilon^3_\mu)\,(p^2_\alpha\epsilon^2_\beta - p^2_\beta\epsilon^2_\alpha)\,(p^4_\alpha\epsilon^4_\beta - p^4_\beta\epsilon^4_\alpha)\\ +&A^{0,1}(t,s)\,(p^1_\mu\epsilon^1_\nu - p^1_\nu\epsilon^1_\mu)\,(p^4_\mu\epsilon^4_\nu - p^4_\nu\epsilon^4_\mu)\,(p^3_\alpha\epsilon^3_\beta - p^3_\beta\epsilon^3_\alpha)\,(p^2_\alpha\epsilon^2_\beta - p^2_\beta\epsilon^2_\alpha) \end{aligned} \tag{3}$$
- and
  $$\begin{aligned} &A^{0,2}(t,u)\,(p^1_\mu\epsilon^1_\nu - p^1_\nu\epsilon^1_\mu)\,(p^3_\nu\epsilon^3_\alpha - p^3_\alpha\epsilon^3_\nu)\,(p^2_\alpha\epsilon^2_\beta - p^2_\beta\epsilon^2_\alpha)\,(p^4_\beta\epsilon^4_\mu - p^4_\mu\epsilon^4_\beta)\\ +&A^{0,2}(s,u)\,(p^1_\mu\epsilon^1_\nu - p^1_\nu\epsilon^1_\mu)\,(p^2_\nu\epsilon^2_\alpha - p^2_\alpha\epsilon^2_\nu)\,(p^3_\alpha\epsilon^3_\beta - p^3_\beta\epsilon^3_\alpha)\,(p^4_\beta\epsilon^4_\mu - p^4_\mu\epsilon^4_\beta)\\ +&A^{0,2}(t,s)\,(p^1_\mu\epsilon^1_\nu - p^1_\nu\epsilon^1_\mu)\,(p^3_\nu\epsilon^3_\alpha - p^3_\alpha\epsilon^3_\nu)\,(p^4_\alpha\epsilon^4_\beta - p^4_\beta\epsilon^4_\alpha)\,(p^2_\beta\epsilon^2_\mu - p^2_\mu\epsilon^2_\beta) \end{aligned} \tag{4}$$

Explicit parameterization of 4 photon S matrices

- $$\begin{aligned}
&\left(A^{2,1}(s,t)+A^{2,1}(t,u)+A^{2,1}(u,s)\right)\times \\
&\Big[\left(p^1_a\epsilon^1_b-p^1_b\epsilon^1_a\right)p^2_a\left(p^2_\mu\epsilon^2_\nu-p^2_\nu\epsilon^2_\mu\right)p^3_b\left(p^3_\nu\epsilon^3_\alpha-p^3_\alpha\epsilon^3_\nu\right)\left(p^4_\alpha\epsilon^4_\mu-p^4_\mu\epsilon^4_\alpha\right) \\
&+\left(p^2_a\epsilon^2_b-p^2_b\epsilon^2_a\right)p^1_a\left(p^1_\mu\epsilon^1_\nu-p^1_\nu\epsilon^1_\mu\right)p^4_b\left(p^4_\nu\epsilon^4_\alpha-p^4_\alpha\epsilon^4_\nu\right)\left(p^3_\alpha\epsilon^3_\mu-p^3_\mu\epsilon^3_\alpha\right) \\
&+\left(p^3_a\epsilon^3_b-p^3_b\epsilon^3_a\right)p^4_a\left(p^4_\mu\epsilon^4_\nu-p^4_\nu\epsilon^4_\mu\right)p^1_b\left(p^1_\nu\epsilon^1_\alpha-p^1_\alpha\epsilon^1_\nu\right)\left(p^2_\alpha\epsilon^2_\mu-p^2_\mu\epsilon^2_\alpha\right) \\
&+\left(p^4_a\epsilon^4_b-p^4_b\epsilon^4_a\right)p^3_a\left(p^3_\mu\epsilon^3_\nu-p^3_\nu\epsilon^3_\mu\right)p^2_b\left(p^2_\nu\epsilon^2_\alpha-p^2_\alpha\epsilon^2_\nu\right)\left(p^1_\alpha\epsilon^1_\mu-p^1_\mu\epsilon^1_\alpha\right)\Big]
\end{aligned} \tag{5}$$

- The most general local S matrices are given by the form listed above with $A^{0,1}$, $A^{0,2}$ and $A^{1,2}$ polynomials of s, t and u. We have counted the data in such S matrices above- our photon S matrix has 7 degrees of freedom. The most general S matrices - not necessarily local - are also given by the forms above allowing for more general (not necessarily polynomial) dependences of the unknown functions.

Checks for our parameterization of 4 photon scattering

- The tree level scattering of 4 photons in type 1 theory (or in type II theory on D branes) has a single index structure - the structure that follows from the Lagrangian structure

$$L^{ss}_{4V} \propto \frac{1}{16}\left(\mathrm{Tr}(F^4) - \frac{1}{4}(\mathrm{Tr}(F^2))^2\right) \tag{6}$$

  which itself can be obtained by expanding the Born Infeld action to quartic order in $F_{\mu\nu}$. Consequently this scattering amplitude can be cast into our general form with $A^{2,1} = 0$ and $A^{0,2} = -\frac{1}{4}A^{0,1}$. The actual expression for $A^{0,1}$ is a well knonwn Veneziano type function.
- We have also recast the formula for tree level scattering in the open bosonic string into our general form. The final result is more complicated - and we do not write it here, but simply note that it involves all three of our structures.

S matrices for 4 identical gravitons

- I now present our results for the case of real interest to us - the most general local parity invariant S matrix for 4 gravitons.
- For $d \geq 7$ the most general S matrix turns out to be parameterized by 6 Z_2 invariant, one function that enjoys no permutation symmetry and two functions that are completely permutation symmetric. (i.e. functions that are symmetric under u goes to t interchange) $A^{0,1}(t,u)$ and a single S_3 invariant function $A^{2,1}(s,t,u)$, or a total of 29 degrees of freedom.
- In more detail we completely symmetric function of s, t and u that parameterizes descendents of the 6 derivative (Riemann^3) term

$$R^{pqrs}R_{pq}{}^{tu}R_{rstu} + 2R^{pqrs}R_{p}{}^{t}{}_{r}{}^{u}R_{qtsu} \tag{7}$$

This function has one degree of freedom.

S matrices for 4 identical Gravitions

- Going up in dimension we have 5 Z_2 invariant functions and one 0-sym function that parameterize descendents of 4 derivative terms obtained from various contractions of 4 Reimann tensors. This gives a total of 21 degrees of freedom.
- Continuing to climb in dimension, we have two Z_2 invariant functions - or 6 degrees of freedom - parameterizing descendents of degree 10 structures built out of 2 derivatives acting on the product of 4 Reimann tensors.
- Finally, we have a single completely symmetric function (one degree of freedom)parameterizing descendents of a 12 derivative terms (4 derivatives of 4 Reimanns)
- Note: If we set $g_{\mu\nu}(k) = \epsilon_\mu(k)\epsilon_\nu(k)e^{ik.x}$ with $k^2 = 0$ then it turns out that R_{abmn} evaluated to linearized order is proportional to $F_{ab}(k)F_{mn}(k)$ where $F_{mn} = k_m\epsilon_n - k_n\epsilon_n$. In our Lagrangian terms below we will sometimes replace R_{abmn} with $F_{ab}F_{mn}$.

Explicit parameterization of the general 4 graviton S matrix

- Explicitly, the most general 4 gravition S matrix is given by the sum of

$$S_1 = 3B^{0,0}(s,t,u)\left(\epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3 \wedge \epsilon_4 \wedge p_1 \wedge p_2 \wedge p_3\right)^2 \quad (8)$$

with $B^{0,0}(s,t,u)$ completely symmetric (this is from descendents of the Reimann^3 structure) and

-

$$B^{0,1}(s,t)\left[\left(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p\right)\left(p^2_p\epsilon^2_q - p^2_q\epsilon^2_p\right)\left(p^3_r\epsilon^3_s - p^3_s\epsilon^3_r\right)\left(p^4_r\epsilon^4_s - p^4_s\epsilon^4_r\right)\right.$$
$$\left.\left(p^1_a\epsilon^1_b - p^1_b\epsilon^1_a\right)\left(p^2_b\epsilon^2_c - p^2_c\epsilon^2_b\right)\left(p^3_c\epsilon^3_d - p^3_d\epsilon^3_c\right)\left(p^4_d\epsilon^4_a - p^4_a\epsilon^4_d\right)\right]$$
$$+ B^{0,1}(s,u)\,[3 \leftrightarrow 4] + B^{0,1}(t,s)\,[2 \leftrightarrow 3] + B^{0,1}(t,u)\,[2 \leftrightarrow 3 \ \text{then} \ 2 \leftrightarrow 4]$$
$$+ B^{0,1}(u,t)\,[2 \leftrightarrow 4] + B^{0,1}(u,s)\,[2 \leftrightarrow 4 \ \text{then} \ 2 \leftrightarrow 3] \quad (9)$$

where $B^{0,1}$ has no special symmetry property; this term is from descendents of $\mathrm{Tr}(F^1F^2)\mathrm{Tr}(F^3F^4)\mathrm{Tr}(F^1F^2F^3F^4)$

Explicit parameterization of the gravity S matrix

- $$B^{0,2}(t,u)\left[\left(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p\right)\left(p^2_p\epsilon^2_q - p^2_q\epsilon^2_p\right)\left(p^3_r\epsilon^3_s - p^3_s\epsilon^3_r\right)\left(p^4_r\epsilon^4_s - p^4_s\epsilon^4_r\right)\right.$$
$$\left.\left(p^1_a\epsilon^1_b - p^1_b\epsilon^1_a\right)\left(p^3_b\epsilon^3_c - p^3_c\epsilon^3_b\right)\left(p^2_c\epsilon^2_d - p^2_d\epsilon^2_c\right)\left(p^4_d\epsilon^4_a - p^4_a\epsilon^4_d\right)\right]$$
$$+ B^{0,2}(s,u)\,[3 \leftrightarrow 2] + B^{0,2}(s,t)\,[2 \leftrightarrow 4] \tag{10}$$

  where
  $$B^{0,2}(t,u) = B^{0,2}(u,t) \tag{11}$$
  From descendents of $\mathrm{Tr}(F^1F^2)\mathrm{Tr}(F^3F^4)\mathrm{Tr}(F^1F^3F^2F^4)$.

- $$B^{0,3}(s,u)\left[\left(p^1_a\epsilon^1_b - p^1_b\epsilon^1_a\right)\left(p^2_b\epsilon^2_c - p^2_c\epsilon^2_b\right)\left(p^3_c\epsilon^3_d - p^3_d\epsilon^3_c\right)\left(p^4_d\epsilon^4_a - p^4_a\epsilon^4_d\right)\right.$$
$$\left.\left(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p\right)\left(p^2_q\epsilon^2_r - p^2_r\epsilon^2_q\right)\left(p^3_r\epsilon^3_s - p^3_s\epsilon^3_r\right)\left(p^4_s\epsilon^4_p - p^4_p\epsilon^4_s\right)\right]$$
$$+ B^{0,3}(t,u)\,[3 \leftrightarrow 2] + B^{0,3}(s,t)\,[3 \leftrightarrow 4] \tag{12}$$
  $$B^{0,3}(s,u) = B^{0,3}(u,s) \tag{13}$$
  (from descendents of $\mathrm{Tr}(F^1F^2F^3F^4)\mathrm{Tr}(F^1F^2F^3F^4)$)

Explicit parameterization of the Gravity S matrix

- $$B^{0,4}(s,t)\left[\left(p^1_a\epsilon^1_b - p^1_b\epsilon^1_a\right)\left(p^2_b\epsilon^2_c - p^2_c\epsilon^2_b\right)\left(p^3_c\epsilon^3_d - p^3_d\epsilon^3_c\right)\left(p^4_d\epsilon^4_a - p^4_a\epsilon^4_d\right)\right.$$
$$\left.\left(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p\right)\left(p^3_q\epsilon^3_r - p^3_r\epsilon^3_q\right)\left(p^2_r\epsilon^2_s - p^2_s\epsilon^2_r\right)\left(p^4_s\epsilon^4_p - p^4_p\epsilon^4_s\right)\right]$$
$$+ B^{0,4}(s,u)\,[3\leftrightarrow 4] + B^{0,4}(u,t)\,[2\leftrightarrow 4] \tag{14}$$
$$B^{0,4}(s,t) = B^{0,4}(t,s) \tag{15}$$
from descendents of $\mathrm{Tr}(F^1F^2F^3F^4)\mathrm{Tr}(F^1F^3F^2F^4)$

- $$B^{0,5}(t,u)\left[\left(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p\right)\left(p^2_p\epsilon^2_q - p^2_q\epsilon^2_p\right)\left(p^3_r\epsilon^3_s - p^3_s\epsilon^3_r\right)\left(p^4_r\epsilon^4_s - p^4_s\epsilon^4_r\right)\right.$$
$$\left.\left(p^1_a\epsilon^1_b - p^1_b\epsilon^1_a\right)\left(p^2_a\epsilon^2_b - p^2_b\epsilon^2_a\right)\left(p^3_c\epsilon^3_d - p^3_d\epsilon^3_c\right)\left(p^4_c\epsilon^4_d - p^4_d\epsilon^4_c\right)\right]$$
$$+ B^{0,5}(s,u)\,[3\leftrightarrow 2] + B^{0,5}(s,t)\,[2\leftrightarrow 4] \tag{16}$$
$$B^{0,5}(t,u) = B^{0,5}(u,t) \tag{17}$$
from descendents of $\mathrm{Tr}(F^1F^2)\mathrm{Tr}(F^3F^4)\mathrm{Tr}(F^1F^2)\mathrm{Tr}(F^3F^4)$

Explicit prameterization of the four graviton S matrix

- $$B^{0,6}(s,u)\left[(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p)\,(p^4_p\epsilon^4_q - p^4_q\epsilon^4_p)\,(p^2_r\epsilon^2_s - p^2_s\epsilon^2_r)\,(p^3_r\epsilon^3_s - p^3_s\epsilon^3_r)\right.$$
$$\left.(p^1_a\epsilon^1_b - p^1_b\epsilon^1_a)\,(p^2_a\epsilon^2_b - p^2_b\epsilon^2_a)\,(p^3_c\epsilon^3_d - p^3_d\epsilon^3_c)\,(p^4_c\epsilon^4_d - p^4_d\epsilon^4_c)\right]$$
$$+ B^{0,6}(t,u)\,[3 \leftrightarrow 2] + B^{0,6}(s,t)\,[3 \leftrightarrow 4] \tag{18}$$

$$B^{0,6}(s,u) = B^{0,6}(u,s) \tag{19}$$

from descendents of $\mathrm{Tr}(F^1F^2)\mathrm{Tr}(F^3F^4)\mathrm{Tr}(F^1F^4)\mathrm{Tr}(F^2F^3)$

- This completes the listing of the S matrices of denscendents of 6 and 8 derivative terms. We now turn to the listing of S matrices that follow from descendents of the two 10 derivative and one 12 derivative terms.

Explicit prameterization of the general 4 graviton S matrix

- $$+ \left(B^{2,1}(s,u)\,(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p)\,(p^2_q\epsilon^2_r - p^2_r\epsilon^2_q)\,(p^3_r\epsilon^3_s - p^3_s\epsilon^3_r)\,(p^4_s\epsilon^4_p - p^4_p\epsilon^4_s)\right.$$
$$B^{2,1}(t,u)\,(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p)\,(p^3_q\epsilon^3_r - p^3_r\epsilon^3_q)\,(p^2_r\epsilon^2_s - p^2_s\epsilon^2_r)\,(p^4_s\epsilon^4_p - p^4_p\epsilon^4_s)$$
$$\left. + B^{2,1}(t,s)\,(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p)\,(p^3_q\epsilon^3_r - p^3_r\epsilon^3_q)\,(p^4_r\epsilon^4_s - p^4_s\epsilon^4_r)\,(p^2_s\epsilon^2_p - p^2_p\epsilon^2_s)\right)$$
$$\left((p^1_a\epsilon^1_b - p^1_b\epsilon^1_a)\,p^2_a\,(p^2_\mu\epsilon^2_\nu - p^2_\nu\epsilon^2_\mu)\,p^3_b\,(p^3_\nu\epsilon^3_\alpha - p^3_\alpha\epsilon^3_\nu)\,(p^4_\alpha\epsilon^4_\mu - p^4_\mu\epsilon^4_\alpha)\right.$$
$$+ (p^2_a\epsilon^2_b - p^2_b\epsilon^2_a)\,p^1_a\,(p^1_\mu\epsilon^1_\nu - p^1_\nu\epsilon^1_\mu)\,p^4_b\,(p^4_\nu\epsilon^4_\alpha - p^4_\alpha\epsilon^4_\nu)\,(p^3_\alpha\epsilon^3_\mu - p^3_\mu\epsilon^3_\alpha)$$
$$+ (p^3_a\epsilon^3_b - p^3_b\epsilon^3_a)\,p^4_a\,(p^4_\mu\epsilon^4_\nu - p^4_\nu\epsilon^4_\mu)\,p^1_b\,(p^1_\nu\epsilon^1_\alpha - p^1_\alpha\epsilon^1_\nu)\,(p^2_\alpha\epsilon^2_\mu - p^2_\mu\epsilon^2_\alpha)$$
$$\left. + (p^4_a\epsilon^4_b - p^4_b\epsilon^4_a)\,p^3_a\,(p^3_\mu\epsilon^3_\nu - p^3_\nu\epsilon^3_\mu)\,p^2_b\,(p^2_\nu\epsilon^2_\alpha - p^2_\alpha\epsilon^2_\nu)\,(p^1_\alpha\epsilon^1_\mu - p^1_\mu\epsilon^1_\alpha)\right) \tag{20}$$

$$B^{2,1}(s,u) = B^{2,1}(u,s) \tag{21}$$

from descendents of $\mathrm{Tr}(F^1F^2F^3F^4)F^1_{ab}\mathrm{Tr}(p^2_aF^2p^3_bF^3F^4)$.

Explicit parameterization of the general 4 graviton S matrix

- $$\begin{aligned}
&\big(B^{2,2}(t,u)\,(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p)\,(p^2_p\epsilon^2_q - p^2_q\epsilon^2_p)\,(p^3_r\epsilon^3_s - p^3_s\epsilon^3_r)\,(p^4_r\epsilon^4_s - p^4_s\epsilon^4_r)\\
&+ B^{2,2}(s,u)\,(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p)\,(p^3_p\epsilon^3_q - p^3_q\epsilon^3_p)\,(p^2_r\epsilon^2_s - p^2_s\epsilon^2_r)\,(p^4_r\epsilon^4_s - p^4_s\epsilon^4_r)\\
&+ B^{2,2}(t,s)\,(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p)\,(p^4_p\epsilon^4_q - p^4_q\epsilon^4_p)\,(p^3_r\epsilon^3_s - p^3_s\epsilon^3_r)\,(p^2_r\epsilon^2_s - p^2_s\epsilon^2_r)\big)\\
&\big((p^1_a\epsilon^1_b - p^1_b\epsilon^1_a)\,p^2_a\,(p^2_\mu\epsilon^2_\nu - p^2_\nu\epsilon^2_\mu)\,p^3_b\,(p^3_\nu\epsilon^3_\alpha - p^3_\alpha\epsilon^3_\nu)\,(p^4_\alpha\epsilon^4_\mu - p^4_\mu\epsilon^4_\alpha)\\
&+ (p^2_a\epsilon^2_b - p^2_b\epsilon^2_a)\,p^1_a\,(p^1_\mu\epsilon^1_\nu - p^1_\nu\epsilon^1_\mu)\,p^4_b\,(p^4_\nu\epsilon^4_\alpha - p^4_\alpha\epsilon^4_\nu)\,(p^3_\alpha\epsilon^3_\mu - p^3_\mu\epsilon^3_\alpha)\\
&+ (p^3_a\epsilon^3_b - p^3_b\epsilon^3_a)\,p^4_a\,(p^4_\mu\epsilon^4_\nu - p^4_\nu\epsilon^4_\mu)\,p^1_b\,(p^1_\nu\epsilon^1_\alpha - p^1_\alpha\epsilon^1_\nu)\,(p^2_\alpha\epsilon^2_\mu - p^2_\mu\epsilon^2_\alpha)\\
&+ (p^4_a\epsilon^4_b - p^4_b\epsilon^4_a)\,p^3_a\,(p^3_\mu\epsilon^3_\nu - p^3_\nu\epsilon^3_\mu)\,p^2_b\,(p^2_\nu\epsilon^2_\alpha - p^2_\alpha\epsilon^2_\nu)\,(p^1_\alpha\epsilon^1_\mu - p^1_\mu\epsilon^1_\alpha)\big)
\end{aligned} \tag{22}$$

$$B^{2,2}(t,u) = B^{2,2}(u,t) \tag{23}$$

from descendents of $\mathrm{Tr}(F^1F^2)\mathrm{Tr}(F^3F^4)F^1_{ab}\mathrm{Tr}(p^2_aF^2p^3_bF^3F^4)$

Explicit prameterization of the general 4 graviton S matrix

- $$\begin{aligned} &\left(B^{4,1}(s,t)+B^{4,1}(t,u)+B^{4,1}(u,s)\right)\times \\ &\left[\left(p^1_a\epsilon^1_b-p^1_b\epsilon^1_a\right)p^2_a\left(p^2_\mu\epsilon^2_\nu-p^2_\nu\epsilon^2_\mu\right)p^3_b\left(p^3_\nu\epsilon^3_\alpha-p^3_\alpha\epsilon^3_\nu\right)\left(p^4_\alpha\epsilon^4_\mu-p^4_\mu\epsilon^4_\alpha\right)\right. \\ &\left(p^1_p\epsilon^1_q-p^1_q\epsilon^1_p\right)p^2_p\left(p^2_\beta\epsilon^2_\gamma-p^2_\gamma\epsilon^2_\beta\right)p^3_q\left(p^3_\gamma\epsilon^3_\delta-p^3_\delta\epsilon^3_\gamma\right)\left(p^4_\delta\epsilon^4_\beta-p^4_\beta\epsilon^4_\delta\right) \\ &\left.+(1\leftrightarrow 2)+(1\leftrightarrow 3)+(1\leftrightarrow 4)\right] \end{aligned} \tag{24}$$

  $$B^{4,1}(s,t)=B^{4,1}(u,t)=B^{4,1}(t,s)=B^{4,1}(u,s)=B^{4,1}(s,u)=B^{4,1}(t,u) \tag{25}$$

  from descendents of
  $F^1_{pq}\mathrm{Tr}(p^2_pF^2p^3_qF^3F^4)F^1_{ab}\mathrm{Tr}(p^2_aF^2p^3_bF^3F^4)$

Checks of our parameterization of the S matrix

- First, the 4 graviton S matrix from the Einstein Lagrangian, which is given by

$$A_{4h}^{EG} = \frac{-4\kappa^2}{stu}\Big(\frac{1}{2}\epsilon_2.\epsilon_3\left(s\epsilon_1.k_3\epsilon_4.k_2 + t\epsilon_1.k_2\epsilon_4.k_3\right) + \frac{1}{2}\epsilon_1.\epsilon_4\left(s\epsilon_2.k_4\epsilon_3.k_1 + t\epsilon_2.k_1\epsilon_3.k_4\right)$$
$$+\frac{1}{2}\epsilon_2.\epsilon_4\left(s\epsilon_1.k_4\epsilon_3.k_2 + u\epsilon_1.k_2\epsilon_3.k_4\right) + \frac{1}{2}\epsilon_1.\epsilon_3\left(s\epsilon_2.k_3\epsilon_4.k_1 + u\epsilon_2.k_1\epsilon_4.k_3\right)$$
$$+\frac{1}{2}\epsilon_3.\epsilon_4\left(t\epsilon_1.k_4\epsilon_2.k_3 + u\epsilon_1.k_3\epsilon_2.k_4\right) + \frac{1}{2}\epsilon_1.\epsilon_2\left(t\epsilon_3.k_2\epsilon_4.k_1 + u\epsilon_3.k_1\epsilon_4.k_2\right)$$
$$-\frac{1}{4}st\epsilon_1.\epsilon_4\epsilon_2.\epsilon_3 - \frac{1}{4}su\epsilon_1.\epsilon_3\epsilon_2.\epsilon_4 - \frac{1}{4}tu\epsilon_1.\epsilon_2\epsilon_3.\epsilon_4\Big)^2 \tag{26}$$

- This turns out to be proportional to $\frac{1}{stu}$ times the S matrix generated the Lagrangian

$$L_{4h}^{EG} \propto \frac{1}{32}\left(R_{pqrs}R^{pqrs}\right)^2 - \frac{1}{2}R^{pqrs}R_{pqr}{}^{t}R^{uvw}{}_{s}R_{uvwt} + \frac{1}{16}R^{pqrs}R_{pq}{}^{tu}R_{tu}{}^{vw}R_{rsvw}$$
$$-\frac{1}{4}R^{pqrs}R_{pq}{}^{tu}R_{rt}{}^{vw}R_{suvw} - R^{pqrs}R_{p}{}^{t}{}_{r}{}^{u}R_{tvws}R_{q}{}^{v}{}_{u}{}^{w} + \frac{1}{2}R^{pqrs}R_{p}{}^{t}{}_{r}{}^{u}R_{t}{}^{v}{}_{u}{}^{w}R_{qvsw} \tag{27}$$

which, in turn, is easily written as a linear combination of the six 4 Reimann structures listed above.

Checks of our parameterization of the 4 graviton S matrix

- Next, the 4-graviton amplitude in Type *II* superstring theory is proportional (in the sense of index structure) to the S matrix for Einstein gravity, and so can also be easily written in our basis.

$$A_{4h}^{ss} = h(s,t,u,\alpha')A_{4h}^{EG} \tag{28}$$

- The tree level S matrices for the heterotic string and the bosonic string are more complicated, but also can each be written as a linear combination of the last 9 structures we discussed above. The first structure - descendents of the 6 derivative term - never appears in tree level string amplitudes. It would be interesting to check whether this structure appears in string loop amplitudes. We have not yet tried this.

Bounding data

- Recall that the CEMZ programme for constraining 3 graviton scattering had 2 steps. The first step was to use symmetry considerations to minimally parameterize the S matrix. We are now done with the analogous step for the 4 graviton S matrix. As you can see the result here is much more complicated; as opposed to 3 numbers it is given in terms of 10 unknown functions of s and t with 29 (infinite functions worth) degrees of freedom.
- We now turn to the second step of the programme, namely to use a physical principle to constrain the parameters that appear in the S matrix. Our work here is, so far, less complete and more conjectural, but I will describe how far we have reached.

Chaos Bound

- Recall the following result proved by Maldacena, Shenker and Stanford. Consider a large N CFT. Consider the (ordinary time ordered) four point function of 4 (to start with identical) operators inserted at the following points in the x, t plane. The first two operators are inserted at the point $t = 0$, $x = 1$ but then boosted respectively with boost parameters $e^{\frac{\tau}{2}}$ and $e^{-\frac{\tau}{2}}$. The next two operators are first placed at $t = 0$ and $x = -1$ and then boosted with the same two boosts.
- MSS considered the limit $N \to \infty$ first and then $\tau \to \infty$. They demonstrated that the four point function described above is allowed to grow as $\tau \to \infty$, but no faster than e^{τ}.
- We will now examine the consequences of this result for the holographic dual of such a CFT

Consequences of chaos bound for local bulk scalars

- Let us now consider a situation in which the holographic dual to our CFT has a scalar field with a local Lagrangian - and more generally couplings that would generate a local S matrix.
- Such lagrangians are paremeterized in precisely the manner described earlier in this talk. Heemskirk, Penedones, Polchinski and Sully used the usual AdS/CFT dictionary to explicitly construct the boudary 4 point function that arises out of any given bulk Lagrangian.
- Using their results one can verify the following result. Consider a term in the bulk Lagrangian that would lead to a flat space S matrix that scales in the Regge limit (large s, fixed t) like s^{m+1}. The four point function that follows from the same bulk term scales like $e^{m\tau}$.
- It follows immediately that any local bulk term that leads to an S matrix that grows faster than s^2 in the Regge limit violates the chaos bound and so must be unphysical.

Conjecture for bounds on Regge Scattering

- The observations above lead us to make the following conjecture
- Classical theories whose S matrices grow faster than s^2 in the Regge limit are unphysical.
- That something like the above should be true has been suggested - perhaps a bit implicitly - by many people including MSS, Caron-Huot, Zhibeodov ...
- The conjecture above is currently best motivated for scalars. However in what follows I will assume that it holds also for vectors and gravitions.
- Note that this bound is saturated by scattering in Einstein gravity, and that α' effects in string theory change the power 2 to a power less than 2.
- We are currently working on using AdS/CFT and the chaos bound to directly gather more evidence for this conjecture for vectors and tensors. In the rest of this talk we will explore the consequences of this conjecture.

Scalars

- It is easy to verify that there are exactly three local scalar S matrices that obey the conjectured bound on growth of Regge amplitudes
- These S matrices and their corresponding Lagrangian structures are

$$a_0 + a_2(st + tu + us) + a_3 stu \tag{29}$$

They come from the local Lagrangian

$$a_0\phi^4 + a_2\left(\partial_\mu\partial_\nu\phi\partial_\mu\phi\partial_\nu\phi\phi\right) + a_3\left(\partial_\mu\partial_\nu\partial_\alpha\phi\partial_\mu\phi\partial_\nu\phi\partial_\alpha\phi\right) \tag{30}$$

- (30) is also precisely the terms that characterize that part of the 4 point function that is undetermined by Caron Huot's formula (see e.g. a paper by Zhibedeov from a year ago). This is not a concidence, as the chaos bound was a key phyiscal input into Caron Huot's formula.

Photons

- Using our explicit parameterization of vector S matrices, it is not difficult to use our explicit parameterization of photon S matrices to ennumerate all local photon 4 point S matrices that grow no faster than s^2 in the Regge limit.
- We find these are given in terms of four constants a, b, c and d by $A^{01}(t,u) = a$, $A^{0,2}(t,u) = b + cs$, $A^{1,0} = d$ corresponding to the four parameter set of Lagrangians

$$a(\mathrm{Tr}F^2)^2 + b\mathrm{Tr}F^4 + c\mathrm{Tr}\left(\partial_\mu F F \partial_\mu F F\right) + dF_{ab}\mathrm{Tr}\left(\partial_a F \partial_b F F\right) \tag{31}$$

- Note that this allowed set of Lagrangians includes the expansion of the Born Infeld action to quadratic order
- In analogy with the scalar case we expect (31) to parameterize the ambiguity in the large N version of Caron-Huots formula for vectors.

Gravitons

- Once again we can use our explicit parameterization of graviton S matrices to list the most general local S matrix that grows no faster than s^2 in the Regge limit. We find that there is only one such S matrix namely

$$a\left(\epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3 \wedge \epsilon_4 \wedge p_1 \wedge p_2 \wedge p_3\right)^2$$

- This S matrix comes from the Lagrangian term

$$a\left(R^{pqrs} R_{pq}{}^{tu} R_{rstu} + 2 R^{pqrs} R_{p}{}^{t}{}_{r}{}^{u} R_{qtsu}\right)$$

Gravitions: Implications

- Let us summarize. The results of CAMS together with our conjecture for the Regge growth of S matrices together imply that the most general local classical gravitational theory in flat spaceis given by the Lagrangian

$$\int \sqrt{g}\left[R + a\left(R^{pqrs}R_{pq}{}^{tu}R_{rstu} + 2R^{pqrs}R_{p}{}^{t}{}_{r}{}^{u}R_{qtsu}\right) + \mathcal{O}(R^5_{abcd})\right]$$

- In the rest of this talk we will discuss this result

Gravitions: Discussion

- $$\int \sqrt{g}\left[R + a\left(R^{pqrs}R_{pq}{}^{tu}R_{rstu} + 2R^{pqrs}R_{p}{}^{t}{}_{r}{}^{u}R_{qtsu}\right) + \mathcal{O}(R^5_{abcd})\right]$$
- It turns out that the term proportional to a vanishes identially for $d \leq 5$. In $d = 6$ this term is a total derivative. The term is classically nontrivial only for $d \geq 7$. This fact is already apparent from the form of its S matrix.
- It follows that our conjecture implies that the 4 graviton Einstein Lagranian is the unique local gravition S matrix for $d \leq 6$.
- It is intriguing that our considerations do not lead to uniqueness for $d \geq 7$. I think it is likely that some other physical consideration rules out the term proportional to a even in these high dimensions. This is a topic for future work.

Discussions and Conclusions

- In this talk we first presented a complete classification of 4 graviton (and four photon) S matrices. We then presented a conjecture about the allowed growth of S matrices in classical theories. We then used this conjecture to completely classify allowed classical theories of gravity, upto Lagrangian terms of order Riemann^5 or higher that do not impact 4 graviton scattering.
- It would be very nice to understand our conjecture better - and if possible to replace it with a clear physical argument. Relatedly, it would be interesting to understand the physical principles that lead to this result (assuming its validity).

Discussions and Conclusions

- It would also be interesting to understand the status of the ambiguity of the action in $d \geq 7$. Is this a genuine ambiguity, or does another physical argument set a to zero?
- It would be very interesting to generalize the results of this talk to the scattering of more than 4 gravitions, and complete the process of characterizing the most general classical local theory of gravity consistent with general principles.
- Finally, if all this works out we could get more ambitious and generalize the study of this talk beyond local S matrices, with the hope of establishing the uniqueness of string scattering, as discussed at the beginning of this talk.