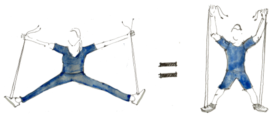
Pulling Yourself Up by Your Bootstraps in Quantum Field Theory

Leonardo Rastelli

Yang Institute for Theoretical Physics
Stony Brook University

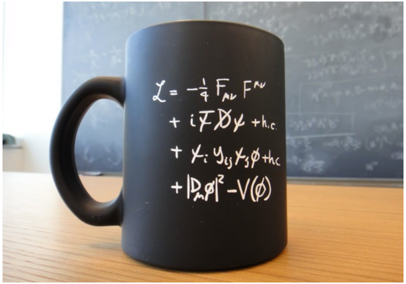


ICTP and SISSA
Trieste, April 3 2019

Quantum Field Theory in Fundamental Physics

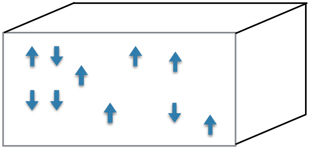
Local quantum fields $\{\varphi_i(x)\}$
$x = (t, \vec{x})$, with $t =$ time, $\vec{x} =$ space

The language of particle physics:
for each particle species, a field



Quantum Field Theory for Collective Behavior

Modelling $N \to \infty$ degrees of freedom in statistical mechanics.
Example: Ising model (uniaxial ferromagnet)



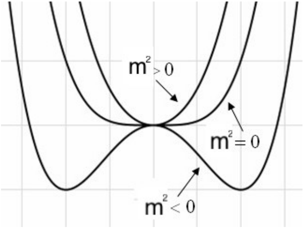
$\sigma_i = \pm 1$, spin at lattice site i

Energy $H = -J \sum_{(ij)} \sigma_i \sigma_j$

Near T_c, field theory description: magnetization $\varphi(\vec{x}) \sim \langle \sigma(\vec{x}) \rangle$,

$$H = \int d^3x \left[\vec{\nabla}\varphi \cdot \vec{\nabla}\varphi + m^2 \varphi^2 + \lambda\, \varphi^4 + \dots \right]$$

$$m^2 \sim T - T_c$$



$$H = \int d^3x \left[\vec{\nabla}\varphi \cdot \vec{\nabla}\varphi + m^2\varphi^2 + \lambda\,\varphi^4 + \ldots \right]$$

The dots stand for higher-order "operators": φ^6, $(\vec{\nabla}\varphi \cdot \vec{\nabla}\varphi)\varphi^2$, φ^8, etc.

They are irrelevant for the large-distance physics at $T \sim T_c$.

Crude rule of thumb:
an operator $\mathcal{O}$ is irrelevant if its scaling weight $[\mathcal{O}] > 3$ ($3 \equiv d$, dimension of space).

Basic assignments: $[\varphi] = \frac{1}{2} \equiv \frac{d}{2} - 1$ and $[\vec{x}] = -1 \Longrightarrow [\vec{\nabla}] = 1$.

So $[\varphi^2] = 1$, $[\varphi^4] = 2$, $[\vec{\nabla}\varphi \cdot \vec{\nabla}\varphi]$ =3, while $[\varphi^8] = 4$ etc.

First hint of universality: critical exponents do not depend on details.
E.g., $C_T \sim |T - T_c|^{-\alpha}$, $\langle\varphi\rangle \sim (T_c - T)^{\beta}$ for $T < T_c$, etc.

QFT $\equiv$ "Theory of fluctuating fields" (Duh!)

Traditionally, QFT is formulated as a theory of local "quantum fields":

$$Z = \int \prod_x d\varphi(x)\, e^{-\frac{H[\varphi(x)]}{g}}$$

In particle physics, $x \in$ space*time* and $g = \hbar$ (quantum)
In statistical mechanics, $x \in$ space and $g = T$ (thermal).

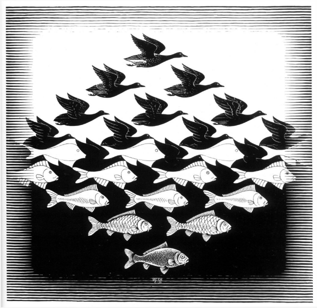
Difficult to calculate when g is not small.

Inadequacy of traditional viewpoint

More *conceptually*, indications that a better framework is needed:

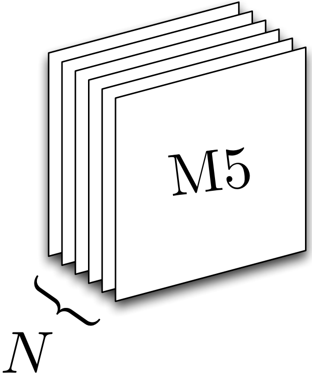
$$\mathcal{T}[\varphi; g] \Leftrightarrow \mathcal{T}'[\varphi'; g'] \,, \qquad g' = \frac{1}{g}$$

$\mathcal{T}[\varphi; g]$



$\mathcal{T}'[\varphi'; g']$

Theories with *no* $H[\varphi(x)]$.
Paradigm: $6d$ $(2,0)$ theory.



Our vision: the Bootstrap

Determining Theory Space using Consistency:

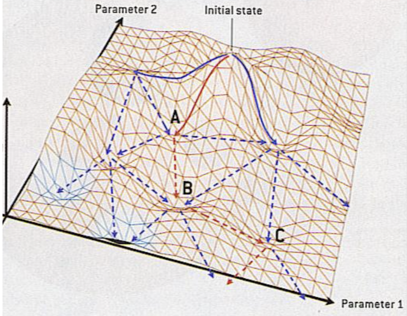
Symmetries & Quantum Mechanics.

Sharp rigorous predictions without resorting to approximations.

Ultimate goal: compile a complete catalogue of consistent QFTs.

[Wikipedia: *Bootstrap* is “a self-starting process that is supposed to proceed without external input”]

A picture of Theory Space



Height $=$ a measure of the number of degrees of freedom.

Stationary points $=$ Theories that look the same at all length scales, called Conformal Field Theories. They are defined by an *abstract* operator algebra.

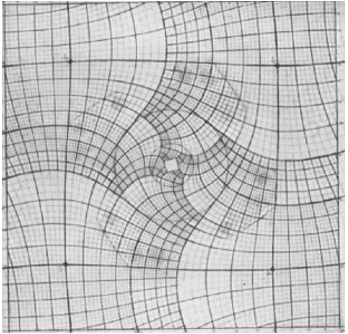
We "flow" from higher to lower points by "integrating out" short-distance d.o.f.

This can be made rigorous in $d = 2$: the "height" is called the c-function.

Scale $\to$ Conformal

Physics simplifies when intrinsic length scales can be neglected:
high/low energy regimes of QFTs and statistical systems near T_c.

Scale invariance is "generically" enhanced to conformal invariance.
A conformal transformation acts *locally* as rotation and dilatation.



Abstract Conformal Field Theory

A CFT is defined by the correlations $\langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle$ of local operators $\{\mathcal{O}_k(x)\}$.

E.g., in Ising CFT we have: the spin operator $\sigma(x)$,
the energy operator $\epsilon(x)$,
and infinitely many more.

Scaling dimensions Δ_i: $\langle \mathcal{O}_i(x) \mathcal{O}_i(y) \rangle = |x-y|^{-2\Delta_i}$,
related to critical exponents, *e.g.*, $\alpha = \frac{3-2\Delta_\epsilon}{3-\Delta_\epsilon}$.

Operator Product Expansion

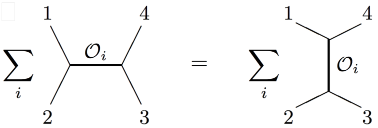
$$\mathrm{OPE}: \quad \mathcal{O}_i(x)\mathcal{O}_j(y) = \sum_k c_{ijk} \, |x-y|^{\Delta_k - \Delta_i - \Delta_j} \left(\mathcal{O}_k(y) + \ldots \right) .$$

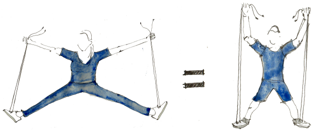
The sum converges (unlike in a general QFT).

Conformal bootstrap

The data $\{\Delta_i, c_{ijk}\}$ determine the theory.

Old aspiration (1970s) Polyakov, Ferrara Gatto Grillo:
Impose crossing symmetry to solve for these data. For a 4-point function:





“Madamina, il catalogo è questo: ...ma in Ispagna son già mille e tre”

Famous success story in $d = 2$. Conformal symmetry is infinite dimensional.

In some cases, only *finite* number of independent $\mathcal{O}_i \Rightarrow$ exact solution of the bootstrap.

Complete catalogue of (unitary) CFTs $\mathcal{M}_p$ with $c < 1$.
Labelled by an integer $p \geq 3$, with

$$c = 1 - \frac{6}{p(p+1)}$$

- $p = 3$, $c = 1/2$, Ising model.
- $p = 4$, $c = 7/10$, tricritical Ising model.
- $p = 5$, $c = 4/5$, three-state Potts model.
- $\ldots$

The modern bootstrap program

Rattazzi Rychkov Tonni Vichi, 2008

Crossing + positive probability $\Rightarrow$ inequalities for $\{\Delta_i, c_{ijk}\}$.

Bootstrap inequalities obtained numerically but perfectly rigorous:
they may not be optimal but they are true.

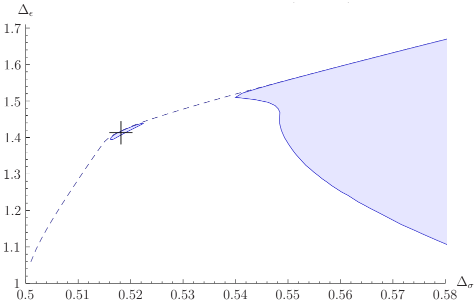
Bootstrap of $3d$ Ising CFT

El-Showk Paulos Poland Rychkov Simmons-Duffin Vichi, Kos Poland S-D

CFT in $d = 3$ with up/down symmetry.

σ odd, ϵ even, $\sigma \times \sigma = \mathbf{1} + \epsilon + \ldots$

Assuming that σ, ϵ are the only relevant operators ($\Delta < 3$)



$3d$ Ising gets cornered. Critical universality explained!

Most precise determination of critical exponents. Rigorous error bars.

Charting Theory Space

The modern bootstrap is a very flexible tool: works for any dimension, any symmetry.

- The CFT genome project:

  Find *the* list of CFTs with a *small number* of relevant operators.
  Some partial results.

Adding symmetries makes theories more rigid and more tractable.

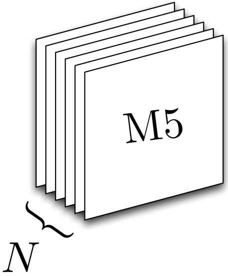
Supersymmetry (fermions $\leftrightarrow$ bosons) is a theorist's favorite.
SuperCFTs play a central role in modern physical mathematics.

- Full classification program for superCFTs.

  They exist only for $d \leq 6$ (Nahm '77).

  Classification facilated by *solvable* subsectors identical to well-studied mathematical structures, such as vertex operator algebras.

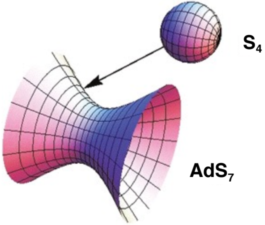
The Great Mother: The Six-dimensional $(2,0)$ Theory



$(2,0)_N$ theory governs low-energy fluctuations of N five-branes in M-theory, a quantum gravity theory in 11 dimensions

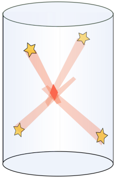
- Maximally (super)symmetric QFT in highest dimension.
- Paradigm of a theory with *no* $H[\varphi]$. For fixed N, intrinsically quantum.
  Surprising that an interacting field theory exists at all in $d = 6$.
  E.g., $[\varphi] = 2$ so $[\varphi^4] = 8 > d$. No obvious microscopic description.
- "Mother" of huge landscape of QFTs in $d < 6$, by compactification.

$(2,0)_{N\to\infty} \cong$ 11d supergravity on



Canonical holographic duality:
Local CFT in flat space $\equiv$ *quantum gravity theory* in Anti-de Sitter, in one higher dim.

For large N, bulk side is weakly coupled: *classical* $11d$ supergravity.
CFT correlators = supergravity "scattering amplitudes" in AdS.



They are entirely fixed by symmetry LR Zhou

Bootstrap of $(2,0)_N$ Theory Beem Lemos LR van Rees

Minimal physical input: $\exists$ a tower of $\mathcal{O}_\Delta^{\rm BPS}$ with $\Delta = 4, 6, \ldots 2N$.
(Δ = conformal dimension)

- Bootstrap is solvable for a subalgebra $\{\mathcal{O}^\chi\} \supset \{\mathcal{O}^{\rm BPS}\}$!

  Protected subalgebra $\{\mathcal{O}^\chi\}$, with meromorphic correlators

$$\langle \mathcal{O}_1^\chi(z_1, \bar{z}_1)\, \mathcal{O}_2^\chi(z_2, \bar{z}_2) \ldots \mathcal{O}_n^\chi(z_n, \bar{z}_n)\rangle = f(z_i)\,.$$

  Isomorphic to the left-moving sector of a $2d$ CFT (= vertex operator algebra).
  $(2,0)_N \to$ well-known W_N algebra! Infinite dimensional symmetry.

  Exact 3-point functions of for any N.
  For $N \to \infty$, striking agreement with supergravity on $AdS_7 \times S^4$.
  One recovers non-linear supergravity purely from algebraic consistency.

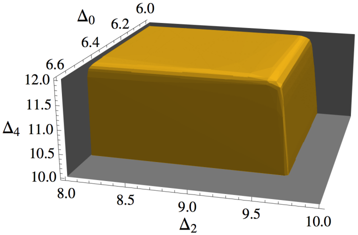
  $1/N$ corrections $\Rightarrow$ quantum M-theory corrections.

- For “non-protected” operators (non-integer Δ), numerical bootstrap.

  For $N = 2$, strong evidence that bootstrap has unique solution!

  Conjecturally true for all N, using more complicated correlators.

  A blueprint for defining and solving the $(2,0)_N$ theory.



4d $\mathcal{N} = 2$ SCFT $\longrightarrow$ 2d Chiral Algebra

Any $4d$ $\mathcal{N} = 2$ SCFT has a protected subsector isomorphic to a $2d$ chiral algebra (*aka* a Vertex Operator Algebra). Beem Lemos Liendo Peelaers LR

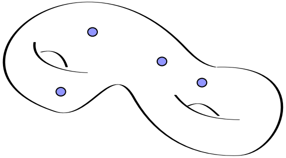
- Always a Virasoro subalgebra with $c_{2d} = -12\, c_{4d}$
  $c_{4d} \equiv$ Weyl^2 conformal anomaly coefficient.
- Global flavor symmetry of SCFT $\rightarrow$ Affine Kac-Moody algebra
- Conjecture (Beem LR): Higgs branch $\equiv$ *associated variety* of the VOA
- Supersymmetric index of the SCFT $\equiv$ vacuum character $\chi(q)$ of the VOA
  Previous conjecture $\rightarrow$ $\chi(q)$ must obey a modular differential equation.

Some examples of VOAs from $4d$ SCFTs

- $SU(2)$ super QCD with $N_f = 4 \Rightarrow \mathfrak{so}(8)_{-2}$ AKM algebra.
- E_6 SCFT $\Rightarrow (\mathfrak{e}_6)_{-3}$ AKM algebra.
- $\mathcal{N} = 4$ SYM $\Rightarrow \mathcal{N} = 4$ super $\mathcal{W}$-algebra,
  with generators given by chiral primaries of dimensions $\{h_i = \frac{r_i+1}{2}\}$,
  $\{r_i\} \equiv$ exponents of gauge group G.
- (A_1, A_{2k}) Argyes-Douglas theories $\Rightarrow$ Virasoro algebra of $(2, 2k+3)$ model

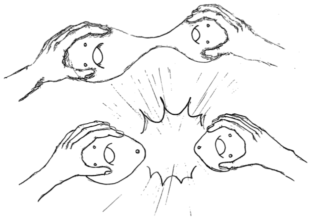
$6 = 4 + 2$: class $\mathcal{S}$(ix) theories Gaiotto

Put $(2,0)$ on $\mathbb{R}^4 \times \mathcal{C}$.
$\mathcal{C} \equiv$ Riemann surface with punctures.
$\Downarrow$
$\mathcal{N} = 2$ SUSY CFT on $\mathbb{R}^4$.



Theory space as a real physical space, the surface $\mathcal{C}$!

Consistency conditions in theory space. E.g., a TQFT valued in VOAs!
Degeneration of $\mathcal{C} \Rightarrow$ Theory $\mathcal{T}[\mathcal{C}]$ splits into decoupled theories. [Roy Sato's drawing]



I've emphasized two heuristic principles:

- Enlarge the view to the whole space of QFTs.
- Bootstrap approach:
  Use internal consistency rules and symmetries to chart theory space, rather than solving detailed "microscopic" models.

(From the Salt Lake Tribune)

Concrete models...

As physicists, we often build detailed dynamical models:

Identify relevant degrees of freedom $\{\varphi_i\}$

$\downarrow$

Write a model $H[\varphi_i]$

$\downarrow$

Solve it

...versus general constraints

A "meta" question:
Which theories are in principle allowed?
Not *anything* goes!

Quantum mechanics + Spacetime symmetries
and Specific symmetries of the problem $\Rightarrow$ very constraining

Ideally, find the complete catalogue of consistent theories.

In some important physical situations, only one theory is possible,
given some minimal physical input

MUST WE NOT THEN RENOUNCE THE OBJECT ALTOGETHER, THROW IT TO THE WINDS AND INSTEAD LAY BARE THE PURELY ABSTRACT?

VASILY KANDINSKY 1911