Star-product:
Higher-Spin Theory vs. String Field Theory

String Field Theory and Related Aspects VI, Trieste

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(based on 1210.7963, 1301.4166 with V. Didenko, Jianwei Mei and some papers to appear with K. Alkalaev and M. Grigoriev)

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Outline

- Star-product, Gaussians, group structure, projectors, sewing Projectors (D-branes)
- Higher-Spin Amplitudes without Higher-Spin theory
- Uniformization of Vasiliev theory

HST vs. SFT

There are two general conjectures relating higher-spin theory and string theory:
(i) Higher-Spin theory is a tensionless limit of String theory
(ii) String theory is a broken phase of Higher-Spin theory
There are more concrete conjectures by Chang Minwalla, Sharma, Yin, 2012 and Gaberidiel and Gopakumar, 2014 and non-stringy conjecture by Klebanov-Polyakov; Sezgin-Sundell that relates Vasiliev theory on AdS_4 to Free/Critical $O(N)$-model.

(i) and (ii) are not well-understood at present, but some computations in HST and SFT are identical thanks to the star-product

Star-product is a tool to handle computations with canonical operators in QFT. It appears naturally in SFT (Bars). Vasiliev HST is a classical theory. Nevertheless the star-product is its essential ingredient. Moreover, it seems that SFT and HST share many interesting (simple) solutions and observables.

Star-product

Let $\hat{Y}^A$ be some canonical operators, e.g. $\hat{Y}^A = \{\hat{q}^m, \hat{p}_n\}$

$$[\hat{Y}^A, \hat{Y}^B] = 2i\mathcal{A}^{AB}$$

Replace noncommutative algebra of $\hat{Y}^A$ with the algebra of commuting generating elements Y^A while deforming the usual dot-product into non-commutative star-product, which contains information about operator product and some additional information, which specifies the ordering of operators

Star-product

$$[\hat{Y}^A, \hat{Y}^B] = 2i\mathcal{A}^{AB}$$

exp-formula

$$f(Y) \star g(Y) = f(Y) \exp i \left(\overleftarrow{\partial}_A \Omega^{AB} \overrightarrow{\partial}_B\right) g(Y)$$

$\int$-formula

$$f(Y) \star g(Y) = \int dU\, dV\, f(Y+U) g(Y+V) \exp i(\Omega_{AB} U^A V^B)$$

Symplectic metric : $\mathcal{A}^{AB} = \frac{1}{2}(\Omega^{AB} - \Omega^{BA})$

Ordering : $\mathcal{S}^{AB} = \frac{1}{2}(\Omega^{AB} + \Omega^{BA})$

Relation to $sp(2N)$

Oscillators provide a realization of $sp(2N)$

$$T_{AB} = -\frac{i}{4}\{Y_A, Y_B\}_\star$$

In addition we have

$$[T_{AB}, Y_C] = Y_A\mathcal{A}_{BC} + Y_B\mathcal{A}_{AC}$$

$sp(2N) \vdash$Heisenberg algebra

T^{AB}, Y^A, 1 are treated as even elements

Ortho-symplectic algebra, $osp(1|2N)$

T^{AB} are even and Y^A are odd, but bosonic

Trace vs. super-trace

$Y = \{q^m, p_n\}$ is $\mathbb{Z}_2$-graded so we can think of it either as of algebra or as of super-algebra, while all elements are bosonic

Trace

$$tr(f) = \int d^{2N}Y\, f(Y)$$

needs f to be integrable

Super-trace

$$str(f) = f(0)$$

works nice for many reasonable functions, reduces to a trace for even functions

Gaussian solutions of interest

$$\Phi(A,\vec{\xi},q) = \exp i\left(\tfrac{1}{2}YAY + Y\xi + q\right)$$

SFT

There are quite a few solutions known

- Vacuum
- Perturbative states
- Sliver
- Wedge
- Butterfly
- D-branes
- ...

HST

There are few solutions known at present

- Didenko-Vasiliev Black hole (Generalized by Sundell and Iazeolla)
- Boundary-to-bulk propagators
- Cosmological (Sezgin, Sundell, Iazeolla)
- that's all

Gaussians and star-product

$$\Phi(A,\vec{\xi},q)=\exp i\left(\tfrac{1}{2}YAY+Y\xi+q\right)$$

Gaussians and star-product

$$\Phi(A, \vec{\xi}, q) = \exp i \left(\tfrac{1}{2} YAY + Y\xi + q\right)$$

Heisenberg group of plane waves

$$\Phi(\vec{\xi}, q) \star \Phi(\vec{\eta}, p) = \Phi(\vec{\xi} + \vec{\eta}, q + p + \vec{\xi} \cdot \vec{\eta})$$

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Trace vs. super-trace

$$\Phi(\vec{\xi}_1)\star\ldots\star\Phi(\vec{\xi}_n) = \Phi(\sum_i \vec{\xi}_i,\sum_{i<j}\vec{\xi}_i\cdot\vec{\xi}_j)$$

$$str = \exp i(\sum_{i<j}\vec{\xi}_i\cdot\vec{\xi}_j)$$

$$tr = str\times\delta(\sum_i\vec{\xi}_i)$$

$$\Phi(A, \vec{\xi}, q) = \exp i\left(\tfrac{1}{2}YAY + Y\xi + q\right)$$

Hidden Symplectic group

$$\Phi(A) \star \Phi(B) = N(A,B)\Phi(f(A,B))$$

$$f(A,B) = \frac{1}{1+BA}(B-I) + \frac{1}{1+AB}(I+A)$$

$$N(A,B) = \det^{-\frac{1}{2}}|1+AB|$$

$$\Phi(A,\vec{\xi},q) = \exp i\left(\tfrac{1}{2}YAY + Y\xi + q\right)$$

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Cayley Map $C(a) = \frac{1-a}{1+a}$, $a \in Sp(2N)$

The group structure is manifest now

$$f(C(a),C(b)) = C(ab)$$

$$\Phi(A,\vec{\xi},q) = \exp i\left(\tfrac{1}{2}YAY + Y\xi + q\right)$$

Symplectic group

Group element is

$$G(a) = \frac{2^N}{\det^{\frac{1}{2}}|1+a|}\exp i\left(\tfrac{1}{2}YC(a)Y\right)$$

$$G(a)\star G(b) = G(ab)$$

Gaussians and star-product

$$\Phi(A,\vec{\xi},q)=\exp i\left(\tfrac{1}{2}YAY+Y\xi+q\right)$$

$SpH(2N)=Sp(2N)\ltimes H_N$

$$(a,\vec{u},x)\diamond(b,\vec{v},y)=(ab,\vec{u}+a\vec{v},x+y+\vec{u}a\vec{v})$$

Generalized Cayley map $(A,\vec{\xi},q)\Longleftrightarrow(a,\vec{u},q)$

$$A=\frac{1-a}{1+a}$$

$$\vec{\xi}=\pm 2\left(\frac{1}{1+a}\right)\cdot\vec{u}$$

$$q=x+\vec{u}\cdot\frac{1}{2}\left(\frac{1-a}{1+a}\right)\cdot\vec{u}\,.$$

Gaussians and star-product

$$\Phi(A,\vec{\xi},q) = \exp i\left(\tfrac{1}{2}YAY + Y\xi + q\right)$$

$SpH(2N) = Sp(2N) \ltimes H_N$

$$(a,\vec{u},x)\diamond(b,\vec{v},y) = (ab,\vec{u}+a\vec{v},x+y+\vec{u}a\vec{v})$$

The group law of $SpH(2N)$ is respected now by the star-product

$$\Phi(A,\vec{\xi},q) = \exp i\left(\tfrac{1}{2}YAY + Y\xi + q\right)$$

Projectors $\Phi \star \Phi = \Phi$, i.e. $A^3 = A$ or $A^2 = I$

Cayley map fails to be invertible for projectors $C^{-1}(A) = \frac{1-A}{1+A}$ does not exist, formally we are approaching the boundary at infinity of SpH.

The star-product is still well-defined.

A gives rise to two matrix projectors

$$(I+A)(I-A) = 0$$

$$\Phi(A, \vec{\xi}, q) = \exp i \left(\tfrac{1}{2} YAY + Y\xi + q\right)$$

Sewing rules

one remove $\pm$ eigen spaces from $A(B)$ and glue them together into a new projector
$A \circ B = (A+B)^{-1}(2I + B - A)$

$$\begin{aligned} \text{Still a}\sqrt{I} : &\qquad (A \circ B)^2 = I \\ \text{Associativity} : &\qquad A \circ (B \circ C) = (A \circ B) \circ C \\ \text{Forgetful} : &\qquad A \circ B \circ C = A \circ C \end{aligned}$$

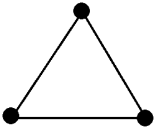
$$(1+A)Y \star \Phi(A \circ B) = \Phi(A \circ B) \star (I - B)Y = 0$$

Amplitude for $A^2 = I$ projectors

$$\Phi(A, \vec{\xi}, q) = \exp i \left(\tfrac{1}{2} YAY + Y\xi + q\right)$$

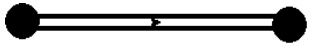
$$Tr(\Phi_1 \star \ldots \star \Phi_n) = \prod_i \frac{1}{|A_i + A_{i+1}|^{1/4}} \exp i \sum_j (Q_j + P_j)$$

$$Q_i = \frac{1}{8} \xi_i \left(A_{i+1} \circ A_i + A_i \circ A_{i-1}\right) \xi_i$$



$\langle OOJ \rangle$

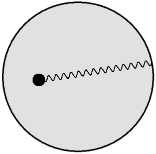
$$P_i = \frac{1}{4} \xi_i \left(I + A_{i+1} \circ A_i\right) \xi_{i+1}$$



$\langle JJ \rangle$

HST Amplitudes

$$\Phi(A,\vec{\xi},q) = \exp i\left(\tfrac{1}{2}YAY + Y\xi + q\right) =$$



In $4d$ Vasiliev HST it turned out that boundary-to-bulk propagators are Gaussians, where $q = \log K$, ξ encodes the spin degrees of freedom of boundary single-trace operators, $A = D\log K$ is a vector pointing from the bulk point to the boundary where the operator is inserted. Sundell and Colombo suggested that

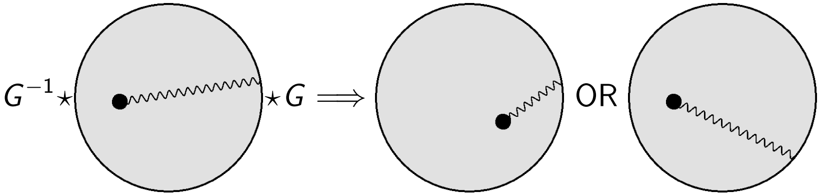
$$\langle J...J\rangle = str(\Phi \star ...\Phi)$$

should compute the correlation function of an infinite multiplet of conserved currents $J_s = \phi\partial^s\phi + ...$ one can built of a free scalar/fermion.

Large transformations

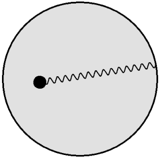
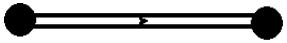
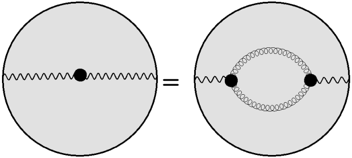
$$\Phi(A,\vec{\xi},q) = \exp i\left(\tfrac{1}{2}YAY + Y\xi + q\right)$$

$$\delta\Phi = [\Phi,\xi]_\star \qquad \xi \in so(3,2) \in HS$$



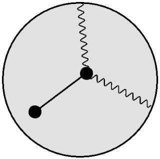
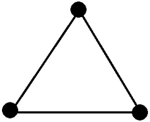
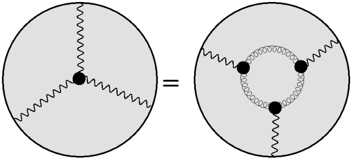
$$A \to (\alpha A+\beta)(\gamma A+\delta)^{-1} \qquad \begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix} \in Sp(4)$$

Diagrammatics: two-point

Vasiliev eq.	CFT	Amplitude
		
propagator	$\langle J_s J_s \rangle$	$str(\Phi \star \Phi)$

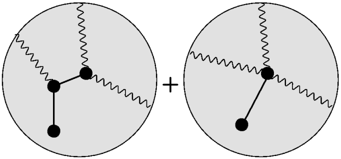
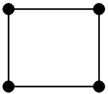
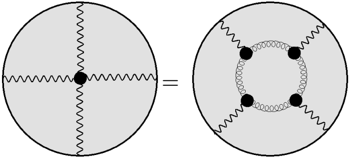
$$\langle JJ \rangle = \frac{1}{|x_{12}|^2} \exp(P_{12})$$

Diagrammatics: three-point

Vasiliev eq.	CFT	Amplitude
		
2nd-order, Yin, Giombi	$\langle J_s J_s J_s \rangle$	$str(\Phi \star \Phi \star \Phi)$, Colombo, Sundell

$$\langle JJJ \rangle = \frac{1}{|x_{12}||x_{23}||x_{31}|} \cos(Q^2_{13}+Q^3_{21}+Q^1_{32}) \cos(P_{12}) \cos(P_{23}) \cos(P_{31})$$

Diagrammatics: four-point

Vasiliev eq.	CFT	Amplitude
		
3nd-order	$\langle J_s J_s J_s J_s \rangle$	$str(\Phi \star \Phi \star \Phi \star \Phi)$

$$\langle JJJJ \rangle = \frac{1}{|x_{12}||x_{23}||x_{34}||x_{41}|} \cos(Q^2_{13} + Q^3_{24} + Q^4_{31} + Q^1_{43}) \cos(P_{12}) \cos(P_{23}) \cos(P_{34}) \cos(P_{41})$$

$$+ \text{ permutations}$$

Vasiliev HS theory vs. Amplitudes

Amplitudes: unintegrated

all HS algebra invariants are traces $tr(\Phi_1 \star ... \star \Phi_n)$ or decorated Wilson loops, which are in fact the same.
These are essentially nonlocal since they are unintegrated and do not depend on the interaction point.

this should match

Vasiliev HS theory: integrated

The HS algebra is deformed, structure constants go over into structure functions. These vertices are integrated and hence local

$$S = \int_{AdS} \phi G^{-1}\phi + \phi^3 + ...$$
$$\delta\phi = d\xi + [\phi, \xi] + \phi^2\xi + ...$$

AdS/CFT dictionary

CFT	AdS	
$\langle J ... J \rangle$	$Tr(\Phi \star ... \star \Phi)$	correlators
$[Q, J]$	$\delta\Phi = [\Phi, \xi]$	symmetries
$Q\langle J ... J \rangle = 0$	$\delta\ Tr(\Phi \star ... \star \Phi) \equiv 0$	Ward iden.

Effective action for Vasiliev theory

$$S = \sum_N a_N Tr(\Phi^N)$$

Higher-spin theory is based on formal consistency and gauge symmetry rather than any simple geometric/algebraic principle. It relies on a very subtle effect of star-product algebra — nontriviality of the theory depends on class of functions. At present it is difficult to detach the star-product realization from the theory to reveal its invariant meaning. The general structure is that of A_∞.

$$d\Psi = \mathcal{V}_2(\Psi, \Psi) + \mathcal{V}_3(\Psi, \Psi, \Psi) + \dots$$

$$\text{Flat connection}: \quad dW + W \star W = 0$$

$$\text{Compatibility}: \quad dT_a + [W, T_a]_\star = 0$$

$$\text{(Super)-algebra}: \quad [T_a, T_b]_{\star,\pm} = f^c{}_{ab} T_c$$

$$\delta W = d\xi + [W, \xi]_\star$$

$$\delta T_a = [T_a, \xi]_\star$$

Know-how

- The algebra that T_a form, $osp(1|2)$
- The associative algebra W and T_a take values in
- The vacuum to expand over (AdS)

Just $osp(1|2)$

$$
\begin{array}{rllll}
\text{Definitions}: & \{e,e\}=-2E & \{f,f\}=2F & \{e,f\}=H & \\
\text{Relations}: & [H,e]=e & [H,f]=-f & [E,f]=e & [F,e]
\\
\text{Consequences}: & [H,E]=+2E & [H,F]=-2F & [E,F]=H &
\end{array}
$$

Minimal set of relations

$$
\begin{array}{ll}
[\{e,f\},e]=e & [\{e,f\},f]=-f \\
[\{e,e\},f]=-2e & [\{f,f\},e]=2f
\end{array}
$$

Four relations for two generators, could be better

Magic of $osp(1|2)$

Casimir has a square root

$$\Upsilon = [e, f] + \frac{1}{2}$$

$$\{\Upsilon, e\} = 0 \qquad \{\Upsilon, f\} = 0$$

$$[\Upsilon, E] = 0 \qquad [\Upsilon, F] = 0 \qquad [\Upsilon, H] = 0$$

Υ^2 is a Casimir operator

Truly minimal set of relations

$$\{\Upsilon, S_\alpha\} = 0 \qquad S_\alpha = (e, f)$$

Essential part of any Vasiliev system

$$dW + W \star W = 0$$
$$dS_\alpha + [W, S_\alpha]_\star = 0$$
$$S_\nu \star S_\alpha \star S^\nu = S_\alpha$$

usually $\Upsilon = S_\alpha \star S^\alpha + 1$ called $B \star \varkappa$

Action for $osp(1|2)$: $\qquad str(S^4 + S^2)$

suggested by Prokushkin, Segal, Vasiliev

Conclusions

1. SFT and HST share many interesting solutions together, all of them being related to Gaussians in star-product algebra
2. Generic Gaussians form a group, $SpH(2N)$, which leads to explicit formulas for Amplitudes
3. Projectors are at infinity of $SpH(2N)$, the star-product simulates Wick theorem
4. All HS amplitudes can be computed explicitly
5. Vasiliev theory has a simple algebraic meaning