Off-shell Amplitudes in Superstring Theory

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PLAN

1. Motivation

Roji Pius, Arnab Rudra, A.S., 1311.1257, 1401.7014, 1404.6254

2. The construction of off-shell amplitudes

A.S., To appear

Motivation

Conventional approach to string perturbation theory has limited use in the study of mass renormalization or vacuum shift.

We shall begin by outlining the origin of these problems.

LSZ formula for S-matrix elements in QFT

$$\lim_{k_i^2 \to -m_{i,p}^2} G^{(n)}_{a_1 \cdots a_n}(k_1, \cdots k_n) \prod_{i=1}^{n} \{Z_i^{-1/2} \times (k_i^2 + m_{i,p}^2)\}$$

$G^{(n)}$: n-point Green's function

$a_1, \cdots a_n$: quantum numbers, $k_1, \ldots k_n$: momenta

$m_{i,p}$: physical mass of the i-th external state

– given by the locations of the poles of two point function in the $-k^2$ plane.

Z_i: wave-function renormalization factors, given by the residues at the poles.

In contrast, on-shell string amplitudes compute 'truncated Greens function on classical mass-shell'

$$\lim_{k_i^2 \to -m_i^2} G^{(n)}_{a_1 \cdots a_n}(k_1, \cdots k_n) \prod_{i=1}^{n} (k_i^2 + m_i^2) \, .$$

m_i: <u>tree level</u> mass of the i-th external state.

$k_i^2 \to -m_i^2$ condition is needed to make the vertex operators conformally invariant.

String amplitudes:

$$\lim_{k_i^2 \to -m_i^2} G^{(n)}_{a_1 \cdots a_n}(k_1, \cdots k_n) \prod_{i=1}^{n} (k_i^2 + m_i^2) \, ,$$

The S-matrix elements:

$$\lim_{k_i^2 \to -m_{i,p}^2} G^{(n)}_{a_1 \cdots a_n}(k_1, \cdots k_n) \prod_{i=1}^{n} \{ Z_i^{-1/2} \times (k_i^2 + m_{i,p}^2) \}$$

The effect of Z_i can be taken care of. Witten

The effect of mass renormalization is more subtle.

$\Rightarrow$ String amplitudes compute S-matrix elements directly if $m_{i,p}^2 = m_i^2$ but not otherwise.

– Includes BPS states, massless gauge particles and all amplitudes at tree level.

Problem with vacuum shift

Example: In many compactifications of SO(32) heterotic string theory on Calabi-Yau 3-folds, one loop correction generates a Fayet-Iliopoulos term.

Effect: Generate a potential of a charged scalar ϕ of the form

$$\mathbf{c}(\phi^*\phi - \mathbf{K}\,\mathbf{g}^2)^2$$

c, K: positive constants, g: string coupling

Dine, Seiberg, Witten; Atick, Dixon, A.S.; Dine, Ichinose, Seiberg

Atick, A.S.; Witten; D'Hoker, Phong; Berkovits, Witten

Conventional approach does not tell us how to carry out systematic perturbation expansion around the correct vacuum at $|\phi| = \mathbf{g}\sqrt{\mathbf{K}}$

– not described by a world-sheet CFT

To address these problems we need to work with off-shell amplitudes.

We can think of two routes:

1. String field theory

– well developed for bosonic string theory.

Witten, Zwiebach, $\cdots$

For superstrings / heterotic strings it looks difficult beyond tree level.

Witten; Zwiebach; Berkovits; Berkovits, Okawa, Zwiebach; Erler, Konopka, Sachs; $\cdots$

2. Pragmatic approach: Generalize Polyakov prescription without worrying about its string field theory origin

– well developed for bosonic string theory.

Vafa; Cohen, Moore, Nelson, Polchinski; Alvarez Gaumé, Gomez, Moore, Vafa; Polchinski; Nelson

In this approach we need to choose local coordinate system on the world-sheet around every puncture where off-shell vertex operators are inserted.

The goal of part 2 of the talk will be to generilize this to super/heterotic string theory.

Off-shell amplitude:

$$\mathbf{A}(\mathbf{1},\cdots\mathbf{n}) = \int_{\mathbf{M_{g,n}}} \mathbf{W}(\mathbf{1},\cdots\mathbf{n})$$

$M_{g,n}$: moduli space of genus g Riemann surfaces with n-punctures

W(1,$\cdots$n): a world-sheet correlation function

In the on-shell limit this reduces to Polyakov amplitude.

But off-shell, A(1,$\cdots$, n) depends on the choice of local coordinates at the punctures.

This is not very different from the situation in a gauge theory where off-shell Green's functions of charged fields are gauge dependent.

Nevertheless the renormalized masses and S-matrix elements computed from these are gauge invariant.

Goal: Prove that the story is similar in string theory.

It turns out that we can achieve our goal only if we impose some additional restrictions on the choice of local coordinate system at the punctures.

Pius, Rudra, A.S.

– gluing compatibility

– constraints on choice of local coordinates near separating type degeneration.

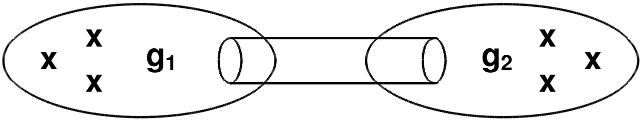
Consider a genus g_1, m-punctured Riemann surface and a genus g_2, n-punctured Riemann surface.

Take one puncture from each of them, and let w_1, w_2 be the local coordinates around the punctures at $w_1 = 0$ and $w_2 = 0$.

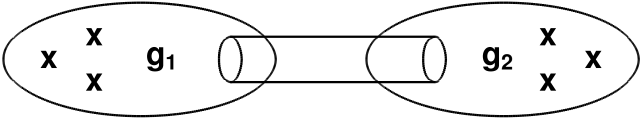
Glue them via the identification (plumbing fixture)

$$w_1 w_2 = e^{-s+i\theta}, \quad 0 \le s < \infty, \quad 0 \le \theta < 2\pi$$

– gives a family of new Riemann surfaces of genus $g_1 + g_2$ with (m+n-2) punctures.



Gluing compatibility: Choice of local coordinates at the punctures of the genus $g_1 + g_2$ Riemann surface must agree with the one induced from the local coordinates at the punctures on the original Riemann surfaces.



(Follows automatically if the choice of local coordinate system is inherited from bosonic string field theory in the Siegel gauge.)

Rastelli, Zwiebach

Gluing compatibility allows us to divide the contributions to off-shell Green's functions into <u>1-particle reducible</u> (1PR) and <u>1-particle irreducible (1PI)</u> contributions.

Two Riemann surfaces joined by plumbing fixture

$\Updownarrow$

Two amplitudes joined by a propagator

Riemann surfaces which <u>cannot</u> be obtained by plumbing fixture of other Riemann surfaces contribute to <u>1PI</u> amplitudes.

1PI amplitudes do not include degenerate Riemann surfaces and hence are free from poles.

Once this division has been made we can apply the usual field theory manipulations to analyze amplitudes.

Example: Two point function

– can be used to partially resum the perturbation series and calculate mass renormalization.

Similar analysis can be used to compute S-matrix elements using LSZ procedure.

Results:

Pius, Rudra, A.S.

1. The renormalized masses of physical states and S-matrix elements are independent of the choice of local coordinate system.

2. Wave-function renormalization factors and the renormalized masses of unphysical states depend on the choice of local coordinates at the punctures.

Application 2: Shifting the vacuum

Suppose we have a scalar field ϕ with tree level potential

$$\mathbf{A}\,\phi^{\mathbf{4}} + \cdots$$

Suppose that loop correction generates a -ve mass^2 term

$$-\mathbf{C}\,\mathbf{g}^{\mathbf{2}}\phi^{\mathbf{2}} + \cdots$$

Physically we expect minima at

$$\phi^{\mathbf{2}} = \frac{\mathbf{1}}{\mathbf{2}}\frac{\mathbf{C}}{\mathbf{A}}\,\mathbf{g}^{\mathbf{2}} + \cdots\,.$$

Question: How do we compute physical quantities in this vacuum?

$\lambda \equiv$ **vacuum expectation value of** ϕ **(unknown)**

$\Gamma^{(n)}$**: String amplitudes in the original vacuum (known)**

$$\Gamma^{(n)} = G^{(n)} \prod_{i=1}^{n} (k_i^2 + m_i^2)$$

m_i**: tree level masses**

$\Gamma_\lambda^{(n)}$**: String amplitudes in the shifted vacuum (unknown)**

Prescription

Lee; Bardakci, Halpern

$$\Gamma^{(n)}_{\lambda}(k_1, a_1; \cdots k_n, a_n)$$
$$= \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \Gamma^{(n+m)}(k_1, a_1; \cdots k_n, a_n; \overbrace{0, \phi; \cdots ; 0, \phi}^{\longleftarrow m \longrightarrow})$$

Determine λ by demanding vanishing of

$$\Gamma^{(1)}_{\lambda}(0, \phi) = \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \Gamma^{(1+m)}(\overbrace{0, \phi; \cdots ; 0, \phi}^{\longleftarrow m+1 \longrightarrow})$$
$$= 4\,A\,\lambda^3 - 2\,C\,g^2\lambda + \cdots$$

– requires cancellation between different loop orders leading to

$$\lambda = \sum_{n=0}^{\infty} A_n g^{2n+1}$$

Results:

Pius, Rudra, A.S.

1. The shift λ depends on the choice of local coordinates.

2. However the physical quantities like S-matrix elements are independent of the choice of local coordinates.

What about heterotic strings / superstrings?

We need to generalize the definition of off-shell amplitude.

– would require choice of local superconformal coordinate system around each puncture. Belopolsky; Witten

The result would depend on the choice of the superconformal coordinate system

– need to show that the renormalized masses and S-matrix elements are independent of this choice.

This procedure should be straightforward, but all the details have not been worked out yet.

Also for practical computations this procedure is complicated.

We shall follow a related but slightly different approach using picture changing operators (PCO)

– operators which need to be inserted into the correlation function to soak up certain ghost charges.

$$\mathbf{X}(\mathbf{z}) = \{\mathbf{Q_B}, \xi(\mathbf{z})\}$$

$\xi(\mathbf{z})$: fermion arising from ‘bosonization’ of supersymmetry ghosts

Friedan, Martinec, Shenker; Verlinde, Verlinde

$\mathbf{Q_B}$: BRST charge

In this formalism the amplitudes in heterotic / superstring theories take the same form as in bosonic string theory.

$$\mathbf{A(1,\cdots n) = \int_{M_{g,n}} W(1,\cdots n)}$$

$\mathbf{M_{g,n}}$: moduli space of genus g Riemann surfaces with n-punctures

W(1,$\cdots$ n): a world-sheet correlation function with insertions of certain number of PCO's.

The off-shell amplitudes now depend on the <u>choice of local coordinates</u> as well as the <u>locations of the PCO's</u>.

Goal: Find a way to systematically analyze the dependence on these spurious data, and show that physical quantities do not depend on them.

– try to generalize the procedure for bosonic string theory. Nelson; Zwiebach

Introduce a new space $P_{g,n}$

– a fiber bundle over $M_{g,n}$ with the data on local coordinates and PCO locations as fibers.

Choice of local coordinates and PCO's

$\leftrightarrow$ choice of a <u>section</u> of $P_{g,n}$.

The off-shell amplitude is an integral over this section.

Question: Are the renormalized masses and S-matrix elements independent of this section?

Some additional issues to be addressed:

1. Spurious poles

– related to supermoduli space not being holomorphically projected.

2. Ramond sector external states

Taking clue from bosonic string theory, we first solve a slightly more general problem. Nelson; Zwiebach

For a given set of external off-shell states collectively called ϕ, construct a p-form $\omega_p(\phi)$ on $P_{g,n}$ that can be integrated along any p-dimensional subspace of $P_{g,n}$.

Desired property:

$$\omega_p(Q_B\phi) \propto d\omega_{p-1}$$

Genus g, n-point amplitude

$$\int_{\text{section}} \omega_{6g-6+2n}$$

If R is the region bounded by the two sections then the difference in the integral over the two sections is

$$\int_{\mathbf{R}} \mathbf{d}\omega_{\mathbf{6g-6+2n}} \propto \int_{\mathbf{R}} \omega_{\mathbf{6g-5+2n}}(\mathbf{Q_B}\phi)$$

– vanishes for on-shell states

– can be manipulated as in bosonic string theory for off-shell states to show that the renormalized masses and S-matrix elements are not affected by this term.

How do we construct such a $\omega_{\mathbf{p}}(\phi)$?

– requires specifying the contraction of $\omega_{\mathbf{p}}(\phi)$ with the tangent vectors of $\mathbf{P_{g,n}}$.

We shall state the results without proof.

$\omega_0(\phi)$: Correlation function of external vertex operators and picture changing operators on the Riemann surface.

In defining ω_p we need to insert additional operators into the correlation function, depending on which tangent vector we contract ω_p with.

For contraction with tangent vectors along the base $M_{g,n}$ or deformations of local coordinates, the additional insertions involve integrals of b or $\bar{b}$ ghosts weighted by Beltrami differentials.

– same as in bosonic string theory.

What is the effect of contracting ω_p with $\partial/\partial z_i$?

z_i: Location of the i-th PCO.

Answer: Replace the i-th PCO $X(z_i)$ by $-\partial\xi(z_i)$.

With this change ω_p satisfies $\omega_p(Q_B\phi) \propto d\omega_{p-1}(\phi)$.

Note 1: In some cases instead of working with a single arrangement of PCO's we need to average over several arrangements to maintain modular invariance of the arrangements.

Note 2. The PCO locations must be chosen in a gluing compatible manner so that near separating type degeneration the arrangement of the PCO's agree with the arrangement on the components.

Spurious poles

The correlation function used for defining ω_p diverges at points where no vertex operators or PCO's coincide.

Verlinde, Verlinde

$$f(\{z_i\}, \{w_j\}) = 0$$

z_i: location of PCO's

w_k: locations of vertex operators

– a real codimension two subspace on the section

– appears even for on-shell amplitudes

– related to the fact that the gauge choice for the world-sheet gravitino breaks down at these points.

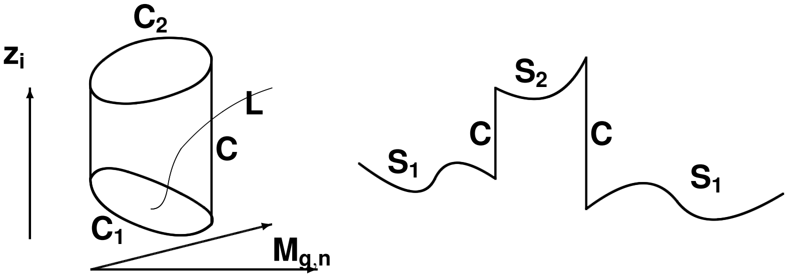
We could try to avoid it by judicious choice of section

– impossible for high enough genus when the supermoduli space is not holomorphically projected.

Donagi, Witten

How to integrate through these poles?

Our prescription: Use 'vertical segment'



Integrate over a section S_1 outside C_1, then along C and then along a section S_2 inside C_2.

L: Path of the spurious pole.

We intercept the spurious pole along the vertical segment.

Along the vertical segment we have to contract $\omega_{6g-6+2n}$ with $\partial/\partial z_i$

$\Rightarrow$ PCO at z_i is replaced by $-\partial\xi(z_i)$.

Thus the vertical integration inserts

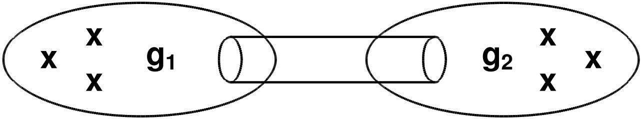
$$-\int_u^v \partial\xi(z_i) dz_i = \xi(u) - \xi(v)$$

– depends only on the initial and final points and has no singularity or ambiguity.

Of course a different choice of integration cycle will give a different result for off-shell amplitudes but this can be handled using $\omega_p(Q_B\phi) \propto d\omega_{p-1}(\phi)$.

Problem in Ramond sector

Consider the plumbing fixture procedure:



The picture numbers at the two punctures which are being glued must add up to -2.

For NS sector states this leads to the natural choice of picture number -1 at each puncture.

Since Ramond sector states carry integer+1/2 units of picture number, it is impossible to treat the two punctures symmetrically.

Our solution: break the symmetry arbitrarily by taking one vertex operator in the $-1/2$ picture and the other one in $-3/2$ picture.

To do this systematically we have to decide, at the beginning, some 'coloring' of the external states which allows us to decide on which side we have $-1/2$ picture vertex operator and on which side we have $-3/2$ picture vertex operator.

Stick to this rule for all degenerations (different genera etc.)

The resulting off-shell amplitudes are not symmetric in all the external legs, but we expect that physical quantities like renormalized masses and S-matrix elements will have all the required symmetries.

Conclusions

A full fledged string field theory is needed for addressing issues beyond perturbation theory.

But the ad hoc procedure for defining off-shell amplitudes seems to be sufficient for resolving problems arising in perturbative string theory.

– mass renormalization and vacuum shift.