ANALYTIC SOLUTIONS IN SFT at HIGHER GHOST NUMBERS and COLLECTIVE VACUUM FOR HIGHER SPINS

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1. 1. General structure of the solution. Bell polynomials: a brief review

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We are considering a cubic open superstring field theory. Ghost numbers of string field components are not fixed and can be arbitrary, and there are no midpoint inverse picture changing insertions. The standard $b_0\Psi = 0$ gauge condition is replaced with the s.c. "ghost cohomology" constraint which is more stringent, since the original string field lives in a larger space.

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The ansatz for the pure ghost analytic solution that we propose is

$$\Psi = \Psi^{(+)} + \Psi^{(-)}$$

$$\Psi^{(+)} = \sum_{N=1}^{\infty} \sum_{n=0}^{N-2} \lambda_N^n c e^{\chi+N\phi} B_n^{[\alpha_n,\beta_n,\gamma_n]}(\phi,\chi,\sigma)$$

$$\Psi^{(-)} = \sum_{N=1}^{\infty} \sum_{n=0}^{N-2} \lambda_N^n c e^{\chi-(N+2)\phi} B_n^{[\alpha_n,\beta_n,\gamma_n]}(\phi,\chi,\sigma)$$

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where ϕ, χ and σ are bosonized superconformal ghosts for the $b-c$ and $\beta-\gamma$ systems:

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$$b = e^{-\sigma}, c = e^{\sigma}, \gamma = e^{\phi-\chi}, \beta = e^{\chi-\phi}\partial\chi$$

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$B_n^{[\alpha,\beta,\gamma]}(\phi, \chi, \sigma)$ are the **Bell polynomials of degree n**:

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$$\begin{aligned} B_n^{[\alpha,\beta,\gamma]}(\phi, \chi, \sigma) \equiv B_n(x_1, ...x_n) \\ x_k = \alpha\partial^k\phi + \beta\partial^k\chi + \gamma\partial^k\sigma \\ k = 1, ..., n \end{aligned}$$

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where $B_n(x_1, ...x_n)$ are **complete** Bell polynomials in $x_1, ...x_n$ (definition will be given below); $\alpha_n, \beta_n, \gamma_n$ are some numbers (to be determined in the process of the solution) and the coefficients λ_{nN} are defined through triangular recursion relations:

$$\lambda_N^n = \sum_{N_1, N_2=1}^{N_1+N_2=N-2} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \rho_{N|N_1;N_2}^{n|n_1;n_2} \lambda_{N_1}^{n_1} \lambda_{N_2}^{n_2} \quad (1)$$

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Our purpose is to determine the coefficients: $\rho_{N|N_1,N_2}^{n|n_1;n_2}$ by directly computing the star product, i.e. the relevant correlators

$$<< \Psi, Q\Psi >>=< \Psi(0) I \circ Q\Psi(0) >$$

and

$$<< \Psi, \Psi \star \Psi >>=< f_1^3 \circ \Psi(0) f_2^3 \circ \Psi(0) f_3^3 \circ \Psi(0) >$$

where $I(z) = -\frac{1}{z}$ and

$$f_k^n(z) = e^{\frac{i\pi(k-1)}{n}} (\frac{1 - iz}{1 + iz})^{\frac{2}{n}} \quad (2)$$

maps the worldsheets of n interacting strings putting them together on a single disc.

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Some basic facts about Bell polynomials

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The standard definition of the *complete* Bell polynomials $B_n(x_1, ...x_n)$ is given by

$$B_n(x_1, ...x_n) = \sum_{k=1}^{n} B_{n|k}(x_1, ...x_{n-k+1}) \tag{3}$$

where $B_{n|k}(x_1, ...x_{n-k+1})$ are the *incomplete* Bell polynomials defined according to

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$$B_{n|k}(x_1, ...x_{n-k+1}) = \sum_{p_1,...p_{n-k+1}} \frac{n!}{p_1!...p_{n-k+1}!} x_1^{p_1} (\frac{x_2}{2!})^{p_2} ... (\frac{x_{n-k+1}}{(n-k+1)!})^{p_{n-k+1}}$$

with the sum taken over all the combinations of non-negative p_j satisfying

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$$\sum_{j=1}^{n-k+1} p_j = k; \sum_{j=1}^{n-k+1} j p_j = n$$

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In number theory, Bell polynomials are known to satisfy a number of useful and beautiful identities and properties

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Just to mention a couple of examples,

$$B_{n|k}(1, ...1) = S(n, k)$$

are the second kind Stirling numbers

$$B_{n|k}(0!, 1!, ..., (n-k)!)$$

can be expressed in terms of combinations of Bernoulli numbers

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Also, for Taylor series of a function $f(x) = \sum_n \frac{a_n x^n}{n!}$ one has

$$e^f(x) = \sum_n B_n(a_1, ..a_n)\frac{x^n}{n!}$$

so vertex operators in string theory are typically given by combinations of Bell polynomials in the expansion modes.

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If one identifies $x_n = \partial^n \phi(z)$, where $\phi(z)$ is some scalar field, one obtains Bell polynomials in derivatives of ϕ (note that in the particular case $\phi(z) = z^2$ this would reduce to Hermite polynomials in z).

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Other useful objects to define are the Bell generators

$$H_n(y|x_1, ..., x_n) = \sum_{k=1}^{n} B_{n|k}(x_1, ..., x_{n-k+1})y^n$$

and more generally

$$G_n(y_1...y_n|x_1,...,x_n) = \sum_{k=1}^{n} B_{n|k}(x_1,...,x_{n-k+1})y_1...y_k$$

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In the context of two-dimensional CFT, one can think of Bell polynomials as higher derivative generalizations of the Schwarzian derivative, appearing in the global conformal transformation law for the stress tensor. That is, under $z \to f(z)$ one has

$$T(z) \to (\frac{df}{dz})^2 T(f(z)) + \frac{c}{12} S(f(z))$$

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where the Schwarzian derivative:

$$S(f(z)) = (\frac{f''(z)}{f'(z)})' - \frac{1}{2}(\frac{f''(z)}{f'(z)})^2$$

can be expressed in terms of the second order Bell polynomials in the log of f', with $x_k \equiv \frac{d^{k-1}}{dz^{k-1}} log(f')$:

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$$S(f(z)) = B_{2|1}(log(f'), \frac{dlog(f')}{dz}) - \frac{1}{2}B_{2|2}(log(f'))$$
$$\equiv -2H_2(-\frac{1}{2}|log(f')) \equiv -2B_2(-\frac{1}{2}log(f'))$$

•

where

$$B_n(g(x)) \equiv B_n(\partial g, ...\partial^n g)$$
$$= B_n(x_1, ...x_n)|_{x_k=\partial^k g; k=1,...,n}$$

for any function $g(x)$.

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This is point is of importance as the higher order Bell polynomials will naturally enter the global transformation law for the string fields of our ansatz for analytic solution

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2. SFT ansatz: global conformal transformation

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The major difficulty in calculating the star product is that generic string fields behave in an extremely cumbersome way under global conformal transformations $f_k^n(z)$ Therefore the straightforward calculation of the star product is generally beyond the reach.

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On the other hand, the string field operators entering our ansatz (Bell polynomials multiplied by exponents) turn out to transform in a relatively simple and compact way, forming an invariant subspace of operators under global conformal transformations.

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Our strategy to find the global transformation for Ψ is the following:
(1) to find the infinitezimal transformation
(2) to deduce the global transformation reproducing the infinitezimal one and preserving its form under composition of two transformations

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Denoting

$$e^{[\alpha,\beta,\gamma]} \equiv e^{\alpha\phi+\beta\chi+\gamma\sigma}$$

and

$$h^{[\alpha,\beta,\gamma]} = \frac{1}{2}(-\alpha^2+\beta^2+\gamma^2) - \alpha - \frac{\beta}{2} - \frac{3\gamma}{2}$$

the straightforward computation of the infinitezimal transformation, using the OPE with the stress tensor gives

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$$\begin{aligned} &\delta_\epsilon(B_n^{[\alpha_n,\beta_n,\gamma_n]}e^{[\alpha,\beta,\gamma]}) \\ = \epsilon\partial(B_n^{[\alpha_n,\beta_n,\gamma_n]}e^{[\alpha,\beta,\gamma]}) &+ \partial\epsilon(n+h^{[\alpha,\beta,\gamma]})B_n^{[\alpha_n,\beta_n,\gamma_n]}e^{[\alpha,\beta,\gamma]} \\ &+\sum_{k=2}^{n+1}\frac{n!}{(n-k+1)!k!}\partial^k\epsilon(z)[kh^{[\alpha_n,\beta_n,\gamma_n]}+n-k+1 \\ +(\alpha_n^2-\alpha_n\alpha-\beta_n^2+\beta_n\beta-\gamma_n^2+\gamma_n\gamma)]&B_{n-k+1}^{[\alpha_n,\beta_n,\gamma_n]}e^{[\alpha,\beta,\gamma]}(z) \end{aligned}$$

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Using the fact that

$$B_n(log(f'(z)))|_{f(z)=z+\epsilon(z)} = \partial^n\epsilon(z) + O(\epsilon^2)$$

and the binomial property of $B_n(f)$:

$$B_n(f(x)+g(x)) \equiv B_n(\partial(f+g),...,\partial^n(f+g)) = \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}B_k(f)B_{n-k}(g)$$

we find the string field components transform under $z \to f(z)$ according to:

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$$B_n^{[\alpha_n,\beta_n,\gamma_n,]}e^{[\alpha,\beta,\gamma]}(z) \to (\frac{df}{dz})^{n+h^{[\alpha,\beta,\gamma]}}B_n^{[\alpha_n,\beta_n,\gamma_n]}e^{[\alpha,\beta,\gamma]}(f(z))$$
$$+\sum_{k=2}^{n+1}\frac{n!}{k!(n-k+1)!}(\frac{df}{dz})^{n-k+1+h^{[\alpha,\beta,\gamma]}}B_{k-1}(\lambda(k,n,h^{[\alpha_n,\beta_n,\gamma_n]})log(f'(z)))$$
$$\times B_{n-k+1}^{[\alpha_n,\beta_n,\gamma_n]}e^{[\alpha,\beta,\gamma]}(f(z))$$

with the weight factor λ given by

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$$\lambda(k,n,h^{[\alpha_n,\beta_n,\gamma_n]}) = kh^{[\alpha_n,\beta_n,\gamma_n]} + n - k + 1$$
$$+\alpha_n^2 - \alpha_n\alpha - \beta_n^2 + \beta_n\beta - \gamma_n^2 + \gamma_n\gamma$$

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The next step is to compute the commutator Ψ with the BRST charge:

$$Q = \oint dz\{cT - bc\partial c - \frac{1}{2}\gamma\psi_m\partial X^m - \frac{1}{4}b\gamma^2\} \quad (4)$$

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Since we are looking for the pure ghost solution, Ψ carries $b-c$ ghost number 1 and $<< Q\Psi, \Psi >>$ has to carry $b-c$ ghost number 3, the correlator will only be contributed by the commutator of Ψ with $Q = \oint dz\{cT - bc\partial c\}$.

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The straightforward computation of the relevant terms of $Q(I \circ \Psi)$ gives

$$Q(I \circ B_n^{[\alpha,\beta,\gamma,]} e^{[\alpha,\beta,\gamma]})(w)|_{w=-\frac{1}{z}}$$

$$= \sum_{k=1}^{n+1} \frac{n!}{k!} \times [(k-\delta_1^k) h^{[\alpha_n,\beta_n,\gamma_n]} + \delta_1^k h^{[\alpha,\beta,\gamma,]}$$

$$+(1-\delta_1^k)(\alpha_n^2 - \alpha_n\alpha - \beta_n^2 + \beta_n\beta - \gamma_n^2 + \gamma_n\gamma)]$$

$$\times w^{2(h^{[\alpha,\beta,\gamma,]}+n-k+1)} B_{k-1}(x_1, ...x_k)|_{x_j=(-1)^j 2(k-1)! w^j; j=1,...,k}$$

$$\times \{ \sum_{l=1}^{n-k+1} \sum_{m=0}^{n-k+1-l} \frac{(-1)^m}{(n-k-l+2)!(l+m)!} \times [(l-\delta_1^l)\tilde{h}^{[\alpha_n,\beta_n,\gamma_n]} + \delta_1^l \tilde{h}^{[\alpha,\beta,\gamma,]}$$

$$+(1-\delta_1^l)(\alpha_n^2 - \alpha_n\alpha - \beta_n^2 + \beta_n\beta - \gamma_n^2 + \gamma_n\gamma)]$$

$$\times$$

$$\partial^{l+m} c B_{001|\alpha_n\beta_n\gamma_n}^{n-k-l-m+2|n-k-l+2} e^{[\alpha,\beta,\gamma]}(w)\} + \sum_{k=1}^{n} \frac{(-1)^k}{(k-1)!} [\frac{\partial^{k+1} c}{k+1} B_{001|\alpha_n\beta_n\gamma_n}^{n-m|n} e^{[\alpha,\beta,\gamma]}](w)$$

$$+\mathrm{k}^{-1} \partial^k c \partial (B_{001|\alpha_n\beta_n\gamma_n}^{n-m|n} e^{[\alpha,\beta,\gamma]}(w)) + c\partial(B_n^{[\alpha_n,\beta_n,\gamma_n,]} e^{[\alpha,\beta,\gamma]})(w)$$

•

Here $B_{pqr|\alpha\beta\gamma}^{m|n}$ are the conformal dimension m polynomials in bosonized ghost fields appearing in the OPE of Bell polynomials with exponential fields, defined according to:

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$$B_n^{[\alpha_n,\beta_n,\gamma_n,]}(z_1)e^{[p,q,r]}(z_2) = \sum_{m=0}^{n} \frac{:B_{pqr|\alpha\beta\gamma}^{m|n}(z_1)e^{[p,q,r]}(z_2):}{(z_1-z_2)^{n-m}}$$

(note the upper script for $B^{m|n}$ chosen here in order not to confuse them with the incomplete Bell polynomials)

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It is straightforward to compute the manifest expressions for $B_{pqr|\alpha\beta\gamma}^{m|n}$. We get

$$B_{pqr|\alpha\beta\gamma}^{m|n} = \frac{(-1)^{n-m}n!}{(n-m)!m!} \sum_{k=1}^{n} \sum_{l=max(1;k-m)}^{min(n-m;k)} B_{n-m|l}(0!,1!,...,(n-m-l)!)B_{m|k-l}^{[\alpha,\beta,\gamma]}$$

Here $B_{m|k-l}^{\{\alpha,\beta,\gamma\}}$ are the incomplete Bell polynomials in the ghost fields.

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The numerical coefficients $B_{n-m|l}(0!,1!,...,(n-m-l)!)$ given by the values of incomplete Bell polynomials $B_{n-m|l}(x_1,...x_{n-m-l+1})$ at $x_j=(j-1)!(j=1,...;n-m-l+1)$ and coincide with $(n-m)$'th order expansion coefficients of $log^l(1+x)$ around $x=0$.

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We are now prepared to compute the SFT correlators relevant to our ansatz solution

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To compute the correlators, the following OPE's are of importance:

$$B_n^{[\alpha,\beta,\gamma]}(z)e^{[p,q,r]}(w)=\sum_{k=1}^{n}\sum_{l=o}^{k}\sum_{m=l}^{n-k+l}(z-w)^{-m}$$
$$\times\frac{n!}{m!(n-m)!}B_{m|l}(0!,1!,...(m-l)!):B^{[\alpha,\beta,\gamma]}_{n-m|k-l}(z)e^{[p,q,r]}(w):$$

•

$$B^{[\alpha,\beta,\gamma]}_{n_1|k_1}(z)B^{[p,q,r]}_{n_2|k_2}(w) = \sum_{l=0}^{min(k_1,k_2)} \sum_{m_1=l}^{n_1-k_1+l} \sum_{m_2=l}^{n_2-k_2+l} (z-w)^{-m_1-m_2}\frac{n_1!n_2!}{(n_1-m_1)!(n_2-m_2)!}$$

$$\Lambda_{Bell}(m_1,m_2|l) : B^{[\alpha,\beta,\gamma]}_{n_1-m_1|k_1-l}(z)B^{[p,q,r]}_{n_2-m_2|k_2-l} : (w)$$

where the generalized Bell numbers $\Lambda_{Bell}(m_1, m_2|l)$ are defined as follows. Let $0 < p_1 \leq p_2 ... \leq p_l$ and $0 < q_1 \leq p_2 ... \leq q_l$ be the ordered length l partitions of m_1 and m_2. Then

$$\Lambda_{Bell}(m_1,m_2|l) = m_1!m_2! \sum_{m_1|p_1,...,p_l}^{partitions} \sum_{m_2|q_1,...,q_l}^{partitions}$$

$$\sum_{p_{i_k};q_{j_k};i_k,j_k=1,...,l}^{pairings} \frac{(p_{i_1}+q_{j_1}-1)!...(p_{i_l}+q_{j_l}-1)!}{p_1!...p_l!q_1!...q_l!r_{p_1}!...r_{p_l}!r_{q_1}!...r_{q_l}!}$$

where

$r_{p,q}$ are multiplicities of p and q entering the partitions. We furthermore impose the *weak cohomology constraints on* Ψ: :

$$\Gamma(z)\Psi_-(w) \sim O(z-w)^0$$
$$:\Gamma\Psi_- :\approx 0$$

and similarly for Γ^{-1}, Ψ_+ to ensure that SF components of different ghost numbers are not related by picture-changing (up to terms irrelevant to correlators). Here Γ, Γ^{-1} are the direct and inverse picture changing operators This fixes $\beta_n = 0, \gamma_n = 1, n \leq N+1$

With these identities, the computation of

$$<< \Psi, Q\Psi >>=< \Psi(0) I \circ Q\Psi(0) >$$

and

$$<< \Psi, \Psi \star \Psi >>=< f_1^3 \circ \Psi(0) f_2^3 \circ \Psi(0) f_3^3 \circ \Psi(0) >$$

is straightforward and leads to the following $\rho^{n|n_1;n_2}_{N|N_1;N_2}$ coefficients in the recurrence relation for λ^N_n defining the analytic solution:

$$\rho^{n|n_1;n_2}_{N|LN_1;N_2} = \frac{(\kappa_3)^{n|n_1;n_2}_{N|N_1;N_2}}{(\kappa_2)^n_N}$$

where

$$(\kappa_2)^{n|n_1;n_2}_{N|L} = \sum_{k=1}^{n+1}\sum_{l=1}^{n-k+1}\sum_{m=0}^{n-k-l}\sum_{L_1=0}^{n}\frac{(n!)^2}{(n-k-l-m+2)!(l+m-1)!(n-L_1)!}$$
$$\{[(k-\delta_1^k)h^{[n,0,0]}+\delta_1^k h^{[-(N+2),0,1]}+(1-\delta_1^k)((N+2)(n+1)-1)]$$
$$\times[(l-\delta_1^l)\tilde{h}^{[n,0,0]}+\delta_1^l \tilde{h}^{[-(N+2),0,1]}+(1-\delta_1^l)((N+2)(n+1)-1)]$$
$$\times\sum_{k_1=1}^{n}\sum_{k_2=1}^{l+m-1}\sum_{k_3=1}^{n-k-l+2}\sum_{l_1=1}^{min(L_1;k_1-1)}\sum_{l_2=1}^{min(m,k_3-1)}(-1)^{k+l_1+l_2+L_1}$$
$$(2+n(N+2))^{l_1}B_{L_1|l_1}(0!,...,(L_1-l_1)!)B_{L_2|l_2}(0!,...,(L_2-l_2)!)$$
$$\times\sum_{q=1}^{k_3-l_3}\sum_{M=1}^{n+2-k-l-m-k_3+l_2-q}(nN-1)^q B_{M|q}(0!,...,(M-q)!)$$
$$\times\sum_{Q=k_3-l_2-q}^{n-L_1-k_1-l_1-k_3+l_2+q}\frac{(-1)^{Q+n-L_1}(n-L_1)!}{Q!(n-L_1-Q)!}\Lambda_{Bell}(Q;n+2-k-l-m-M|k_3-l_3-q)$$
$$\times[n^{k_2}\delta_{k_2}^{k_1-k_3-l_1+l_2+q}\Lambda_{Bell}(n-L_1-Q;l+m-1|k_2)$$
$$-(l+m-1)n^{k_2-1}\delta_{k_2-1}^{k_1-k_3-l_1+l_2+q}\Lambda_{Bell}(n-L_1-Q;l+m-2|k_2)]\}$$

$$(\kappa_3)^{n|n_1;n_2}_{N|N_1;N_2} = \sum_{k=1}^{n+1}\sum_{k_1=1}^{n_1+1}\sum_{k_2=1}^{n_2+1}\frac{n!n_1!n_2!}{(n-k+1)!(n_1-k_1+1)!(n_2-k_2+1)!k!k_1!k_2!}$$

$$\times[(k-\delta_1^k)h^{[n,0,1]}+\delta_1^k h^{[N,0,1]}+(1-\delta_1^k)(n^2-nN)]$$

$$\times[(k_1-\delta_1^{k_1})h^{[n_1,0,1]}+\delta_1^k h^{[-N_1-2,0,1]}+(1-\delta_1^{k_1})(n_1^2+n_1(N_1+2))]$$

$$\times[(k_2-\delta_1^{k_2})h^{[n_2,0,1]}+\delta_1^{k_2} h^{[-N_2-2,0,1]}+(1-\delta_1^{k_2})(n_2^2+n_2(N_2+2))]$$

$$(\frac{2}{3})^{h^{[N,1,1]}+h^{[-N_1-2,0,1]}+h^{[-N_2-2,0,1]}+n+n_1+n_2-k-k_1-k_2}\times\frac{(\Gamma(\frac{4}{3}))^3}{\Gamma(\frac{4}{3}-k)\Gamma(\frac{4}{3}-k_1)\Gamma(\frac{4}{3}-k_2)}$$

$$\times\sum_{m=1}^{n-k+1}\sum_{m_1=1}^{(n_1-k_1+1)}\sum_{m_2=1}^{(n_2-k_2+1)}\sum_{s_1=0}^{m}\sum_{s_2=0}^{m-s_1}\sum_{t_1=0}^{m_1}\sum_{t_2=0}^{m_1-t_1}\sum_{u_1=0}^{m_2}\sum_{u_2=0}^{m_2-u_1}$$

$$\sum_{L_1=s_1}^{(n-k+1-m+s_1)}\sum_{L_2=s_2}^{(n-k+1-m+s_2-L_1)}\sum_{M_1=t_1}^{(n_1-k_1+1-m_1+t_1)}$$

$$\sum_{M_2=t_2}^{(n_1-k_1+1-m_1+t_2-M_1)}\sum_{P_1=u_1}^{(n_2-k_2+1-m_2+u_1)}\sum_{P_2=u_2}^{(n_2-k_2+1-m_2+u_2-P_1)}$$

$$\{B_{L_1|s_1}(0!,1!,...,(L_1-s_1)!)B_{L_2|s_2}(0!,1!,...,(L_2-s_2)!)$$

$$B_{M_1|t_1}(0!,1!,...,(M_1-t_1)!)B_{M_2|t_2}(0!,1!,...,(M_2-t_2)!)$$

$$B_{P_1|u_1}(0!,1!,...,(P_1-u_1)!)B_{P_2|u_2}(0!,1!,...,(P_2-u_2)!)$$

$$(-(N_1+2)n-1)^{L_1}(-(N_2+2)n-1)^{L_2}(-n_1N+1)^{M_1}$$
$$(-n_1(N_2+2)-1)^{M_2}(-n_2N-1)^{P_1}(n_2(N_1+2)+1)^{P_2}$$
$$(\sqrt{3})^{N(N_1+2)-(N_1+2)(N_2+1)+2-L_1-M_1-M_2-P_2}(2\sqrt{3})^{N(N_2+2)+1-L_2-P_1}\}$$
$$\times(-nn_1+1)^{\frac{1}{2}(m+m_1-m_2-s_1-s_2-t_1-t_2+u_1+u_2)}(-nn_2+1)^{\frac{1}{2}(m+m_2-m_1-s_1-s_2+t_1+t_2-u_1-u_2)}$$
$$\times(-n_1n_2+1)^{\frac{1}{2}(-m+m_1-m_2+s_1+s_2-t_1-t_2+u_1+u_2)}\sum_{R_1=\frac{1}{2}(m+m_1-m_2-s_1-s_2-t_1-t_2+u_1+u_2)}^{n-k+1+\frac{1}{2}(m+m_1-m_2-s_1-s_2-t_1-t_2+u_1+u_2)}$$
$$\sum_{R_2=\frac{1}{2}(m+m_2-m_1-s_1-s_2+t_1+t_2-u_1-u_2)}^{n_1-k_1+1+\frac{1}{2}(m+m_2-m_1-s_1-s_2+t_1+t_2-u_1-u_2)}\sum_{R_3=\frac{1}{2}(-m+m_1-m_2+s_1+s_2-t_1-t_2+u_1+u_2)}^{n_2-k_2+1+\frac{1}{2}(-m+m_1-m_2+s_1+s_2-t_1-t_2+u_1+u_2)}$$
$$(\sqrt{3})^{-R_1+R_3}(2\sqrt{3})^{-R_1-R_3}\Lambda_{Bell}(R_1;R_2|\frac{1}{2}(m+m_1-m_2-s_1-s_2-t_1-t_2+u_1+u_2))$$
$$\times\Lambda_{Bell}(R_1;R_3|\frac{1}{2}(m+m_2-m_1-s_1-s_2+t_1+t_2-u_1-u_2))$$
$$\times\Lambda_{Bell}(R_2;R_3|\frac{1}{2}(-m+m_1-m_2+s_1+s_2-t_1-t_2+u_1+u_2))\}$$

4. Bell polynomials and higher spin algebras

Our main conjecture is that the solution discussed in this talk is related to collective higher-spin vacuum, in the spirit similar to relating the solution by Erler, Schnabl, et.al. to tachyon vacuum.

In general, our hope is that the higher spin algebras can be realized as operator algebras in string theory.

The first insightful hint, relating Bell polynomials to free field realizations of higher spin algebras in AdS, comes from $c = 1$ model, i.e. one-dimensional non-critical string theory.

The one-dimensional string compactified on S^1 has no standard massless modes (like a photon) but does have a $SU(2)$ multiplet of massless states existing at non-standard ghost numbers and discrete mumentum values (Klebanov, A. Polyakov, Witten, 1992):

The $SU(2)$ symmetry at self-dual radius $R = \frac{1}{\sqrt{2}}$ is realized by the operators:

$$T_{\pm} = \oint dz e^{\pm iX\sqrt{2}}; T_0 = \oint dz \partial X$$

The SU(2) multiplet of discrete states can be constructed by acting with the lowering T_- of $SU(2)$ on the highest weight vectors given by tachyonic primaries

$V_l = e^{(ilX+(l-1)\varphi)\sqrt{2}}$: (with integer l)

$$U_{l|m} = T_-^{l-m} V_l$$

Manifest expressions for $U_{l|m}$ vertex operators are complicated, however, their structure constants have been deduced by I. Klebanov, A. Polyakov and E.Witten in 1991 by using symmetry arguments. One has

$$U_{l_1|m_1}(z)U_{l_2|m_2}(w) \sim (z-w)^{-1} C(l_1, l_2, l_3|m_1, m_2, m_3) f(l_1, l_2) U_{l_3,m_3}$$

where the $SU(2)$ Clebsch-Gordan coefficients are fixed by the symmetry while the function of Casimir eigenvalues $f(l_1, l_2)$ is nontrivial and was deduced to be given by

$$f(l_1, l_2) = \frac{\sqrt{l_1+l_2}(2l_1+2l_2-2)!}{\sqrt{2l_1 l_2}(2l_1-1)!(2l_2-1)!}$$

(I. Klebanov, A. Polyakov,Mod.Phys.Lett. A6 (1991) 3273-3281, E. Witten, Nucl.Phys. B373 (1992) 187-213)

Remarkably, these structure constants coincide exactly with those of higher spin algebra in AdS_3 in a certain basis, computed by E. Fradkin and V. Linetsky in 1989, in , what appeared at that time a completely different context (E. Fradkin, V. Linetsky, Mod.Phys.Lett. A4 (1989) 2635-2647) On the other hand, the explicit structure of the vertex operators for the discrete states realizing this algebra is given by

$$U_{l|m} \sim \sum_{\frac{1}{2}(l(l-1)-m(m-1))|p_1,\dots,p_{l-m}} B_{p_1}(-iX\sqrt{2})\dots B_{p_{l-m}}(-iX\sqrt{2})e^{\sqrt{2}(imX+(l-1)\varphi)}$$

with the sum taken over ordered partitions of $\frac{1}{2}(l(l-1)-m(m-1))|p_1,\dots,p_{l-m}$

This is a relatively simple example of Bell polynomials multiplied by exponentials realizing the higher spin algebras in AdS_d in terms of vertex operator algebras in $d-1$-dimensional string theory. More complicated examples, such as the v.o. realizations of HS algebras in AdS_5, can also be constructed (D.P., in preparation). Our main conjecture is that the OSFT solutions of the type:

$$\Psi = \sum_{N,n_1,\dots,n_k} \lambda_N^{n_1\dots n_k} B_{n_1}(\phi,\chi,\sigma)\dots B_{n_k}(\phi,\chi,\sigma)(c\xi e^{N\phi} + ce^{-(N+2)\phi})$$

can be related to vacuum configurations of k-row higher spin fields with mixed symmetries.

In general, the space of these solutions would form an "enveloping" of higher-spin algebra

From the onshell prospective, another hint at the higher spins comes from the structure of the vertex operators for higher spin fields in Vasiliev's formalism.

Namely, consider open string vertex operators for Vasiliev type two-row higher spin gauge fields $\Omega_m^{a_1...a_{s-1}|b_1...b_t}(x) \equiv \Omega^{s-1|t}(x)(0 \leq t \leq s-1)$ where m is the curved d-dimensional space index and a, b indices (corresponding to rows of lengths $s-1$ and t) label d-dimensional tangent space. In case of $t = s-3$ the expression for the spin s operator particularly simplifies and is given by:

$$V_{s-1|s-3}(p) \equiv= \Omega_m^{a_1...a_{s-1}|b_1...b_{s-3}}(p) V^m_{a_1...a_{s-1}|b_1...b_{s-3}}(p)$$
$$= \Omega_m^{a_1...a_{s-1}|b_1...b_{s-3}}(p) \oint dz e^{-s\phi}\psi^m \partial\psi_{b_1}\partial^2\psi_{b_2}...\partial^{s-3}\psi_{b_{s-3}}\partial X_{a_1}...\partial X_{a_{s-2}} e^{ipX}$$

at minimal negative picture $-s$. The manifest expressions for the spin s operators with $0 \leq t < s-3$ are generally more complicated, however, at their canonical pictures equal to $-2s+t+3$, they can be related to the operator $V^m_{a_1...a_{s-1}|b_1...b_{s-3}}$

$$: \Gamma^{s-t-3} \Omega_m^{a_1...a_{s-1}|b_1...b_t}(p) V^m_{a_1...a_{s-1}|b_1...b_t} : (p)$$
$$= \Omega_m^{a_1...a_{s-1}|b_1...b_{s-3}}(p) V^m_{a_1...a_{s-1}|b_1...b_{s-3}}(p)$$

where $\Gamma =: e^{\phi} G :$ is the picture-changing operator satisfying $: \Gamma^m \Gamma^n :=: \Gamma^{m+n} : + \{Q_{brst}, ...\}$, G is the full matter+ ghost worlsheet supercurrent and $: \Gamma^n :\sim: e^{n\phi} G \partial G ... \partial^{n-1} G :$ This particularly entails a set of generalized torsion zero constraints relating the space-time extra fields in the frame-like formalism for the higher spins:

$$\Omega s-1|s-3(x) \sim \partial^{s-3-t}\Omega^{s-1|s-3}(x)$$

Corresponding operators at positive ghost numbers can be obtained by homotopy mapping (D.P., 2013)

The operators for for the frame-like fields of spin $s \geq 3$ are the elements of ghost cohomologies $H_{-s} \sim H_{s-2}$ (refs). The structure of their OPE:

$$H_{s_1} \otimes H_{s_2} \sim \sum_{k=|s_1-s_2|}^{s_1+s_2-2} H_k \tag{5}$$

coincides with the general structure of the HS algebra in AdS

5. Conclusion and discussion

- We have considered a pure ghost SFT solution at higher ghost numbers, a simplest in the family of other solutions (involving multiple Bell polynomials) which are still to be found

- Combined together, they presumably describe the ghost part of collective higher spin vacuum with mixed symmetries

- A concept of Bell polynomials being a free field realization of HS algebra in AdS needs to be elaborated

- Plenty of work ahead!