The Ramond Sector of Heterotic String Field Theory

Hiroshi Kunitomo (YITP, Kyoto U.)

2014/07/28 SFT2014 @SISSA

arXiv:1312.7197, 1407.0801
1. Introduction

♠ **WZW-type Superstring field theory**
   ◇ A formulation with *no explicit picture changing operator* and working well for NS sector:
      ○ Open superstring [Berkovits (1995)]
      ○ Hetetrotic string [Okawa and Zwiebach (2004), Berkovits, Okawa and Zwiebach (2004)]

However,

◇ Difficult to construct an action for R sector

For Open superstring:
   ○ *EOMs* (non-covariant action) [Berkovits (2001)]
   ○ An action with a constraint [Michishita (2005)]

---

**We construct the EOMs for heterotic string field theory**
CONTENTS

1. Introduction
2. NS EOM
3. Including R sector
4. First-order EOMs
5. Gauge symmetry
6. Conclusion and discussion
2. **NS EOM** [Berkovits Okawa Zwiebach(2004)]

- **Closed string products**:

  The $n$-string product is defined by linear maps:

  \[
  \{B_1, \cdots, B_n\} \rightarrow [B_1, \cdots, B_n],
  \]

  with $|[B_1, \cdots, B_n]| = \sum_i |B_i| + 1$ and $(G, P) = (\sum_i G_i - 2n + 3, \sum_i P_i)$.

- **Fundamental relations** ($L_\infty$-algebra):

  \[
  0 = Q[B_1, \cdots, B_n] + \sum_{i=1}^n (-1)^{|B_1|+\cdots+|B_{i-1}|} [B_1, \cdots, QB_i, \cdots, B_n]
  \]

  \[
  + \sum_{\{i_l, j_k\}} \sigma(i_l, j_k) [B_{i_1}, \cdots, B_{i_l}, [B_{j_1}, \cdots, B_{j_k}]].
  \]
• Derivations:

Operator $X = \eta$ or $\delta$ satisfies $[Q, X] = 0$ and

$$X[B_1, \cdots, B_n] = (-1)^X \sum_{i=1}^{n} (-1)^{X(B_1+\cdots+B_{i-1})} [B_1, \cdots, XB_i, \cdots, B_n]. \quad (3)$$

• NS String Field:

$V$ is Grassmann odd with $(G, P) = (1, 0)$.

• Pure-gauge string field: $B_Q = G(V)$ [cf. $A_Q = e^{-\Phi}(Qe^{\Phi})$ in OSSFT]

$$G(V) = QV + \frac{\kappa}{2} [V, QV] + \frac{\kappa^2}{3!} \left( [V, (QV)^2] + [V, [V, QV]] \right) \cdots,$$

satisfying (pure gauge solution of closed bosonic string EOM)

$$QG(V) + \sum_{n=2}^{\infty} \frac{\kappa^{n-1}}{n!} [(G(V))^n] \equiv 0.$$
\[
Q_G B = QB + \sum_{n=1}^{\infty} \frac{\kappa^n}{n!}[G^n, B],
\]

\[
[B_1, \cdots, B_n]_G = \sum_{m=0}^{\infty} \frac{\kappa^m}{m!}[G^m, B_1, \cdots, B_n],
\]

satisfying the same form of relations with (2).

In contrast, \(X\) neither commutes with \(Q_G\) nor acts as derivation on \([\cdots]_G\).

\[
Q_G(XB) - (-1)^X X(Q_G B) = -\kappa[XG, B]_G,
\]  

(4)

\[
X[B_1, \cdots, B_n]_G = \sum_{i=1}^{n} (-1)^X(1+B_1+\cdots+B_n)[B_1, \cdots, XB_i, \cdots, B_n]_G
\]

\[+(-1)^X \kappa[XG, B_1, \cdots, B_n]_G.\]  

(5)
• **Useful (MC) equations** :

\[
\delta G(V) = Q_G B_\delta(V), \quad (6a)
\]

\[
\eta G(V) = Q_G B_\eta(V). \quad (6b)
\]

• **NS EOM** :

\[
\eta G(V) = 0. \quad (7)
\]

• **Gauge symmetry** :

EOM (7) has symmetry under [cf. \(A_\delta = e^{-\Phi}(\delta e^\Phi)\)]

\[
B_\delta(V) = Q_G \Lambda_0 + \eta \Lambda_1, \quad (8)
\]

since

\[
\delta(\eta G(V)) = -\kappa [\eta G(V), \eta \Lambda_1]_G. \quad (9)
\]

In particular, EOM is *invariant* under \(\Lambda_0\)-gauge tf. because of (6a):

\[
\delta_{\Lambda_0} G(V) = Q_G (Q_G \Lambda_0) = 0. \quad (10)
\]
• Important identities:

\[ Q_G(\eta G) \equiv 0, \quad (11a) \]
\[ \eta(\eta G) \equiv 0. \quad (11b) \]

The first identity (11a) is a result of (6b).
3.2 Including R sector [HK arXiv:1312.7197]

• R string field:

Ψ is Grassmann odd with \((G,P) = (1,1/2)\).

• EOMs:

At the leading order in \(\Psi\) (but full order in \(V\)), it is natural to assume

\[
\eta G(V) + \frac{\kappa}{2}[(\eta \Psi)^2]_G = 0, \tag{12a}
\]

\[
Q_G \eta \Psi = 0, \tag{12b}
\]

which are consistent with (11a) as

\[
Q_G \left( \eta G + \frac{\kappa}{2}[(\eta \Psi)^2]_G \right) = -\kappa[\eta \Psi, Q_G \eta \Psi]_G = 0.
\]
EOMs (12) have to be consistent also with (11b) but

\[
\eta \left( \eta G + \frac{\kappa}{2}[(\eta \Psi)^2]_G \right) = -\frac{\kappa}{2}[(\eta \Psi)^2, \eta G]_G
\]

\[
= -\frac{\kappa}{2}[(\eta \Psi)^2, \left( \eta G + \frac{\kappa}{2}[(\eta \Psi)^2]_G \right)]_G + \frac{\kappa^2}{4}[(\eta \Psi)^2, [(\eta \Psi)^2]_G]_G
\]

\[
\neq 0.
\]

Thus EOMs (12) have to be corrected. Since RHS is higher-order in \( \Psi \), this is possible order by order in \( \Psi \) under the following ansatz.

- **G-ansatz** :

  Recall that the \( \Lambda_0 \)-gauge tf.

  \[
  B_\delta(V) = Q_G \Lambda_0, \tag{13}
  \]

  keeps NS EOM (7) invariant since

  \[
  \delta G(V) = Q_G B_\delta = Q_G (Q_G \Lambda_0). \tag{14}
  \]
We assume that this is also true in the full EOMs. That is, the transformation

\[ B_\delta(V) = Q_G \Lambda_0, \quad \delta \Psi = 0, \]  

keeps the full EOMs \textit{invariant}, which is achieved by

\[ G \text{-ansatz} \]

\[ V \text{ only appears through } G(V) \text{ in } Q_G \text{ and } [\cdots]_G \text{ in the correction terms.} \]

- General form:

From (11a), we can guess that the full EOMs have the form

\[ \eta G + \frac{\kappa}{2} [(B_{-\frac{1}{2}})^2]_G + Q_G B_{-1} = 0, \]

\[ (16a) \]

\[ Q_G B_{-\frac{1}{2}} = 0, \]

\[ (16b) \]

with

\[ B_{-1/2} = \eta \Psi + O(\Psi^3), \quad B_{-1} = O(\Psi^4). \]

\[ (17) \]
We can determine $B_{-1/2}$ and $B_{-1}$ order by order in $\Psi$ from

$$\eta \left( \eta G + \frac{\kappa}{2} [(B_{-1/2})^2]_G + Q_G B_{-1} \right) = 0. \quad (18)$$

For example, in the next order in $\Psi$, the possible terms are

$$B_{-1/2}^{(3)} = \alpha \kappa^2 [\Psi, (\eta \Psi)^2]_G, \quad B_{-1}^{(4)} = \beta \kappa^3 [\Psi, (\eta \Psi)^3]_G. \quad (19)$$

(18) gives

$$0 = \eta \left( \frac{\kappa}{2} [(\eta \Psi)^2]_G + \alpha \kappa^3 [\eta \Psi, [\Psi, (\eta \Psi)^2]_G]_G + Q_G \left( \beta \kappa^3 [\Psi, (\eta \Psi)^3]_G \right) \right)$$

$$= (\alpha - 4\beta) \kappa^3 [\eta \Psi, [(\eta \Psi)^3]_G]_G + \frac{\kappa^3}{4} (1 - 24\beta) [(\eta \Psi)^2, [(\eta \Psi)^2]_G]_G + \cdots. \quad (20)$$
Hence $\alpha = 1/3!$ and $\beta = 1/4!$. EOMs at the next-leading order in $\Psi$ are

$$\eta G + \frac{\kappa}{2}[(\eta \Psi)^2]_G + \frac{\kappa^3}{3!}[\eta \Psi, [\Psi, (\eta \Psi)^2]_G]_G + Q_G \left( \frac{\kappa^3}{4!} [\Psi, (\eta \Psi)^3]_G \right) = 0, \quad (21)$$

$$Q_G \left( \eta \Psi + \frac{\kappa^2}{3!} [\Psi, (\eta \Psi)^2]_G \right) = 0. \quad (22)$$

We can also determine the gauge transformations.
3.3 First-order EOMs [HK arXiv:1407.0801]

- **First-order EOMs:**

  EOMs are also written in the first-order (w.r.t $Q$ and $\eta$) form as

  \[
  \hat{Q}\hat{B} + \sum_{m=2}^{\infty} \frac{\kappa^{(m-1)}}{m!} [\hat{B}^m] = 0, \tag{23}
  \]

  with

  \[
  \hat{Q} = Q + \eta, \quad \hat{B} = \sum_{n=0}^{\infty} B_{-n/2}. \tag{24}
  \]

  Eq. (23) holds at each picture number independently, and thus an infinite number of equations for an infinite number of independent string fields $B_{-n/2}$ with $n = 0, 1, 2, \cdots$.

  Eq. (23) satisfies the consistency equation (cf. Bosonic SFT)

  \[
  \hat{Q}\left(\hat{Q}\hat{B} + \sum_{n=2}^{\infty} \frac{\kappa^{n-1}}{n!} [\hat{B}^n]\right) = - \sum_{m=1}^{\infty} \frac{\kappa^m}{m!} [\hat{B}^m, \left(\hat{Q}\hat{B} + \sum_{n=2}^{\infty} \frac{\kappa^{n-1}}{n!} [\hat{B}^n]\right)]. \tag{25}
  \]
Let us show that these first-order equations are equivalent to the previous EOMs by considering (23) in details.

Expanding (23) into different pictures, we obtain the following equations.

At $P = 0$,

$$QB_0 + \sum_{n=2}^{\infty} \frac{\kappa^{n-1}}{n!} [(B_0)^n] = 0,$$

which can be solved by

$$B_0 = G(V).$$

(27)

We substitute this in the following equations and define $\tilde{B} = \sum_{n=1}^{\infty} B_{-n/2}$.

At $P = -1/2$ and $-1$,

$$QB_{-1/2} = 0, \quad \eta G + \frac{\kappa}{2} [(B_{-1/2})^2]_G + Q_G B_{-1} = 0.$$

(28)

These have the same form as the previous EOMs (16).
At $P \leq -3/2$, (Subsidiary equations)

$$\eta B_{-n/2} + Q_G B_{-(n+2)/2} + \sum_{m=2}^{\infty} \frac{\kappa^{m-1}}{m!} [\tilde{B}^m]_G \big|_{-(n+2)/2} = 0, \quad (n \geq 1). \quad (29)$$

If (29) are solved for $B_{-n/2}$ in terms of $\Psi$ (and implicitly $V$), eqs. (28) give the previous EOMs.

We can show order by order in $\Psi$ that this is indeed the case under

**Ansatz**

$B_{-1/2} \neq 0$ but $B_{-n/2} = 0 \ (n \geq 2)$ at the linearized level.
Then the nontrivial subsidiary equation at the linearized level is only

$$\eta B_{-1/2}^{(1)} = 0,$$

which can be solved using $\Psi$ as

$$B_{-1/2}^{(1)} = \eta \Psi. \quad (31)$$

Eqs. (28) now become the leading order EOMs (12):

$$Q_G \eta \Psi = 0, \quad \eta G + \frac{\kappa}{2}[(\eta \Psi)^2]_G = 0. \quad (32)$$

**Next order**:

The linearized solution (31) make all the $B_{-n/2}$ non-trivial through the next order subsidiary equations:

$$\eta B_{-n/2}^{(n+2)} = -\frac{\kappa^{n+1}}{(n+2)!}[(\eta \Psi)^{n+2}]_G, \quad (n \geq 1). \quad (33)$$
can be solved by

\[ B^{(n+2)}_{-n/2} = \frac{\kappa^{n+1}}{(n + 2)!} [\Psi, (\eta \Psi)^{n+1}]_G, \quad (n \geq 1), \]  

(34)

under EOMs (28) up to the \( \eta \)-exact terms which can be gauged away using the invariance under the following gauge invariance.

- **Gauge symmetry** :

EOM (23) is invariant under

\[ \delta \hat{B} = \hat{Q} \hat{\sigma} + \sum_{m=2}^{\infty} \frac{\kappa^{m-1}}{m!} [\hat{B}^m, \hat{\sigma}], \]  

(35)

with \( \hat{\sigma} = \sum_{n=-1}^{\infty} \sigma_{-n/2} \). Parameters \( \sigma_{-n/2} \) are independent, except for

\[ \sigma_{1/2} = Q_G \Lambda_{1/2}, \]  

(36)

which is constrained by \( P = 1/2 \) component of (35): \( 0 = Q_G \sigma_{1/2} \).
The ambiguity from $\eta$-exact terms can be gauged away using $\sigma_{-n/2}$ with $n \geq 0$ since (35) has the form

$$\delta B_{-n/2} = \eta\sigma_{-(n-2)/2} + \cdots, \quad (n \geq 1).$$  \hspace{5cm} (37)$$

Note: Ambiguity of $B_{-1/2}$ cannot be gauged away but absorbed into the redefinition of $\Psi$.

- **Next to next order**:

The subsidiary equations at next-to-next order are given as

$$\eta B_{-(n+2)/2} = -Q_G B_{-(n+2)/2} - \sum_{m=1}^{n+1} [(\eta \Psi)^m, B_{-(n-m+2)/2}]_G,$$  \hspace{5cm} (38)

where $B_{-(n+2)/2}$ is given by (34).
Eqs. (38) can be solved as

\[
B^{(n+4)}_{-n/2} = -\frac{\kappa^{n+3}}{(n+4)!}[\Psi, Q_G \Psi, (\eta \Psi)^{n+2}]_G - \frac{\kappa^{n+3}}{(n+4)!}(n + 3)[\Psi, (\eta \Psi)^{n+1}, [\Psi, \eta \Psi]_G]_G \\
+ \frac{\kappa^{n+3}}{(n+4)!} \sum_{k=0}^{n} \binom{n + 3}{k}[\Psi, (\eta \Psi)^k, [\Psi, (\eta \Psi)^{n+2-k}]_G]_G,
\]

(39)

where the ambiguity from \( \eta \)-exact terms again gauged away.

It has been proved that the procedure can be repeated order by order in \( \Psi \), and eqs. (28) explicitly give the second-order EOMs (16).
3.4 Gauge symmetry

The symmetry of the second-order EOMs can also be obtained from (35) as follows.

Expanding (35) into different picture numbers, we obtain, at $P = 0$,

$$\delta B_0 = Q_G \left( \sigma_0 - \kappa \left[ B_{-1/2}, \Lambda_{1/2} \right]_G \right) - \kappa \left[ Q_GB_{-1/2}, \Lambda_{1/2} \right]_G. \quad (40)$$

This gives the transformation of $V$ as

$$B_\delta(V) = \sigma_0 - \kappa \left[ B_{-1/2}, \Lambda_{1/2} \right]_G + Q_G \Lambda_0. \quad (41)$$

Here, however, $\sigma_0$ is not independent parameter, as will be shown shortly, since it is needed to fix the ambiguity.
Remaining tfs. at $P \leq -1/2$ can be studied order by order in $\Psi$.

The ansatz for $B_{-n/2}$ requires that the only $\sigma_0$ is non-zero at the linearized level. From (35) at $P = -1$, $0 = \eta \sigma_0^{(0)}$, we obtain, at the leading order in $\Psi$,

$$\sigma_0^{(0)} = \eta \Lambda_1. \quad (42)$$

Then eq. (35) at $P = -1/2$ gives

$$(\delta B_{-1/2})^{(1)} = \eta \left( Q_G \Lambda_{1/2} - \kappa [\Psi, \eta \Lambda_1]_G \right) - \kappa^2 [\eta G, \Psi, \eta \Lambda_1]_G, \quad (43)$$

from which we can read

$$\delta^{(0)} \Psi = Q_G \Lambda_{1/2} - \kappa [\Psi, \eta \Lambda_1]_G + \eta \Lambda_{3/2}. \quad (44)$$
The $\sigma_0^{(0)}$ makes $\sigma_{-n/2}$ non-trivial through (35) at $P \leq -1$, 

\[ 0 = \eta \sigma_{-(n-2)/2} + \frac{\kappa^n}{n!} [(\eta \Psi)^n, \sigma_0^{(0)}]_G \]

\[ = \eta \left( \sigma_{-(n-2)/2} - \frac{\kappa^n}{n!} [\Psi, (\eta \Psi)^{n-1}, \eta \Lambda_1]_G \right) - \frac{\kappa^n}{n!} [\eta G, \Psi, (\eta \Psi)^{n-1}, \eta \Lambda_1]_G, \]

which requires

\[ \sigma_{-(n+2)/2} = \frac{\kappa^{n+2}}{(n+2)!} [\Psi, (\eta \Psi)^{n+1}, \eta \Lambda_1]_G, \quad (n \geq 0). \]

Ambiguity from $\eta$-exact terms can again be gauged away using the symmetry of the gauge tfs.
Eq. (35) at $P = -1/2$ gives the next order $t_f$.

$$
\delta^{(2)} \bar{B}^{(1)}_{-1/2} + \delta^{(0)} \bar{B}^{(3)}_{-1/2} = (\delta B_{-1/2})^{(3)} + \frac{\kappa^3}{2} \left[[\eta \Psi]^2\right]_G, \Psi, \eta \Lambda_1]_G.
$$

We can obtain

$$
\delta^{(2)} \Psi = \frac{\kappa^3}{3!} [\Psi, Q_G \Psi, \eta \Psi, \eta \Lambda_1]_G - \frac{\kappa^3}{3!} [\Psi, [\Psi, \eta \Psi, \eta \Lambda_1]_G]_G + \frac{\kappa^3}{3} [[\Psi, \eta \Psi]_G, \Psi, \eta \Lambda_1]_G
$$

$$
- \frac{\kappa^2}{3} [\Psi, \eta \Psi, Q_G \Lambda_2]_G + \frac{\kappa^2}{3!} [\Psi, \eta \Psi, \eta \Lambda_3]_G.
$$
4. Summary and discussion

Summary

♠ We have constructed EOM and gauge tfs. for Heterotic SFT including R sector order by order in $\Psi$.

♠ Their explicit forms can be systematically obtained using the first-order formulation.

We have also partially constructed

♣ An action with a constraint (in principle),

and explicitly computed

♣ Four point amplitudes with external fermions.
Remaining tasks

★ Constructing an action for the R sector

- Action with the constraint $Q G \Xi = B_{-1/2}$
  
  General prescription?

- Action deriving the first-order EOM.
  
  $\hat{Q}$-cohomology? Democratic formulation?

★ Constructing EOM for the type II superstring