Constructing an NS-NS closed string field theory

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SFT and related aspects VI, 2014

based on work with T. Erler and I. Sachs [arXiv:1312.2948, 1403.0940]

Motivation

- Lots of candidates for open super string field theory:
  - RNS based: Witten, Modified, WZW-like, Democratic, ...
  - Pure spinor based: Berkovits
- Closed SFTs are harder to construct, but it is possible:
  - Bosonic CSFT [B. Zwiebach '93]
  - Heterotic SFT [Y. Okawa, B. Zwiebach '04]
- Recursive construction of Witten's OSSFT [T. Erler, S. K., I. Sachs '13]
- Main tools: large Hilbert space and homological methods
- Goals:
  - Adapt methods to closed NS-NS string
  - Find a covariant field theory in small Hilbert space

Homotopy algebras and moduli spaces

Geometric construction for bosonic closed string:

[B. Zwiebach '93, T. Kimura, J. Stasheff, A. Voronov '95]

- String background yields a TCFT on (restricted) state space $\mathcal{H}$
- Choose a decomposition of moduli space $\mathcal{M}$ solving master equation
- TCFT maps decomposition to a solution S of master equation
- restricted state space: $b_0^- = L_0^- = 0$

Algebraic properties of solution $S = \frac{1}{2!} V_{ij}^{(2)} \Phi^i \Phi^j + \frac{1}{3!} V_{ijk}^{(3)} \Phi^i \Phi^j \Phi^k + \cdots$:

- Generates all gauge transformations via $\delta_\Lambda = ((S, \Lambda), \cdot)$
- Vertices $V^{(n)}$ satisfy axioms of cyclic L_∞ algebra for some symplectic structure ω_{ij}.
- Vertices $V^{(n)}$ must preserve the constraints.

Homotopy algebras and moduli spaces

The cobar construction (for A_∞):

- The vertices $V^{(n)}$ can be used to define a sequence of maps $m_{n-1} : \mathcal{H}^{\otimes(n-1)} \to \mathcal{H}$ with

$$\langle e_j, m_{n-1}(e_{i_1}, e_{i_2}, \ldots, e_{i_{n-1}})\rangle = (-1)^j V^{(n)}_{ji_1\cdots i_{n-1}}$$

  for some basis e_i of $\mathcal{H}$.
- We extend the maps m_n to maps $M_n : T\mathcal{H} \to T\mathcal{H}$ by virtue of

$$M_n = \sum_{r,s} \mathbf{1}^{\otimes r} \otimes m_n \otimes \mathbf{1}^{\otimes s}.$$

- The A_∞ relations can then be written very compactly in the form $[M, M] = 0$, with $M = \sum_n M_n$.

A_∞ vs. L_∞ algebras

- Depending on the symmetry properties of the vertices we obtain A_∞ (only cyclic) or L_∞ (totally symmetric) algebras.
- The cobar construction for L_∞ is obtained by replacing $T\mathcal{H}$ by $S\mathcal{H}$.
- L_∞ can be reduced to A_∞ using symmetrization/universal envelope. [T. Lada and M. Markl '94]

Preparing the input data for the L_∞ case:

- Take the bosonic algebra, $L_n : S\mathcal{H} \to S\mathcal{H}$, and construct its universal envelope M_n.
- The maps M_n now define an A_∞ algebra and we can regard all maps as $T\mathcal{H} \to T\mathcal{H}$.
- The final result has to be symmetrized to obtain an L_∞ structure again.

Review of the open string case

- Given initial data Q, $m_2 = *$ and a contracting homotopy $\xi\circ$ for $[\eta, \cdot]$, define for $n \geq 0$ recursively ($M_1 = Q$, $\mu_1 = 0$):

$$M_{n+2} = \frac{1}{n+1}\left([Q, \mu_{n+2}] + [M_2, \mu_{n+1}] + \cdots + [M_{n+1}, \mu_2]\right)$$
$$\mu_{n+2} = \xi \circ m_{n+2}$$
$$m_{n+3} = \frac{1}{n+1}\left([m_2, \mu_{n+2}] + [m_3, \mu_{n+1}] + \cdots + [m_{n+2}, \mu_2]\right)$$

- The maps M_n are η-closed and define a cyclic A_∞-structure on $\mathcal{H}$ and the S-matrix reduces to perturbative string amplitudes.
  [T. Erler, S. K., I. Sachs '13]

Review of the open string case

- Recursion relations are equivalent to a solution to an ODE:

$$M(t) = \sum_n t^n M_{n+1}, \qquad \mu(t) = \sum_n t^n \mu_{n+1}, \qquad m(t) = \sum_n t^n m_{n+2}$$

$$\frac{d}{dt} M(t) = [M(t), \mu(t)], \quad [\eta, \mu(t)] = m(t), \quad \frac{d}{dt} m(t) = [m(t), \mu(t)]$$

$$M(0) = Q, \qquad m(0) = m_2$$

- These differential equations are the starting point for various generalizations by reinterpreting them as gauge transformations of a free theory.

Constructing OSSFT via gauge transformations

- Define picture deficit of map M_n as $\mathrm{def}(M_n) = n - 1 - \mathrm{pic}(M_n)$.
- Introduce a formal variable s and write $M(s) = \sum_k s^k M^{[k]}$ and $\mu(s) = \sum_k s^k \mu^{[k]}$ with $\mathrm{def}(\mu^{[k]}) = \mathrm{def}(M^{[k]}) = k$.
- The original system of ODEs is now:

$$\frac{\partial}{\partial t} M(t,s) = [M(t,s), \mu(t,s)], \qquad [\eta, \mu(t,s)] = \frac{\partial}{\partial s} M(t,s)$$
$$M(0,s) = Q + s m_2, \qquad M(t,s) = M(t) + s m(t)$$

- Commutators preserve picture deficit, $[\eta, \cdot]$ increases it by 1, so the ODEs preserve s as counting picture deficit, so that $M(t,0)$ contains only maps with no unintegrated odd moduli.
- Can use arbitrary initial conditions subject to $[M(0,s), M(0,s)] = 0$, e.g. the initial maps are A_∞ and have no PCOs.

Recursion formula for the solution

A choice of contracting homotopy allows us to solve for $\mu = \xi \circ \frac{\partial}{\partial s} M$. So that the ODE to solve is now,

$$\dot{M}(t) = \left[M(t), \xi \circ \frac{\partial}{\partial s} M(t) \right], \qquad M(t) - M(0) = \int_0^t dt' \left[M(t'), \xi \circ \frac{\partial}{\partial s} M(t') \right].$$

The evolution of $M^{[k]}$ is only sourced by terms with strictly greater picture deficit.
If $\xi \circ \frac{\partial}{\partial s} M_1(0) = 0$, it follows that $\xi \circ \frac{\partial}{\partial s} M_1(t) = 0$. Consequently, $M_1(t) = M_1(0)$. Hence, M_n is only sourced by terms with lower number of inputs and the ODE can be solved in a finite number of steps at each number of inputs.

$$\begin{aligned} M^{[0]}(t) - M^{[0]}(0) &= \int_0^t dt' \, \left[M^{[0]}(t'), \xi \circ M^{[1]}(t') \right] \\ M^{[1]}(t) - M^{[1]}(0) &= \int_0^t dt' \, \left(\left[M^{[0]}(t'), 2\xi \circ M^{[2]}(t') \right] + \left[M^{[1]}(t'), \xi \circ M^{[1]}(t') \right] \right) \\ &\vdots \end{aligned}$$

At fixed number of inputs n, the RHS will be eventually 0 after at most n steps. Then, we can go back and find the desired product $M_n^{[0]}(t)$.

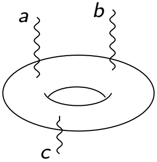
Constructing OSSFT via gauge transformations

Abstract characterization via vector fields:

- Formal (Q-)manifold of all η-closed, s-extended A_∞ structures: $\mathcal{A}$
- Field redefinitions: $\delta_R M = [M, R]$ with $[\eta, R] = 0$
- Integrating out odd moduli:
  $\delta_\xi M = [M, \xi \circ \frac{d}{ds} M]$ with $\xi \circ$ a contracting homotopy for $[\eta, .]$
- Commutators:
  $[\delta_\xi, \delta_{\xi'}] = \delta_R$, $[\delta_R, \delta_\xi] = \delta_{R'}$, $[\delta_R, \delta_{R'}] = \delta_{R''}$ and $\delta_\xi - \delta_{\xi'} = \delta_R$.
- Given construction of OSSFT equivalent to integrating along flow of some δ_ξ.

Review of NS-NS closed string theory

Perturbative Polyakov prescription to calculate S-matrix:

$\int_{\mathfrak{M}}$ 

$= \langle a, b|S|c\rangle = \int \langle d\tau\, \mathcal{V}_a \mathcal{V}_b \mathcal{V}_c \rangle_S$

$$\mathcal{V}_a(z,\bar{z}) = \delta(\gamma)\delta(\bar{\gamma})c\bar{c}V_M(z,\bar{z})$$
$$\delta(\gamma) = e^{-\phi},\ \gamma = \eta e^{\phi}, \beta = \partial\xi e^{-\phi}$$
$$d\tau = \prod_i b(\mu_i) \prod_\alpha \delta(\beta(s_\alpha))$$

- $\mathcal{V}_a$ represent vertex operators in Siegel gauge, V_M is a weight $(\frac{1}{2}, \frac{1}{2})$ superconformal primary.
- Bosonization of the $\beta\gamma$-system enlarges the (small) Hilbert space to the large Hilbert space that includes ξ_0.
- Ghost number and picture number are defined by $j_{\rm gh} = -bc - \xi\eta$ and $j_{\rm pic} = \partial\phi + \xi\eta$. They are anomalous: for $g = 0$, total ghost number must be 6, total picture 2 in each sector.
- To saturate missing picture some of the VOs can be in $(-1,-1)$ and some in $(0,0)$ picture: Use PCOs $X(z) = [Q, \xi(z)]$ and $\bar{X}(\bar{z}) = [Q, \bar{\xi}(\bar{z})]$ for this.

Application: NS-NS closed string field theory

Important properties for this SFT:

- String field Φ in small Hilbert space: $\eta\Phi = \bar{\eta}\Phi = 0$, picture $(-1,-1)$, ghost number 2, level matching $b_0^-\Phi = L_0^-\Phi = 0$
- Sympectic form $\omega(e_i, e_j) = \langle e_i(\infty) c_0^- e_j(0) \rangle_S = \langle \xi_0 e_i(\infty) c_0^- e_j(0) \rangle_L$
- BRST charge Q, induced by world-sheet BRST current j_B

Some general remarks:

- η has trivial cohomology since $[\eta, \xi(z)] = 1$, same for $\bar{\eta}$.
- BPZ even modes of ξ with $[\eta, \xi] = 1$ give rise to contracting homotopy $\xi\circ$, $K_n : \mathcal{H}^{\otimes n} \to \mathcal{H}$:

$$\xi \circ K_n = \frac{1}{n+1} \left(\xi K_n + (-1)^{|K|} K_n \xi \right)$$

Differences to the open string case

- Holomorphic and antiholomorphic pictures: introduce formal variables s and $\bar{s}$ counting picture deficits:

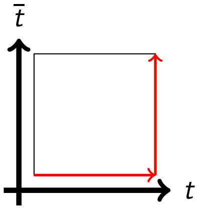
$$L = \sum_{k,l} s^k \bar{s}^l L^{[k,l]}, \qquad \mu = \sum_{k,l} s^k \bar{s}^l \mu^{[k,l]}$$

- Homotopies $\xi\circ$ and $\bar{\xi}\circ$ must preserve the $b_0^- = L_0^- = 0$ constraints, choose $\xi = \xi_0, \bar{\xi} = \bar{\xi}_0$.
- Can integrate out odd-moduli in holomorphic and antiholomorphic directions independently:

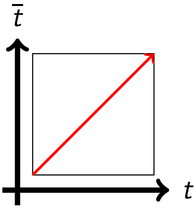
$$\delta L \equiv [L, \xi_0 \circ \frac{\partial}{\partial s} L], \qquad \bar{\delta} L \equiv [L, \bar{\xi}_0 \circ \frac{\partial}{\partial \bar{s}} L]$$

- Initial data $L = Q + s\bar{s} L_2^{(0,0)} + s^2 \bar{s}^2 L_3^{(0,0)} + \cdots$, with $L_N^{(0,0)}$ being the bosonic closed string products.

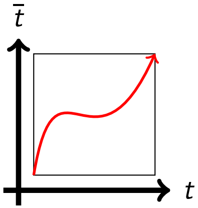
Integrating the flow



Naive choice



Symmetric choice



Generic choice

- Generic flow given by $a\delta + b\bar{\delta}$ with a and b arbitrary functions of the path parameter.
- Naive choice: Apply the open string construction to the holomorphic, then to the antiholomorphic sector.
- Symmetric choice: Treat both chiralities in the same way.

Calculating 3-vertices for the symmetric choice

Our starting point is the integral form of the flow equation along $\delta + \bar{\delta}$:

$$L(t) - L(0) = \int_0^t dt' \, [L(t'), \mu(t') + \bar{\mu}(t')], \text{where } \mu = \xi_0 \circ \frac{\partial}{\partial s} L, \bar{\mu} = \bar{\xi}_0 \circ \frac{\partial}{\partial \bar{s}} L$$

$$L(0) = Q + s\bar{s} L_2^{(0,0)} + \cdots$$

We note that $\mu_1(t) = \bar{\mu}_1(t) = 0$ as for the open string. Hence, concentrating at the BRST-charge $L_1^{[0,0]}(t)$ and the two product $L_2^{[0,0]}(t)$, we find that

$$L_1^{[0,0]}(t) - Q = 0,$$

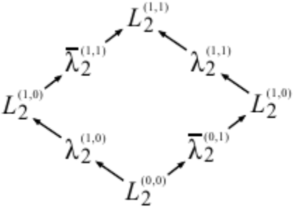
$$L_2^{[0,0]}(t) = \left[Q, \int_0^t dt' \, (\xi_0 \circ L_2^{[1,0]}(t') + \bar{\xi}_0 \circ L_2^{[0,1]}(t')) \right]$$

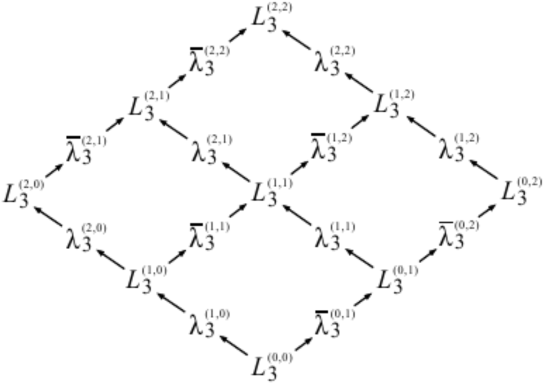
$$L_2^{[1,0]}(t) = \left[Q, \int_0^t dt' \, (2\xi_0 \circ L_2^{[2,0]}(t') + \bar{\xi}_0 \circ L_2^{[1,1]}(t')) \right], \qquad L_2^{[0,1]}(t) = \text{c.c.}$$

$$L_2^{[1,1]}(t) - L_2^{(0,0)} = \left[Q, \int_0^t dt' \, (2\xi_0 \circ L_2^{[2,1]}(t') + 2\bar{\xi}_0 \circ L_2^{[1,2]}(t')) \right] = 0$$

Note the triangular structure: L is always a polynomial in $s, \bar{s}$, so that at fixed number of inputs, the picture deficit is always bounded from above and the recursion terminates after a finite number of steps: $L_2^{[k,l]} = 0$, if $k > 1$ or $l > 1$

Calculating 3-vertices for the symmetric choice





Calculating 3-vertices for the symmetric choice

- If we follow the recursion, we find that the 3-vertex must take the form

$$L_2^{[0,0]}(t) = \int_0^t dt' \left[Q, \xi_0 \circ \int_0^{t'} dt'' \left[Q, \bar{\xi_0} \circ L_2^{(0,0)} \right] + \text{c.c.} \right] = \frac{t^2}{2} \left(X_0 \circ \bar{X}_0 \circ L_2^{(0,0)} + \text{c.c.} \right),$$
$$X_0 \circ = [Q, \xi_0 \circ], \bar{X}_0 \circ = [Q, \bar{\xi}_0 \circ].$$

- The cubic term in the action looks like

$$S_3 = \frac{1}{3!} \omega(\Phi, \frac{1}{3} X_0 \bar{X}_0 L_2^{(0,0)}(\Phi, \Phi) + \frac{2}{3} L_2^{(0,0)}(X_0 \Phi, \bar{X}_0 \Phi)).$$

- At the quartic level, the generic expression contains already 91 inequivalent terms. The individual terms cannot be expressed using ω, but only through the symplectic form for the large Hilbert space, $\omega(\cdot, \cdot) = \omega_L(\cdot, \xi_0 \cdot)$.

Uniqueness of the construction

Define $\mathfrak{A}$ as the Q-manifold of $\eta, \bar{\eta}$-closed, $s, \bar{s}$-extended A_∞-structures subject to the constraints $b_0^- = L_0^- = 0$.

Locally on $\mathfrak{A}$ uniqueness can be seen as follows:

- On $\mathfrak{A}$ we have an integrable distribution with basis δ, $\bar{\delta}$ and δ_R, schematically:

$$[\delta, \bar{\delta}] = \delta_R, \qquad [\delta, \delta_R] = \delta_{R'}, \qquad [\bar{\delta}, \delta_R] = \delta_{R'}, \qquad [\delta_R, \delta_{R'}] = \delta_{R''}$$

- The object of interest is the leaf space of δ_R: $\mathfrak{S} = \mathfrak{A}/(\text{field red.})$
- On $\mathfrak{S}$ the vector fields $\pi_*\delta$, $\pi_*\bar{\delta}$ commute. By Frobenius theorem, we can choose coordinates $t, \bar{t}$ on $\mathfrak{S}$, s.t. $\frac{\partial}{\partial t} = \pi_*\delta$, $\frac{\partial}{\partial \bar{t}} = \pi_*\bar{\delta}$.
- Since any path in $\mathfrak{A}$ descends to a path in $\mathfrak{S}$ via π, it follows that if the end points of two paths have the same $(t, \bar{t})$ coordinates, then the structures must be related by a field redefinition.

Conclusions and outlook

- Constructed classical field theory of NS-NS closed strings in small Hilbert space
- Recursive construction, but large growth of terms in higher vertices
- Theory satisfies a classical BV-master equation.

- Geometric interpretation in terms of decomposition of supermoduli spaces of NS-punctured discs and spheres?
- Obtain given construction from gauge-fixing a WZW-like theory?
- How to incorporate the Ramond sector?
- Quantization: quantum BV-equation, non-minimal solutions, gauge fixing

Thank you!