On gauge invariant observables for identity-based marginal solutions in bosonic and super string field theory

Isao Kishimoto

Niigata University, Japan

July 31, 2014

String Field Theory and Related Aspects VI, SFT2014
SISSA, Trieste (Italy)
This talk is based on

I. K. and Tomohiko Takahashi,
“Comments on observables for identity-based marginal solutions in Berkovits’ superstring field theory,”

and references therein:

cf. Erler, Ishibashi and Maccaferri’s talks
In my previous talk@SFT2012, we have constructed [IKT(2012)] nontrivial solutions around identity-based marginal solutions in modified cubic SSFT using the “$G'K'Bc\gamma$” algebra.

Some time ago in [KT(2005)], we constructed identity-based marginal solution $\Phi_J$ in Berkovits’ WZW-like SSFT. Recently, Erler has constructed the tachyon vacuum solution $\Phi^E_T$ in Berkovits’ WZW-like SSFT.

We will construct the tachyon vacuum solution $\Phi_T$ around the identity-based marignal solution $\Phi_J$, using a version of the extended $KBc$ algebra in the framework of Berkovits’ WZW-like SSFT.
It is difficult to evaluate vacuum energy and gauge invariant overlaps for “identity-based” solutions because of singular property of the identity state: $\langle I | (\cdots) | I \rangle$, corresponding to zero width in the sliver frame, is indefinite at least naively.

Instead of straightforward calculations, there have been various studies about the gauge invariants indirectly for identity-based solutions in bosonic SFT. (2001～, cf. Takahashi’s talks)

On the other hand, gauge invariants for “wedge-based” solutions using $KBc$ algebra and its extention have been evaluated exactly. ([Schnabl(2005)]～)
In our previous paper [KT(2013)], we have evaluated the gauge invariant overlaps (GIO), $\langle \Psi \rangle_V$, for identity-based marginal solution $\Psi_J$ in bosonic SFT:

$$\langle \Psi_J \rangle_V = \langle \Psi^{ES}_T \rangle_V - \langle \Psi_T \rangle_V$$

using the relationship to (wedge-based) tachyon vacuum solutions:

$$\Psi_J = \Psi^{ES}_T - \Psi_T + \int_0^1 Q_{\Psi_T} \Lambda_t dt.$$ 

In fact, we can show that $\Psi^{ES}_T$ [Erler-Schnabl(2009)] and the sum of $\Psi_J$ [Takahashi-Tanimoto(2001)] and $\Psi_T$ [IKT(2012)] are gauge equivalent:

$$\Psi_J + \Psi_T = g^{-1} \Psi^{ES}_T g + g^{-1} Q_B g, \quad g = \text{P exp} \left( \int_0^1 \Lambda_t dt \right)$$

$$\rightarrow \quad S[\Psi_J; Q_B] = S[\Psi^{ES}_T; Q_B] - S[\Psi_T; Q_\Psi J] = 0$$
Figure: Identity/wedge-based solutions in bosonic SFT
Using this relation among identity-based marginal solution $\Phi_J$ and wedge-based tachyon vacuum solutions, $\Phi^E_T$ and $\Phi_T$, the GIO for identity-based marginal solution can be evaluated:

$$\langle \Phi_J \rangle \nu = \langle \Phi^E_T \rangle \nu - \langle \Phi_T \rangle \nu.$$ 

Actually, $\Phi^E_T$ and $\log(e^{\Phi_J} e^{\Phi_T})$ are gauge equivalent and we have

$$S[\Phi_J; \hat{Q}] = S[\Phi^E_T; \hat{Q}] - S[\Phi_T; \hat{Q} \Phi_J] = 1/(2\pi^2) - 1/(2\pi^2) = 0.$$ 

The energy for the identity-based marginal solution $\Phi_J$ is zero, which agrees with the previous result as a consequence of $\xi$ zeromode counting [KT(2005)].
Figure: Identity/wedge-based solutions in Berkovits’ WZW-like SSFT including GSO(−) sector
Contents

1 Introduction and summary

2 Tachyon vacuum around the identity-based marginal solution
   - Identity-based marginal solutions
   - Deformed algebra
   - Tachyon vacuum solution
   - Energy and gauge invariant overlaps

3 Evaluation of observables for identity-based marginal solutions
   - Interpolation
   - On gauge equivalence relations
   - Energy and gauge invariant overlaps

4 Conclusions
Identity-based marginal solutions

Identity-based marginal solution in Berkovits’ WZW-like SSFT:

\[ \Phi_J = \tilde{V}_L^a(F_a) I, \]

\[ \tilde{V}_L^a(f) \equiv \int_{C_L} \frac{dz}{2\pi i} f(z) \frac{1}{\sqrt{2}} c \gamma^{-1} \psi^a(z), \quad \gamma^{-1}(z) = e^{-\phi} \xi(z). \]

\[ F_a(-1/z) = z^2 F_a(z), \quad C_L: \text{a half unit circle: } |z| = 1, \text{ Re } z \geq 0 \]

\[ I: \text{the identity state}, \quad \psi^a: \text{matter worldsheet fermion}. \]

\[ \psi^a(y) \psi^b(z) \sim \frac{1}{y-z} \frac{1}{2} \Omega_{ab}, \quad J^a(y) \psi^b(z) \sim \frac{1}{y-z} f^{ab}_c \psi^c(z), \]

\[ J^a(y) J^b(z) \sim \frac{1}{(y-z)^2} \frac{1}{2} \Omega_{ab} + \frac{1}{y-z} f^{ab}_c J^c(z), \]

\[ \Omega_{ab} = \Omega_{ba}, \quad f^{ab}_c \Omega^{cd} + f^{ad}_c \Omega^{cb} = 0, \quad f^{ab}_c = -f^{ba}_c, \quad f^{ad}_b f^{cd}_e + f^{bc}_d f^{ad}_e + f^{ca}_d f^{bd}_e = 0. \]

EOM in the NS sector is satisfied:

\[ \eta_0 \left( e^{-\Phi_J} Q_B e^{\Phi_J} \right) = 0. \]
By expanding the NS action $S[\Phi; Q_B]$ of Berkovits’ WZW-like SSFT around $\Phi_J$ as

$$e^{\Phi} = e^{\Phi_J} e^{\Phi'},$$

we have

$$S[\Phi; Q_B] = S[\Phi_J; Q_B] + S[\Phi'; Q_{\Phi_J}],$$

where $S[\Phi'; Q_{\Phi_J}]$ is given by a deformed BRST operator:

$$Q_{\Phi_J} = Q_B - V^a(F_a) + \frac{1}{8} \Omega^{ab} C(F_a F_b).$$

$V^a(F_a)$ and $C(F_a F_b)$ are given by integrations along the whole unit circle:

$$V^a(f) \equiv \oint \frac{dz}{2\pi i} \frac{1}{\sqrt{2}} f(z)(c J^a(z) + \gamma \psi^a(z)), \quad C(f) \equiv \oint \frac{dz}{2\pi i} f(z) c(z).$$
Deformed algebra

A version of the extended $KBc$ algebra with

$Q_B \rightarrow Q' \equiv Q\Phi_f$

$L_n \rightarrow L'_n = \{Q', b_n\} = L_n - \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} F_{a,k} J_{n-k}^a + \frac{1}{8} \Omega^{ab} \sum_{k \in \mathbb{Z}} F_{a,n-k} F_{b,k}$

$F_{a,n} \equiv \oint \frac{d\sigma}{2\pi} e^{i(n+1)\sigma} F_a(e^{i\sigma})$,  $F_{a,n} = -(-1)^n F_{a,-n}$

Relations among string fields $K', B, c, \gamma$:

$B^2 = 0$, $c^2 = 0$, $Bc + cB = 1$, $BK' = K'B$, $K'c - cK' = Kc - cK \equiv \partial c$, $\gamma B + B\gamma = 0$, $c\gamma + \gamma c = 0$, $K'\gamma - \gamma K' = K\gamma - \gamma K \equiv \partial \gamma$, $\hat{Q}'B = K'$, $\hat{Q}'K' = 0$, $\hat{Q}'c = cK'c - \gamma^2 = cKc - \gamma^2 = c\partial c - \gamma^2$, $\hat{Q}'\gamma = \hat{Q}\gamma = c\partial \gamma - \frac{1}{2}(\partial c)\gamma$

where $\hat{Q}' \equiv Q'\sigma_3$, $\hat{Q} \equiv Q_B\sigma_3$ and $\sigma_i$ are Pauli matrices (CP factor) for the GSO$(-)$ sector

$B = \frac{\pi}{2} B_1^L I\sigma_3$, $c = \frac{2}{\pi} \hat{U}_1 \tilde{c}(0)|0\rangle\sigma_3$, $\gamma = \sqrt{\frac{2}{\pi}} \hat{U}_1 \tilde{\gamma}(0)|0\rangle\sigma_2$  $K' = \frac{\pi}{2} K_1^L I$,  $K = \frac{\pi}{2} K_1^L I$
For the string fields $\gamma^{-1}, \zeta, V$, we have

\[
\begin{align*}
\gamma^{-1} \gamma &= \gamma \gamma^{-1} = 1, & \gamma^{-1} B + B \gamma^{-1} &= 0, & \gamma^{-1} c + c \gamma^{-1} &= 0, \\
K' \gamma^{-1} - \gamma^{-1} K' &= K \gamma^{-1} - \gamma^{-1} K \equiv \partial \gamma^{-1}, \\
\hat{Q}' \gamma^{-1} &= \hat{Q} \gamma^{-1} = c \partial \gamma^{-1} + \frac{1}{2} (\partial c) \gamma^{-1}, & \hat{Q}' \zeta &= \hat{Q} \zeta = c V + \gamma
\end{align*}
\]

where

\[
\begin{align*}
\gamma^{-1} &= \sqrt{\frac{\pi}{2}} \hat{U}_1 \tilde{\gamma}^{-1} (0) |0\rangle \sigma_2, & \zeta &= \gamma^{-1} c = \sqrt{\frac{2}{\pi}} \hat{U}_1 \tilde{\gamma}^{-1} \tilde{c}(0) |0\rangle i \sigma_1, \\
V &= \frac{1}{2} \gamma^{-1} \partial c = \sqrt{\frac{\pi}{2}} \hat{U}_1 \frac{1}{2} \tilde{\gamma}^{-1} \tilde{\partial c}(0) |0\rangle i \sigma_1.
\end{align*}
\]

$K'$, $B$, $c$, $\gamma$, $\gamma^{-1}$, $\zeta$, $V$ and $\hat{Q}'$ have the same algebraic structure as that of the extended $KBc$ algebra with $\hat{Q}$ [Erler(2013)].
From the result in [Erler(2013)] and the above algebra, we can immediately construct a solution $\Phi_T$ in the theory with $\hat{Q}'$:

$$
e^{\Phi_T} = 1 - c \frac{B}{1 + K'} + q \left( \zeta + (\hat{Q}' \zeta) \frac{B}{1 + K'} \right)
$$

($q$ is a nonzero constant.) Actually, $\Phi_T$ satisfies

$$
e^{-\Phi_T} \hat{Q}' e^{\Phi_T} = c - (\hat{Q}' c) \frac{B}{1 + K'} = (c + \hat{Q}'(Bc)) \frac{1}{1 + K'}
$$

which is in the small Hilbert space, and therefore the EOM in the NS sector holds:

$$\hat{\eta}(e^{-\Phi_T} \hat{Q}' e^{\Phi_T}) = 0$$

($\hat{\eta} \equiv \eta_0 \sigma_3$)
Expanding the action $S[\Phi'; \hat{Q}']$ around the solution $\Phi_T$ as $e^{\Phi'} = e^{\Phi_T} e^{\Phi''}$, we have a new BRST operator $\hat{Q}'_{\Phi_T}$:

$$\hat{Q}'_{\Phi_T} \Xi = \hat{Q}' \Xi + (e^{-\Phi_T \hat{Q}' e^{\Phi_T}}) \Xi - (-1)^{|\Xi|} \Xi (e^{-\Phi_T \hat{Q}' e^{\Phi_T}}).$$

Note that $e^{-\Phi_T \hat{Q}' e^{\Phi_T}}$ is the tachyon vacuum solution on the marginally deformed background in the modified cubic SSFT.

→ We can find a homotopy operator $\hat{A}'$ for $\hat{Q}'_{\Phi_T}$:

$$\hat{A}' \Xi = \frac{1}{2} (A' \Xi + (-1)^{|\Xi|} \Xi A'),$$

such as $\{\hat{Q}'_{\Phi_T}, \hat{A}'\} = 1$, $(\hat{A}')^2 = 0$, where $A' \equiv \frac{B}{1 + K'}$ is a homotopy state in the small Hilbert space.

→ No physical open string state around the solution $\Phi_T$

~ the tachyon vacuum solution in the marginally deformed background.
Figure: Identity/wedge-based solutions and BRST operators around them. $\hat{Q}_{\Phi_T}'$ has no cohomology in the small Hilbert space.
Energy and gauge invariant overlaps

NS action $S[\Phi'; \hat{Q}_{\Phi_J}]$ around $\Phi_J$:

$$S[\Phi'; \hat{Q}_{\Phi_J}] = -\int_0^1 dt \; \text{Tr} \left[ (\hat{\eta}(g(t)^{-1}\partial_t g(t))) \left( g(t)^{-1}\hat{Q}_{\Phi_J} g(t) \right) \right].$$

g(t): an interpolating string field s.t. $g(0) = 1$ and $g(1) = e^{\Phi'}$.

We take an interpolating string field as $g_T(t) = 1 + t(e^{\Phi_T} - 1)$ and the integrand in the action for the solution: $S[\Phi_T; \hat{Q}_{\Phi_J}]$, can be manipulated in the same way as the Erler solution, with $Q_B \rightarrow Q' \equiv Q_{\Phi_J}$, $L_n \rightarrow L'_n$. As a result, we have

$$\text{Tr} \left[ (\hat{\eta}(g_T(t)^{-1}\partial_t g_T(t))) \left( g_T(t)^{-1}\hat{Q}_{\Phi_J} g_T(t) \right) \right]$$

$$= -\frac{2q^2 t^2(1-t)(2q^2 t - 1)}{(1-t + q^2 t^2)^3} \int_0^1 d\theta (1 - \theta) \chi(\theta) + \frac{q^2 t(1-t)}{(1-t + q^2 t^2)^2} \int_0^1 d\theta \chi(\theta)$$

where

$$\chi(\theta) = \text{Tr} \left[ B(\hat{\eta}(cV)) e^{-\theta K'} cV e^{-(1-\theta)K'} \right].$$
We use the result for the modified cubic SSFT in the marginally deformed background [IKT(2012)]:

\[
e^{-\alpha K'} = e^{-\alpha \frac{\pi}{2} C \hat{U}_{\alpha+1} T \exp\left(\frac{\pi}{4} \int_{-\alpha}^{\alpha} du \int_{-\infty}^{\infty} dv f_a(v) \tilde{J}^a(i v + \frac{\pi}{4} u)\right)} |0\rangle,
\]

\[
f_a(v) = \frac{F_a(\tan(i v + \frac{\pi}{4}))}{2\pi \sqrt{2 \cos^2(it + \frac{\pi}{4})}}, \quad C = \frac{\pi}{2} \int_{-\infty}^{\infty} dv \Omega^{ab} f_a(v) f_b(v),
\]

where \(T\) is an ordering symbol with respect to the real part of the argument of \(\tilde{J}^a\).

Finally, the trace can be evaluated as

\[
\text{Tr} \left[ B(\hat{\eta}(cV)) e^{-\theta K'} cV e^{-\left(1-\theta\right)K'} \right] = -\frac{1}{4} \left\langle (\eta_0 \gamma^{-1}(\frac{\pi}{2})) \gamma^{-1}(\frac{\pi}{2}(1 - 2\theta)) \right\rangle_{\xi \eta \phi} \left\langle B_1^L c\partial c(\frac{\pi}{2}) c\partial c(\frac{\pi}{2}(1 - 2\theta)) \right\rangle_{bc}
\]

\[
\times e^{-\frac{\pi}{2} C} \left\langle \exp\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} du \int_{-\infty}^{\infty} dv f_a(v) J^a(2iv + u)\right) \right\rangle_{\text{mat}}
\]
The last factor in the trace, which comes from the matter sector, is 1:

\[ e^{-\frac{\pi}{2}C} \left\langle \exp \left( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} du \int_{-\infty}^{\infty} dv f_a(v) J^a(2iv + u) \right) \right\rangle_{\text{mat}} = 1 \]

as proved in [IKT(2012)], and so the trace becomes the same result as the case of the Erler solution.

Consequently, the vacuum energy is unchanged from the case in the original background without marginal deformations, namely,

\[ E = -S[\Phi_T; \hat{Q}_{\Phi_J}] = -1/(2\pi^2). \]
Next, we will evaluate the gauge invariant overlap (GIO) in Berkovits’ WZW-like SSFT. We define the GIO $\langle \Phi \rangle_{\mathcal{V}}$ as

$$\langle \Phi \rangle_{\mathcal{V}} \equiv \text{Tr}[\mathcal{V}(i)\Phi].$$

$\mathcal{V}(i)$: a midpoint insertion of a primary closed string vertex operator with picture #: $-1$, ghost #: 2, conformal dim.: $(0, 0)$, BRST invariant in the small Hilbert space: $[Q_B, \mathcal{V}(i)] = 0$, $[\eta_0, \mathcal{V}(i)] = 0$. Then, $\forall \Lambda, \Xi$

$$\langle \hat{Q} \Lambda \rangle_{\mathcal{V}} = 0, \quad \langle \hat{\eta} \Lambda \rangle_{\mathcal{V}} = 0, \quad \langle \Lambda \Xi \rangle_{\mathcal{V}} = (-)^{|\Lambda||\Xi|} \langle \Xi \Lambda \rangle_{\mathcal{V}}.$$

Infinitesimal gauge transformation: $\delta_{\Lambda} e^\Phi = (\hat{Q} \Lambda_0) e^\Phi + e^\Phi \hat{\eta} \Lambda_1$ ($\Lambda_0, \Lambda_1$: gauge parameter string field with the picture # 0, 1.) Namely,

$$\delta_{\Lambda} \Phi = \frac{\text{ad}_\Phi}{e^{\text{ad}_\Phi} - 1} \hat{Q} \Lambda_0 + \frac{-\text{ad}_\Phi}{e^{-\text{ad}_\Phi} - 1} \hat{\eta} \Lambda_1 = \sum_{n=0}^{\infty} \frac{B_n}{n!} (\text{ad}_\Phi)^n (\hat{Q} \Lambda_0 + (-1)^n \hat{\eta} \Lambda_1)$$

$$(\text{ad}_B(A) \equiv [B, A] = BA - AB)$$
Thanks to the above properties, the GIO is invariant under this gauge transformation: $\langle \delta \Phi \rangle_{\mathcal{V}} = 0$.

Inserting $1 = \{ Q_B, \xi Y(i) \}, (Y(z) = c \partial \xi e^{-2\phi}(z)$ the inverse picture changing operator) the GIO can be rewritten as

$$\langle \Phi \rangle_{\mathcal{V}} = \text{Tr}[\mathcal{V}(i)\{ Q_B, \xi Y(i) \}\Phi] = \text{Tr}[\xi Y \mathcal{V}(i)\sigma_3 \hat{Q}\Phi] = \text{Tr}[\xi Y \mathcal{V}(i)\sigma_3 \hat{Q}'\Phi]$$

$$= \text{Tr}[\xi Y \mathcal{V}(i)\sigma_3 e^{-\Phi} \hat{Q}' e^{\Phi}].$$

GIO for the tachyon vacuum $\Phi_T$: $\langle \Phi_T \rangle_{\mathcal{V}} = \text{Tr} \left[ \xi Y \mathcal{V}(i)\sigma_3 c\frac{1}{1 + K'} \right]$.

In a similar way to the calculation of the vacuum energy, we obtain an expression of the GIO in the marginally deformed background:

$$\langle \Phi_T \rangle_{\mathcal{V}} = \frac{e^{-\pi C}}{\pi} \left\langle \xi Y \mathcal{V}(i\infty) c(\frac{\pi}{2}) \exp \left( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} du \mathcal{J}(u) \right) \right\rangle_{C_{\pi}}$$

$$\mathcal{J}(u) = \int_{-\infty}^{\infty} dv f_a(v) J^a(iv + u)$$
Contents

1 Introduction and summary

2 Tachyon vacuum around the identity-based marginal solution
   • Identity-based marginal solutions
   • Deformed algebra
   • Tachyon vacuum solution
   • Energy and gauge invariant overlaps

3 Evaluation of observables for identity-based marginal solutions
   • Interpolation
   • On gauge equivalence relations
   • Energy and gauge invariant overlaps

4 Conclusions
Let us calculate two observables, the vacuum energy and the GIO, for the identity-based marginal solutions.

An interpolation: $\Phi_J(t) = t\Phi_J$ s.t. $\Phi_J(0) = 0$, $\Phi_J(1) = \Phi_J$. $\Phi_J(t)$: a replacement of weighting function: $F_a(z) \rightarrow tF_a(z)$ in $\Phi_J$. 

EOM is satisfied: $\hat{\eta}(e^{-\Phi_J(t)}\hat{Q}e^{\Phi_J(t)}) = 0$

A new BRST operator $Q_{\Phi_J(t)}$ for the theory around $\Phi_J(t)$:

$$Q_{\Phi_J(t)} = Q_B - tV^a(F_a) + \frac{t^2}{8}\Omega^{ab}C'(F_aF_b)$$

Following the same procedure as before, we define a string field $K'(t) \equiv \hat{Q}_{\Phi_J(t)}B$ ($\hat{Q}_{\Phi_J(t)} \equiv Q_{\Phi_J(t)}\sigma_3$). Then, we can construct a tachyon vacuum solution $\Phi_T(t)$ as

$$e^{\Phi_T(t)} = 1 - c\frac{B}{1 + K'(t)} + q\left(\zeta + (\hat{Q}_{\Phi_J(t)}\zeta)\frac{B}{1 + K'(t)}\right)$$
It satisfies the EOM around the identity-based solution $\Phi_J(t)$:

$$\hat{\eta}(e^{-\Phi_T(t)} \hat{Q} \Phi_J(t) e^{\Phi_T(t)}) = 0.$$ 

In particular, $\Phi_T(t)$ satisfies $\Phi_T(1) = \Phi_T$ and $\Phi_T(0) = \Phi_T^E$ (the Erler solution (2013)) because $Q\Phi_J(1) = Q\Phi_J$ and $Q\Phi_J(0) = Q_B$.

Using the above string fields, we define a string field $\tilde{\Phi}_T(t)$ with the parameter $t$ as $e^{\tilde{\Phi}_T(t)} \equiv e^{\Phi_J(t)} e^{\Phi_T(t)}$ and then we find a relation:

$$e^{-\tilde{\Phi}_T(t)} \hat{Q} e^{\tilde{\Phi}_T(t)} = e^{-\Phi_J(t)} \hat{Q} e^{\Phi_J(t)} + e^{-\Phi_T(t)} \hat{Q} \Phi_J(t) e^{\Phi_T(t)}.$$

Hence, $\tilde{\Phi}_T(t)$ satisfies the EOM of the original theory:

$$\hat{\eta}(e^{-\tilde{\Phi}_T(t)} \hat{Q} e^{\tilde{\Phi}_T(t)}) = 0.$$
Expanding around the solution $\tilde{\Phi}_T(t)$ in the theory with $\hat{Q}$, we have the theory with the deformed BRST operator $\hat{Q} \tilde{\Phi}_T(t)$:

$$\hat{Q} \tilde{\Phi}_T(t) \Xi = \hat{Q} \Xi + (e^{-\tilde{\Phi}_T(t)} \hat{Q} e^{\tilde{\Phi}_T(t)}) \Xi - (-1)^{|\Xi|} \Xi (e^{-\tilde{\Phi}_T(t)} \hat{Q} e^{\tilde{\Phi}_T(t)})$$

$$= \hat{Q} \Phi_J(t) \Xi + (e^{-\Phi_T(t)} \hat{Q} \Phi_J(t) e^{\Phi_T(t)}) \Xi - (-1)^{|\Xi|} \Xi (e^{-\Phi_T(t)} \hat{Q} \Phi_J(t) e^{\Phi_T(t)})$$.

The last expression implies that $\hat{Q} \tilde{\Phi}_T(t)$ is the same as the BRST operator $\hat{Q}' \Phi_T(t)$ in the theory around $\tilde{\Phi}_T(t)$, which is a tachyon vacuum solution in the theory around $\Phi_J(t)$.

Following the previous results with appropriate replacement, we find that there exists a homotopy state: $A'(t) \equiv \frac{B}{1+K'(t)}$ such as

$$\hat{Q} \tilde{\Phi}_T(t) A'(t) = 1$$, which implies that

there is no cohomology for $\hat{Q} \tilde{\Phi}_T(t)$ in the small Hilbert space.
Figure: Interpolating identity/wedge-based solutions and BRST operators around them. With $e^{\tilde{\Phi}_T(t)} \equiv e^{\Phi_J(t)} e^{\Phi_T(t)}$, a BRST operator around $\tilde{\Phi}_T(t)$, $\hat{Q}'_{\Phi_T(t)} = \hat{Q}_{\tilde{\Phi}_T(t)}$ has no cohomology in the small Hilbert space.
Differentiating an identity: 
\[ \hat{Q}(e^{-\tilde{\Phi}_T(t)}\hat{Q}e^{\tilde{\Phi}_T(t)}) + (e^{-\tilde{\Phi}_T(t)}\hat{Q}e^{\tilde{\Phi}_T(t)})^2 = 0 \]
with respect to \( t \), we have

\[ \hat{Q} \tilde{\Phi}_T(t) \frac{d}{dt} (e^{-\tilde{\Phi}_T(t)}\hat{Q}e^{\tilde{\Phi}_T(t)}) = 0. \]

Therefore, there exists a state \( \Lambda_t \) in the small Hilbert space such as

\[ \frac{d}{dt} (e^{-\tilde{\Phi}_T(t)}\hat{Q}e^{\tilde{\Phi}_T(t)}) = \hat{Q} \tilde{\Phi}_T(t) \Lambda_t. \]

Integrating the above, we have

\[ e^{-\tilde{\Phi}_T(1)}\hat{Q}e^{\tilde{\Phi}_T(1)} = e^{-\Phi^E_T} \hat{Q}e^{\Phi^E_T} + \int_0^1 \hat{Q} \tilde{\Phi}_T(t) \Lambda_t dt. \]

This relation implies that \( \tilde{\Phi}_T(1) = \log(e^{\Phi_J}e^{\Phi_T}) \) is gauge equivalent to the Erler solution \( \Phi^E_T \).
On gauge equivalence relations

Here, we discuss some gauge equivalence relations in terms of the NS sector of Berkovits’ WZW-like SSFT.

A gauge transformation of the superstring field $\Phi$ by group elements $h(t)$ and $g(t)$ with one parameter $t$ s.t. $g(0) = h(0) = 1$ is

$$e^{\Phi(t)} = h(t) e^\Phi g(t), \quad Q_B h(t) = \eta_0 g(t) = 0. \quad (1)$$

For the string fields, $\Phi$ and $\Phi(t)$, “one-form” string fields are defined by $\Psi \equiv e^{-\Phi} Q_B e^\Phi$ and $\Psi(t) \equiv e^{-\Phi(t)} Q_B e^{\Phi(t)}$. From (1), these turn out to be related by a transformation:

$$\Psi(t) = g(t)^{-1} Q_B g(t) + g(t)^{-1} \Psi g(t), \quad \eta_0 g(t) = 0. \quad (2)$$

It is the same form as gauge transformations in the modified cubic SSFT.

Conversely, given the relation (2), we find that the relation (1) holds for $h(t) = e^{\Phi(t)} g(t)^{-1} e^{-\Phi}$. In fact, $Q_B (e^{\Phi(t)} g(t)^{-1} e^{-\Phi}) = 0$ holds from (2).
Differentiating (2) with respect to $t$ and integrating it again, we find another relation between $\Psi$ and $\Psi(t)$:

$$\Psi(t) = \Psi + \int_0^t Q_{\Phi(t')} \Lambda(t') \, dt', \quad \eta_0 \Lambda(t) = 0, \quad (3)$$

where $\Lambda(t) = g(t)^{-1} \frac{d}{dt} g(t)$ and $Q_{\phi}$ is a modified BRST operator associated with $\psi \equiv e^{-\phi} Q_B e^{\phi}$: $Q_{\phi} \lambda = Q_B \lambda + \psi \lambda - (-1)^{|\lambda|} \psi \lambda$.

Conversely, supposing that the equations (3) for a given $\Lambda(t)$ hold, we find the relations (2) hold for the group element $g(t)$ such as $g(0) = 1$:

$$g(t) = P \exp \left( \int_0^t \Lambda(t') \, dt' \right),$$

where $P \exp$ means a $t$-ordered exponent.

Consequently, the above relations (1), (2) and (3) are all equivalent. (We can include internal Chan-Paton factors in these relations.)
Energy and gauge invariant overlaps

From the gauge equivalence:

$$\log(e^{\Phi_J} e^{\Phi_T}) \sim \Phi^E_T$$

we can analytically evaluate gauge invariants for the identity-based marginal solutions.

The value of the action:

$$S[\Phi_J; \hat{Q}] + S[\Phi_T; \hat{Q}_{\Phi_J}] = S[\Phi^E_T; \hat{Q}]$$

$$\therefore \quad E = -S[\Phi_J; \hat{Q}] = S[\Phi_T; \hat{Q}_{\Phi_J}] - S[\Phi^E_T; \hat{Q}] = 0$$

The result agrees with the previous one derived from $\xi$ zeromode counting:

$$S[\Phi_J, Q_B] = -\int_0^1 dt \ Tr[(\eta_0 \Phi_J) (e^{-t\Phi_J} Q_B e^{t\Phi_J})] = 0$$
Evaluation of the GIO for the identity-based marginal solution $\langle \Phi_J \rangle_V$:

Using an identity,

$$e^{-\hat{\Phi}_T(1)} \hat{Q} e^{\hat{\Phi}_T(1)} = e^{-\Phi_J} \hat{Q} e^{\Phi_J} + e^{-\Phi_T} \hat{Q} \Phi_J e^{\Phi_T},$$

we have obtained

$$e^{-\Phi_J} \hat{Q} e^{\Phi_J} = e^{-\Phi_T^E} \hat{Q} e^{\Phi_T^E} - e^{-\Phi_T} \hat{Q} \Phi_J e^{\Phi_T} + \int_0^1 \hat{Q} \tilde{\Phi}_T(t) \Lambda_t dt.$$ 

It leads to a relation for the GIOs:

$$\langle \Phi_J \rangle_V = \langle \Phi_T^E \rangle_V - \langle \Phi_T \rangle_V$$

$$= \frac{1}{\pi} \left\langle \xi Y \nu(i\infty) e^{\frac{\pi}{2}} \left\{ 1 - e^{-\pi c} \exp \left( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} du \mathcal{J}(u) \right) \right\} \right\rangle_{C_{\pi}}.$$
In the same way as the case of bosonic SFT [KT(2013)], it can be rewritten as a difference between two disk amplitudes with the boundary deformation by taking

\[ F_a(z; s) = \frac{2\lambda_a s(1 - s^2)}{\arctan \frac{2s}{1-s^2}} \frac{1 + z^{-2}}{1 - s^2(z^2 + z^{-2}) + s^4} \]

for the function \( F_a(z) \) in \( \Phi_J \), which satisfies

\[ \int_{C_L} \frac{dz}{2\pi i} F_a(z; s) = \frac{2\lambda_a}{\pi}, \]

\[ F_a(z; s) \to 4\lambda_a \{ \delta(\theta) + \delta(\pi - \theta) \}, \quad (s \to 1, \quad z = e^{i\theta}). \]

Because the form of weighting function \( F_a \) except the half-integration mode can be changed by a kind of gauge transformation [KT(2005)], we have the same value of the GIO for a fixed value of \( \lambda_a \), which corresponds to a marginal deformation parameter).
Namely, for the limit $s \to 1$, the GIO is expressed as

$$\langle \Phi_J \rangle_V = \frac{1}{\pi} \left\langle \xi Y V(i\infty)c(\frac{\pi}{2}) \left\{ 1 - e^{-\pi C} \exp\left( \frac{\sqrt{2}}{\pi} \lambda_a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} du J^a(u) \right) \right\} \right\rangle_{C \pi} .$$

In this expression, $C$ becomes divergent for the function $\lim_{s \to 1} F_a(z; s)$ and then it cancels the contact term divergence due to singular OPE among the currents.

This expression of the GIO corresponds to the result in [Ellwood(2008)] for a wedge-based marginal solution.
Conclusions

We have applied the method in [IKT(2012)] for cubic (S)SFT to the Erler solution $\Phi^E_T$ for Berkovits’ WZW-like SSFT.

- We have constructed a tachyon vacuum solution $\Phi_T$ around the identity-based marginal solution $\Phi_J$ [KT(2005)] in SSFT with an extended $KBc$ algebra in the marginally deformed background.

- Around $\Phi_T$, we have obtained a homotopy operator and evaluated vacuum energy and gauge invariant overlap (GIO) for it. The energy is the same value as that on the original background, but the GIO is deformed by the marginal operators.
Conclusions

Using the above, we have extended our computation for bosonic SFT [KT(2013)] to superstring: We have evaluated the energy and the GIO for the identity-based marginal solution $\Phi_J$ in the framework of Berkovits’ WZW-like SSFT.

- The gauge equivalence relation between $\Phi_T^E$ and $\log(e^{\Phi_J} e^{\Phi_T})$ is essential. It is derived from the vanishing cohomology in the small Hilbert space around the interpolating tachyon vacuum solutions.

- The energy for $\Phi_J$ vanishes and it is consistent with our previous result using $\xi$ zeromode counting.

- The GIO for $\Phi_J$ is expressed by a difference of those of the tachyon vacuum solutions on the undeformed and deformed backgrounds.

We hope that this approach to identity-based solutions will be useful to deeply understand bosonic and super SFT.