

# ON $p$ -ADIC STRING THEORY

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# 1. Introduction: $p$ -adic mathematical physics

- This talk is related to  $p$ -adic strings
- Application of  $p$ -adic numbers in modeling some physical systems started in 1987 by construction of  $p$ -adic scattering string amplitudes (Volovich, ...)
- $p$ -Adic mathematical physics


# 1. Introduction: applications in various parts of modern mathematical physics and related fields

- $p$ -adic and adelic string theory
- $p$ -adic and adelic quantum mechanics and (quantum) field theory
- $p$ -adic and adelic gravity and (quantum) cosmology
- $p$ -adic and adelic space-time structure at the Planck scale
- $p$ -adic and adelic dynamical systems
- $p$ -adic stochastic processes
- $p$ -adic aspects of information theory
- $p$ -adic structure of the genetic code
- $p$ -adic wavelets
- spin glasses,  $p$ -adic unordered systems, ...

# 1. Introduction: international interdisciplinary journal on $p$ -adics


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# 1. Introduction: international conferences on $p$ -adic mathematical physics



2<sup>nd</sup> International Conference on  $p$ -Adic Mathematical Physics  
Belgrade, 2005

## 2. $p$ -Adic numbers, adeles and their functions

- $p$ -Adic numbers discovered by Kurt Hensel (1861-1941) in 1897.
- Any  $p$ -adic number ( $x \in \mathbb{Q}_p$ ) has a unique canonical representation, ( $p = 2, 3, 5, 7, 11, \dots =$  a prime number)

$$x = p^\nu \sum_{n=0}^{+\infty} x_n p^n, \quad \nu \in \mathbb{Z}, \quad x_n \in \{0, 1, \dots, p-1\}$$

$$-1 = 2 + 2 \cdot 3 + 2 \cdot 3^2 + \dots + 2 \cdot 3^n + \dots \quad (p = 3)$$



## 2. $p$ -Adic numbers, adeles and their functions: some topological properties of $p$ -adic numbers

- $p$ -adic distances  $d_p(x, y) = |x - y|_p$  are non-Archimedean (ultrametric):  $d_p(x, y) \leq \max\{d_p(x, z), d_p(z, y)\}$
- open ball of radius  $r$  and centre  $a$ :

$$B_a(r^-) = \{x \in \mathbb{Q}_p : |x - a|_p < r\}$$

- closed ball of radius  $r$  and centre  $a$ :

$$B_a(r) = \{x \in \mathbb{Q}_p : |x - a|_p \leq r\}$$

- balls are simultaneously closed and open



All triangles are isosceles.



There is no partial intersection of balls.

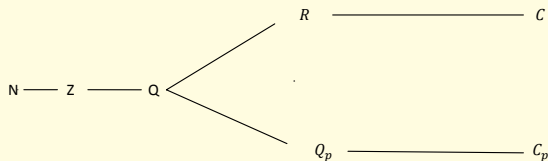


Any point of a ball is its center.



## 2. $p$ -Adic numbers, adeles and their functions

- Ostrowski theorem: Any nontrivial norm on  $\mathbb{Q}$  is equivalent to usual absolute value or to  $p$ -adic norm, where  $p$  is any prime number ( $p = 2, 3, 5, 7, 11, \dots$ ).
- $p$ -adic norm of  $x \in \mathbb{Q}$ :  $|x|_p = |p^\nu \frac{a}{b}|_p = p^{-\nu}$ ,  $\nu \in \mathbb{Z}$ .
- Completion of  $\mathbb{Q}$  with respect to  $p$ -adic distance gives the field  $\mathbb{Q}_p$  of  $p$ -adic numbers, in analogous way to construction of the field  $\mathbb{R}$  of real numbers.
- $\mathbb{Q}$  is dense in  $\mathbb{Q}_p$  and  $\mathbb{R}$ .
- $\mathbb{Q}_p$  is totally disconnected space.



## 2. $p$ -Adic numbers, adeles and their functions

- There are mainly two kinds of analysis of  $p$ -adic variable:
  - (i)  $p$ -adic valued functions of  $p$ -adic variable
  - (ii) complex (real) valued functions of  $p$ -adic variable.
- Results of measurements are (rational) subset of real numbers –  $p$ -adic number cannot be directly measured
- Analysis of complex (real) valued functions of  $p$ -adic (and real) variables is necessary to connect models with measurements.

## 2. $p$ -Adic numbers, adeles and their functions

- Real and  $p$ -adic numbers are unified by adeles. An **adele**  $\alpha$  is an infinite sequence

$$\alpha = (\alpha_\infty, \alpha_2, \alpha_3, \dots, \alpha_p, \dots), \quad \alpha_\infty \in \mathbb{R}, \quad \alpha_p \in \mathbb{Q}_p$$

where for all but a finite set  $\mathcal{P}$  of primes  $p$  one has that  $\alpha_p \in \mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ , i.e.  $p$ -adic integers.

- Space of adeles

$$\mathbb{A} = \bigcup_{\mathcal{P}} A(\mathcal{P}), \quad A(\mathcal{P}) = \mathbb{R} \times \prod_{p \in \mathcal{P}} \mathbb{Q}_p \times \prod_{p \notin \mathcal{P}} \mathbb{Z}_p.$$



C. Chevalley (1909 - 1984)



A. Weil (1906 - 1998)

## 2. $p$ -Adic numbers, adeles and their functions

Some connections of  $p$ -adic and real norms of rational numbers

$$|x|_{\infty}^c \times \prod_{p \in \mathbb{P}} |x|_p^c = 1, \text{ if } x \in \mathbb{Q}^{\times}$$

$$\chi_{\infty}(x) \times \prod_{p \in \mathbb{P}} \chi_p(x) = 1, \text{ if } x \in \mathbb{Q}$$

$$\chi_{\infty}(x) = \exp(-2\pi i x), \quad \chi_p(x) = \exp 2\pi i \{x\}_p$$

$$\prod_{p \in \mathbb{P}} \Omega(|x|_p) = \begin{cases} 1, & x \in \mathbb{Z}, \\ 0, & x \in \mathbb{Q} \setminus \mathbb{Z} \end{cases}$$

$$\Omega(|x|_p) = \begin{cases} 1, & |x|_p \leq 1, \\ 0, & |x|_p > 1 \end{cases}$$

## 2. $p$ -Adic numbers, adeles and their functions

- Numerical result of any measurement is a (real) rational number with some uncertainty (error): i.e.  $x = \bar{x} \pm \Delta x$
- Numerical result of the corresponding theoretical model has to be equal to this rational number from the measurement.
- Note that  $p$ -adic number is not result of measurement!
- Then the question: How  $p$ -adic numbers and  $p$ -adic analysis can be related to applications in physics and sciences?

## 2. $p$ -Adic numbers, adeles and their functions

- $p$ -Adic side of some phenomena emerges at a deeper level.
- Recall standard theoretical models

*classical models* :  $\mathbb{R} \rightarrow \mathbb{R}$

*quantum models* :  $\mathbb{R} \rightarrow \mathbb{C} \rightarrow \mathbb{R}$

- $p$ -Adic applications

*classical models*(*genetic code, ...*) :  $\mathbb{Q}_p \rightarrow \mathbb{R}$

*quantum models*(*string theory, ...*) :  $\mathbb{Q}_p \rightarrow \mathbb{C} \rightarrow \mathbb{R}$

### 3. $p$ -Adic strings

Volovich, Freund, Witten, Vladimirov, Arefeva, B. D., ...

String amplitudes (1987):

- standard crossing symmetric Veneziano amplitude

$$\begin{aligned} A_\infty(a, b) &= g_\infty^2 \int_{\mathbb{R}} |x|_\infty^{a-1} |1-x|_\infty^{b-1} d_\infty x \\ &= g_\infty^2 \frac{\zeta(1-a)}{\zeta(a)} \frac{\zeta(1-b)}{\zeta(b)} \frac{\zeta(1-c)}{\zeta(c)} \end{aligned}$$

- $p$ -adic crossing symmetric Veneziano amplitude

$$\begin{aligned} A_p(a, b) &= g_p^2 \int_{\mathbb{Q}_p} |x|_p^{a-1} |1-x|_p^{b-1} d_p x \\ &= g_p^2 \frac{1-p^{a-1}}{1-p^{-a}} \frac{1-p^{b-1}}{1-p^{-b}} \frac{1-p^{c-1}}{1-p^{-c}} \end{aligned}$$

where  $a = -s/2 - 1$  and  $a, b, c \in \mathbb{C}$  and  $a + b + c = 1$ .

### 3. $p$ -Adic strings

- $p$ -Adic strings are strings which world-sheet is  $p$ -adic.
- Freund-Witten (adelic) product formula for ordinary and  $p$ -adic strings

$$A(a, b) = A_\infty(a, b) \prod_p A_p(a, b) = g_\infty^2 \prod_p g_p^2 = \text{const.}$$

- Ordinary and  $p$ -adic strings are at the equal footing
- Amplitude for real string  $A_\infty(a, b)$ , which is a special function, can be presented as product of inverse  $p$ -adic amplitudes, which are elementary functions.



### 3. $p$ -Adic strings – Effective Lagrangian

$\geq 1988$  : Freund, Witten, Frampton, Okada, ...,  $\geq 2000$  :  
Vladimirov, Volovich, Sen, Ghoshal, Zwiebach, Moeller,  
Schnabl, Barnaby, Kamran, Minahan, Yang, Biswas, Arefeva,  
Koshelev, Vernov, Joukovskaya, B.D., ...

- There is an effective field description of scalar open and closed  $p$ -adic strings. The corresponding Lagrangians are very simple and exact. They describe not only four-point scattering amplitudes but also all higher ones at the tree-level.
- The exact tree-level Lagrangian for effective scalar field  $\varphi$  which describes open  $p$ -adic string tachyon is

$$\mathcal{L}_p = \frac{m_p^D}{g_p^2} \frac{p^2}{p-1} \left[ -\frac{1}{2} \varphi p^{-\frac{\square}{2m_p^2}} \varphi + \frac{1}{p+1} \varphi^{p+1} \right]$$

where  $\square = -\partial_t^2 + \nabla^2$  is the  $D$ -dimensional d'Alembertian and metric with signature  $(- + \dots +)$ .

### 3. $p$ -Adic strings

$$\mathcal{L}_p = \frac{m_p^D}{g_p^2} \frac{p^2}{p-1} \left[ -\frac{1}{2} \varphi p^{-\frac{\square}{2m_p^2}} \varphi + \frac{1}{p+1} \varphi^{p+1} \right]$$

- Lagrangian with  $p$ -adic world-sheet B.D.

$$\mathcal{L}_p = \frac{m^D}{g^2} \frac{p^2}{p-1} \left[ \frac{1}{2} \varphi \int_{\mathbb{R}} \left( \int_{\mathbb{Q}_p \setminus \mathbb{Z}_p} \chi_p(u) |u|_p^{\frac{k^2}{2m^2}} du \right) \tilde{\varphi}(k) \chi_\infty(kx) d^4 k \right. \\ \left. + \frac{1}{p+1} \varphi^{p+1} \right]$$

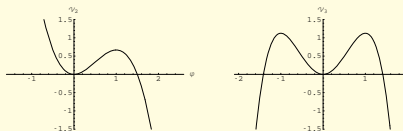
$$\int_{|x|_p > 1} \chi_p(u) |u|_p^s du = -p^s$$

- The corresponding potential  $\mathcal{V}(\varphi)$  for this Lagrangian is  $\mathcal{V}_p(\varphi) = -\mathcal{L}_p(\square = 0)$ , which the explicit form is

$$\mathcal{V}_p(\varphi) = \frac{m^D}{g^2} \left[ \frac{1}{2} \frac{p^2}{p-1} \varphi^2 - \frac{p^2}{p^2-1} \varphi^{p+1} \right].$$

It has local minimum  $\mathcal{V}_p(0) = 0$ . If  $p \neq 2$  there are two local maxima at  $\varphi = \pm 1$  and there is one local maximum  $\varphi = +1$  when  $p = 2$ .

- The 2-adic string potential  $\mathcal{V}_2(\varphi)$  (on the left) and 3-adic potential  $\mathcal{V}_3(\varphi)$  (on the right) of the figure.



### 3. $p$ -Adic strings – equation of motion

- The equation of motion is

$$p^{-\frac{\square}{2m_p^2}} \varphi = \varphi^p$$

- It has trivial solutions  $\varphi = 0$  and  $\varphi = 1$ , and  $\varphi = -1$  for  $p \neq 2$ .
- There are also solutions in any direction  $x^i$  and time  $t$

$$\varphi(x^i) = p^{\frac{1}{2(p-1)}} \exp\left(-\frac{p-1}{2p \ln p} m^2 (x^i)^2\right)$$

$$\varphi(t) = p^{\frac{1}{2(p-1)}} \exp\left(\frac{p-1}{2p \ln p} m^2 t^2\right)$$

### 3. $p$ -Adic strings

- There is also  $D$ -dimensional solution

$$\varphi(x) = p^{\frac{D}{2(p-1)}} \exp\left(-\frac{p-1}{2p \ln p} m^2 x^2\right), \quad x^2 = -t^2 + \sum_{i=1}^{D-1} x_i^2$$

- The above solutions can be obtained using identity

$$e^{A \partial_t^2} e^{B t^2} = \frac{1}{\sqrt{1-4AB}} e^{\frac{B t^2}{1-4AB}}, \quad 1-4AB > 0.$$

- $p$ -Adic realization of tachyon condensation

### 3. $p$ -Adic strings – Lagrangian for $p$ -adic string sector

B.D.

Consider

$$L = \sum_{n=1}^{+\infty} C_n \mathcal{L}_n = m^D \sum_{n=1}^{+\infty} \frac{C_n}{g_n^2} \frac{n^2}{n-1} \left[ -\frac{1}{2} \phi n^{-\frac{D}{2m^2}} \phi + \frac{1}{n+1} \phi^{n+1} \right]$$

where  $\frac{C_n}{g_n^2} \frac{n^2}{n-1} = (-1)^{n-1}$ . Take into account

$$\sum_{n=1}^{+\infty} (-1)^{n-1} \frac{1}{n^s} = (1 - 2^{1-s}) \zeta(s), \quad s = \sigma + i\tau, \quad \sigma > 0$$

which has analytic continuation to the entire complex  $s$  plane without singularities.

### 3. $p$ -Adic strings – Lagrangian for $p$ -adic string sector

- The corresponding Lagrangian is

$$L = m^D \left[ -\frac{1}{2} \phi \left( 1 - 2^{1 - \frac{\square}{2m^2}} \right) \zeta \left( \frac{\square}{2m^2} \right) \phi + \phi - \frac{1}{2} \log(1 + \phi)^2 \right].$$

- The potential is

$$V(\phi) = -L(\square = 0) = m^D \left[ \frac{1}{4} \phi^2 - \phi + \frac{1}{2} \log(1 + \phi)^2 \right],$$

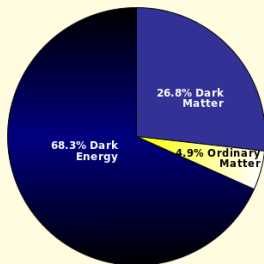
which has one local maximum  $V(0) = 0$  and one local minimum at  $\phi = 1$ . It is singular at  $\phi = -1$ .

- The equation of motion is

$$\left( 1 - 2^{1 - \frac{\square}{2m^2}} \right) \zeta \left( \frac{\square}{2m^2} \right) \phi = \frac{\phi}{1 + \phi}, \quad \phi = 0, \quad \phi = 1.$$

## 4. Possible gravitational and cosmological aspects

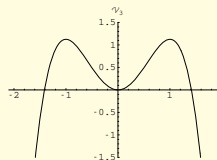
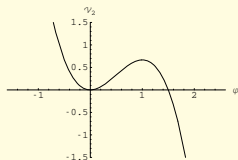
- If Einstein general theory of relativity is theory of gravity for the Universe as a whole, then there is only about 5% of ordinary matter, about 27% of dark matter and about 68% of dark energy.
- Conjecture: Dark matter and dark energy have  $p$ -adic origin.
- Modified gravity may be nonlocal.





## 4. Possible gravitational and cosmological aspects

- $p$ -Adic inflation



## 5. Concluding remarks

- $p$ -Adic numbers are not results of measurements, but can be useful in physical models.
- $p$ -Adic strings are strings with world-sheet labeled by  $p$ -adic numbers.
- If there exist ordinary strings then there should exist also  $p$ -adic strings.
- $p$ -Adic string theory is simpler than ordinary string theory, and seems to be useful for ordinary string theory.
- $p$ -Adic strings may play significant role in description of dark side of the universe.

## 5. Concluding remarks – some review references

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