PhD Entrance Examination
Physics and Chemistry of Biological Systems
June 2014

Solve **one** of the following problems (no extra credit is given for attempts to solve more than one problem). Write out solutions clearly and concisely. State each approximation used. Diagrams welcome. Number page, problem, and question clearly. All essays/solutions should be written in English. Do not write your name on the problem sheet, but use extra envelope.

Problem n. 1 – One-dimensional random-walk

Consider a random-walker on the one-dimensional lattice with equi-spaced sites located at coordinates $x = \{-\infty, ..., -2, -1, 0, +1, +2, ..., \infty\}$. The walker starts at $x = 0$, and then jumps to $x = +1$ or $x = -1$ with equal probability $p = 1/2$ (*unbiased* random-walk). It then performs a new jump to one of its neighbours and so on. Consider one such walker moving by N jumps.

1. Derive the analytical expression for the probability distribution $\pi_N(N_+)$ that the walker has moved N_+ sites along the positive direction. *Hint:* It might help to calculate first $\pi_N(N_+)$ for specific values of N.

2. By defining the *total* displacement $D \equiv N_+ - N_-$ of the walker along the lattice (N_- being the total number of sites the walker has moved along the negative direction) use $\pi_N(N_+)$ to calculate explicitly the average momenta $\langle D \rangle$ and $\langle D^2 \rangle$.

3. From $\pi_N(N_+)$, derive the probability distribution $\phi_N(D)$ for the variable D. By using the Stirling's approximation $\log N! \approx \log \sqrt{2\pi N} + N \log N - N$, calculate the corresponding approximate expression for large N.

4. Assume now that the walker is *biased* to move along the positive direction, namely it jumps to the positive-neighbour site with probability $p > 1/2$ and to the negative-neighbour one with probability q, with $p + q = 1$. Show how to modify $\pi_N(N_+)$ in order to correctly describe the process.

5. Again, calculate the corresponding momenta $\langle D \rangle$ and $\langle D^2 \rangle$. In particular, discuss briefly the small- and large-N behaviours of $\langle D^2 \rangle$.

Problem n. 2 – Tristable molecule

A molecule can exist in three metastable isomers, namely A, B, and C.
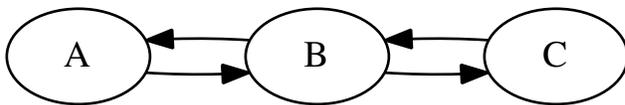
The interconversion rates are:

- $k_{A \to B} = 0.05 \text{ s}^{-1}$

- $k_{B \to A} = 100 \text{ s}^{-1}$

- $k_{B \to C} = 100 \text{ s}^{-1}$

- $k_{C \to B} = 0.01 \text{ s}^{-1}$

Direct interconversion from A to C and viceversa is forbidden. Assuming the system is at equilibrium, answer to the following questions:

1. Which is the most stable among the 3 isomers?

2. Find the probability to observe each of the 3 isomers at equilibrium.

Now consider the case where the system is initialized in isomer A at time $t = 0$ s:

3. Which is the most likely observed isomer at time $t = 10^{-4}$ s, $t = 1$ s, and $t = 10^4$ s, respectively?

4. Repeat point 3 assuming that the system is initialized in isomer B.

5. Repeat point 3 assuming that the system is initialized in isomer C.

Now assume that isomer C converts irreversibly to product D with rate $k_{C \to D} = 10^4 \text{ s}^{-1}$.
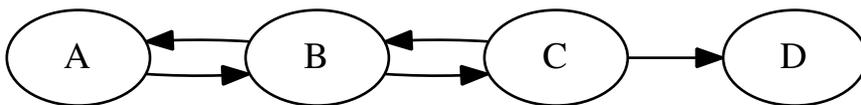
Similarly to the previous points, also assume that the molecule is initialized in isomer A at time $t = 0$ s.

6. In the limit of large t, discuss how the probability of observing the molecule in isomer A decreases with time. In the same limit, discuss how the probability of observing the molecule in product D increases with time.

7. Find the exact time-dependence of the probability of observing the molecule in isomer A, for any time.

Problem n. 3 – A model protein on a square lattice

Consider a model protein composed by 4 residues on a square lattice. The protein is assumed to be fully flexible and behaving as a self avoiding walk, namely each site can be occupied only by one residue at a time. The primary sequence of the protein is ABBA. Two residues interact only when they occupy neighbouring sites and the interaction energies are $\varepsilon_{AA} = -1$, $\varepsilon_{AB} = 0$ and $\varepsilon_{BB} = 2$.
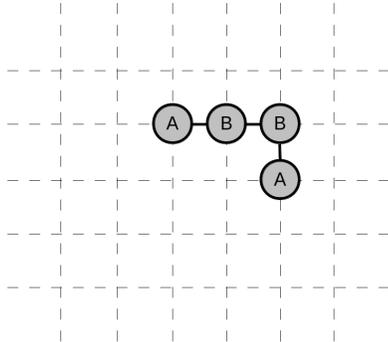
1. Compute the average energy of the system as a function of the temperature T

2. Compute the average end-to-end distance d as a function of T. Draw a graph of the result. What is the value of d for $T \to 0$ and for $T \to \infty$?

3. Repeat point 1 and 2 for a protein of primary sequence ABBB

Assume now that a protein of primary sequence BABB has to diffuse through the gate separating two chambers depicted in figure
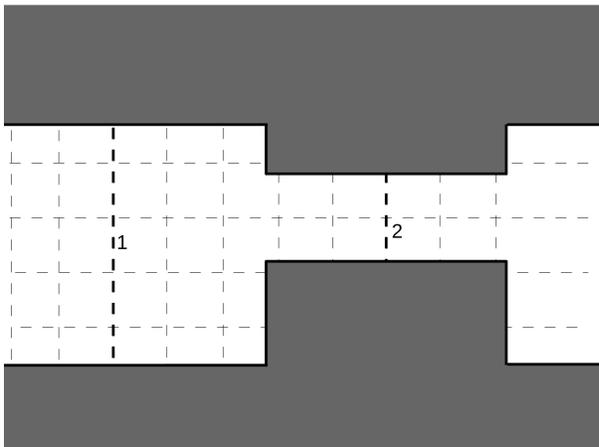
4. Compute, as a function of T, the ratio of the probabilities of observing the residue "A" on the grid lines labeled by 1 and 2. Draw a graph of the result, and discuss the $T \to 0$ and $T \to \infty$ limits. Would you expect to observe a similar behaviour for a real protein that has to translocate through a nonochannel? Why?

Problem 4. Orbital correlation in pericyclic reactions

Cycloaddition reactions between two ethene molecules (2+2 addition) and or between one ethane and one cis-butadiene molecule (Diels-Alder, 4+2 addition) lead to the formation of cyclic molecules.

According to the Hueckel molecular orbital theory calculate the energy and the shape of the molecular orbital of the ethane and cisbutadiene and determine which orbitals are occupied.

Considering the symmetry elements of reagents' and product's orbitals scketch the molecular orbitals diagram of reagents and products and sketch a state correlation diagram. Dicscuss which are the resulting products and which orbitals react to form them.

One of the two reactions can readily occurs, while the other is more difficult. On the basis of the correlation diagram can you qualitatively explain why one of the reactions is difficult to occur? How is it possible to promote this reaction?

Character Tables

Character table for D_{2h} point group

	E	C_2 (z)	C_2 (y)	C_2 (x)	i	σ (xy)	σ (xz)	σ (yz)	linear, rotations	quadratic
A_g	1	1	1	1	1	1	1	1		x^2, y^2, z^2
B_{1g}	1	1	-1	-1	1	1	-1	-1	R_z	xy
B_{2g}	1	-1	1	-1	1	-1	1	-1	R_y	xz
B_{3g}	1	-1	-1	1	1	-1	-1	1	R_x	yz
A_u	1	1	1	1	-1	-1	-1	-1		
B_{1u}	1	1	-1	-1	-1	-1	1	1	z	
B_{2u}	1	-1	1	-1	-1	1	-1	1	y	
B_{3u}	1	-1	-1	1	-1	1	1	-1	x	

Character table for C_{2v} point group

	E	C_2 (z)	$σ_v$(xz)	$σ_v$(yz)	linear, rotations	quadratic
A_1	1	1	1	1	z	x^2, y^2, z^2
A_2	1	1	-1	-1	R_z	xy
B_1	1	-1	1	-1	x, R_y	xz
B_2	1	-1	-1	1	y, R_x	yz