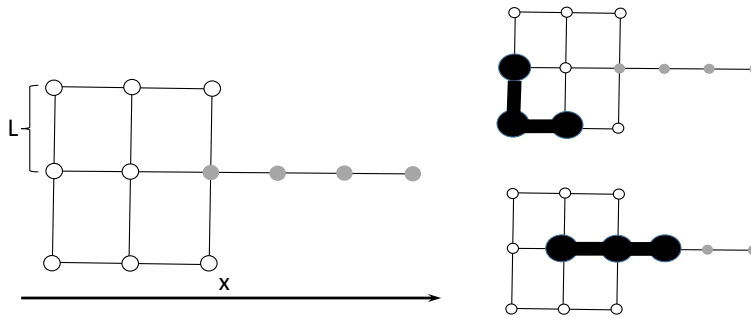# Entrance Examination
Physics and Chemistry of Biological Systems
April 2019

Solve **one** of the following problems (no extra credit is given for attempts to solve more than one problem). Write out solutions clearly and concisely. State each approximation used. Diagrams welcome. Number page, problem, and question clearly. All essays/solutions should be written in English. Do not write your name on the problem sheet, but use extra envelope.

Problem 1. Thermodynamics of a small polymer on a lattice

Consider a polymer with three beads. The beads can occupy only the sites of a lattice, depicted in figure, composed by a chamber (white circles) and a channel (grey circles). Two possible configurations of the polymer are shown in the right panels.

- Compute the probability of observing at least one bead of the polymer in a channel site.

- Compute the probability of observing the central bead of the polymer as a function of x (see Figure).

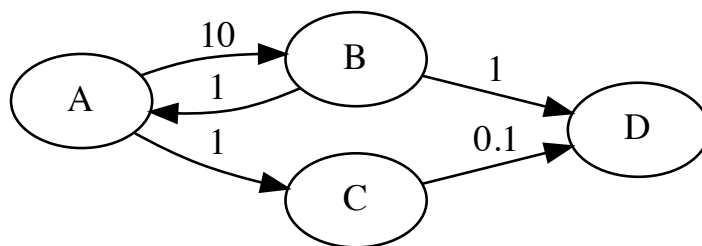Assume now that the beads of the polymer interact with a harmonic potential:

$$V = \frac{1}{2L^2} \sum_{i=1}^{3} \sum_{j=i+1}^{3} \left((x_i - x_j)^2 + (y_i - y_j)^2 \right) \tag{1}$$

where (x_i, y_i) is the position of bead i and L is the length of the bonds of the polymer (which coincides with the distance between two lattice sites, see Figure).

- At a temperature $T = 2$ compute the probability of observing the central bead of the polymer as a function of x.

- Compute the probability of observing at least one bead of the polymer in a channel site as a function of the temperature.

- Compute the entropy as a function of T.

Problem 2. Transitions in a four-state molecule

A molecule can be found in four different conformations, A, B, C, and D. Transitions rates between these four conformations are shown in the picture and are expressed in s^{-1}.

Assuming the molecule is initially prepared in A, find:

1. In the long time limit (t→ ∞), find the probability of observing the molecule in each of the conformations.

2. In the short time limit (t≪0.01 s), find the probability of each conformation to be observed as a function of time.

3. Imagine that you would like the population of D to remain as low as possible, and you have the chance to block a single transition (that is: set to zero one of the transition rates in the figure). Which transition would you block? And which one would you block to make the population of D as large as possible?

4. Find the probability of observing the molecule in A as a function of time.

5. Assuming $t \gg 100$ s, which is the *second* most probable conformation in which the molecule can be found.

6. Now consider all possible transitions starting in A and ending in D. Is it more likely to observe a transition passing through B or through C? Find the probability for the two cases.

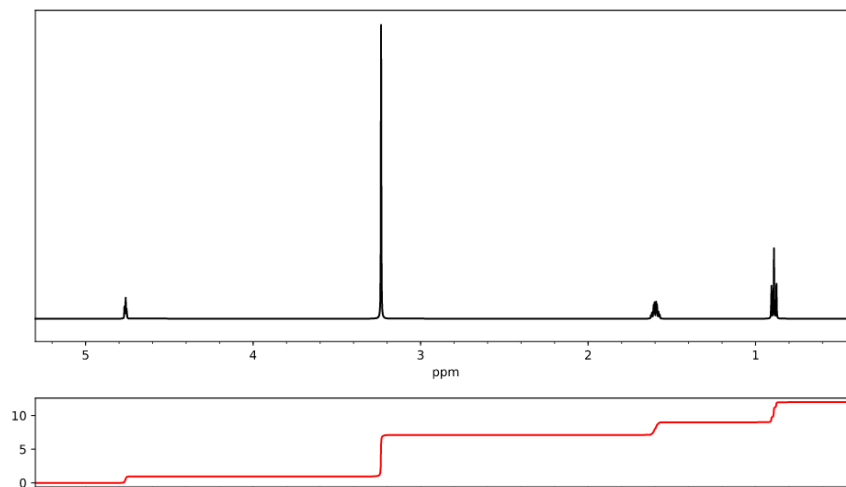## Problem 3. Solution of charges in canonical equilibrium

Consider an overall neutral solution of many oppositely charged particles at temperature, T. The particles have valence Z and concentration c.

Take one particular ion as reference and take its position as the origin of the Cartesian reference system. Assume that at any time, the charge distribution, $\rho(\vec{x})$, around the reference ion is locally adjusted (canonical equilibrium) to the electrostatic potential, $\rho(\vec{x})$, created by the set of charged particles.

1. Write down the Poisson equation relating $\rho(\vec{x})$ and $\phi(\vec{x})$ and, assuming spherical symmetry, recast the equation focussing on the radial part.

2. Assume that the temperature T is large enough to allow for linearising the Poisson equation and solve it in this limit.

3. Discuss how $\phi(r)$ varies with the distance from the reference ion, r, and explain the qualitatively different behaviour ensuing from the presence of several mobile charged particles in solution.

4. Discuss the quantitative aspects of the effect at room temperature for a 0.1M solution of monovalent salt (e.g. NaCl), a typical reference case for biological systems. Useful constants: $\epsilon_0 = 8.85 \ 10^{-12} F/m$, $\epsilon_{water} \sim 80\epsilon_0$, $e = 1.6 \ 10^{-19} C$, $\kappa_B = 1.38 \ 10^{-23} J/K$, $N_{Avogadro} = 6.02 \ 10^{23}$.

5. Can you argue what are the more general conditions which allow for linearising the Poisson equation?

Problem 4. Synthesis and analysis of a small organic molecule

A small organic compound of molecular formula $C_5H_{12}O_2$ has the following ^{1}H-NMR spectrum in CDCl$_3$. The integral of the spectrum is shown below the spectrum.

Draw the structure of the molecule, discuss the spectral parameters (chemical shifts, fine structure and couplings) and explain how you derive the structure from these parameters.

Explain why some (which?) couplings are not seen in the spectrum.

Indicate the name of the compound.

Describe the mechanism of a possible synthesis of the compound starting from appropriate aldehydes and/or alcohols in acidic environment.

Problem 5. Coupled Langevin equations of charged particles with different temperatures

Consider the pair of Langevin equations for two elastically bonded particles with electric charges $+q$ and $-q$ in an external electric field $\vec{E}$ in two dimensions:

$$\gamma \dot{\vec{x}}_1 = -k\,(\vec{x}_1(t) - \vec{x}_2(t)) + \vec{\eta}_1(t) - q\vec{E}\,,$$
$$\gamma \dot{\vec{x}}_2 = -k\,(\vec{x}_2(t) - \vec{x}_1(t)) + \vec{\eta}_2(t) + q\vec{E}\,. \tag{2}$$

Here:

- k is the elastic force constant;

- γ is the friction constant for the two particles;

- We assume that each particle is coupled to its own thermostat, with given temperatures $T_1 \neq T_2$. The stochastic forces $\vec{\eta}_1(t)$ and $\vec{\eta}_2(t)$ obey thermal averages:

  (a) $\langle \vec{\eta}_1(t) \rangle = \langle \vec{\eta}_2(t) \rangle = 0$,

  (b) $\langle \vec{\eta}_1(t) \cdot \vec{\eta}_1(t') \rangle = 4\,\gamma\,\kappa_B\,T_1\,\delta(t-t')$ and $\langle \vec{\eta}_2(t) \cdot \vec{\eta}_2(t') \rangle = 4\,\gamma\,\kappa_B\,T_2\,\delta(t-t')$,

  (c) $\langle \vec{\eta}_1(t) \cdot \vec{\eta}_2(t') \rangle = 0$.

  κ_B is the Boltzmann constant and $\delta(t)$ is the Dirac δ-function.

Answer the following questions:

1. Assuming $q = 0$, derive the expression for the distribution function $P = P(\vec{\xi})$ of spatial vectors $\vec{\xi} \equiv \vec{x}_1 - \vec{x}_2$. In particular, may you comment briefly on its functional form?

2. By neglecting the electrostatic interaction between the two particles, derive $P = P(E; \vec{\xi})$ for the general case $q \neq 0$ and $\vec{E} = E\hat{x}$ with E independent of time. In particular, plot the probability that $\vec{\xi} = 0$ as a function of E.

3. By taking $T_1 = T_2$ and again neglecting the electrostatic interaction between the two particles, repeat point (1) by assuming that $\vec{E}$ is also changing randomly and satisfies the correlation functions:

   (a) $\langle \vec{E}(t) \rangle = 0$,

   (b) $\langle \vec{E}(t) \cdot \vec{E}(t') \rangle = E_0^2\,\tau_E\,\delta(t - t')$ where τ_E is the characteristic microscopic time scale for the electric field,

   (c) $\langle \vec{E}(t) \cdot \vec{\eta}_1(t') \rangle = \langle \vec{E}(t) \cdot \vec{\eta}_2(t') \rangle = 0$.