A class of Laplacians on the quantum group $\mathrm{SU}_q(2)$ and on the quantum sphere S_q^2

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S.I.S.S.A. – May 2011

Introduction

- The idea of this talk is to describe a formulation to introduce Hodge duality and corresponding Laplacian operators on the quantum group $\mathrm{SU}_q(2)$ and on its homogeneous space S^2_q.
- It comes as an evolution of a more general research work, started here in Trieste, aimed to study models of gauged Laplacians on quantum Hopf bundles.

It is based on:

1. G.Landi, C.Reina, AZ, *Gauged Laplacians on quantum Hopf bundles*, C.M.P. 287 (2009),
2. AZ, *Laplacians and gauged Laplacians on a quantum Hopf bundle*, Quantum groups and noncommutative spaces (2011),
3. G.Landi, AZ, *Calculi, Hodge operators and Laplacians on a quantum Hopf fibration* in press on Rev.Math.Phys.
4. AZ, *(A class of) Hodge duality operators over* $\mathrm{SU}_q(2)$, arXiv:1104.0425(math.QA)

A classical setting

- With G a Lie group, $K \subset G$ a closed subgroup, the projection $\pi : G \to G/K$ has the structure of a principal bundle, with vertical fields given by $\mathfrak{k} = \mathrm{Lie}(K)$.
- SU(2) is a matrix Lie group

$$g = \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix}, \quad (u, v) \in \mathbb{C}^2 : \bar{u}u + \bar{v}v = 1 \sim S^3;$$

- The Lie algebra $\mathfrak{su}(2) \sim \mathbb{R}^3 \sim T_e(\mathrm{SU}(2))$ is represented by left (right) L_a $(R_a) \in \mathfrak{X}(\mathrm{SU}(2))$ invariant vector fields $(a = x, y, z)$

$$[L_a, L_b] = \epsilon_{abc} L_c,$$
$$[R_a, R_b] = -\epsilon_{abc} R_c$$

- SU(2) contains a U(1) subgroup generated by L_z

$$\mathrm{U}(1) \ni h = \exp\left[\frac{is}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = \begin{pmatrix} e^{is/2} & 0 \\ 0 & e^{-is/2} \end{pmatrix};$$

  the quotient of its right action $r_k(g) = g \cdot k$ is $\mathrm{SU}(2)/\,\mathrm{U}(1) \sim S^2$.

- With $\rho : K \to \mathrm{GL}(W)$ a representation of the gauge group K, $\mathcal{E} = G \times_{\rho(K)} W$ is an associated bundle.
- Equivariant and horizontal W-valued forms are

$$\Omega^r_{\mathrm{hor}}(G, W)_{\rho(\mathrm{K})} = \{\phi \in \Omega^r(G, W) = \Omega^r(G) \otimes W : \\ r_k^*(\phi) = \rho^{-1}(k)\phi, \quad i_{L_K}\phi = 0\}.$$

- There is a canonical isomorphism

$$\Gamma^{(r)}(G/\mathrm{K}, \mathcal{E}) \simeq \Omega^r_{\mathrm{hor}}(G, W)_{\rho(\mathrm{K})}$$

with $\Gamma^{(r)}(G/\mathrm{K}, \mathcal{E})$ the W-valued section forms of $\mathcal{E}$ (matter fields)
- The $\Gamma^{(r)}(G/\mathrm{K}, \mathcal{E})$ are projective $\mathcal{A}(G/K)$-modules, with a corresponding set of connections (covariant derivative)
- For any integer n there is a representation of the gauge group U(1),

$$\rho_{(n)} : \mathrm{U}(1) \to \mathbb{C}^*, \qquad \rho_{(n)}(e^{i\alpha}) = e^{in\alpha}$$

- Horizontal and ρ_n-equivariant r-forms on SU(2) give the line bundles

$$\mathcal{L}_n^{(r)} = \{\phi \in \Omega^r(\mathrm{SU}(2)) : i_{L_z}\phi = 0, \\ \mathrm{r}_k^*(\phi) = \rho_{(n)}^{-1}(k)\phi \Leftrightarrow L_z(\phi) = -\frac{in}{2}\phi \Big\}.$$

- If G semi-simple, K compact, the Laplacian coming from the C.-K. metric on G is $\Box_G = C_{\mathfrak{g}}$ (the quadratic Casimir in $\mathfrak{g} = \mathrm{Lie}(G)$)
- Coupling $\Box_{G/K}$ to the monopole connection, it results (ϕ an equivariant map on G)

$$\Box_{\nabla_0}\phi = (\Box_G - C_{\mathfrak{k}})\phi = (C_{\mathfrak{g}} - C_{\mathfrak{k}})\phi$$

On the Hopf bundle:

1. The Cartan-Killing metric on $SU(2)$ is the round metric $g = 2(\omega_- \otimes \omega_+ + \omega_+ \otimes \omega_-) + \omega_z \otimes \omega_z$

$$\Box_{S^3} x = \star \mathrm{d} \star \mathrm{d} x = C_{\mathfrak{su}(2)} x = \left(\frac{1}{2}\{L_+ L_- + L_- L_+\} + L_z L_z \right) x$$

2. The corresponding Laplacian gauged via the Dirac monopole connection is

$$\begin{aligned} \Box_{\nabla_0}\phi &= \star \nabla_0 \star \nabla_0 \phi \\ &= \frac{1}{2}\{L_+ L_- + L_- L_+\}\phi = (C_{\mathfrak{su}(2)} - L_z^2)\phi \end{aligned}$$

on $\phi \in \mathcal{L}_n^{(0)}$. Its spectrum on Wigner D-functions is $\Box_{\nabla_0} D^J_{-\frac{n}{2},s} = (J(J+1) - \frac{n^2}{4}) D^J_{-\frac{n}{2},s}$ $(J = 0, 1/2, 1, 3/2, \ldots)$, $(k, s = -J, \ldots, +J)$

- Is a similar relation valid, if we consider Hopf fibrations given by quantum groups over quantum homogeneous spaces?
- A formulation of a topological principal bundles where a Hopf algebra $\mathcal{H}$ (the gauge group) *coacts* on a total space algebra $\mathcal{P}$ has been developed by [BM,H,D].
- In order to have a consistent notion of vertical vectors, horizontal forms, and connections, differential calculi on $\mathcal{P}$ and $\mathcal{H}$ must fullfill a condition of compatibility.
- It is then possible to define *coequivariant* elements in $\mathcal{P}$, and associate bundles with sections, which are elements of finite projective modules over the base ${}^{\mathcal{H}}\mathcal{P}$ of the bundle.
- Covariant derivatives on such projective modules are in one to one correspondence with *strong* connections on the principal bundle $\mathcal{A}({}^{\mathcal{H}}\mathcal{P}) \hookrightarrow \mathcal{A}(\mathcal{P})$.
- In [LRZ08, Z, LZ10] we applied this formalism to the *quantum Hopf fibration*:

$$\mathcal{P} = \mathrm{SU}_q(2), \ \mathcal{H} = \mathrm{U}(1), \ {}^{\mathcal{H}}\mathcal{P} = \mathrm{S}^2_q.$$

- It is possible to introduce *compatible* Laplacian operators on both $\mathrm{SU}_q(2)$ and S^2_q, to couple $\Box_{\mathrm{S}^2_q}$ with gauge connections; and to find a covariant derivative ∇ s.t.

$$\Box_\nabla \phi = \star\nabla \star \nabla\phi = (\Box_{\mathrm{SU}_q(2)} - V)\phi :$$

the operator $\Box_\nabla$ extends to line bundles elements ϕ the action of the Laplacian $\Box_{\mathrm{S}^2_q}$. The Laplacian $\Box_{\mathrm{SU}_q(2)}$ is a function of the *quantum Casimir*, V is its vertical part.

Given this setting, I shall focus on the formulation of compatible Hodge operators on the exterior algebras $\Omega(\mathrm{SU}_q(2))$ and $\Omega(\mathrm{S}^2_q)$. To describe it, let us recall some basics of the classical formulation.

1 Hodge operators and Laplacians on classical Lie groups and homogeneous spaces

- Consider G as a *compact, connected,* N-dim. Lie group given as the real form of a complex connected Lie group.
- The group manifold is parallelizable. The infinitesimal generators of the natural *left* and *right* actions of G on itself:

$$l_g(p) = g \cdot p, \qquad r_g(p) = p \cdot g$$

give a *global* basis for the differential calculus.

$$\mathrm{d}\phi = \sum_a (L_a \cdot \phi)\omega^a = \sum_b (R_b \cdot \phi)\eta^b \qquad \text{on } \phi \in \mathcal{A}(G)$$

$$[L_a, L_b] = f_{ab}^{\ \ c} L_c \qquad [R_a, R_b] = -f_{ab}^{\ \ c} R_c$$

- The exterior algebra $\Omega(G) = (\oplus_{k=1}^N \Omega^k(G), \wedge, \mathrm{d} : \Omega^k(G) \to \Omega^{k+1}(G), \mathrm{d}^2 = 0)$ is given by free $\mathcal{A}(G)$-bimodules on the basis of left (right) – invariant $\{\omega^a\}$ $(\{\eta^b\})$ 1-forms.

Braiding and contractions

- The braiding of the calculus is the *standard flip*

$$\sigma(\omega^a \otimes \omega^b) = \omega^b \otimes \omega^a, \qquad \sigma^2 = 1$$

$$\omega^{a_1} \wedge \ldots \wedge \omega^{a_k} \; = \; A^{(k)}(\omega^{a_1} \otimes \ldots \otimes \omega^{a_k}) \; = \; \sum_{\pi \in S_k} (-1)^\pi \omega^{\pi(a_1)} \otimes \ldots \otimes \omega^{\pi(a_k)}$$

so that

1. $A^{(k)}(\omega^{a_1} \wedge \ldots \wedge \omega^{a_k}) = k!(\omega^{a_1} \wedge \ldots \wedge \omega^{a_k}) \qquad \Rightarrow \qquad [A^{(k)}, *] = 0$
2. $\dim\, \Omega^k(G) = \dim\, \Omega^{N-k}(G) = \begin{pmatrix} N \\ k \end{pmatrix}$

- Set $\mathcal{A}(G)$-bimodule bilinear contraction maps

$$\mathfrak{G} : \Omega^1(G) \times \Omega^1(G) \to \mathcal{A}(G)$$

and by recurrence $(k < s)$, $\mathfrak{G} : \Omega^{\otimes k}(G) \times \Omega^{\otimes s}(G) \to \Omega^{\otimes s-k}(G)$

$$\mathfrak{G}(\omega^{a_1} \otimes \ldots \otimes \omega^{a_k}, \omega^{b_1} \otimes \ldots \otimes \omega^{b_s}) \; := \; \left(\Pi_{i=1}^k \mathfrak{G}(\omega^{a_i}, \omega^{b_i})\right) \omega^{b_{k+1}} \otimes \ldots \otimes \omega^{b_s}$$

which descend to consistent contraction maps:

$$\mathfrak{G} : \Omega^k(G) \times \Omega^s(G) \to \Omega^{s-k}(G)$$

- Given an hermitian volume form, set the operator

$$\mu = m\,\omega^1 \wedge \ldots \wedge \omega^N = \mu^* \in \Omega^N(G) \qquad \text{(volume)}$$

$$\mathfrak{G}_\mu(\phi) := \mathfrak{G}(\phi, \mu) \qquad \Omega^k(G) \to \Omega^{N-k}(G)$$

- One has, for symmetric contractions:

$$\mathfrak{G}(\omega^a, \omega^b) = \mathfrak{G}(\omega^b, \omega^a) \quad \Leftrightarrow \quad \mathfrak{G}_\mu^2(\phi) = (-1)^{k(N-k)}\{m^2 k!(N-k)!\det \mathfrak{G}\}\,\phi$$

- The operator $\mathfrak{G}_\mu$ is not (yet) an Hodge operator. It has to be *real*:

$$\mathfrak{G}(\omega_a, \omega_b)^* = \mathfrak{G}(\omega_b^*, \omega_a^*) \quad \Leftrightarrow \quad \mathfrak{G}_\mu(\omega^{a*}) = (\mathfrak{G}_\mu(\omega^a))^*$$

$$\Rightarrow \quad \mathfrak{G}_\mu(\phi^*) = (\mathfrak{G}_\mu(\phi))^* :$$

- $\mathfrak{G}(\omega^a, \omega^b)$ are then components of (the inverse of) a real metric tensor on G, with Hodge operator:

$$\mathfrak{G}_\mu(\mu) := sgn(\mathfrak{G}) \quad \Leftrightarrow \quad N! m^2 \det \mathfrak{G} := sgn(\mathfrak{G})$$

$$\star(\phi) := \frac{1}{k!}\mathfrak{G}_\mu(\phi) \quad \Rightarrow \quad \star^2(\xi) = sgn(\mathfrak{G})(-1)^{k(N-k)}\phi \qquad (\phi \in \Omega^k(G))$$

- It gives a scalar product on the exterior algebra $\Omega(G)$:

$$\langle \phi, \phi' \rangle \;:=\; \int_\mu \phi^* \wedge (\star\phi') \;=\; \frac{1}{k!}\mathfrak{G}(\phi^*, \phi')$$

- The Laplacian operator corresponding to a metric tensor is

$$\Box : \Omega^k(G) \to \Omega^k(G), \qquad\qquad \Box\, x := \star \circ \mathrm{d}(\star \mathrm{d}x)$$

- On the basis of the principal bundle $\pi : G \to G/K$ the exterior algebra is

$$\Omega^r(G/K) = \{\phi \in \;\Omega^r(G) \;\; : i_{L_V}\phi = 0, \qquad r_k^*(\phi) = \phi \;\Leftrightarrow\; L_V(\phi) = 0\}.$$

They are no longer free $\mathcal{A}(G/K)$-bimodules.

- Given the volume form $\check{\mu}$ on $\Omega(G/K)$, set the scalar product as a restriction

$$\langle \phi, \phi' \rangle_{G/K} := \langle \phi, \phi' \rangle_G$$

- For metric tensors on G which are K–equivariant, the equation

$$\langle \phi, \phi' \rangle_{G/K} = \int_{\check{\mu}} \phi^* \wedge (\star_{G/K}\phi')$$

uniquely defines the Hodge operator corresponding to the projection on G/K of the (inverse of) the metric $\mathfrak{G}$.

Let us try to generalise this path to the quantum group setting

The quantum algebras $\mathrm{SU}_q(2)$ **and** S^2_q

- As quantum group $\mathrm{SU}_q(2)$ consider the Hopf $(S, \varepsilon, \Delta, *)$, polynomial *unital* $*$-algebra (with $0 < q < 1$)

$$U = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix}, \qquad \begin{array}{l} ac = qca \quad ac^* = qc^*a \quad cc^* = c^*c \\ a^*a + c^*c = aa^* + q^2cc^* = 1. \end{array}$$

- Its dually paired Hopf universal envelopping algebra is $\mathcal{U}_q(\mathfrak{su}(2)) = \{K^\pm, E, F = E^*\}$

$$K^\pm E = q^\pm E K^\pm \qquad K^\pm F = q^\mp F K^\pm \qquad [E, F] = \frac{K^2 - K^{-2}}{q - q^{-1}}$$

The Podleś quantum sphere S_q^2

- Given $\mathcal{A}(\mathrm{U}(1)) := \mathbb{C}[z, z^*]/ < zz^* - 1 >$. The map

$$\pi : \mathcal{A}(\mathrm{SU}_q(2)) \to \mathcal{A}(\mathrm{U}(1)), \qquad \pi \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} := \begin{pmatrix} z & 0 \\ 0 & z^* \end{pmatrix}$$

  is a surjective Hopf $*$-algebra homomorphism.

- U(1) is then a quantum subgroup of $\mathrm{SU}_q(2)$ with right coaction:

$$\delta_R := (\mathrm{id} \otimes \pi) \circ \Delta \, : \, \mathcal{A}(\mathrm{SU}_q(2)) \to \mathcal{A}(\mathrm{SU}_q(2)) \otimes \mathcal{A}(\mathrm{U}(1))$$

- The algebra of the standard Podleś sphere is given by the *coinvariants* for this coaction

$$\mathcal{A}(\mathrm{S}_q^2) = \{b \in \mathcal{A}(\mathrm{SU}_q(2)) : \delta_R(b) = b \otimes 1\}$$

- This coaction allows to define S_q^2-bimodules: $(n \in \mathbb{Z})$

$$\mathcal{L}_n := \{x \in \mathrm{SU}_q(2) \quad : \quad \delta_R(x) = x \otimes z^{-n}\}$$

Differential calculi

The theory of covariant differential calculi over compact quantum groups is due to Woronowicz. The two most known examples on $\mathrm{SU}_q(2)$ are

- A left-covariant 3D $*$-calculus (i.e. $\mathrm{d}(x^*) = (\mathrm{d}x)^*$)

$$\mathrm{d}x = \sum_a (X_a \triangleright x)\omega_a, \qquad (a = \pm, z)$$

- A bicovariant $4D_+$ $*$-calculus:

$$\mathrm{d}x = \sum_a (L_a \triangleright x)\omega_a = \sum_a \omega_a (R_a \triangleright x) = \sum_a \eta_a (x \triangleleft R_a) \qquad (a = \pm, 0, z)$$

  with quantum tangent spaces $(X_a, R_a = -S^{-1}(L_a)) \in \mathcal{U}(\mathfrak{su}(2))$.

- The centre of $\mathcal{U}_q(\mathfrak{su}(2))$ is generated by the quantum Casimir

$$C_q = \frac{qK^2 - 2 + q^{-1}K^{-2}}{(q - q^{-1})^2} + FE - \frac{1}{4} \;=\; L_0 + [\frac{1}{2}] - \frac{1}{4}.$$

A $4D$ exterior algebra over $\mathrm{SU}_q(2)$

- A bicovariant calculus has a canonical braiding:

$$\sigma \; : \; \omega \otimes \eta \; = \; \eta \otimes \sigma$$

with $\mathrm{spec}\,(\sigma^2) = (1, q^{\pm 2})$. It is $\sigma^2 \neq 1$.

- the corresponding anti-symmetrisers give isomorphic exterior algebras

$$\Omega^k_{\pm}(\mathrm{SU}_q(2)) = \Omega^{\otimes k}(\mathrm{SU}_q(2)) / \ker \; A^{(k)}_{\pm},$$

$$\dim \; \Omega^k_{\pm}(\mathrm{SU}_q(2)) = \dim \; \Omega^{4-k}_{\pm}(\mathrm{SU}_q(2)) = \begin{pmatrix} 4 \\ k \end{pmatrix}$$

- Their spectral decomposition, $A_{\pm}(\omega) = \lambda^{\pm}_{\omega}\omega$ coincide:

$$\mathrm{spec}(A^{(2)}_{\pm}) = (1 + q^{\pm 2}),$$
$$\mathrm{spec}(A^{(3)}_{\pm}) = 2(1 + q^2 + q^{-2}),$$
$$\mathrm{spec}(A^{(4)}_{\pm}) = 2(q^4 + 2q^2 + 6 + 2q^{-2} + q^{-4}).$$

- Anti-symmetrisers do not commute with the hermitian conjugation

- As a volume $\mu = \mu^* = i\, m\, \omega_- \wedge \omega_+ \wedge \omega_0 \wedge \omega_z = \check{\mu} = \check{\mu}^*,\ m \in \mathbb{R}$.
- Since $[A^{(k)}, *] \neq 0$, set a U(1)-coequivariant sesquilinear form

$$\Gamma : \Omega^1_{inv}(\mathrm{SU}_q(2)) \times \Omega^1_{inv}(\mathrm{SU}_q(2)) \to \mathbb{C}; \quad \Gamma(\omega_a, \omega_b) = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \nu & \epsilon \\ 0 & 0 & \xi & \gamma \end{pmatrix}$$

- Set [LZ10] the Hodge operators $L^{\pm} : \Omega^k_{inv}(\mathrm{SU}_q(2)) \to \Omega^{4-k}_{inv}(\mathrm{SU}_q(2))$ by

$$L^{\pm}(\omega) := \frac{1}{\lambda^{\pm}_{\omega}} \left(\Gamma(\omega, \mu)\right)^*.$$

On 0,1,3,4-forms it is $L^+(\omega) = L^-(\omega)$.

- Set a sesquilinear inner product by the implicit relation

$$\mu\{\omega, \omega'\}_{\pm} := \omega^* \wedge L^{\pm}(\omega')$$

In the classical setting, it would be $\{\omega, \omega'\} = \frac{1}{k!}\Gamma^*(\omega', \omega)$.

- Towards obtaining an Hodge duality, we use it to impose both a hermitianity and reality condition on Γ:

  a) A contraction map Γ is hermitian provided (on 1-forms $L^- = L^+$):

$$\{\omega_a, \omega_b\} = \Gamma(\omega_a, \omega_b) \qquad \Leftrightarrow \qquad \Gamma(\omega_a, \omega_b)\mu = \omega_a^* \wedge L(\omega_b)$$

  b) A contraction map is real provided, on 1-forms:

$$L(\omega_a^*) = (L(\omega_a))^*$$

- Are such $L^\pm$ Hodge operators? Not yet
- We expect their actions to be compatible with the $*$ hermitian conjugation on the whole $\Omega(\mathrm{SU}_q(2))$.

We fix this compatibility, enlarging the reality condition above path 1 We require, on higher order forms

$$\lambda^{\pm}_{\omega^*}(L^{\pm}(\omega^*)) = \lambda^{\pm}_{\omega}(L^{\pm}(\omega))^* :$$

For the corresponding contractions:

- the parameters satisfy the constraints

$$\begin{aligned}
&(\alpha \neq 0,\ \beta = q^2\alpha,\ \epsilon = \xi \neq 0)\ \in\ \mathbb{R}, \\
&2\nu + (q^2 - q^{-2})\epsilon = 0, \\
&2(\epsilon^2 - \gamma\nu) + (q - q^{-1})^2(2q^2\alpha^2 + \epsilon^2) = 0.
\end{aligned}$$

- for this class of contractions: $\{\omega, \omega'\} = 0$ if $\lambda^{\pm}_{\omega} \neq \lambda^{\pm}_{\omega'}$;

- for this class of contractions: $\{\omega, \omega'\} = \frac{\lambda^{\pm}_{\omega^*}}{\lambda^{\pm}_{\omega}\lambda^{\pm}_{\omega'}}\,\Gamma(\omega, \omega')$ if $\lambda^{\pm}_{\omega} = \lambda^{\pm}_{\omega'}$;

- the *signature* of the contraction is constant: $\Gamma(\mu, \mu) > 0$.

- The square of the Hodge operators is not necessarily diagonal, its spectrum does not show the classical degeneracy path 2 We require, on higher order forms

$$\lambda^{\pm}_{\omega}(L^{\pm}(\omega^*)) = \lambda^{\pm}_{\omega^*}(L^{\pm}(\omega))^*.$$

For the corresponding contractions:

- the parameters satisfy the constraints

$$\begin{aligned}&(\alpha \neq 0,\ \beta = q^2\alpha,\ \nu = 0,\ \gamma \neq 0,\ \epsilon \neq 0,\ \xi = \epsilon)\ \in\ \mathbb{R},\\ &\epsilon^2 = (q - q^{-1})^2 \alpha\,\beta,\\ &(q + q^{-1})\epsilon + (q - q^{-1})\gamma = 0;\end{aligned}$$

- it is $\Gamma(\mu, \mu) < 0$;

- the spectra of the corresponding operators $(L^{\pm})^2$ have the same degeneracy of the anti-symmetrisers;

- and more: the spectra of $L^+L^- = L^-L^+$ have the classical degeneracy, i.e. they are constant on each $\Omega^k_{\pm}(\mathrm{SU}_q(2))$;

- The eigenvalues of L^+L^- depend on the order of the forms.

- The inner product on the left invariant forms is consistently enlarged to the whole $\Omega(\mathrm{SU}_q(2))$ via the requirement of

$$\text{left covariance}: \qquad \langle x\,\omega, x'\,\omega'\rangle^L := h(x^* x')\{\omega, \omega'\}$$
$$\text{right covariance}: \qquad \langle \omega\,x, \omega'\,x'\rangle^R := h(x^* x')\{\omega, \omega'\}$$

with $h : \mathcal{A}(\mathrm{SU}_q(2)) \to \mathbb{C}$ the Haar state of $\mathrm{SU}_q(2)$.

A $3D$ exterior algebra on S^2_q

- The inclusion $\mathcal{A}(\mathrm{S}^2_q) \hookrightarrow \mathcal{A}(\mathrm{SU}_q(2))$ is a topological principal bundle

- From the 4dim. on $\mathrm{SU}_q(2)$ it is possible to induce compatible calculi on $\mathrm{U}(1)$.

- The induced *left-invariant* exterior algebra is given by

$$\Omega^k(\mathrm{S}^2_q) = \left\{\phi \in \Omega^k(\mathrm{SU}_q(2)) : \, i_{L_z}\phi = 0, \quad \delta_R^{(k)}(\phi) = \phi \otimes 1\right\}$$

- Assume a volume form $\tilde{\mu} = \tilde{m}\,\omega_- \wedge \omega_+ \wedge \omega_0, \ \tilde{m} \in \mathbb{R}$

- The restriction to $\Omega(\mathrm{S}^2_q)$ of the inner product given by the Hodge operators above gives
  1. $\langle\tilde{\mu},\tilde{\mu}\rangle_{\mathrm{S}^2_q} = 0$ for $L^{\pm}$ (path 2) ($\nu = 0$),
  2. $\langle\tilde{\mu},\tilde{\mu}\rangle_{\mathrm{S}^2_q} > 0$ for $L^{\pm}$ (path 1),
- The equations (we omit to write the $\pm$ dependence)

$$\tilde{L} : \Omega^k(\mathrm{S}^2_q) \to \Omega^{3-k}(\mathrm{S}^2_q) \qquad \langle\phi,\phi'\rangle^L_{\mathrm{S}^2_q} = \int_{\tilde{\mu}} \phi^* \wedge (\tilde{L}\phi')$$

$$\tilde{R} : \Omega^k(\mathrm{S}^2_q) \to \Omega^{3-k}(\mathrm{S}^2_q) \qquad \langle\phi,\phi'\rangle^R_{\mathrm{S}^2_q} = \int_{\tilde{\mu}} \phi^* \wedge (\tilde{R}\phi')$$

  uniquely define a left and right $\mathcal{A}(\mathrm{S}^2_q)$-linear Hodge operators.
- The corresponding Laplacians are:

$$\Box^L_{\mathrm{S}^2_q}\, b = \tilde{L}\,\mathrm{d}\,\tilde{L}\,\mathrm{d}\, b = \left\{\alpha\, L_+L_- + q^2\alpha\, L_-L_+ + \nu\, L_0L_0\right\} \triangleright b$$

$$\Box^R_{\mathrm{S}^2_q}\, b = \tilde{R}\,\mathrm{d}\,\tilde{R}\,\mathrm{d}\, b = \left\{q^2\alpha\, R_+R_- + \alpha\, R_-R_+ + \nu\, R_0R_0\right\} \triangleright b$$

- They both are the restriction to S^2_q of the Laplacians on $\mathrm{SU}_q(2)$,
  Their actions on $\mathcal{A}(\mathrm{S}^2_q)$ coincide.
  For $2q\,\alpha = 1$, $\quad \nu = q^{-2}(q - q^{-1})^4$, $\Box_{\mathrm{S}^2_q} = (D^2 - [\tfrac{1}{2}])$, [BL10]

Interlude: gauged Laplacians on a quantum Hopf fibration

- There is an equivalence

$$\begin{aligned} \text{r.p.m.}: \quad & \mathcal{L}_n^{(0)} \sim \mathcal{P}_n \cdot (\mathcal{A}(\mathrm{S}_q^2))^{\otimes |n|+1} = \mathcal{E}_n \\ \text{l.p.m.}: \quad & \mathcal{L}_n^{(0)} \sim (\mathcal{A}(\mathrm{S}_q^2))^{\otimes |n|+1} \cdot \check{\mathcal{P}}_n = \mathcal{F}_n \\ & \check{P}_n^2 = \check{P}_n, \qquad \check{P}_n^\dagger = \check{P}_n, \qquad \check{\mathcal{P}}_n = \mathcal{P}_{-n} \end{aligned}$$

- There is a complete equivalence among connections on the Hopf bundle and covariant derivatives on $\mathcal{F}_n$.
- Such an equivalence does *not* exist with $\mathcal{E}_n$. The definition of connections on the q.H.b. breaks this left-right symmetry.
- For a gauged Laplacian we need an Hodge operator which is compatible with a right-acting projector on $\mathcal{F}_n$

$$\star^{\mathcal{R}} : \Omega^k(\mathrm{S}_q^2) \otimes_{\mathcal{A}(\mathrm{S}_q^2)} \mathcal{F}_n \to \Omega^{3-k}(\mathrm{S}_q^2) \otimes_{\mathcal{A}(\mathrm{S}_q^2)} \mathcal{F}_n$$

so we extend the right $\mathcal{A}(\mathrm{S}_q^2)$-linear Hodge operator $\check{R}$ on $\Omega(\mathrm{S}_q^2)$:

$$\Box_\nabla^{\mathcal{R}} : \mathcal{F}_n \to \mathcal{F}_n, \qquad \Box_\nabla^{\mathcal{R}} \phi := \star^{\mathcal{R}} \nabla (\star^{\mathcal{R}} \nabla \phi).$$

- There is a set of covariant derivatives ∇_s on $\mathcal{F}_n$, $(s \in \mathbb{R})$ which converge $(q \to 1)$ to the monopole Dirac connection:

$$\Box^{\mathcal{R}}_{\nabla_s}\phi = q^{-2n}\left\{\alpha\left(q^2 R_+ R_- + R_- R_+\right) + \nu\left(R_0 + sq^{-n} - [\frac{|n|}{2}][1 - \frac{|n|}{2}]\right)^2\right\} \triangleright \phi$$

- There is then a value $s = s(n)$ such that:

$$q^{2n}\left(\Box^{\mathcal{R}}_{\nabla_s}\phi\right) = \left\{\alpha(q^2 R_+ R_- \, + \, R_- R_+) \, + \, \nu R_0^2\right\} \triangleright \phi$$

- Apart the global factor, this gauged Laplacian $\Box^{\mathcal{R}}_{\nabla_s}$ extends to $\mathcal{L}_n^{(0)}$ the action of the Laplacian on S^2_q.

- It is

$$q^{2n}\left(\Box^{\mathcal{R}}_{\nabla_s}\phi\right) = \left(\Box^{\mathcal{R}}_{\mathrm{SU}_q(2)} \, - \, V\right)\phi$$

  with V the vertical part of the Laplacian on $\mathrm{SU}_q(2)$

- The spectrum of the gauged Laplacian $\Box^{\mathcal{R}}_{\nabla_s}$ is not invariant for $n \to -n$. This symmetry is restored in the limit $q \to 1$

- This analysis clearly depends on the differential calculi over $\mathcal{A}(\mathrm{SU}_q(2))$. If the calculus on $\mathrm{SU}_q(2)$ is the left-covariant 3-dim. one, a similar pattern can be recovered [LRZ08].

We construct now other Hodge operators...

- Fix a left-invariant ordered basis of 1-forms

$$\vartheta := i\, m\, \omega_- \otimes \omega_+ \otimes \omega_0 \otimes \omega_z;$$

  it is $\mu := A^{(4)}_+(\vartheta) = A^{(4)}_-(\vartheta) := \check{\mu} = \mu^* = \check{\mu}^*$.

- On the *eigenform* left-invariant basis of the antisymmetrisers we set the operators $T^{\pm} : \Omega^k_{\pm inv}(\mathrm{SU}_q(2)) \longrightarrow \Omega^{4-k}_{\pm inv}(\mathrm{SU}_q(2))$ by

$$T^{\pm}(\omega) := \frac{\lambda^{\pm}_{\omega^*}}{\lambda^{\pm}_{\omega}} \left(\mathfrak{A}^{(4-k)}_{\pm}(\Gamma(\omega, B^{\pm}_{k,4-k}\vartheta))\right)^*$$

- The operators $T^{\pm}$ are given by the shuffles $B^{\pm}_{k,N-k} := \sum_{p_j \in S(k,N-k)} \mathrm{sign}(p_j)\mathfrak{P}^{\pm}(p_j)$, where $S(k, 4-k) \subset P(4)$, are those permutations p_j of 4 elements such that $p_j(1) < \ldots < p_j(k)$ and $p_j(k+1) < \ldots < p_j(4)$.

- Operators $L^{\pm}$ and $T^{\pm}$ do coincide in the classical setting

- We consider as before U(1)-coequivariant contractions Γ

- It is clear that $T^{\pm}(1) = \mu$,
- and also $T^{\pm}(\omega_a) = L^{\pm}(\omega_a)$ on 1-forms.

We link the symmetry requirements on the contractions to the spectra of the corresponding Hodge operators, as in the classical setting. We fix a reality and an hermitianity condition for Γ:

- A contraction map is $T^{\pm}$-real provided, on 1-forms:

$$T(\omega_a^*) = (T(\omega_a))^*;$$

- It is $T^{\pm}$-hermitian provided

$$(T^{\pm})^2(\omega_a) = -((T^{\pm})^2(1))\omega_a = -(T^{\pm}(\mu))\omega_a$$

- It is $T^{\pm}$-maximally hermitian provided it is real, hermitian, and the squares of the corresponding Hodge operators have the degeneracy of the anti-symmetrisers.

$T^{\pm}$-maximally hermitian contractions do exist, and coincide:
they are given by the conditions coming from path 2 above.

For the corresponding Hodge operators

- $T^{\pm} = L^{\pm}$ on 0,1,2,3-forms, so that
- the reality condition on higher order forms is a consequence of the requirement of maximal hermitianity: it is

$$\lambda^{\pm}_{\omega}(T^{\pm}(\omega^{*})) = \lambda^{\pm}_{\omega^{*}}(T^{\pm}(\omega))^{*}$$

- Their squares on $\Omega^{k}(\mathrm{SU}_q(2))$ satisfy:

$$(T^{\pm})^2(\omega) = (-1)^{k(4-k)} \left(\frac{\lambda^{\pm}_{\omega}}{\lambda^{\pm}_{\omega^{*}}} (T^{\pm})^2(1) \right) \omega$$

- To clarify this relation define the quantum determinant Γ as

$$\det_q \Gamma := \Gamma(i\,\omega_- \otimes \omega_+ \otimes \omega_0 \otimes \omega_z, i\,\omega_- \wedge \omega_+ \wedge \omega_0 \wedge \omega_z)$$

- It is $\det_q \Gamma < 0$: so fix (up to a sign) the scale parameter $m \in \mathbb{R}$ imposing

$$(T^{\pm})^2(1) = \mathrm{sgn}\,\Gamma \qquad \Leftrightarrow \qquad m^2(\alpha\,\beta\,\epsilon^2) = 1 :$$

- This choice allows to write the relation above as

$$(T^{\pm})^2(\omega) = (-1)^{k(4-k)} \left((\operatorname{sgn}\Gamma) \frac{\lambda^{\pm}_{\omega}}{\lambda^{\pm}_{\omega^*}} \right) \omega.$$

It appears as a natural generalisation of the classical relation to the quantum setting, where the braiding σ associated to the calculus on $\mathrm{SU}_q(2)$ has a non trivial spectrum.

- Moreover:

$$T^+T^-(\omega) = T^-T^+(\omega) = (-1)^{k(4-k)}(\operatorname{sgn}\Gamma)\,\omega.$$

This relation shows, in a quantum setting with $\sigma^2 \neq 1$, the closest similarity to the classical.

- Extend to operators $\star^{L,R} : \Omega^k(\mathrm{SU}_q(2)) \to \Omega^{4-k}(\mathrm{SU}_q(2))$ by:

$$\star^L_{\pm}(x\,\omega) := x\,T^{\pm}(\omega), \qquad \star^R_{\pm}(\omega\,x) := (T^{\pm}\omega)x,$$

so that the above relation is written as

$$\star^{L,R}_{\pm}\,\star^{L,R}_{\mp}\,(\phi) = (-1)^{k(4-k)}(\operatorname{sgn}\Gamma)\phi, \qquad \phi \in \Omega^k(\mathrm{SU}_q(2)).$$

Real and maximally hermitian contractions?!

- We associate a *quantum metric* to them

$$g_q(\omega_a, \omega_b) := \Gamma(\omega_a^*, \omega_b)$$

- The set of metrics $\mathfrak{G}$ is parametrised by $(\mathbb{R}/0) \times \mathbb{Z}_2$

$$g_{q\,ab} = \begin{pmatrix} 0 & q^2 a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm(1-q^2)a \\ 0 & 0 & \pm(1-q^2)a & \pm(q^2+1)a \end{pmatrix}$$

with $a = -\alpha \neq 0$.

- Real and σ-symmetric $g_I : \Omega^{\otimes 2}(\mathrm{SU}_q(2)) \to \mathcal{A}(\mathrm{SU}_q(2))$ metrics were introduced in [Hec00, Hec03]. They satisfy the conditions:

  1. g_I is an homomorphism of the $\mathcal{A}(\mathrm{SU}_q(2))$-bimodule $\Omega^{\otimes 2}(\mathcal{A}(\mathrm{SU}_q(2)))$;
  2. g_I is non-degenerate;
  3. $g_I \circ \sigma^{\pm} = g_I$ (symmetry condition);
  4. It holds $(g_I \otimes 1) \circ \sigma_2^{\pm} = (1 \otimes g_I) \circ \sigma_1^{\mp}$ on $\Omega^{\otimes 3}(\mathrm{SU}_q(2))$;

  σ-metrics are called real provided $(g_I(\phi, \phi'))^* = g_I(\phi'^*, \phi^*)$, and left-covariant provided $\Delta \circ g_I = (1 \otimes g_I)\Delta_L^{(2)}$.

- The set $\mathfrak{G}_\sigma$ of real, U(1)-coequivariant, left-invariant, σ-metrics is a subset of the set of quantum metrics $\mathfrak{G}_\sigma \subset \mathfrak{G}$ (corresponding to + sign above)
- The Laplacian coming from a quantum metric in $\mathfrak{G}_\sigma$ is

$$\begin{aligned}\Box^R x = \Box^L x &= \alpha\{L_+L_- + q^2 L_-L_+ + (1+q^2)L_zL_z - 2(q^2-1)L_0L_z\}\triangleright x \\ &= 2q\,\alpha\, L_0\triangleright x,\end{aligned}$$

- its spectrum comes from $L_0\triangleright\phi_{n,J,l} = [J][J+1]\phi_{n,J,l}$ ($n \in \mathbb{N}, J = |n|/2 + \mathbb{Z}, l = 0, \ldots 2J$)
- The action of this Laplacian reduces in the classical limit to the action of the quadratic Casimir C of the Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$.
- It is well known that C coincides with the Laplacian corresponding to the Cartan-Killing metric on SU(2): we then assume that a metric $\tilde{g}_q \in \mathfrak{G}_\sigma$ reduces in the classical limit to a Cartan-Killing metric on SU(2).
- We recover also that although real and maximally hermitian contractions have the same signature in the quantum setting, corresponding metrics in $\mathfrak{G}_\sigma$ and $\mathfrak{G}\backslash\mathfrak{G}_\sigma$ have different signatures in the classical limit.
- The spectrum of Laplacians corresponding to metrics in $\mathfrak{G}\backslash\mathfrak{G}_\sigma$ moreover ranges on the whole real line: it seems then meaningless to use them to model energy levels of a stable quantum system.

Satisfied?! Not completely.

- The inner product on $SU_q(2)$ cannot be projected onto S^2_q, where it is degenerate. Following the procedure described above it is not possible to define a consistent Hodge operator on $\Omega(SU_q(2))$.
- The restriction of the Laplacian to S^2_q is $\Box f = 2q\,\alpha(FE \triangleright f)$. It is the square of the Dirac operator in [DS03,SW04], corresponding to the 2-dim. left-covariant calculus on S^2_q.
- It is also true that the quotient $\Omega^1(S^2_q)/\ker(\Gamma)$ is the left-covariant FODC on S^2_q.
- Is it true that the whole 2-dim exterior algebra $\Omega(S^2_q)$ can be obtained as the quotient of the 3-dim one via a differential bi-ideal generated by $\ker(\Gamma)$?
- Is then the metric projectable on this 2-dim exterior algebra?