On the many Dirichlet Laplacians on a non-convex polygon and their approximations by point interactions

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Introduction

Point perturbations of the Laplacian have been defined by Berezin and Faddeev as self-adjoint extensions of the Laplacian restricted to smooth functions with compact support disjoint from a finite set in $\mathbb{R}^d$, $d \leq 3$. Such operators have attracted a lot of attention and have been used in a wide range of applications, mainly in Quantum Mechanics. In the case of a confined quantum system, point perturbations of the Dirichlet Laplacian on a bounded domain can be defined in a similar way (see [Casper, Sweers 1994], [Blanchard, Figari, Mantile 2007], [Exner, Mantile 2008], ...). Of course in this case the perturbations can not be placed at the boundary, since functions in the domain of the Dirichlet Laplacian vanish there.

Question: How to put point-like perturbations on the boundary?

Naive idea: move the points supporting the perturbation towards the boundary while increasing the interactions strengths, so to compensate the vanishing of the functions.

However it is not clear how to implement this procedure: there is no universal behavior for the functions in the operator domain in a neighborhood of the boundary.
For example if $\Omega \subset \mathbb{R}^2$ is a plane bounded domain which either is convex or has a regular boundary, then the Friedrichs-Dirichlet Laplacian Δ^F_Ω on $L^2(\Omega)$ has domain $\mathscr{D}(\Delta^F_\Omega) = H^2(\Omega) \cap H^1_0(\Omega)$. By the (dense) inclusions

$$C_0^\infty(\Omega) \subset H^2(\Omega) \cap H^1_0(\Omega) \subset C_0(\Omega)$$

there is no minimal vanishing rate for $u(\mathrm{x})$ as x approaches the boundary.

This is not true if Ω is a **non-convex** polygon. Then $\mathscr{D}(\Delta_\Omega^F)$ is strictly larger than $H^2(\Omega) \cap H_0^1(\Omega)$ and

$$\forall u \in \mathscr{D}(\Delta_\Omega^F)\,, \quad u(\mathsf{x}) \sim \xi_u\, s_{\mathsf{v}}(\mathsf{x}) \|\mathsf{x}-\mathsf{v}\|^{\pi/\omega}\,, \quad \|\mathsf{x}-\mathsf{v}\| \ll 1$$

where v is any vertex at a non-convex corner, $\omega > \pi$ is the measure of the interior angle at v, and $0 < s_{\mathsf{v}}(\mathsf{x}) \leq 1$.
This suggests that it is possible to renormalize the value of u at v by considering the limit

$$\lim_{\mathsf{x}\to\mathsf{v}} \|\mathsf{x}-\mathsf{v}\|^{-\frac{\pi}{\omega}} s_{\mathsf{v}}(\mathsf{x})^{-1} u(\mathsf{x})\,.$$

Such a procedure works and the limit operator, as the n points supporting the perturbations converge to the n non-convex vertices, turns out to be a well defined self-adjoint operator: it is a self-adjoint extension of the closed symmetric operator given by the Laplace operator on $H^2(\Omega) \cap H_0^1(\Omega)$.

Self-adjoint extensions of restrictions

Let

$$A : \mathscr{D}(A) \subseteq \mathscr{H} \to \mathscr{H}\,, \quad -A > 0\,,$$

be a semi-bounded self-adjoint operator and let

$$\tau : \mathscr{D}(A) \to \mathbb{C}^n\,.$$

be a continuous (w.r.t. the graph norm on $\mathscr{D}(A)$) and surjective linear map such that $\mathscr{K}(\tau)$ is dense in $\mathscr{H}$. Then

$$S : \mathscr{D}(S) \subseteq \mathscr{H} \to \mathscr{H}\,, \quad S := A|\mathscr{K}(\tau)$$

is a semi-bounded, closed symmetric operator with deficiency indices (n, n). Its Friedrichs' extension coincides with A.

One has the following

Theorem. [AP 2001, AP 2008]
1. The set of all self-adjoint extensions of S is parametrized by the bundle $p : \mathsf{E}(\mathbb{C}^n) \to \mathsf{P}(\mathbb{C}^n)$, where $\mathsf{P}(\mathbb{C}^n)$ is the set of orthogonal projectors on $\mathbb{C}^n$ and $p^{-1}(\Pi)$ is the set of symmetric operators on $\mathbb{C}^n_\Pi := \mathscr{R}(\Pi)$.
2. Defining

$$G_z : \mathbb{C}^n \to \mathscr{H} \,, \quad G_z := \left(\tau(-A+\bar{z})^{-1}\right)^* \,, \quad z \in \rho(A) \,,$$

one has

$$\mathscr{R}(G_z) \cap \mathscr{D}(A) = \{0\}$$

and, if $A^{\Pi,\Theta}$ denotes the self-adjoint extension corresponding to $(\Pi,\Theta)\in \mathsf{E}(\mathbb{C}^n)$, then

$$A^{\Pi,\Theta}:\mathscr{D}(A^{\Pi,\Theta})\subseteq\mathscr{H}\to\mathscr{H}\,,\quad A^{\Pi,\Theta}\phi:=A\phi_0\,,$$

$$\mathscr{D}(A^{\Pi,\Theta}):=\{\phi=\phi_0+G_0\xi_\phi\,,\ \phi_0\in\mathscr{D}(A)\,,\,\xi_\phi\in\mathbb{C}^n_\Pi\,,\,\Pi\tau\phi_0=\Theta\xi_\phi\}\ .$$

Moreover the resolvent of $A^{\Pi,\Theta}$ is given by the Kreĭn's type formula

$$(-A^{\Pi,\Theta}+z)^{-1}=(-A+z)^{-1}+G_z\Pi(\Theta+z\Pi G_0^*G_z\Pi)^{-1}\Pi G_{\bar z}^*\,.$$

Since $S = A|\mathscr{K}(\tau) = A|\mathscr{K}(\tau_M)$, where $\tau_M := M\tau$ and $M \in GL(\mathbb{C}^n)$, $GL(\mathbb{C}^n)$ acts on the bundle $\mathsf{E}(\mathbb{C}^n)$ by

$$\alpha : GL(\mathbb{C}^n) \times \mathsf{E}(\mathbb{C}^n) \to \mathsf{E}(\mathbb{C}^n)\,, \quad \alpha(M, (\Pi, \Theta)) = (\Pi_M, \Theta_M)\,,$$

where α is defined in such a way that

$$A_M^{\Pi_M,\Theta_M} = A^{\Pi,\Theta}\,,$$

with $A_M^{\Pi,\Theta}$ the extension corresponding to $(\Pi, \Theta) \in \mathsf{E}(\mathbb{C}^n)$ in the case one uses the map τ_M.
The action α is explicitly given in the following
Lemma.

$$\mathbb{C}^n_{\Pi_M} = (M^*)^{-1}(\mathbb{C}^n_\Pi)\,, \qquad \Theta_M = \Pi_M M \Theta M^* \Pi_M\,.$$

Finally one has a simple convergence result:

Lemma. Let the sequence $\tau_N : \mathscr{D}(A) \to \mathbb{C}^n$ be converging, with respect to the norm on bounded linear operators, to $\tau : \mathscr{D}(A) \to \mathbb{C}^n$ as $N \uparrow \infty$. Given $\Pi \in \mathsf{P}(\mathbb{C}^n)$, let the sequence $\Theta_N : \mathbb{C}^n_\Pi \to \mathbb{C}^n_\Pi$ be converging to $\Theta : \mathbb{C}^n_\Pi \to \mathbb{C}^n_\Pi$ as $N \uparrow \infty$. Let A_N^{Π,Θ_N} and $A^{\Pi,\Theta}$ denote the corresponding self-adjoint extensions of $S_N = A|\mathscr{K}(\tau_N)$ and $S = A|\mathscr{K}(\tau)$ respectively. Then A_N^{Π,Θ_N} converges in norm resolvent sense to $A^{\Pi,\Theta}$ as $N \uparrow \infty$.

Dirichlet Laplacians on non-convex polygons

Let $\Omega \subset \mathbb{R}^2$ be a plane bounded open curvilinear polygon with boundary Γ. By the Cacciopoli's type estimate

$$\forall u \in C_0^\infty(\Omega)\,, \qquad \|u\|_{H^2(\Omega)} \leq c_\Omega \, \|\Delta u\|_{L^2(\Omega)}\,,$$

the restriction of Δ to $C_0^\infty(\Omega)$ is closable and its closure is the symmetric operator

$$\Delta_\Omega^\circ : \mathscr{D}(\Delta_\Omega^\circ) \subset L^2(\Omega) \to L^2(\Omega)\,, \qquad \Delta_\Omega^\circ u := \Delta_\Omega u\,.$$

$$\mathscr{D}(\Delta_\Omega^\circ) = \{u \in H^2(\Omega) \,:\, u(x) = 0\,,\ x \in \Gamma\} = H^2(\Omega) \cap H_0^1(\Omega)\,.$$

Question: is Δ_Ω° self-adjoint? Equivalently: does Δ_Ω° coincide with Δ_Ω^F, the Friedrichs' extension of $\Delta|C_0^\infty(\Omega)$?

Answer (Birman and Skvortsov 1962): the deficiency indices of Δ_Ω° are both equal to the number of non-convex corners of Ω.

Example. Take $\Omega = \{0 < r < 1\,,\ 0 < \theta < \omega\}$. The function

$$g(r, \theta) = \frac{1}{\sqrt{\pi}} \left(\frac{1}{r^{\pi/\omega}} - r^{\pi/\omega} \right) \sin \frac{\pi}{\omega}\,\theta,$$

solves the boundary value problem

$$\begin{aligned} \Delta_\Omega\, g(x) &= 0\,, \quad x \in \Omega\,, \\ \lim_{x \to x_0} g(x) &= 0\,, \quad x_0 \in \Gamma \backslash \{0\}\,. \end{aligned}$$

Notice that $g \in L^2(\Omega)$ if and only if $\omega \in (\pi, 2\pi]$, i.e. when Ω is non-convex.

From now on we suppose that Ω is a plane bounded open curvilinear polygon which coincides with a plane polygon in the neighborhood its n non-convex corners. We denote by $\mathsf{v}_1, \ldots, \mathsf{v}_n$ the vertices at the non-convex corners, by ω_k the measure of the angle at v_k and we pose $\beta_k := \pi/\omega_k$.

Lemma. Let

$$\tau_\Omega^{\mathsf{V}} : \mathscr{D}(\Delta_\Omega^F) \to \mathbb{C}^n, \quad \left(\tau_\Omega^{\mathsf{V}} u\right)_k := \frac{\sqrt{\pi^3}}{4}\,(2+\beta_k)\,\lim_{R\downarrow 0}\ \frac{1}{R^{\beta_k}}\,\langle u\rangle_{\mathsf{W}_k^R}\,,$$

where $\langle u\rangle_{\mathsf{W}_k^R}$ denotes the mean of u over the wedge W_k^R of radius R centered at v_k and with opening angle ω_k. Then τ_Ω^{V} is well defined, continuous, surjective and

$$\mathscr{K}(\tau_\Omega^{\mathsf{V}}) = \mathscr{D}(\Delta_\Omega^\circ) \equiv H^2(\Omega)\cap H_0^1(\Omega)\,.$$

Moreover

$$G_z : \mathbb{C}^n \to L^2(\Omega)\,, \qquad G_z := \left(\tau_\Omega^{\mathsf{V}}(-\Delta_\Omega^F + \bar{z})^{-1}\right)^* \,.$$

is given by

$$G_z\xi = \sigma\cdot\xi + (-\Delta_\Omega^F + z)^{-1}(-\Delta_\Omega + z)\sigma\cdot\xi\,,$$

where

$$\sigma \equiv (\sigma_1, \cdots, \sigma_n)\,, \quad \sigma_k = f u_k^-\,, \quad u_k^-(r_k, \theta_k) = \frac{1}{\sqrt{\pi}}\,\frac{1}{r_k^{\beta_k}}\,\sin\beta_k\theta_k\,,$$

$0 < r_k < R$, R sufficiently small, $0 < \theta_k < \omega_k$, and f is a radial bump functions, $f(r_k) = 1$ when $0 < r_k < R/3$ and $f(r_k) = 0$ when $r_k > 2R/3$.

For notational convenience we pose

$$g \equiv (g_1, \ldots, g_n)\,, \quad g_k := \sigma_k + (-\Delta_\Omega^F)^{-1} \Delta_\Omega \sigma_k\,.$$

Using the extension theorem with

$$A = \Delta_\Omega^F\,, \quad \tau = \tau_\Omega^V\,,$$

one defines the self-adjoint operator

$$\Delta_\Omega^{\Pi,\Theta} := A^{\Pi,\tilde{\Theta}}\,, \quad \tilde{\Theta} = \Theta - \Pi\Lambda\Pi\,,$$

$$\Lambda_{ij} = \langle (-\Delta_\Omega^F)^{-1} \Delta_\Omega \sigma_i, \Delta_\Omega \sigma_j \rangle_{L^2(\Omega)} (1 - \delta_{ij})\,.$$

Theorem. Any self-adjoint extension of Δ°_Ω is of the kind

$$\Delta_\Omega^{\Pi,\Theta} : \mathscr{D}(\Delta_\Omega^{\Pi,\Theta}) \subset L^2(\Omega) \to L^2(\Omega)\,, \quad \Delta_\Omega^{\Pi,\Theta} u := \Delta_\Omega u\,,$$

$$\mathscr{D}(\Delta_\Omega^{\Pi,\Theta}) :=$$
$$\{u \in L^2(\Omega)\ :\ u = u_0 + g\cdot\xi_u\,,\ u_0 \in \mathscr{D}(\Delta_\Omega^F)\,,\ \xi_u \in \mathbb{C}^n_\Pi\,,\ \Pi\hat\tau^{\mathsf{V}}_\Omega u = \Theta\xi_u\}\,,$$

$$(\hat\tau^{\mathsf{V}}_\Omega u)_k = (\tau^{\mathsf{V}}_\Omega(u - (\xi_u)_k\, g_k))_k\,.$$

Moreover

$$(-\Delta_\Omega^{\Pi,\Theta} + z)^{-1} = (-\Delta_\Omega^F + z)^{-1} + G_z\Pi\,(\Theta + \Pi\,\Gamma_z\Pi)^{-1}\Pi\, G^*_{\bar z}\,,$$

$$(\Gamma_z)_{ij} = \left(z\|\sigma_i\|^2_{L^2(\Omega)} + \|(-\Delta_\Omega^F)^{-\frac12}\Delta_\Omega\sigma_i\|^2_{L^2(\Omega)}\right)\,\delta_{ij}$$
$$-\langle(-\Delta_\Omega^F + z)^{-1}(-\Delta_\Omega + z)\sigma_i, (-\Delta_\Omega + z)\sigma_j\rangle_{L^2(\Omega)}$$

Remark Let $\hat{\gamma}_\Omega$ be the unique continuous extensions to $\mathscr{D}(\Delta_\Omega^{max})$ of the continuos linear map

$$\gamma_\Omega : \mathscr{D}(\Delta_\Omega^{max}) \cap H^1(\Omega) \to H^{1/2}(\Gamma)\,,$$

$$\forall u \in C^\infty(\bar{\Omega})\,, \quad [\gamma_\Omega u](\mathrm{x}) = u(\mathrm{x})\,, \quad \mathrm{x} \in \Gamma\,.$$

Then g_k, $1 \le k \le n$, belong to $\mathscr{K}(\hat{\gamma}_\Omega)$, and so all the self-adjoint extensions of Δ_Ω° have domains contained in $\mathscr{K}(\hat{\gamma}_\Omega)$, the only one with domain contained in $\mathscr{K}(\gamma_\Omega)$ being the Friedrichs' Laplacian Δ_Ω^F. Thus we can interpret the set of all self-adjoint extensions of Δ_Ω° different from Δ_Ω^F as the set of self-adjoint, non-Friedrichs' Dirichlet Laplacians on $L^2(\Omega)$.

Remark Notice that if both Π and Θ are diagonal, then both the boundary conditions $\Pi\hat{\tau}_\Omega^{\mathsf{V}} u = \Theta\xi_u$ are local, i.e. they do not couple values of u at different vertices.

Example. If Ω is the non-convex wedge W of radius R and opening angle $\omega \in (\pi, 2\pi]$ then

$$g(r,\theta) = \frac{1}{\sqrt{\pi}} \left(\frac{1}{r^\beta} - \frac{r^\beta}{R^{2\beta}} \right) \sin \beta\theta \,, \quad \beta = \frac{\pi}{\omega} \,,$$

and $G_z : \mathbb{C} \to L^2(\mathrm{W})$ acts as the multiplication by the function

$$\begin{aligned} & g_z(r,\theta) \\ =& \frac{1}{\sqrt{\pi}} \left(\frac{\sqrt{z}}{2} \right)^\beta \Gamma\,(1-\beta) \left(J_{-\beta}(\sqrt{z}\, r) - \frac{J_{-\beta}(\sqrt{z}\, R)}{J_\beta(\sqrt{z}\, R)} J_\beta(\sqrt{z}\, r) \right) \sin \beta\theta \,, \end{aligned}$$

where $\mathrm{Re}(\sqrt{z}) > 0$, $\Gamma(x)$ denotes Euler's gamma function at x and J_ν denotes the Bessel function of order ν.

Then the set of non-Friedrichs' Dirichlet Laplacians on W is parametrized by $\theta \in \mathbb{R}$ and any of such extensions has resolvent R_z^θ with kernel

$$R_z^\theta(x, y) = R_z^F(x, y) + \left(\theta + \left(\frac{z}{4} \right)^\beta \frac{\Gamma(-\beta)}{\Gamma(\beta)} \frac{J_{-\beta}(\sqrt{z}\, R)}{J_\beta(\sqrt{z}\, R)} \right)^{-1} g_z(x) g_z(y)\,,$$

where R_z^F denotes the resolvent of the Friedrichs-Dirichlet Laplacian.

Friedrichs-Dirichlet Laplacians with point interactions

Given $\mathrm{Y} = \{\mathrm{y}_1, \dots, \mathrm{y}_n\} \subset \Omega$, we consider the linear, continuous surjective map with a dense (in $L^2(\Omega)$) kernel

$$\tau_\Omega^{\mathrm{Y}} : \mathscr{D}(\Delta_\Omega^F) \to \mathbb{C}^n , \qquad (\tau_\Omega^{\mathrm{Y}} u)_k := u(\mathrm{y}_k) , \quad k = 1, \dots, n .$$

Defining

$$G_z^{\mathrm{Y}} : \mathbb{C}^n \to L^2(\Omega) , \qquad G_z^{\mathrm{Y}} := \left(\tau_\Omega^{\mathrm{Y}} (-\Delta_\Omega^F + \bar{z})^{-1} \right)^* ,$$

one has

$$(G_z^{\mathrm{Y}} \xi)(\mathrm{x}) = \sum_{i=1}^{n} g_\Omega(z; \mathrm{x}, \mathrm{y}_i) \xi_i$$

where $g_\Omega(z;\cdot,\cdot)$ is the Green's function of $-\Delta^F_\Omega + z$, i.e.

$$g_\Omega(z;\mathrm{x},\mathrm{y}) = g(z;\mathrm{x},\mathrm{y}) - h_\Omega(z;\mathrm{x},\mathrm{y})\,,$$

$$g(0;\mathrm{x},\mathrm{y}) = \frac{1}{2\pi}\,\ln\frac{1}{\|\mathrm{x}-\mathrm{y}\|}\,,$$

$$g(z;\mathrm{x},\mathrm{y}) = \frac{1}{2\pi}\,K_0(\sqrt{z}\,\|\mathrm{x}-\mathrm{y}\|)\,,\quad \mathrm{Re}(\sqrt{z}) > 0\,,$$

K_0 the Macdonald function, and $h_\Omega(z;\cdot,\mathrm{y})$ solves

$$\begin{cases}(-\Delta^F_\Omega + z)h_\Omega(z;\mathrm{x},\mathrm{y}) = 0\,, & \mathrm{x}\in\Omega\\ h_\Omega(z;\mathrm{x},\mathrm{y}) = g(z;\mathrm{x},\mathrm{y})\,, & \mathrm{x}\in\Gamma\,.\end{cases}$$

Using the extension theorem with

$$A = \Delta_\Omega^F\,, \quad \tau = \tau_\Omega^{\mathrm{Y}}\,,$$

one defines the self-adjoint operator

$$\Delta_{\Omega,\mathrm{Y}}^{\Pi,\Theta} := A^{\Pi,\tilde{\Theta}}\,, \quad \tilde{\Theta} = \Theta - \Pi\Lambda^{\mathrm{Y}}\Pi\,,$$

$$\Lambda_{ij}^{\mathrm{Y}} := g_\Omega(0;\mathrm{y}_i,\mathrm{y}_j)\,(1-\delta_{ij})\,.$$

Theorem. Any self-adjoint extension of $\Delta^{\circ}_{\mathsf{Y},\Omega}$ is of the kind

$$\Delta^{\Pi,\Theta}_{\mathsf{Y},\Omega} : \mathscr{D}(\Delta^{\Pi,\Theta}_{\mathsf{Y},\Omega}) \subset L^2(\Omega) \to L^2(\Omega)\,, \quad \Delta^{\Pi,\Theta}_{\mathsf{Y},\Omega} u := \Delta^{F}_{\Omega} u_0\,,$$

$$\mathscr{D}(\Delta^{\Pi,\Theta}_{\mathsf{Y},\Omega}) :=$$
$$\{u \in L^2(\Omega)\ :\ u = u_0 + G^{\mathsf{Y}}_0 \xi_u\,,\ u_0 \in \mathscr{D}(\Delta^{F}_{\Omega})\,,\ \xi_u \in \mathbb{C}^n_{\Pi}\,,\ \Pi \hat{\tau}^{\mathsf{Y}}_{\Omega} u = \Theta \xi_u\},$$

$$\left(\hat{\tau}^{\mathsf{Y}}_{\Omega} u\right)_k := \lim_{\mathsf{x}\to\mathsf{y}_k} \left(u(\mathsf{x}) - \frac{(\xi_u)_k}{2\pi}\, g_{\Omega}(0;\mathsf{x},\mathsf{y}_k)\right)\,.$$

Moreover

$$(-\Delta^{\Pi,\Theta}_{\mathsf{Y},\Omega} + z)^{-1} = (-\Delta^{F}_{\Omega} + z)^{-1} + G^{\mathsf{Y}}_z \Pi\,(\Theta + \Pi\,\Gamma^{\mathsf{Y}}_z \Pi)^{-1} \Pi\,(G^{\mathsf{Y}}_{\bar{z}})^*\,,$$

$$\left(\Gamma^{\mathsf{Y}}_z\right)_{ij} := \left(\frac{1}{2\pi}\,\left(\ln\left(\frac{\sqrt{z}}{2}\right) - \psi(1)\right) - h_{\Omega}(0;\mathsf{y}_i,\mathsf{y}_i) + h_{\Omega}(z;\mathsf{y}_i,\mathsf{y}_i)\right)\,\delta_{ij}$$
$$- g_{\Omega}(z;\mathsf{y}_i,\mathsf{y}_j)\,(1-\delta_{ij})\,.$$

Approximating non-Friedrichs Dirichlet Laplacians by point perturbations

Let $\{Y_N\}_1^\infty$ denote a sequence of discrete sets $Y_N = \{y_k^N\}_1^n \subset \Omega$ such that, for any $1 \le k \le n$,

$$\lim_{N\uparrow\infty} y_k^N = v_k \,,$$

$$y_k^N \equiv (r_k^N \cos\theta_k^N, r_k^N \sin\theta_k^N)\,,$$

and

$$\inf_N \sin\beta_k\theta_k^N = c > 0\,.$$

Posing

$$\widetilde{\tau}_{\Omega}^{\mathsf{Y}_N} : \mathscr{D}(\Delta_{\Omega}^{F}) \to \mathbb{C}^n\,, \quad \left(\widetilde{\tau}_{\Omega}^{\mathsf{Y}_N} u\right)_k := \frac{u(\mathrm{y}_k^N)}{s_k(\mathrm{y}_k^N)}\,,$$

where

$$s_k = f u_k^+\,, \quad u_k^+(r_k, \theta_k) = \frac{1}{\sqrt{\pi}}\; r_k^{\beta_k}\, \sin \beta_k \theta_k\,,$$

i.e.

$$\tau_{\Omega}^{\mathsf{Y}_N} = M_N\, \widetilde{\tau}_{\Omega}^{\mathsf{Y}_N}\,, \quad (M_N)_{ij} := s_i(\mathrm{y}_i^N)\, \delta_{ij}\,, \tag{1}$$

one has the following

Lemma. There exist $c > 0$ and $0 < \alpha_k < 1 - \beta_k$ such that

$$\left|\left(\widetilde{\tau}_{\Omega}^{\mathsf{Y}_N} u - \tau_{\Omega}^{\mathsf{V}} u\right)_k\right| \le c\, \|\mathrm{y}_k^N - \mathrm{v}_k\|^{\alpha_k} \|\Delta_{\Omega}^{F} u\|_{L^2(\Omega)}\,.$$

Using the extension theorem with

$$A = \Delta_\Omega^F\,, \quad \tau = \widetilde{\tau}_\Omega^{\mathsf{Y}}\,,$$

one defines the self-adjoint operator

$$\tilde{\Delta}_{\Omega,\mathsf{Y}}^{\Pi,\Theta} := A_N^{\Pi,\tilde{\Theta}_N}\,, \quad \tilde{\Theta}_N = \Theta - \Pi M_N^{-1} \Lambda^{\mathsf{Y}_N} M_N \Pi\,,$$

Then, since $\widetilde{\tau}_\Omega^{\mathsf{Y}_N} = M_N\, \widetilde{\tau}_\Omega^{\mathsf{Y}_N}$, one has

$$\tilde{\Delta}_{\Omega,\mathsf{Y}}^{\Pi,\Theta} = \Delta_{\Omega,\mathsf{Y}}^{\Pi_N,\Theta_N}\,, \quad \mathbb{C}^n_{\Pi_N} = M_N^{-1}(\mathbb{C}^n_\Pi)\,, \quad \Theta_N = \Pi_N M_N \Theta M_N \Pi_N\,.$$

We know that $\widetilde{\tau}_\Omega^{\mathsf{Y}_N} \to \tau_\Omega^{\mathsf{V}}$ and one can show that $\tilde{\Theta}_N \to \tilde{\Theta}$. Thus one gets the following

Theorem. Let $(\Pi, \Theta) \in \mathsf{E}(\mathbb{C}^n)$ and define $(\Pi_N, \Theta_N) \in \mathsf{E}(\mathbb{C}^n)$ by

$$\mathbb{C}^n_{\Pi_N} = M_N^{-1}(\mathbb{C}^n_{\Pi}), \qquad \Theta_N = \Pi_N M_N \Theta M_N \Pi_N .$$

Then $\Delta^{\Pi_N,\Theta_N}_{\mathsf{Y}_N,\Omega}$ converges in norm resolvent sense to $\Delta^{\Pi,\Theta}_{\Omega}$ as $N \uparrow \infty$.