Trails in a Non Commutative Land
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Gauge fields over noncommutative manifolds

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Why we are doing this:

put together some Yang–Mills-Higgs systems

with noncommutative spaces

hoping to get interesting stuff

Vortices and gauge fields ; Taubes , ...

The Ginzburg–Landau equations for vortices is related to the four dimensional Yang–Mills equations via reduction:

any $SO(3)$ symmetric solution to the $SU(2)$ Y–M eqs on $\mathbb{R}^2 \times S^2$

determines a solution to the G–L eqs on $\mathbb{R}^2$ and vice versa.

M a compact oriented Riemann surface
(or maybe $\mathbb{R}^2$ with suitable boundary conditions,
e.g. locally bounded and globally square integrable curvatures);

holomorphic vector bundles $\mathcal{E}_0, \mathcal{E}_1$ over M and a holomorphic

$$\mathcal{E}_0 \ \xrightarrow{\phi} \ \mathcal{E}_1$$

self-duality equations

$$\star F_\nabla = -F_\nabla$$

on $M \times S^2$ are vortex equations

$$\star F_{\nabla_0} = \mathrm{id}_{\mathcal{E}_0} - \phi \circ \phi^* \quad \text{and} \quad \star F_{\nabla_1} = -\,\mathrm{id}_{\mathcal{E}_1} + \phi^* \circ \phi$$

on M

Equivariant dimensional reduction :

a systematic procedure for including internal fluxes on S/R (instantons and/or monopoles of R-fields)

which are 'symmetric' (equivariant) under S

Monopoles ; relevant for QHE
F.D. Haldane,
Fractional quantization of the Hall effect: A hierarchy of incompressible quantum fluid states, Phys. Rev. Lett. (1983)

Instantons ; relevant for Spin HE
S.-C. Zhang, J.-P. Hu
A four-dimensional generalization of the quantum Hall effect, Science (2001)

Physics: A brief history of dimensional reduction

Kaluza (1921), Klein (1926): the observed fundamental forces in 4-dimensions can be understood in terms of the dynamics of a simpler higher dimensional theory

Starting from a 5-D theory on $M_5 = M_4 \times S^1$, the product of a curved 4-D space-time M_4 and a circle with radius r (and coordinate $0 \leq y < 2\pi$). Take the line element

$$ds^2_{(5)} = ds^2_{(4)} + \Big(r dy + A(x)\Big)^2,$$

with $A(x) = A_\mu(x) dx^\mu$ a 4-dimensional vector potential

The 5-dimensional Einstein action reduces to

$$\frac{1}{2\pi r}\int_{M_5}\sqrt{-g_{(5)}}\,R_{(5)}\,d^4x\,dy = \int_{M_4}\sqrt{-g_{(4)}}\Big(R_{(4)} - \frac{1}{4}F^2\Big)d^4x,$$

$F = dA$ is a $U(1)$ field strength in 4-dimensions

Matter, e.g. a scalar field Φ, harmonically expanded on S^1,

$$\Phi(x,y) = \sum_{n=-\infty}^{\infty}\phi_n(x)e^{\frac{\mathrm{i}ny}{r}},$$

then the 5-dimensional kinetic term for Φ gives rise to an infinite tower of massive fields $\phi_n(x)$ in M_4, with masses $m_n = \frac{n}{r}$.

A non-abelian generalisation of the K-K idea,

Start from d-dimensional Y–M theory on $M_4 \times S/R$ with gauge group G

If $R \subset G$, integrating over S/R gives a Y–M–H system on M_4, with gauge group K, the centraliser, i.e. $[R, K] = 0$, of R in G

Upon dimensional reduction the internal components of the d-dimensional gauge field A play the rôle of Higgs fields in 4-dim:

$$A(x,y) \quad \longrightarrow \quad \begin{cases} A_\mu(x) & \text{(4-dim gauge fields)} \\ \\ \Phi_a(x) & \text{(4-dim Higgs fields)} \end{cases}$$

A Higgs potential is generated from the d-dimensional Y–M action. Indeed, the full d-dim Y–M action reduces as

$$-\frac{1}{4}\int_{M_d}\sqrt{-g_{(d)}}\ \mathrm{Tr}(F^2)d^4x\,d^{d-4}y =$$

$$= \mathrm{vol}(S/R)\int_{M_4}\sqrt{-g_{(4)}}\ \mathrm{tr}\Big(-\frac{1}{4}F^2+\big(D\Phi\big)^{\dagger}D\Phi - V(\Phi)\Big)d^4x$$

The Higgs potential breaks K dynamically: if $S\subset G$, $V(\Phi)$ breaks K spontaneously to K', the centraliser, $[S,K']=0$, of S in G.

The Ginsburg–Landu action functional

$$GL(A,\Phi) \quad = \quad \int_{\mathbb{R}^2} \mathrm{tr}\Big(\, - \, \frac{1}{4}F^2 \, + \, D\Phi^\dagger D\Phi \, + \, \lambda(\Phi^\dagger\Phi \, - \, 1)^2\Big)$$

with critical points

as mentioned self-duality equation are vortex equations:

$$\star F = \mathrm{id}_{\mathcal{E}_0} - \Phi \circ \Phi^* \qquad \text{and} \qquad D\Phi = 0$$

a caveat:

In the co-set space dimensional reduction programme,

spinors on $M_4 \times S/R$ cannot give a chiral theory on M_4

rather anti-climax, one should admit !!!

Equivariant dimensional reduction

Equivariant dimensional reduction is a systematic procedure for including internal fluxes on S/R (instantons and/or monopoles of R-fields) which are 'symmetric' (equivariant) under S

In general, a one-to-one correspondence between S-equivariant complex vector bundles over M_d

$$B \longrightarrow M_d = M_4 \times S/R,$$

and R-equivariant bundles over M_4,

$$E \longrightarrow M_4,$$

where S acts on the space M_d via the trivial action on M_4 and by the standard left translation action on S/R

In general the reduction yields rise quiver gauge theories on M_4

Including spinor fields, coupling to background equivariant fluxes, can give rise to chiral theories on M_4

Yukawa couplings are induced and the dimensional reduction can give masses to some zero modes of the Dirac operator on S/R

A simple example: Complex projective line

$S = SU(2)$ and $R = U(1)$, giving a 2-dim sphere $S^2 \simeq SU(2)/U(1)$ (or projective line $\mathbf{CP}^1$), and with $G = U(k)$.

An embedding $S \hookrightarrow G$ results into a decomposition

$$U(k) \to \prod_{i=0}^{m} U(k_i),$$

$k = \sum_{i=0}^{m} k_i$, associated with the $m+1$-dim I.R. of $SU(2)$

$\mathbf{g} \in G$ decomposes as:

$$\mathbf{g} = \left(\mathbf{g}_{k_i \times k_i} \right) \ , \qquad \mathbf{g}_{k_i \times k_i} \qquad \text{on} \quad \mathbf{C}^{k_i}$$

each $\mathbf{C}^{k_i}$ transforms under $U(k_i) \subset U(k)$

carries a $U(1)$ charge $p_j = m - 2j$, for $-m \leq p_j \leq m$

The $U(k)$ gauge potential, A on M_d, splits into $k_i \times k_j$ blocks

$$A(x,y) = A(x) + a(y) + \Phi(x)\overline{\beta}(y) + \Phi^\dagger(x)\beta(y),$$

$a = \oplus_{i=0}^{m} a_{m-2i}$, $\qquad a_{m-2i}$ charge $m-2i$ monopole connection

$A(x) = \oplus_{i=0}^{m} A^i(x)$, $A^i(x)$ is a $U(k_i)$ gauge connection on M_4,

and $\Phi(x)$ is a collection of Higgs fields.

$$A(x,y) = \begin{pmatrix} A^0 + a_m\,\mathbf{1}_{k_0} & \phi_1\overline{\beta} & 0 & \cdots & 0 \\ \phi_1^\dagger\beta & A^1 + a_{m-2}\,\mathbf{1}_{k_1} & \phi_2\overline{\beta} & \cdots & 0 \\ 0 & \phi_2^\dagger\beta & \vdots & \ddots & \vdots \\ \vdots & \vdots & 0 & \cdots & \phi_m\overline{\beta} \\ 0 & 0 & 0 & \cdots & A^m + a_{-m}\,\mathbf{1}_{k_m} \end{pmatrix}$$

Dimensional reduction generates a 4-dim Higgs potential,

$$V(\Phi) = \frac{g^2}{2}\,\mathrm{tr}_k\left(\frac{1}{4g^2r^2}\begin{pmatrix} m\mathbf{1}_{k_0} & 0 & \cdots & 0 \\ 0 & (m-2)\mathbf{1}_{k_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & -m\mathbf{1}_{k_m} \end{pmatrix} - [\Phi, \Phi^\dagger]\right)^2,$$

whose minimization gives a vacuum structure

depending on the monopole charges $p_j = m - 2j$

all of the above is related to stability of bundles

a form of Hitchin-Kobayashi correspondence

but

let us

move to noncommutative spaces

interesting consequences: e.g. de-singularization of moduli spaces

A dictionary :

Classical	**Noncommutative**
(locally) compact space	(unital) C^*-algebra
smooth manifold	C^*-algebra with 'smooth' subalgebra
vector bundle	finite projective module
spin structure	spectral triple

....

Symmetries

$\mathsf{U} = (\mathsf{U}, \Delta, S, \varepsilon)$ a Hopf $*$-algebra

$\Delta : \mathsf{U} \to \mathsf{U} \otimes \mathsf{U}$ the coproduct

$S : \mathsf{U} \to \mathsf{U}$ the antipode

$\varepsilon : \mathsf{U} \to \mathbb{C}$ counit

$\mathcal{A}$ a left U-module $*$-algebra: there is a left action $\triangleright$ of U on $\mathcal{A}$,

$$h \triangleright xy = (h_{(1)} \triangleright x)(h_{(2)} \triangleright y),$$

$$h \triangleright 1 = \varepsilon(h)1, \quad (h \triangleright x)^* = S(h)^* \triangleright x^*,$$

notation $\Delta : \mathsf{U} \to \mathsf{U} \otimes \mathsf{U}$, $\qquad \Delta(h) = h_{(1)} \otimes h_{(2)}$.

Connections on bundles

a space

traded with

a noncommutative algebra $\mathcal{A}$ (and a calculus $(\Omega(\mathcal{A}), \mathrm{d})$

a vector bundle

traded with

a finitely generated projective (right) $\mathcal{A}$-module $\mathcal{E}$

a connection:

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1(\mathcal{A}), \qquad \nabla(\eta a) = \nabla(\eta) a + \eta \mathrm{d} a$$

M a smooth manifold; $\mathbb{C}\mathsf{P}^1_q$ the quantum projective line

Characterize vector bundles over the quantum space

$$\underline{M} := \mathbb{C}\mathsf{P}^1_q \times M$$

equivariant under an action of the quantum group $\mathsf{SU}_q(2)$

These are finitely-generated and projective $\mathsf{SU}_q(2)$-equivariant modules over the algebra of functions

$$\mathcal{A}(\underline{M}) = \mathcal{A}(\mathbb{C}\mathsf{P}^1_q) \otimes \mathcal{A}(M)$$

Describe the dimensional reduction of invariant connections

In particular, Yang–Mills gauge theory on $\mathcal{A}(\underline{M})$ is reduced to a type of Yang–Mills–Higgs theory on the manifold M

The equations of motion give q-deformations of known vortex equations, whose solutions possess remarkable properties

In particular de-singularization of moduli spaces

$q \in \mathbb{R}_{>0}$ $\qquad$ $q \simeq q^{-1}$

$\mathcal{A}(\mathrm{SU}_q(2))$:= $*$-algebra generated by a and c, with relations

$$UU^* = U^*U = 1 \qquad U = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix}$$

$$ac = qca, \qquad ac^* = qc^*a, \qquad cc^* = c^*c,$$

$$a^*a + c^*c = aa^* + q^2cc^* = 1$$

Hopf $*$-algebra structure on $\mathcal{A}(\mathrm{SU}_q(2))$:

$$\Delta U = U \otimes U \qquad S(U) = U^* \qquad \varepsilon(U) = 1$$

These dualize classical operations

$\mathcal{A}_1 = \mathcal{A}(\mathsf{SU}(2))$, polynomial functions on SU(2)

$$\Delta : \mathcal{A}_1 \to \mathcal{A}_1 \otimes \mathcal{A}_1; \qquad (\Delta f)(x \otimes y) = f(xy)$$

$$S : \mathcal{A}_1 \to \mathcal{A}_1; \qquad (Sf)(x) = f(x^{-1})$$

$$\varepsilon : \mathcal{A}_1 \to \mathbb{C}; \qquad (\varepsilon f) = f(e)$$

A (right) action: $\alpha : \mathsf{U}(1) \to \mathsf{Aut}(\mathcal{A}(\mathsf{SU}_q(2)))$

$$\alpha_u \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \qquad \text{for} \quad u \in \mathsf{U}(1).$$

The invariant elements form a subalgebra of $\mathcal{A}(\mathsf{SU}_q(2))$, the co-ordinate algebra $\mathcal{A}(\mathsf{S}^2_q)$ of the standard Podleś sphere S^2_q

$$\mathcal{A}(\mathsf{S}^2_q) = \mathcal{A}(\mathsf{SU}_q(2))^{\mathsf{U}(1)}$$

the algebra inclusion

$$\mathcal{A}(\mathsf{S}^2_q) \hookrightarrow \mathcal{A}(\mathsf{SU}_q(2))$$

is a noncommutative principal bundle

As a set of generators for $\mathcal{A}(\mathsf{S}_q^2)$ we may take

$$B_- := ac^*, \qquad B_+ := ca^*, \qquad B_0 := cc^*.$$

A natural complex structure on the 2-sphere S_q^2
for the unique 2-dimensional $\mathsf{SU}_q(2)$-covariant calculus;

S_q^2 becomes a quantum Riemannian sphere
or quantum projective line $\mathbb{C}\mathsf{P}_q^1$

noncommutative manifolds

$\mathsf{SU}_q(2)$ and S_q^2

admit equivariant spectral triples $(\mathcal{A}, \mathcal{H}, D)$

A vector space decomposition

$$\mathcal{A}(\mathsf{SU}_q(2)) = \bigoplus_{n\in\mathbb{Z}} \mathcal{L}_n \ , \qquad (\star)$$

$$\mathcal{L}_n := \mathcal{A}(\mathsf{SU}_q(2)) \boxtimes_{\rho_n} \mathbb{C} \simeq \Big\{ x \in \mathcal{A}(\mathsf{SU}_q(2)) \ \Big| \ \alpha_u(x) = x\,(u^*)^n \Big\}$$

for $u \in \mathsf{U}(1)$

Each $\mathcal{L}_n$ is a finitely-generated projective (right, say) $\mathcal{A}(\mathbb{C}\mathsf{P}^1_q)$-module of rank one

module of $\mathsf{SU}_q(2)$-equivariant sections of a line bundles over the quantum projective line $\mathbb{C}\mathsf{P}^1_q$ with degree (monopole charge) $-n$

Left covariant calculus

$$\Omega^i = \mathcal{A}(\mathsf{SU}_q(2)) \otimes \bigwedge^i\{\beta_+, \beta_-, \beta_z\} \quad 0 \leq i \leq 3$$

$$\Delta_L^{(1)}(\beta_s) = 1 \otimes \beta_s, \qquad s = z, \pm,$$

$\Delta_L^{(1)}$ the left-coaction extended to 1-forms

$\bigoplus \bigwedge^i\{\beta_+, \beta_-, \beta_z\}$ = the q-Grassmann algebra:

$$\begin{aligned}
&\beta_+ \wedge \beta_+ = \beta_- \wedge \beta_- = \beta_z \wedge \beta_z = 0 \\
&\beta_- \wedge \beta_+ + q^{-2}\beta_+ \wedge \beta_- = 0 \\
&\beta_z \wedge \beta_- + q^4\beta_- \wedge \beta_z = 0, \\
&\beta_z \wedge \beta_+ + q^{-4}\beta_+ \wedge \beta_z = 0.
\end{aligned}$$

unique top form: $\beta_- \wedge \beta_+ \wedge \beta_z$

Differential d : $\mathcal{A}(\mathrm{SU}_q(2)) \to \Omega^1(\mathrm{SU}_q(2))$:

$$\mathrm{d}f = (X_+ \triangleright f)\,\beta_+ + (X_- \triangleright f)\,\beta_- + (X_z \triangleright f)\,\beta_z,$$

$$X_z = \frac{1-K^4}{1-q^{-2}}, \qquad X_- = q^{-1/2}FK, \qquad X_+ = q^{1/2}EK$$

E, F, K, K^{-1} generates the q.u.e.a. $\mathcal{U}_q(\mathrm{su}(2))$:

$\mathcal{U}_q(\mathrm{su}(2))$ is a Hopf $*$-algebra of twisted derivation on $\mathrm{SU}_q(2)$:

$$h \triangleright x := x_{(1)} \left\langle h\,,\, x_{(2)} \right\rangle, \qquad x \triangleleft h := \left\langle h\,,\, x_{(1)} \right\rangle x_{(2)}$$

with notation $\Delta(x) = x_{(1)} \otimes x_{(2)}$

$\langle\ ,\ \rangle$ a natural pairing between $\mathcal{U}_q(\mathrm{su}(2))$ and $\mathcal{A}(\mathrm{SU}_q(2))$

The holomorphic calculus on $\mathbb{C}\mathsf{P}^1_q$

By restriction

$$\Omega(\mathcal{A}(\mathsf{S}^2_q)) \simeq \mathcal{A}(\mathsf{S}^2_q) \oplus \left(\mathcal{L}_{-2}\beta_- \oplus \mathcal{L}_2\beta_+\right) \oplus \mathcal{A}(\mathsf{S}^2_q)$$

In particular

$$\Omega^1(\mathcal{A}(\mathsf{S}^2_q)) = \Omega^-(\mathcal{A}(\mathsf{S}^2_q)) \oplus \Omega^+(\mathcal{A}(\mathsf{S}^2_q)) \simeq \mathcal{L}_{-2}\beta_- \oplus \mathcal{L}_2\beta_+$$

a complex structure

$$\mathsf{d} = \partial + \overline{\partial}, \qquad \mathsf{d}f = (X_+ \triangleright x)\beta_+ + (X_- \triangleright x)\beta_-$$

$$\partial f = (X_+ \triangleright x)\beta_+, \quad \overline{\partial} f = (X_- \triangleright x)\beta_-$$

Also, $\qquad \Omega^2(\mathcal{A}(\mathsf{S}^2_q)) = \mathcal{A}(\mathsf{S}^2_q)(\beta_+ \wedge \beta_-) = (\beta_+ \wedge \beta_-)\mathcal{A}(\mathsf{S}^2_q)$

Enlarging the space

For a smooth manifold M, consider $\underline{M} := \mathbb{C}\mathsf{P}^1_q \times M$
with 'coordinate' algebra,

$$\mathcal{A}(\underline{M}) := \mathcal{A}(\mathbb{C}\mathsf{P}^1_q) \otimes \mathcal{A}(M) \ .$$

A coaction of $\mathsf{SU}_q(2)$ on $\mathcal{A}(\underline{M})$;
trivially on $\mathcal{A}(M)$ and with canonical coaction Δ_L on $\mathcal{A}(\mathbb{C}\mathsf{P}^1_q)$:

$$\underline{\Delta} : \mathcal{A}(\underline{M}) \longrightarrow \mathcal{A}(\mathsf{SU}_q(2)) \otimes \mathcal{A}(\underline{M}) \ ,$$

$$b \otimes f \longmapsto m_{13}\Big(\Delta_L(b) \otimes (1 \otimes f)\Big) = b_{(-1)} \otimes b_{(0)} \otimes f$$

for $b \in \mathcal{A}(\mathbb{C}\mathsf{P}^1_q)$, $f \in \mathcal{A}(M)$. Here $\Delta_L(b) = b_{(-1)} \otimes b_{(0)}$

with $b_{(-1)} \in \mathcal{A}(\mathsf{SU}_q(2))$, $b_{(0)} \in \mathcal{A}(\mathbb{C}\mathsf{P}^1_q)$

A $\mathrm{SU}_q(2)$-equivariant right $\mathcal{A}(\underline{M})$-module $\underline{\mathcal{E}}$ carries a coaction

$$\delta : \underline{\mathcal{E}} \longrightarrow \mathcal{A}(\mathrm{SU}_q(2)) \otimes \underline{\mathcal{E}}$$

compatible with the coaction $\underline{\boldsymbol{\Delta}}$ of $\mathcal{A}(\mathrm{SU}_q(2))$ on $\mathcal{A}(\underline{M})$,

$$\delta(\varphi \cdot \underline{f}) = \delta(\varphi) \cdot \underline{\boldsymbol{\Delta}}(\underline{f}) \qquad \text{for all} \quad \varphi \in \underline{\mathcal{E}} \ , \ \underline{f} \in \mathcal{A}(\underline{M})$$

Relate $\mathcal{A}(\mathrm{SU}_q(2))$-equivariant bundles $\underline{\mathcal{E}}$ on the quantum space $\underline{M}$ to U(1)-equivariant bundles E over the manifold M

Proposition 1. *Every finitely-generated* $\mathsf{SU}_q(2)$*-equivariant projective module* $\underline{\mathcal{E}}$ *over* $\mathcal{A}(\underline{M})$ *equivariantly decomposes as*

$$\underline{\mathcal{E}} = \bigoplus_{i=0}^{m} \underline{\mathcal{E}}_{\,i} = \bigoplus_{i=0}^{m} \mathcal{L}_{m-2i} \otimes \mathcal{E}_i$$

(and uniquely up to isomorphism), for some $m \in \mathbb{N}_0$*;*

$\mathcal{E}_i$ *are modules of sections of (usual) vector bundles* E_i *over* M *with trivial* $\mathsf{SU}_q(2)$ *coactions;*

$\mathcal{L}_n$ *are the above modules of sections of* $\mathsf{SU}_q(2)$*-equivariant line bundles over* $\mathbb{C}\mathsf{P}^1_q$*.*

(there are also morphisms $\Phi_i \in \mathsf{Hom}_{\mathcal{A}(\underline{M})}(\underline{\mathcal{E}}_{\,i-1}, \underline{\mathcal{E}}_{\,i})$,
of $\mathcal{A}(\underline{M})$-modules, coming from the $\mathsf{SU}_q(2)$-coaction).

On each $\mathcal{A}(M)$-module $\mathcal{E}_i$ in this decomposition fix an $\mathcal{A}(M)$-valued hermitian structure

$$h_i : \mathcal{E}_i \times \mathcal{E}_i \longrightarrow \mathcal{A}(M) \ .$$

Combined with the hermitian structure on the line bundles $\mathcal{L}_n$ this gives an $\mathcal{A}(\underline{M})$-valued hermitian structure on $\underline{\mathcal{E}}_i$ defined by

$$\underline{h}_i = \widehat{h}_{m-2i} \otimes h_i : \underline{\mathcal{E}}_i \times \underline{\mathcal{E}}_i \longrightarrow \mathcal{A}(\mathbb{C}\mathrm{P}^1_q) \otimes \mathcal{A}(M) \ ,$$

and in turn a left $\mathrm{SU}_q(2)$-covariant hermitian structure on $\underline{\mathcal{E}}$ by

$$\underline{h} = \bigoplus_{i=0}^{m} \underline{h}_i : \underline{\mathcal{E}} \times \underline{\mathcal{E}} \longrightarrow \mathcal{A}(\underline{M}) \ .$$

Lemma 2. *A unitary connection* $\underline{\nabla}$ *on* $(\underline{\mathcal{E}}, \underline{h})$ *decomposes as*

$$\underline{\nabla} = \sum_{i=0}^{m} \Big(\underline{\nabla}_i + \sum_{j<i} \big(\underline{\beta}_{ji} - \underline{\beta}_{ji}^{*} \big) \Big) ,$$

where:

1. *Each* $\underline{\nabla}_i$ *is a unitary connection on* $(\underline{\mathcal{E}}_i, \underline{h}_i)$, *i.e.*

   $$\underline{h}_i(\underline{\nabla}_i\varphi, \psi) + \underline{h}_i(\varphi, \underline{\nabla}_i\psi) = \underline{\mathsf{d}}\Big(\underline{h}_i(\varphi, \psi)\Big) \qquad \textit{for} \quad \varphi, \psi \in \underline{\mathcal{E}}_i \; .$$

2. *For* $j \neq i$,

   $\underline{\beta}_{ji} \in \mathrm{Hom}_{\mathcal{A}(\underline{M})}(\underline{\mathcal{E}}_i, \Omega^1(\underline{\mathcal{E}}_j))$ *is the adjoint of* $-\underline{\beta}_{ij}$, *i.e.*

   $$\underline{h}(\underline{\beta}_{ji}\varphi, \psi) + \underline{h}(\varphi, \underline{\beta}_{ij}\psi) = 0 \qquad \textit{for} \quad \varphi \in \underline{\mathcal{E}}_i \; , \; \psi \in \underline{\mathcal{E}}_j \; .$$

In an analogous way, any element $\underline{A}$ of the space (of anti-hermitian elements) $\mathsf{Hom}^a_{\mathcal{A}(\underline{M})}(\underline{\mathcal{E}}, \Omega^1(\underline{\mathcal{E}}))$ decomposes as

$$\underline{A} = \sum_{i=0}^{m} \left(\underline{A}_i + \sum_{j<i} \left(\underline{A}_{ji} - \underline{A}^*_{ji} \right) \right) ,$$

with

$$\underline{A}_i \in \mathsf{Hom}^a_{\mathcal{A}(\underline{M})}(\underline{\mathcal{E}}_i, \Omega^1(\underline{\mathcal{E}}_i)) , \qquad \underline{A}_{ji} \in \mathsf{Hom}_{\mathcal{A}(\underline{M})}(\underline{\mathcal{E}}_i, \Omega^1(\underline{\mathcal{E}}_j))$$

leading to a decomposition

$$\mathsf{Hom}^a_{\mathcal{A}(\underline{M})}\left(\underline{\mathcal{E}}\,, \Omega^1(\underline{\mathcal{E}})\right) \simeq \bigoplus_{i=0}^{m} \left(\mathsf{Hom}^a_{\mathcal{A}(\underline{M})}\left(\underline{\mathcal{E}}_i\,, \Omega^1(\underline{\mathcal{E}}_i)\right) \oplus \bigoplus_{j<i} \mathsf{Hom}_{\mathcal{A}(\underline{M})}\left(\underline{\mathcal{E}}_i\,, \Omega^1(\underline{\mathcal{E}}_j)\right)\right)$$

$SU_q(2)$-invariant connections and gauge transformations

On $\underline{\mathcal{E}} = \bigoplus_i \underline{\mathcal{E}}_i = \bigoplus_i \mathcal{L}_{m-2i} \otimes \mathcal{E}_i$ a coaction $\Delta_{\underline{\mathcal{E}}}$ of $\mathcal{A}(SU_q(2))$ which combines the natural coaction of $\mathcal{A}(SU_q(2))$ on the modules $\mathcal{L}_{m-2i}$ and the trivial coaction on the modules $\mathcal{E}_i$:

$$\Delta_{\underline{\mathcal{E}}} : \underline{\mathcal{E}} \longrightarrow \mathcal{A}(SU_q(2)) \otimes \underline{\mathcal{E}} .$$

'Adjoint' coactions of $\mathcal{A}(SU_q(2))$ on

the space $\mathcal{C}(\underline{\mathcal{E}})$ of unitary connections,

the group $\mathcal{U}(\underline{\mathcal{E}})$ of gauge transformations,

the spaces $\mathrm{Hom}_{\mathcal{A}(\underline{M})}(\underline{\mathcal{E}}_i, \underline{\mathcal{E}}_j)$ and $\mathrm{Hom}_{\mathcal{A}(\underline{M})}(\underline{\mathcal{E}}_i, \Omega^1(\underline{\mathcal{E}}_j))$.

The corresponding space of coinvariant elements, i.e.

$$\mathcal{C}(\,\underline{\mathcal{E}}\,)^{\mathsf{SU}_q(2)} = \left\{ \underline{\nabla} \in \mathcal{C}(\,\underline{\mathcal{E}}\,) \;\middle|\; \Delta^{\mathcal{C}}(\,\underline{\nabla}\,) = 1 \otimes \underline{\nabla} \right\} ,$$

and similarly for the other spaces and coactions

These spaces of invariants are given in terms of objects defined on M and of canonical (and unique) objects defined on $\mathbb{C}\mathsf{P}^1_q$.

Proposition 3. *There is a bijection*

$$\mathcal{C}(\underline{\mathcal{E}})^{\mathsf{SU}_q(2)} \simeq \mathscr{C}(\underline{\mathcal{E}}) := \prod_{i=0}^{m} \Big(\mathcal{C}(\mathcal{E}_i) \times \mathrm{Hom}_{\mathcal{A}(M)}(\mathcal{E}_i, \mathcal{E}_{i+1})\Big)$$

associate, to any element (∇, ϕ) *of* $\mathscr{C}(\underline{\mathcal{E}})$*, given by:*

- *connections* $\nabla_i \in \mathcal{C}(\mathcal{E}_i)$
- *Higgs fields* $\phi_{i+1} \in \mathrm{Hom}_{\mathcal{A}(M)}(\mathcal{E}_i, \mathcal{E}_{i+1})$,

the $\mathsf{SU}_q(2)$*-invariant unitary connection* $\underline{\nabla} \in \mathcal{C}(\underline{\mathcal{E}})^{\mathsf{SU}_q(2)}$*:*

$$\underline{\nabla} = \sum_{i=0}^{m} \left(\underline{\nabla}_i + \beta_+ \otimes \phi_{i+1} + \beta_- \otimes \phi_{i+1}^* \right) .$$

Here $\underline{\nabla}_i$ *is the unitary connection on* $(\underline{\mathcal{E}}_i, \underline{h}_i)$ *given by*

$$\underline{\nabla}_i = \widehat{\nabla}_{m-2i} \otimes \mathsf{id} + \mathsf{id} \otimes \nabla_i \ ,$$

with

$\widehat{\nabla}_{m-2i}$ *the unique* $\mathsf{SU}_q(2)$*-invariant unitary connection on the hermitian line bundle* $(\mathcal{L}_{m-2i}, \widehat{h}_{m-2i})$ *over* $\mathbb{C}\mathsf{P}^1_q$

β_+ , β_- *the basis holomorphic, anti-holomorphic, 1-forms on* $\mathbb{C}\mathsf{P}^1_q$

Proposition 4. *There is a bijection*

$$\mathcal{U}(\,\underline{\mathcal{E}}\,)^{\mathsf{SU}_q(2)} \simeq \mathscr{U}(\,\underline{\mathcal{E}}\,) := \prod_{i=0}^{m} \mathcal{U}(\mathcal{E}_i) \ ,$$

which associates to

any element $u = (u_0, u_1, \ldots, u_m) \in \mathscr{U}(\,\underline{\mathcal{E}}\,)$

the $\mathsf{SU}_q(2)$*-invariant gauge transformation of* $(\,\underline{\mathcal{E}}\,, \underline{h}\,)$

$$\underline{u} = \sum_{i=0}^{m} \underline{u}_{\,i} \quad \text{with} \quad \underline{u}_{\,i} = 1 \otimes u_i \in \mathcal{U}(\,\underline{\mathcal{E}}_{\,i})^{\mathsf{SU}_q(2)} \simeq \mathbb{C} \otimes \mathcal{U}(\mathcal{E}_i)\,.$$

Integrable connections

M be a complex manifold, with standard complex structure. Combined with the complex structure for the differential calculus on $\mathcal{A}(\mathbb{C}\mathsf{P}^1_q)$ we get a natural complex structure for the calculus on the algebra $\mathcal{A}(\underline{M}) = \mathcal{A}(\mathbb{C}\mathsf{P}^1_q) \otimes \mathcal{A}(M)$.

If $\underline{\nabla}$ is a connection on the $\mathcal{A}(\underline{M})$-module $\underline{\mathcal{E}}$ with equivariant decomposition as before, then the $(0,2)$-component of its curvature $F^{0,2}_{\underline{\nabla}}$ is an element of $\mathrm{Hom}_{\mathcal{A}(\underline{M})}(\underline{\mathcal{E}}, \Omega^{0,2}(\underline{\mathcal{E}}))$

The connection $\underline{\nabla}$ is then integrable if $F^{0,2}_{\underline{\nabla}} = 0$. In this case the pair $(\underline{\mathcal{E}}, \underline{\nabla})$ is a holomorphic vector bundle.

Let $\mathcal{C}(\mathcal{E}_i)^{1,1}$ be the integrable unitary connections on $(\mathcal{E}_i, h_i)$.

Let $\mathscr{C}(\underline{\mathcal{E}})^{1,1} \subset \mathscr{C}(\underline{\mathcal{E}})$ made of

- integrable connections $\nabla_i^{\overline{\partial}} \in \mathcal{C}(\mathcal{E}_i)^{1,1}$

- Higgs fields $\phi_{i+1}^* \in \mathrm{Hom}_{\mathcal{A}(M)}(\mathcal{E}_{i+1}, \mathcal{E}_i)$ s. t.

$$\nabla_{i+1,i}^{\overline{\partial}}(\phi_{i+1}^*) := \phi_{i+1}^* \circ \nabla_{i+1}^{\overline{\partial}} - \nabla_i^{\overline{\partial}} \circ \phi_{i+1}^* = 0 \ .$$

Then, Proposition 3 yields a bijection

$$\mathscr{C}(\underline{\mathcal{E}})^{1,1} \simeq \left(\mathcal{C}(\underline{\mathcal{E}})^{1,1}\right)^{\mathrm{SU}_q(2)}$$

with the space of $\mathrm{SU}_q(2)$-invariant integrable connections.

Yet another ingredient: an integral on $\mathbb{C}\mathsf{P}^1_q$

The Haar state H on $\mathcal{A}(\mathsf{SU}_q(2))$ when restricted to $\mathcal{A}(\mathbb{C}\mathsf{P}^1_q)$ yields a faithful, invariant state, $H(a \triangleleft X) = H(a)\,\epsilon(X)$ for $a \in \mathcal{A}(\mathbb{C}\mathsf{P}^1_q)$ and $X \in \mathcal{U}_q(\mathsf{su}(2))$, with modular automorphism

$$\vartheta(a) = a \triangleleft K^2 \qquad \text{such that} \qquad H(a\,b) = H\Big(\vartheta(b)\,a\Big)$$

With $\beta_- \wedge \beta_+$ the generator of $\Omega^2(\mathbb{C}\mathsf{P}^1_q)$, the linear functional

$$\int_{\mathbb{C}\mathsf{P}^1_q} : \ \Omega^2(\mathbb{C}\mathsf{P}^1_q) \longrightarrow \mathbb{C} \ , \qquad \int_{\mathbb{C}\mathsf{P}^1_q} a\,\beta_- \wedge \beta_+ := H(a)$$

defines a non-trivial ϑ-twisted cyclic two-cocycle τ on $\mathcal{A}(\mathbb{C}\mathsf{P}^1_q)$

$$\tau(a_0, a_1, a_2) := \frac{1}{2} \int_{\mathbb{C}\mathsf{P}^1_q} a_0 \ \mathsf{d}a_1 \wedge \mathsf{d}a_2 \ .$$

i.e. $\qquad \mathsf{b}_\vartheta \tau = 0 \qquad$ and $\qquad \lambda_\vartheta \tau = \tau$

b_ϑ is the ϑ-twisted coboundary operator

$$(\mathsf{b}_\vartheta\tau)(f_0,f_1,f_2,f_3) := \tau(f_0\,f_1,f_2,f_3) - \tau(f_0,f_1\,f_2,f_3) + \tau(f_0,f_1,f_2\,f_3) - \tau\Big(\vartheta(f_3)\,f_0\,,\,f_1\,,\,f_2\Big)$$

λ_ϑ is the ϑ-twisted cyclicity operator

$$(\lambda_\vartheta\tau)(f_0,f_1,f_2) := \tau\Big(\vartheta(f_2)\,,\,f_0\,,\,f_1\Big)$$

With M a connected Kähler manifold of complex dimension d, using this we get an integral

$$\int_{\underline{M}} := \int_{\mathbb{C}\mathsf{P}^1_q} \otimes \int_M \;:\; \Omega^2(\mathbb{C}\mathsf{P}^1_q)\otimes\Omega^{2d}(M) \;\longrightarrow\; \mathbb{C}$$

We set $\int_{\underline{M}}\,\alpha := 0$ whenever $\alpha \notin \Omega^2(\mathbb{C}\mathsf{P}^1_q)\otimes\Omega^{2d}(M)$.

There is also a Hodge operator (as a bimodule map)

$$\underline{\star} := \widehat{\star} \otimes \star \, : \, \Omega^p(\underline{M}) \longrightarrow \Omega^{2(d+1)-p}(\underline{M})$$

Let $\mathcal{C}(\underline{\mathcal{E}})$ be the space of unitary connections on an $\mathrm{SU}_q(2)$-equivariant hermitian $\mathcal{A}(\underline{M})$-module $(\underline{\mathcal{E}}, \underline{h})$.

The Y–M action functional $\mathrm{YM} : \mathcal{C}(\underline{\mathcal{E}}) \to [0, \infty)$ is as usual

$$\mathrm{YM}(\underline{\nabla}) = \left\| F_{\underline{\nabla}} \right\|_{\underline{h}}^2 \tag{5}$$

from a suitable L^2-norm $\|-\|_{\underline{h}}$ on the space $\mathrm{Hom}_{\mathcal{A}(\underline{M})}(\underline{\mathcal{E}}, \Omega^p(\underline{\mathcal{E}}))$

Dimensional reduction of the Yang–Mills action functional

Proposition 6.

The functional $\mathsf{YM}\,|_{\mathcal{C}(\,\underline{\mathcal{E}}\,)^{\mathsf{SU}_q(2)}}$ *on the quantum space* $\underline{M}$, *when restricted to* $\mathsf{SU}_q(2)$*-invariant unitary connections coincides with the Y–M–H functional* $\mathsf{YMH}_{q,m}$ *on* M*:*

$$\mathsf{YMH}_{q,m}(\nabla,\phi) = \sum_{i=0}^{m} \Big(\big\|F_{\nabla_i}\big\|^2_{h_i} + \big(q^2+1\big)\,\big\|\nabla_{i-1,i}(\phi_i)\big\|^2_{h_{i-1,i}} + \big\|\phi^*_{i+1}\circ\phi_{i+1} - q^2\,\phi_i\circ\phi^*_i - q^{m-2i+1}\,[m-2i]_q\ \mathrm{id}_{\mathcal{E}_i}\big\|^2_{h_i}\Big)\ ,$$

with

- $F_{\nabla_i} = \nabla_i^2$, *the curvature of the connection* $\nabla_i \in \mathcal{C}(\mathcal{E}_i)$ *on* M

- $\nabla_{i-1,i}$ *the connection on* $\mathrm{Hom}_{\mathcal{A}(M)}(\mathcal{E}_{i-1},\mathcal{E}_i)$ *induced by* ∇_{i-1} *on* $\mathcal{E}_{i-1}$ *and* ∇_i *on* $\mathcal{E}_i$ *and given by*

$$\nabla_{i-1,i}(\phi_i) = \phi_i\circ\nabla_{i-1} - \nabla_i\circ\phi_i\ .$$

Symbol

$$[x]_q = \frac{q^x - q^{-x}}{q - q^1}$$

Also $\phi_0 := 0 =: \phi_0^*$ *and* $\phi_{m+1} := 0 =: \phi_{m+1}^*$

This functional restricts to a map on gauge orbits

$$\mathrm{YMH}_{q,m} : \mathscr{C}(\underline{\mathcal{E}}) \Big/ \mathscr{U}(\underline{\mathcal{E}}) \to [0, \infty)$$

Characterize stable critical points of the Y–M functional (5) on $\underline{M}$, and study their reduction to configurations on M.

generalized instantons

Lemma 7. *Let* $\underline{\nabla} \in \mathcal{C}(\underline{\mathcal{E}})$ *be a unitary connection such that*

$$\underline{\star}\, F_{\underline{\nabla}} = -F_{\underline{\nabla}} \wedge \Xi \qquad (8)$$

for $\Xi \in \Omega^{2d-2}(\underline{M})$ *a closed form of degree* $2d-2$*.*

Then $\underline{\nabla}$ *is a critical point of the Y–M functional and*

$$\mathsf{YM}(\underline{\nabla}) = \mathsf{Top}_2(\underline{\mathcal{E}}, \Xi) := -\Big(F_{\underline{\nabla}}\,,\ \underline{\star}(F_{\underline{\nabla}} \wedge \Xi)\Big)_{\underline{h}}$$

The functional $\mathrm{Top}_2(\underline{\mathcal{E}}, \Xi)$ does not depend on the choice of $\underline{\nabla}$

It defines a 'topological action' depending only on the $\mathcal{A}(\underline{M})$-module $\underline{\mathcal{E}}$ and the closed form Ξ

Provides an *a priori* lower bound on the Y–M functional

The gauge invariant equation (8) is the Ξ-anti-selfduality eqn

The gauge equivalence classes in $\mathcal{C}(\underline{\mathcal{E}}) \big/ \mathcal{U}(\underline{\mathcal{E}})$ of solutions are

generalized instantons or Ξ-instantons

Holomorphic chain q-vortex equations

(M, ω) a Kähler manifold

A natural closed $(1,1)$-form on $\mathcal{A}(\,\underline{M}\,)$

$$\underline{\omega} = (\beta_- \wedge \beta_+) \otimes 1 + 1 \otimes \omega$$

Set $$\Xi = \frac{\underline{\omega}^{\,d-1}}{(d-1)!} = 1 \otimes \frac{\omega^{d-1}}{(d-1)!} + (\beta_- \wedge \beta_+) \otimes \frac{\omega^{d-2}}{(d-2)!}$$

Recall the generalized instanton equation

$$\underline{\star}\, F_{\underline{\nabla}} = -F_{\underline{\nabla}} \wedge \Xi$$

Proposition 9. *The subspace of* $\mathsf{SU}_q(2)$ *invariant connections*

$$\underline{\nabla}^{\,\overline{\partial}} \in \Big(\mathcal{C}(\,\underline{\mathcal{E}}\,)^{1,1}\Big)^{\mathsf{SU}_q(2)}$$

solving the generalized instanton equation on $\underline{M}$ *corresponds bijectively to the subspace of* $\mathscr{C}(\,\underline{\mathcal{E}}\,)^{1,1}$ *of elements* $(\nabla^{\overline{\partial}}, \phi^*)$ *satisfying the holomorphic chain* q*-vortex equations on* M

$$F^{\omega}_{\nabla_i} = q^2\,\phi_i \circ \phi_i^* - \phi_{i+1}^* \circ \phi_{i+1} + q^{m-2i+1}\,[m-2i]_q\ \mathrm{id}_{\mathcal{E}_i} \qquad (10)$$

for $i = 0, 1, \dots, m$. *Here*

$$F^{\omega}_{\nabla_i} = \star\Big((F^{1,1}_{\nabla_i})^* \wedge \star\omega\Big) \in \mathrm{End}_{\mathcal{A}(M)}(\mathcal{E}_i),$$

the component of the curvature of ∇_i *along the Kähler form* ω

Stability conditions

A hermitian finitely-generated projective $\mathcal{A}(M)$-module $(\mathcal{E}, h)$ has degree and slope given by

$$\mathrm{deg}(\mathcal{E}) = \frac{\mathrm{Top}_1(\mathcal{E}, \omega)}{\mathrm{vol}_\omega(M)} \qquad \text{and} \qquad \mu(\mathcal{E}) = \frac{\mathrm{deg}(\mathcal{E})}{\mathrm{rank}(\mathcal{E})} ,$$

with $\mathrm{rank}(\mathcal{E}) = \mathrm{tr}(\mathrm{id}_\mathcal{E})$

Analogously, the (q, m)-degree of a finitely-generated $\mathrm{SU}_q(2)$-equivariant projective module $\underline{\mathcal{E}}$ over the algebra $\mathcal{A}(\underline{M})$, with equivariant decomposition $\underline{\mathcal{E}} = \bigoplus_{i=0}^m \underline{\mathcal{E}}_i = \bigoplus_{i=0}^m \mathcal{L}_{m-2i} \otimes \mathcal{E}_i$ is

$$\mathrm{deg}_{q,m}(\underline{\mathcal{E}}) = \sum_{i=0}^m \Big(\mathrm{deg}(\mathcal{E}_i) - q^{m-2i+1} \, [m-2i]_q \, \mathrm{rank}(\mathcal{E}_i) \Big)$$

and, its (q,m)-slope is

$$\mu_{q,m}(\underline{\mathcal{E}}) = \frac{\deg_{q,m}(\underline{\mathcal{E}})}{\mathrm{rank}(\underline{\mathcal{E}})}$$

with $\mathrm{rank}(\underline{\mathcal{E}}) = \sum_i \mathrm{rank}(\mathcal{E}_i)$.

Natural topological meaning: the q-integers $[m-2i]_q$ label classes in the $SU_q(2)$-equivariant K-theory $K_0^{\mathcal{U}_q(\mathrm{su}(2))}(\mathbb{C}P^1_q)$.

$SU_q(2)$ acting trivially on M, the (q,m)-degree labels classes in the $SU_q(2)$-equivariant K-theory of $\underline{M} = \mathbb{C}P^1_q \times M$.

Thus the usual assignment of D-brane charges in equivariant K-theory to quiver vortices extends to our q-vortices as well.

The parameters q, m and the topology of the bundles $\mathcal{E}_i$ over M are constrained by the following (semi-)stability criteria.

Proposition 11. *A stable q-quiver bundle on M has slopes constrained by the inequalities:*

(a) $\mu(\mathcal{E}_0) \leq q^{m+1}\,[m]_q$, *with equality iff* $\mathcal{E}_0$ *admits a holomorphic connection* ∇_0 *solving the hermitian Yang–Mills eqn*

$$F^{\omega}_{\nabla_0} = q^{m+1}\,[m]_q\;\mathrm{id}_{\mathcal{E}_0}\,.$$

(b) $\mu(\mathcal{E}_m) \geq -q^{-m+1}\,[m]_q$, *with equality iff* $\mathcal{E}_m$ *admits a holomorphic connection* ∇_m *solving the hermitian Yang–Mills eqn*

$$F^{\omega}_{\nabla_m} = -q^{-m+1}\,[m]_q\;\mathrm{id}_{\mathcal{E}_m}\,.$$

(c) $\mu_{q,m}(\,\underline{\mathcal{E}}\,) \leq 0$, *with equality iff* $\mathcal{E}_i$ *admits a holomorphic connection* ∇_i *solving the hermitian Yang–Mills eqn*

$$F^{\omega}_{\nabla_i} = q^{m-2i+1}\,[m-2i]_q\;\mathrm{id}_{\mathcal{E}_i} \qquad \text{for each} \quad i = 0,1,\dots,m.$$

Examples

Some explicit examples of the q-vortex equations

$$F^{\omega}_{\nabla_i} = q^2\,\phi_i \circ \phi_i^* - \phi_{i+1}^* \circ \phi_{i+1} + q^{m-2i+1}\,[m-2i]_q\ \mathrm{id}_{\mathcal{E}_i}$$

$i = 0, 1, \ldots, m.$

The q-deformations affect stability conditions for the existence of solutions and the structure of the corresponding moduli spaces

1. Deformations of holomorphic triples and stable pairs

A holomorphic triple $(\mathcal{E}_0, \mathcal{E}_1, \phi)$ on a compact Kähler manifold (M, ω) is a pair of holomorphic vector bundles $\mathcal{E}_0, \mathcal{E}_1$ over M and a holomorphic morphism

$$\mathcal{E}_0 \ \xrightarrow{\phi} \ \mathcal{E}_1$$

With $\phi := \phi_1$, we get

$$F^{\omega}_{\nabla_0} = q^2 \left(\mathrm{id}_{\mathcal{E}_0} - q^{-2}\, \phi \circ \phi^* \right) \quad \text{and} \quad F^{\omega}_{\nabla_1} = -\left(\mathrm{id}_{\mathcal{E}_1} - q^2\, \phi^* \circ \phi \right) \ (\diamondsuit)$$

The degrees of the bundles are related by

$$\deg(\mathcal{E}_0) + q^{-2} \deg(\mathcal{E}_1) = q^2 \operatorname{rank}(\mathcal{E}_0) - q^{-2} \operatorname{rank}(\mathcal{E}_1)$$

Much more stringent than the undeformed stability condition

2. q-vortices on Riemann surfaces

M a compact oriented Riemann surface of genus g and scalar curvature κ. Eq. ($\diamondsuit$) describe q-non-abelian vortices on M.

A particular case:

$$\mathcal{E} := \mathcal{E}_0, \quad \nabla := \nabla_0 \quad ; \qquad \mathcal{E}_1 \simeq \mathbb{C}^r \otimes \mathcal{A}(M) \quad r = \text{rank}(\mathcal{E}),$$

the Higgs field $\phi = q^{-1}\varphi$ can be regarded as an element of $\mathbb{C}^r \otimes \mathcal{E}$

Also $\frac{1}{2\pi}\,\text{Top}_1(M,\omega) = c_1(\mathcal{E})$.

A non-empty moduli space of solutions to the q-vortex equations ($\diamondsuit$) is ensured by the stability condition

$$c_1(\mathcal{E}) = \frac{4r}{\kappa}\left(q^2 - q^{-2}\right)(1-g) \qquad \text{for} \quad g \neq 1$$

Since $0 < q < 1$, this degree satisfies the bound

$$c_1(\mathcal{E}) < \frac{4q^2}{\kappa}(1-g).$$

Hence the pair $(\mathcal{E}, \varphi)$ is τ-stable and by the Hitchin–Kobayashi correspondence it is gauge equivalent to a solution of the non-abelian q-vortex equations.

For abelian vortices, $r = 1$, this moduli space is the $|n|$-th symmetric product orbifold of M, i.e. the space of effective divisors on M of degree $n = c_1(\mathcal{E})$.

3. q-instantons

Let (M, ω) be a Kähler surface. Set $\mathcal{E}_0 \simeq \mathcal{E}_1 =: \mathcal{E}$.

Since ϕ is a holomorphic section, $\nabla^{\overline{\partial}}_{0,1}(\phi) = 0$;

we have $\nabla_0 = \nabla_1 =: \nabla$ and both equations in ($\diamondsuit$) simplify to

$$F^{\omega}_{\nabla} = \left(q^2 - 1\right) \mathrm{id}_{\mathcal{E}}$$

a deformation of the hermitian Yang–Mills equation on M, and hence of the standard anti-selfduality equations $\star F_{\nabla} = -F_{\nabla}$. Its gauge equivalence classes of solutions are thus called q-instantons.

The natural $\mathcal{U}(\mathcal{E})$-invariant symplectic form $\omega_{\mathcal{C}}$ on $\mathcal{C}(\mathcal{E})$:

$$\omega_{\mathcal{C}}(\alpha, \alpha') = \frac{1}{2} \int_M \mathrm{tr} \left(\alpha^* \wedge \alpha' \right)^{\omega} \omega^2$$

for $\alpha, \alpha' \in \mathrm{Hom}^a_{\mathcal{A}(M)}(\mathcal{E}, \Omega^1(\mathcal{E}))$.

Then corresponding moment map $\mu_{\mathcal{C}} : \mathcal{C}(\mathcal{E}) \to (\mathrm{Lie}\, \mathcal{U}(\mathcal{E}))^*$ is

$$\mu_{\mathcal{C}}(\nabla) = F^{\omega}_{\nabla} \ .$$

The moduli space of q-instantons on M is the symplectic quotient

$$\mu_{\mathcal{C}}^{-1}\Big((q^2 - 1)\ \mathrm{id}_{\mathcal{E}} \Big) \Big/ \mathcal{U}(\mathcal{E}) \ ,$$

and q-vortices correspond to points of $\mu_{\mathcal{C}}^{-1}\Big((q^2 - 1)\ \mathrm{id}_{\mathcal{E}} \Big)$ which lie inside the Kähler submanifold $\mathcal{C}(\mathcal{E})^{1,1}$ (outside the singularities).

When $M = \mathbb{C}^2$, the constant shift in the moment map condition

$$\text{from} \quad \mu_{\mathcal{C}} = 0 \quad \text{to} \quad \mu_{\mathcal{C}} = (q^2 - 1)\ \mathrm{id}_{\mathcal{E}}$$

induces a shift in the corresponding real ADHM equation.

NS: this modification arises in the equations which determine instantons on a certain noncommutative deformation of $\mathbb{R}^4$

Here we have the same sort of resolution of instanton moduli space via our q-deformed dimensional reduction procedure over the quantum projective line $\mathbb{C}\mathrm{P}^1_q$.

Summing up:

Characterized vector bundles over the quantum space

$$\underline{M} := \mathbb{C}\mathsf{P}^1_q \times M$$

equivariant under an action of the quantum group $\mathsf{SU}_q(2)$

Described the dimensional reduction of invariant connections

In particular, Yang–Mills gauge theory on $\mathcal{A}(\underline{M})$ is reduced to a type of Yang–Mills–Higgs theory on the manifold M

The equations of motion give q-deformations of known vortex equations, whose solutions possess remarkable properties.

Thank you