QUANTUM BROWNIAN MOTION AS A SCALING LIMIT

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PLAN OF THE LECTURES

1, Overview of rigorous derivations of diffusion.

2, Random Schrödinger Hamiltonians

3, Quantum Boltzmann eq. and quantum diffusion

4, Feynman graphs

5, Estimate of the interference and recollision effects

6, Computation of the main term

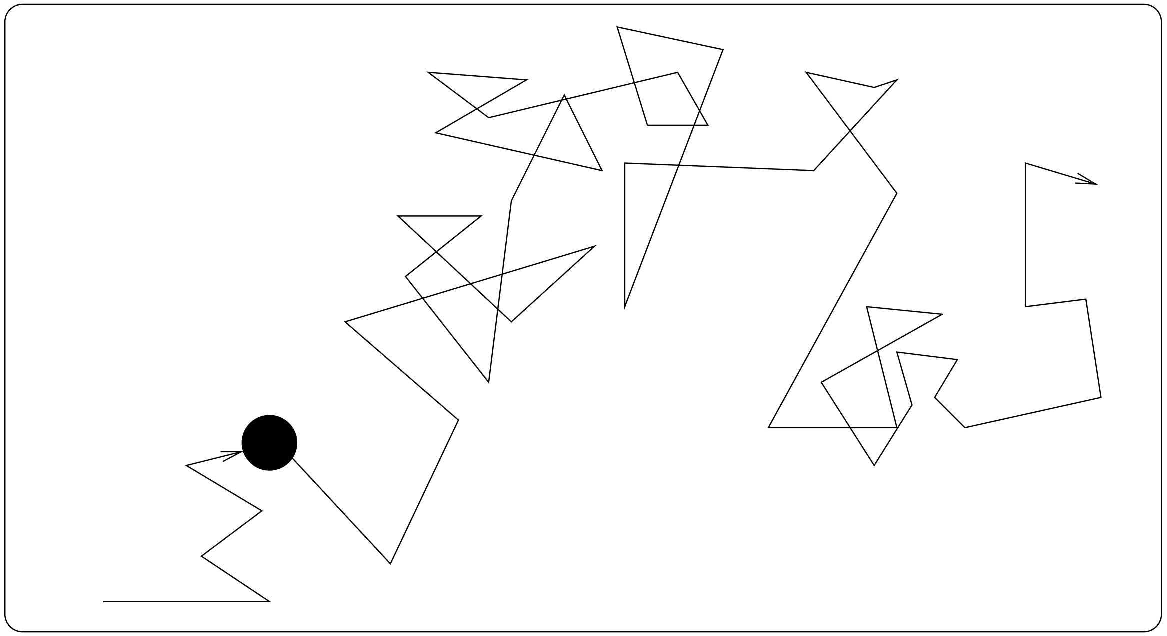
7, $\mathbf{Z}^d$ is harder than $\mathbf{R}^d$

8, Random band matrices

Questions are welcome

DIFFUSION IN CLASSICAL MECHANICS

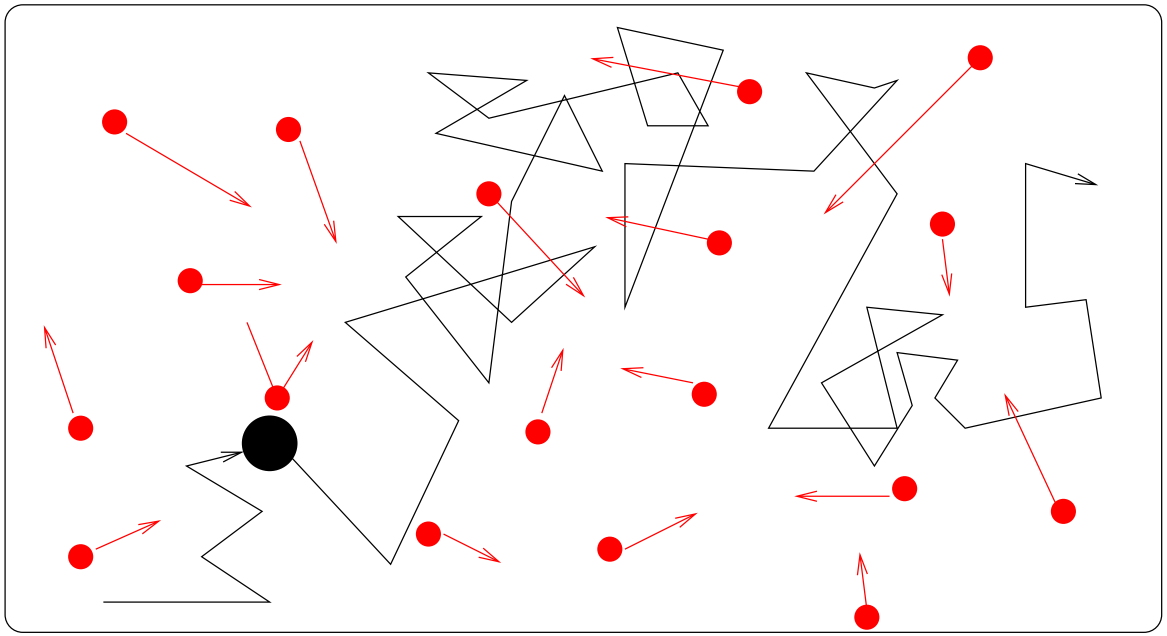
Observation [Brown, 1827]: Light particle (pollen) suspended in water performs an erratic motion.



Brown's microscope picture

The motion never stops. It holds for "live" and "dead" pollens.

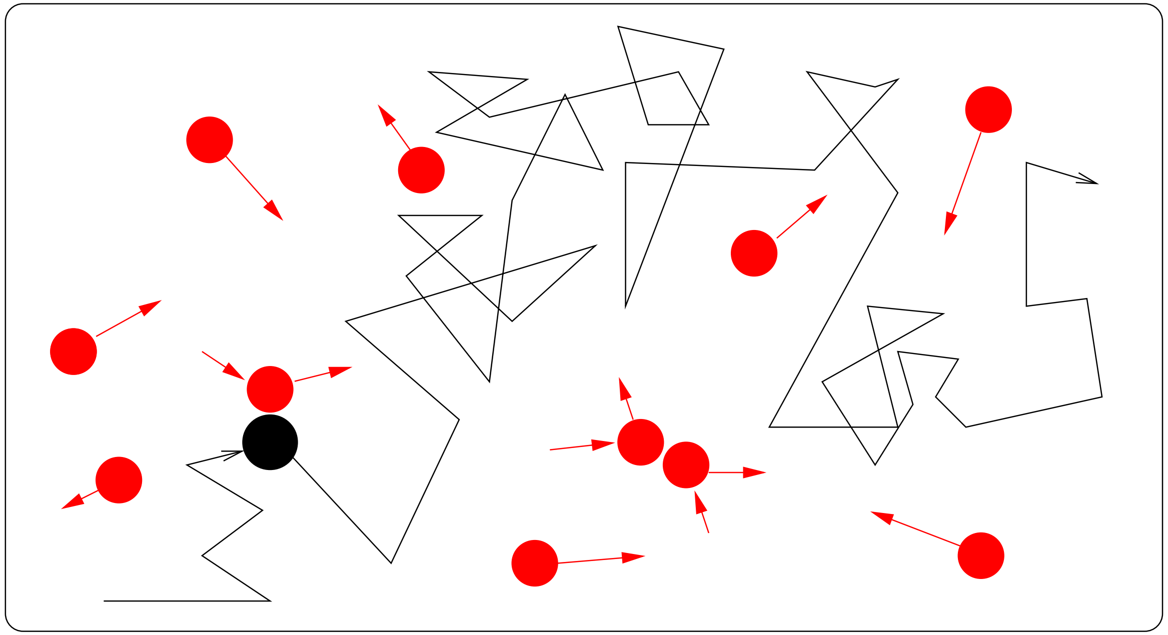
Physical Explanation [Einstein 1905, Perrin 1908]: Molecular-kinetic picture: pollen is constantly kicked by lighter water molecules. Support for Maxwell's and Boltzmann's kinetic theory and the existence of atoms/molecules. Estimate on Avogadro's number.



Einstein's explanation

DETOUR ON BOLTZMANN'S THEORY

Interacting particles at low density (gas).



Many−body modell for nonlinear Boltzmann eq.

Want: equation on the single particle phase space density:

$f_t(x, v)\mathrm{d}x\mathrm{d}v = \#\{\text{Particles at } x + \mathrm{d}x \text{ with vel. } v + \mathrm{d}v \text{ at time } t\}$

Main Assumption (Ansatz of molecular chaos):
Colliding particles are statistically independent before collision

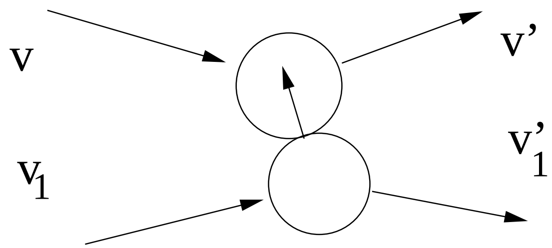
According to Boltzmann, f_t satisfies the *nonlinear Boltzmann eq.*

$$(\partial_t + v \cdot \nabla_x) f_t(x,v) = Q(f_t, f_t)(x,v)$$

$$Q(f,f)(x,v) = \int \sigma(v,v_1)\Big[f(x,v')f(x,v_1') - f(x,v)f(x,v_1)\Big]$$

(v, v_1) pair of incoming velocities, (v', v_1') – outgoing vel. pair.

(v', v_1') is determined by (v, v_1) plus the random contact vector (energy+momentum conservation)



Ingeniuosly combines the particle and fluid picture!
It was strongly debated.

Key conceptual difficulty:

The Hamiltonian dynamics is reversible and deterministic.

How does the irreversible and chaotic nature arise?

Answer: Loss of information is due to **scale separation** and integrating out the microscopic degrees of freedom.

Key technical difficulties: Controlling memory (recollision) and interference (QM) effects.

Remark: Einstein's model is simpler than Boltzmann's, as light particles do not interact. Verifying the Ansatz of molecular chaos is technically easier. Still, Einstein's model addresses the key issue: how does diffusion emerge from Hamiltonian mechanics?

For this talk, we will forget Boltzmann's interacting model

Some stochasticity has to be added to the model:

1. Stochastic Dynamics: E.g. Scaling limit of random walk [Wiener, 1923]. Stochastic microscopic system with no memory.

2. Random Hamiltonian: E.g. Lorenz gas in random scatterers in weak-coupling (van Hove) limit [Kesten/Papanicolaou,'78]

$$\text{time} \quad t \sim \lambda^{-2}, \qquad \text{where } \lambda \text{ is the coupling}$$

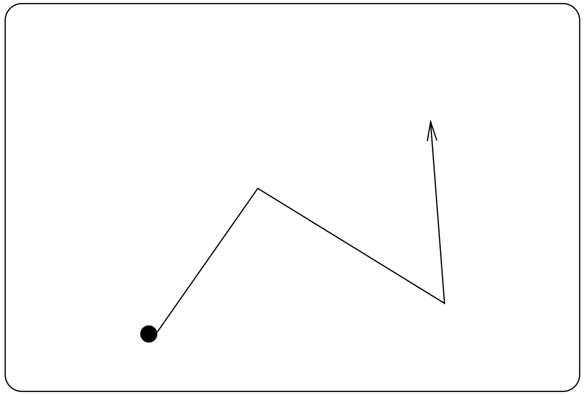
3. Deterministic Hamiltonian with random initial data of many degrees of freedom

E.g: Heavy particle (M) in a light (m) ideal gas in $m/M \to 0$, coll. rate $\to \infty$ limit. [Dürr-Goldstein-Lebowitz '81]

4. Deterministic Hamiltonian with a random initial data of a few degrees of freedom

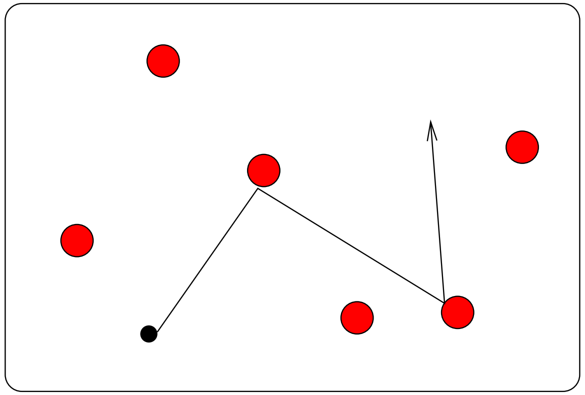
Hard-core periodic Lorenz gas (billiard) [Bunimovich-Sinai, '80]

MICROSCOPIC MODELS FOR DIFFUSION



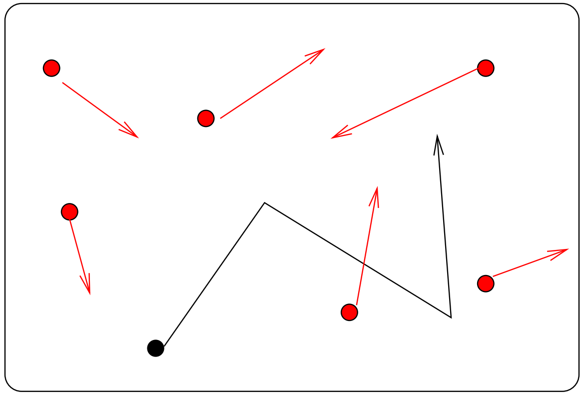
Random walk (Wiener)

Stochastic dynamics



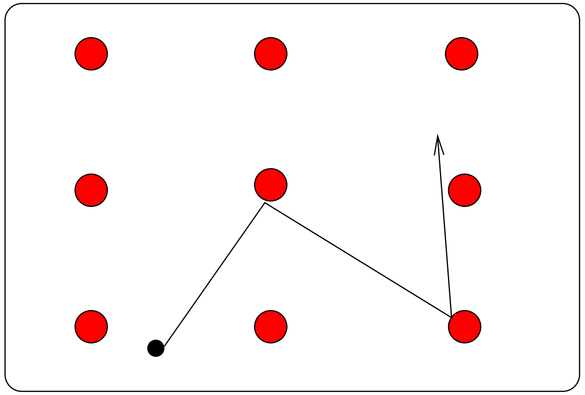
Lorentz gas (random scat.)

Random Hamiltonian, one body



Einstein's model

Deterministic Hamiltonian
many body random data.



Periodic Lorentz gas (billiard)

Deterministic Hamiltonian
one body random data.

	CLASSICAL MECHANICS	QUANTUM MECHANICS
Stochastic dynamics (no or short memory)	Random walk (Wiener)	Random kick model with zero or short time corr. pot. (Pillet, Kang-Schenker)
Hamiltonian particle in a random environment (one body)	Lorenz gas: particle in random scatterers (Kesten-Papanicolaou) (Komorowski-Ryzhik)	Anderson model or quantum Lorenz gas (Spohn, E-Yau, E-Salmhofer-Yau Disertori-Spencer-Zirnbauer)
Hamiltonian particle in a heat bath (randomness in the many-body data)	Einstein's kinetic model (Dürr-Goldstein-Lebowitz)	Electron in phonon or photon bath (E, E-Adami Lukkarinen-Spohn, De Roeck–Fröhlich)
Periodic Lorenz gas (randomness in the one-body initial data)	Sinai billiard (Bunimovich-Sinai)	Ballistic (Bloch waves)
Many-body interacting Hamiltonian	Nonlinear Boltzmann eq (short time: Lanford)	Quantum NL Boltzmann (unsolved)

RECAPITULATION OF STOCHASTIC DYNAMICS

1, Central limit theorem for

$$X_T = \varepsilon^{1/2} \sum_{i=1}^{[T/\varepsilon]} v_i$$

(if $\mathbf{E}v_j = 0$ and $R(i-j) = \mathbf{E}v_i v_j$ is summable).

2, Generator of a Markov process: $f_t(x) = \mathbf{E}_x \varphi(X_t)$ satisfies

$$\partial_t f_t = \mathcal{L} f_t, \qquad f_0 = \varphi, \qquad X_0 = x$$

3, Wiener process and its generator: $\partial_t f_t = \frac{1}{2} \Delta f_t$

4, Random jump process on S^{d-1} and its generator.

$$\partial_t f_t(v) = \int \sigma(v,u) \Big[f_t(u) - f_t(v) \Big] \mathrm{d}u$$

if $\sigma(v,u)$ is the rate of jump from v to u.

If an exponential clock ticks, then

$$\text{Prob}\{\text{The particle from } v \text{ jumps to } u + \mathrm{d}u\} = \frac{\sigma(v,u)\mathrm{d}u}{\int \sigma(v,u)\mathrm{d}u}$$

The transition at an infinitesimal time increment (as $\varepsilon \to 0$).

$$v_{t+\varepsilon} = \begin{cases} u + \mathrm{d}u & \text{with probability} \quad \varepsilon\sigma(v_t, u)\mathrm{d}u \\ v_t & \text{with probability} \quad 1 - \varepsilon \int \sigma(v_t, u)\mathrm{d}u \end{cases}$$

Let

$$f_t(v) := \mathbf{E}_v \varphi(v_t)$$

Then, by Markovity and the jump rate:

$$\partial_t f_t(v) = \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \mathbf{E}_v \Big(f_t(v_\varepsilon) - f_t(v) \Big) = \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \mathbf{E}_v \Big(\mathbf{E}_{v_\varepsilon} \varphi(\widetilde{v}_t) - \mathbf{E}_v \varphi(\widetilde{v}_t) \Big)$$

$$= \int \mathrm{d}u \, \sigma(v,u) \Big[\mathbf{E}_u \varphi(\widetilde{v}_t) - \mathbf{E}_v \varphi(\widetilde{v}_t) \Big]$$

$$= \int \mathrm{d}u \, \sigma(v,u) \Big[f_t(u) - f_t(v) \Big]$$

Note that with probability $1 - \varepsilon \int \sigma(v,u)\mathrm{d}u$ we have $v_\varepsilon = v$.

Thus the generator of the random jump process is

$$(\mathcal{L}f)(v) := \int \mathrm{d}u\, \sigma(v,u)\Big[f(u) - f(v)\Big]$$

and

$$\partial_t f_t = \mathcal{L} f_t, \qquad f_t(v) = \mathbf{E}_v \varphi(v_t), \qquad f_0 = \varphi$$

Here f_t observes v_t, starting from $v_0 = v$.

The dual question is: Suppose $v_0 = v$ is distributed by f_0. What is the distribution of v_t? Answer: f_t, where

$$\partial_t f_t = \mathcal{L}^* f_t = \int \mathrm{d}u\, \sigma(u,v)\Big[f_t(u) - f_t(v)\Big]$$

This evolution equation for the probability density of the jump process, is called *linear Boltzmann equation in velocity space.* It has no spatial structure.

Note that it has two terms; the first term is called the *gain term* the second one is the *loss term.*

CLASSICAL MECHANICS OF A SINGLE PARTICLE

Hamiltonian (energy) function: $H(v,x) := \frac{1}{2}v^2 + U(x)$ on $\mathbf{R}^d \times \mathbf{R}^d$

$$\dot{x}(t) = \partial_v H = v \qquad \dot{v}(t) = -\partial_x H = -\nabla U(x)$$

$$\text{Free evol. } U \equiv 0 \implies x(t) = x_0 + v_0 t$$

Phase space density: $f_t(x,v)$, e.g. $\delta(x - x(t))\delta(v - v(t))$

$$(\partial_t + v \cdot \nabla_x) f_t(x,v) = \nabla U(x) \cdot \nabla_v f_t(x,v)$$

Liouville equation.

Free evolution:

$$(\partial_t + v \cdot \nabla_x) f_t(x,v) = 0 \implies f_t(x,v) = f_0(x - vt, v)$$

LINEAR BOLTZMANN EQUATION

Phenomenological combination of the free evolution and the jump process on the unit sphere of the velocity space (energy conservation – although there is no Hamiltonian behind it!)

$$(\partial_t + v \cdot \nabla_x) f_t(x,v) = \int \sigma(u,v) \Big[f_t(x,u) - f_t(x,v) \Big] \mathrm{d}u$$

It is really the adjoint of the jump process (observe that u and v are interchanged) and it determines:

$$f_t(x,v) = \text{Prob}\{\text{to be at } (x,v) \text{ at time } t\}$$

given the initial probability density $f_0(x,v)$

The free flight + collision process behind the Boltzmann equation is like a random walk.

Theorem [Relatively easy]

Long time evolution of the linear Boltzmann equation is diffusion in position space

$$X_\varepsilon(T) =: \varepsilon^{1/2} \int_0^{T/\varepsilon} v_s \, \mathrm{d}s \to \sqrt{D} W_T \qquad \text{(in distr.)}$$

where W_T is the Wiener process and the diffusion coeffient D is given by the velocity autocorrelation

$$D = \int_0^\infty R(s) \mathrm{d}s, \qquad R(s) := \mathbf{E} v_0 v_s$$

(where $\mathbf{E}$ is with respect to the equilibrium measure of the jump process)

QUANTUM MECHANICS OF A SINGLE PARTICLE

$$H = -\tfrac{1}{2}\Delta_x + U(x) \quad \text{acting on} \quad \psi \in L^2(\mathbf{R}^d)$$

$|\psi(x)|^2$ — position; $|\widehat{\psi}(v)|^2$ – momentum space densities.

Wigner transform of ψ = "quantum phase space density"

$$W_\psi(x,v) := \int \overline{\psi}\Big(x+\frac{z}{2}\Big)\psi\Big(x-\frac{z}{2}\Big)e^{ivz}\mathrm{d}z$$

$$\int W_\psi(x,v)\mathrm{d}v = |\psi(x)|^2, \qquad \int W_\psi(x,v)\mathrm{d}x = |\widehat{\psi}(v)|^2$$

W is real but not positive

SEMICLASSICAL LIOUVILLE EQUATION

$$i\partial_t \psi_t(x) = \Big[-\tfrac{1}{2}\Delta_x + U(\varepsilon x) \Big] \psi_t(x)$$

More familiar in macro coordinates, $(X, T) = (x\varepsilon, t\varepsilon)$,

$$i\varepsilon\partial_T \Psi_T(X) = \Big[-\tfrac{\varepsilon^2}{2}\Delta_X + U(X) \Big] \Psi_T(X)$$

Wigner transform is rescaled:

$$W_\psi^\varepsilon(X, V) := \varepsilon^{-d} W_\psi\Big(\frac{X}{\varepsilon}, V\Big), \qquad \iint W^\varepsilon = 1$$

Theorem: The weak limit of the rescaled Wigner tr.

$$W_T(X, V) := \lim_{\varepsilon \to 0} W^\varepsilon_{\psi_{T/\varepsilon}}(X, V)$$

satisfies the Liouville eq.

$$(\partial_T + V \cdot \nabla_X) W_T(X, V) = \nabla U(X) \cdot \nabla_V W_T(X, V)$$

RANDOM SCHRÖDINGER EQUATION IN $\mathbf{Z}^d$ OR $\mathbf{R}^d$, $d \geq 3$

$$i\partial_t \psi_t(x) = H\psi_t(x), \qquad H = -\Delta_x + \lambda V_x(\omega)$$

Dispersion relation (F. Transform of $-\Delta_x$)

$$e(p) = \sum_{j=1}^{d} (1 - \cos p^{(j)}) \quad [\text{on } \mathbf{Z}^d], \qquad e(p) = p^2 \quad [\text{on } \mathbf{R}^d]$$

Random field on $\mathbf{Z}^d$:

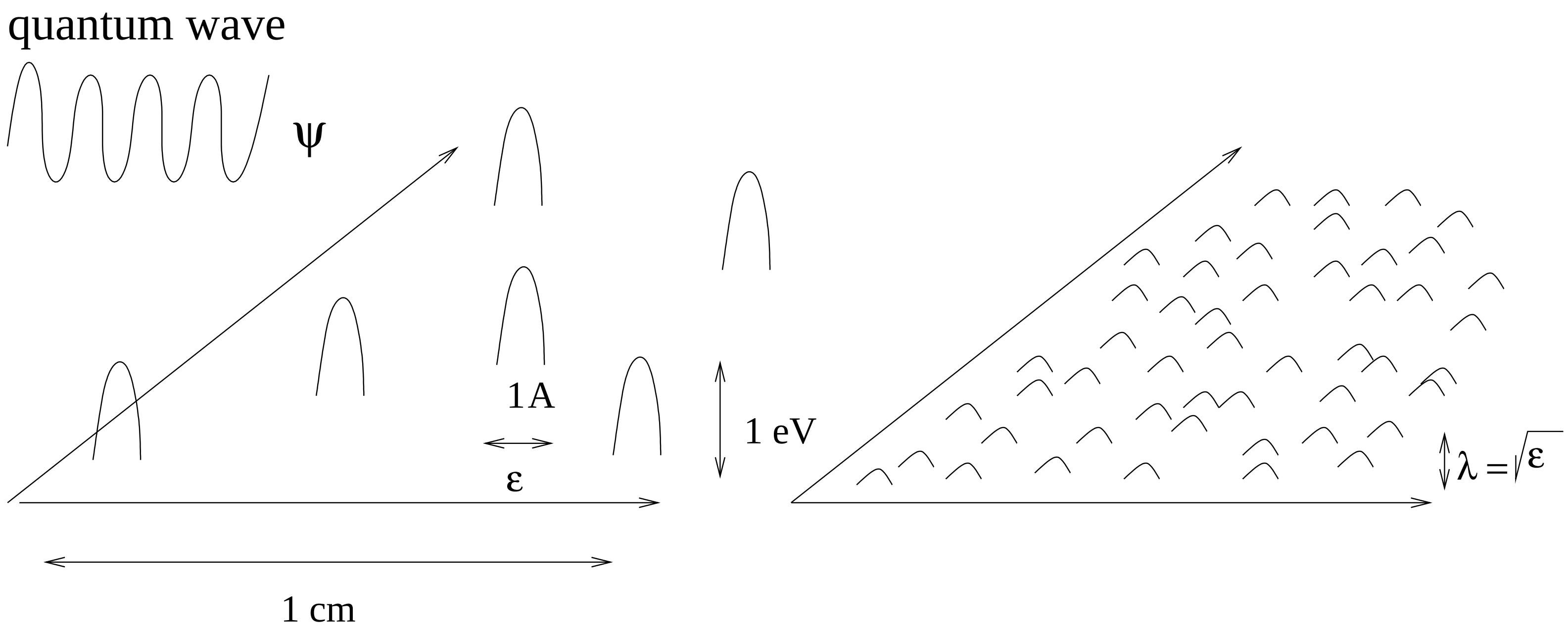
$$\{V_x \ : \ x \in \mathbf{Z}^d\} \quad \text{i.i.d} \qquad \mathbf{E}V_x = 0, \ \mathbf{E}V_x^2 = 1 \qquad [\text{on } \mathbf{Z}^d]$$

Random field on $\mathbf{R}^d$:

$$V_x = \sum_{\alpha \in \mathbf{Z}^d} v_\alpha \, B(x - x_\alpha)$$

v_α is normalized i.i.d., x_α uniform in the unit box around α. B is a "nice" single site potential. [Other potentials are also possible].

$\lambda \ll 1$, $d \geq 3$: presumed extended states regime. OPEN.



Low density scenery

Number of obstacles: ε^{-2}

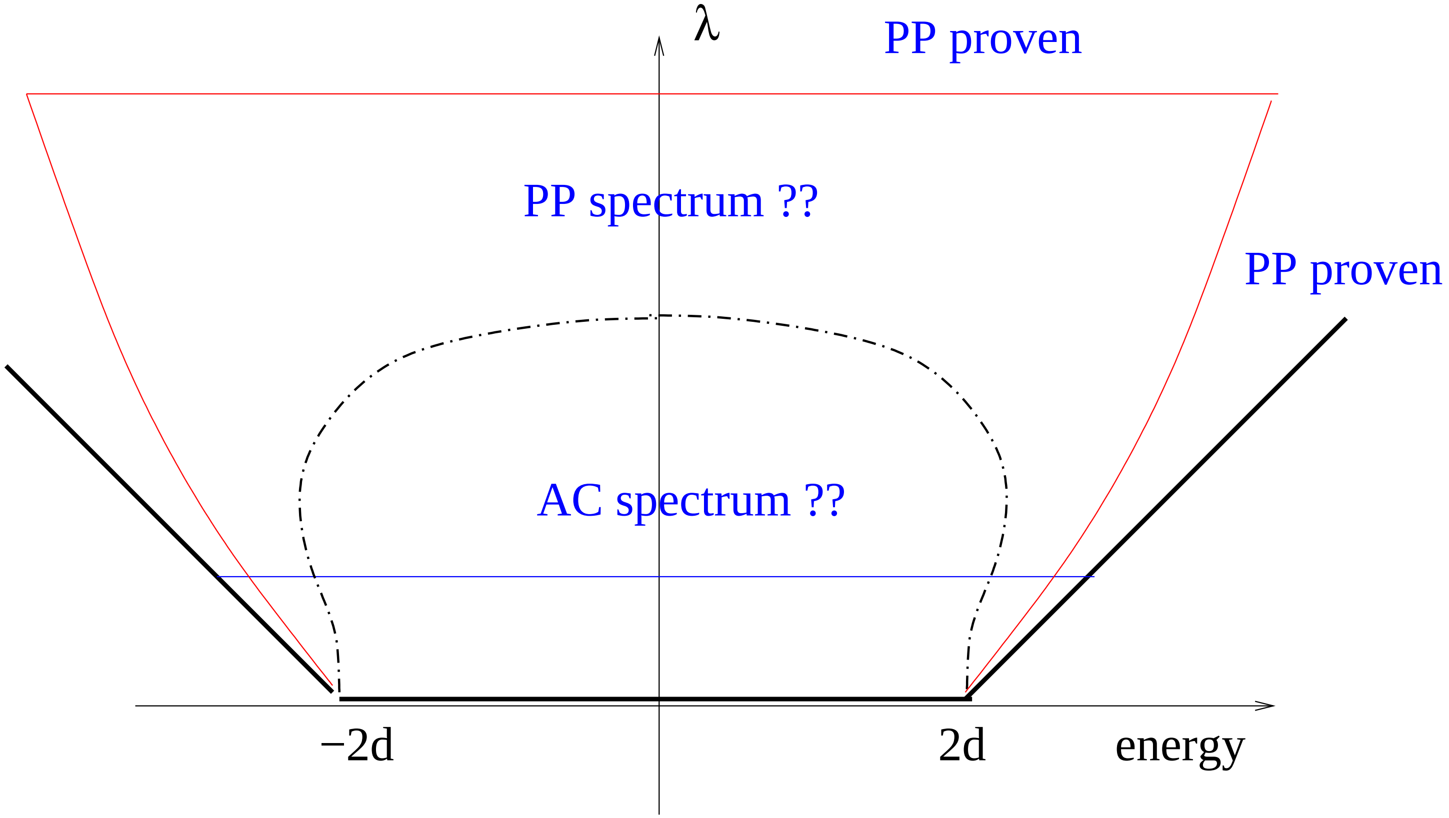
Density: $O(\varepsilon)$

Weak coupling scenery

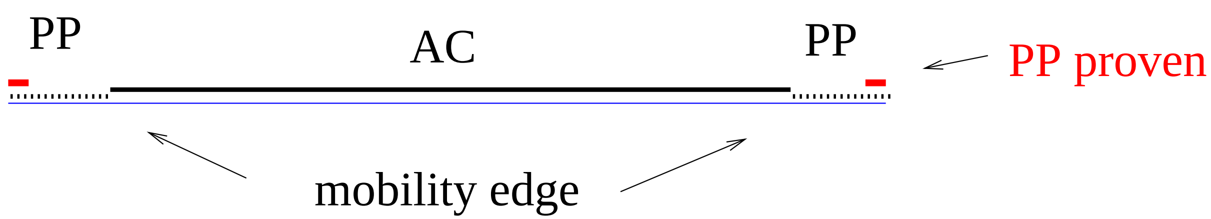
Number of obstacles: ε^{-3}

Density: $O(1)$

PHASE DIAGRAM OF THE $d = 3$ ANDERSON MODEL



Expected spectrum at small, nonzero disorder $0 < \lambda << 1$



HOW FAST DOES THE STATE DELOCALIZE?

Free Schrödinger equation is ballistic:

$$V \equiv 0 \qquad \Longrightarrow \qquad \langle\, x^2 \,\rangle_t := \int \mathrm{d}x |\psi_t(x)|^2 x^2 \sim t^2$$

Quantum Brownian Motion conjecture: For small FIXED λ, in $d \geq 3$, the location of the electron is governed by a heat equation:

$$\text{“}\, \partial_t |\psi_t(x)|^2 \sim \Delta_x |\psi_t(x)|^2 \,\text{”} \qquad \Longrightarrow \qquad \langle\, x^2 \,\rangle_t \sim t \qquad t \gg 1$$

Moreover, if $|s-t| \gg 1$, $|x-y| \gg 1$, then (up to scaling)

$$\Big(|\psi_s(x)|^2, |\psi_t(y)|^2\Big) \mathrm{d}x \mathrm{d}y \sim \mathrm{Prob}(W_s = x + \mathrm{d}x, W_t = y + \mathrm{d}y)$$

Our main result: QBM conjecture holds up to $t \sim \lambda^{-2-\kappa}$

Rate of collision with a weakly coupled potential λV: $\qquad \lambda^2$

The number of collisions: $\qquad n = \lambda^2 t \sim \lambda^{-\kappa} \to \infty$

QBM for all t would include the extended states conjecture

BASIC SCATTERING

$h = -\Delta + \lambda V_0, \qquad V_0$ smooth, cpct supp – single bump

Let ψ_{in} be an incoming travelling wave. Then, as $t \gg 1$,

$$e^{-ith}\psi_{in} = \beta e^{it\Delta}\psi_{in} + \psi_{sc}(t)$$

the decomposition is orthogonal and

$$\|\psi_{sc}\|^2 = O(\lambda^2), \qquad \beta = 1 - O(\lambda^2)$$

Roughly $\qquad \psi_{sc}(x,t) \sim \lambda \, \dfrac{e^{iS(x,t)}}{|x|}, \qquad d = 3$

In the multi-bump scattering model:

$$\Longrightarrow \text{Rate of collision} = O(\lambda^2)$$

$$\text{Total number of collisions} = n = O(\lambda^2 t)$$

SCALINGS IN THE WEAK COUPLING MODEL

Weak coupling: $\lambda \to 0$. Mean free path: $\lambda^{-2} \to \infty$.

Kinetic scaling:

$$t = \frac{T}{\varepsilon}, \;\; x = \frac{X}{\varepsilon}, \quad \varepsilon = \lambda^2$$

Convergence to the linear Boltzmann eq. [Spohn, E-Yau, Chen]

Diffusive scaling:

$$t = \lambda^{-\kappa}(\lambda^{-2}T), \quad x = \lambda^{-\kappa/2}(\lambda^{-2}X), \qquad (\kappa > 0)$$

Convergence to the heat equation. [E-Salmhofer-Yau]

$\equiv$ the long time ($\lambda^{-\kappa}$) behavior of the Boltzmann eq.

Number of collisions $\lambda^{-\kappa} \to \infty$.

Quantum phase-space "density": *Wigner distribution*

$$W_\psi(x,v) := \int e^{ivy}\overline{\psi\Big(x+\frac{y}{2}\Big)}\psi\Big(x-\frac{y}{2}\Big)dy\ .$$

Meaning: "probability" to find the particle at x with velocity v.

Rescaling $$W_\psi^\varepsilon(X,V) := \varepsilon^{-d}W_\psi\Big(\frac{X}{\varepsilon},V\Big), \qquad \int W_\psi^\varepsilon = 1$$

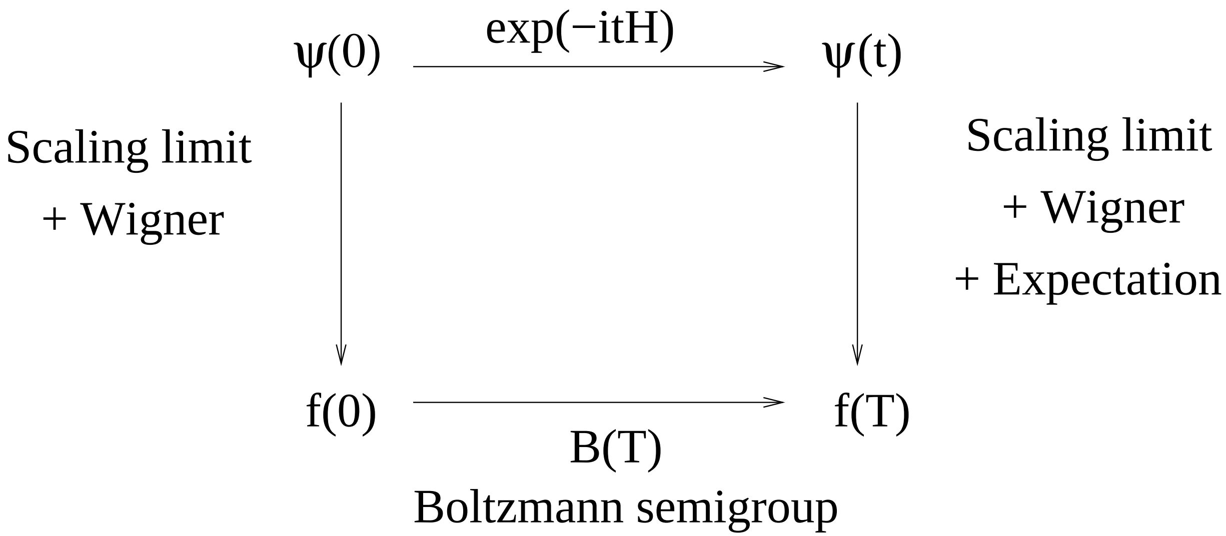
Theorem [Boltzmann equation in the kinetic limit] Let $\varepsilon = \lambda^2$.

$$\mathbf{E}_\omega W^\varepsilon_{\psi_{T/\varepsilon}}(X,V) \rightharpoonup F_T(X,V) \qquad \text{(weakly)}$$

and F satisfies the linear Boltzmann equation for all T

$$\Big(\partial_T + \nabla e(V)\cdot\nabla_X\Big)F_T(X,V) = \int dU\sigma(U,V)\Big[F_T(X,U) - F_T(X,V)\Big]\ .$$

$\sigma(U,V) = \delta(e(U)-e(V))|\widehat{B}(U-V)|^2$ [on $\mathbf{R}^d$] [E-Yau]
$\sigma(U,V) = \delta(e(U)-e(V))$ [on $\mathbf{Z}^d$] [Chen]



The long time ($t = T\varepsilon^{-1}$) Schrödinger evolution is modelled by a finite time (T) Boltzmann evolution on the macroscopic scale.

Weak limit = only macroscopic observables can be controlled:

$$\mathbf{E}\langle \mathcal{O}, W \rangle \to \langle \mathcal{O}, f \rangle \ , \qquad \mathcal{O} = \mathcal{O}(X, V) \in C^{\infty}$$

Detailed short scale information is lost (irreversible).

Effective equation is classical, but quantum features are retained in the collision kernel.

Theorem [Quantum diffusion] (E-Salmhofer-Yau, '05-'07) Let

$$t = \lambda^{-\kappa}\left(\lambda^{-2}T\right) \quad x = \lambda^{-\kappa/2}\left(\lambda^{-2}X\right), \quad \varepsilon = \lambda^{-\kappa/2-2},$$

For $d \geq 3$ and $\kappa < \kappa_0$ we have as $\lambda \to 0$

$$\int_{\{e(v)=e\}} \mathbf{E}\, W_\psi^\varepsilon(T\lambda^{-2-\kappa}, X, v)\, \mathrm{d}v \rightharpoonup f(T, X, e) \qquad \text{(weakly)}$$

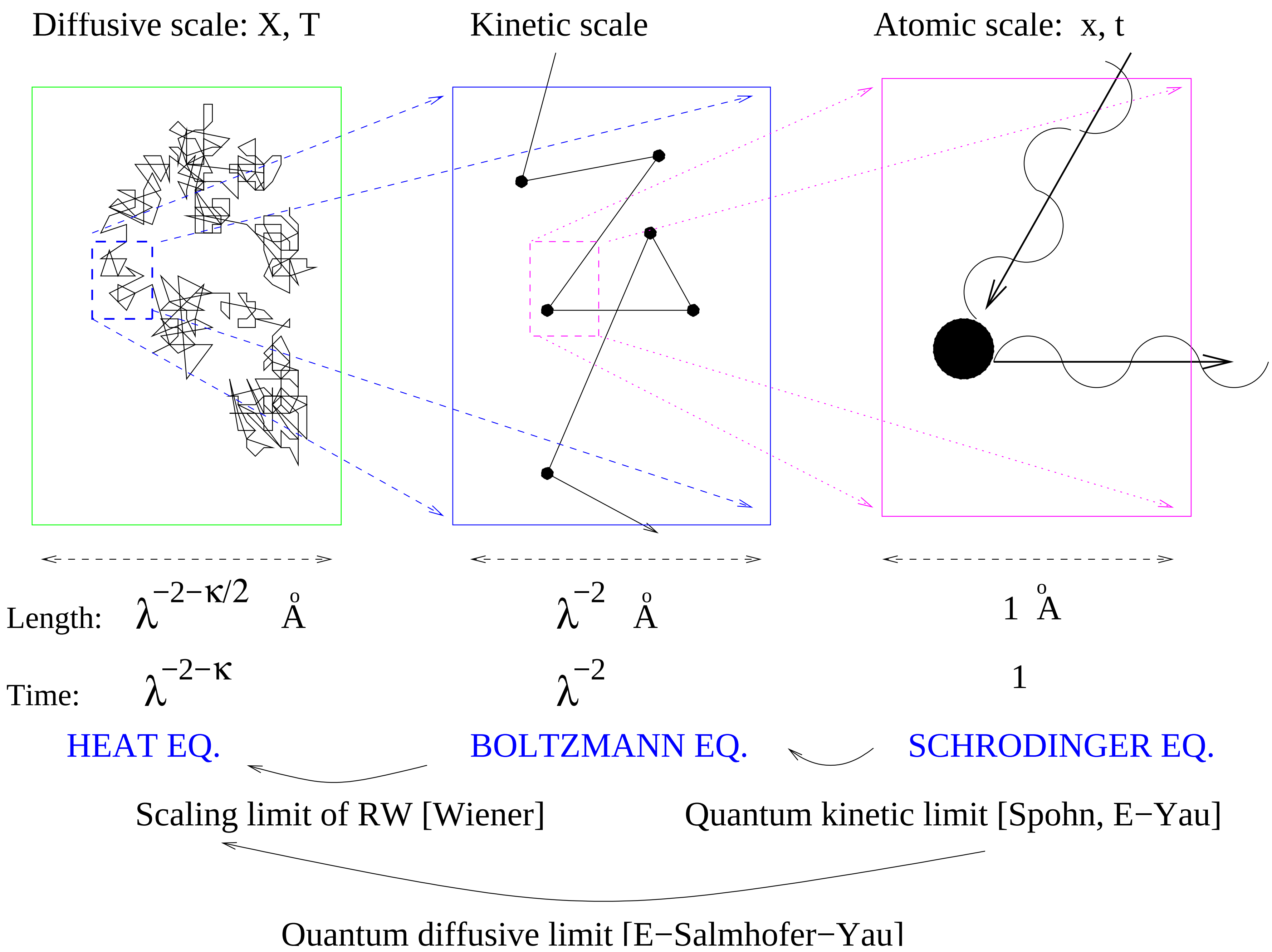
$$\boxed{\partial_T f(T, X, e) = \nabla_X \cdot D(e) \nabla_X f(T, X, e)}$$

$$D(e) = \Big\langle \nabla e(v) \otimes \nabla e(v) \Big\rangle_e, \qquad \langle f(v) \rangle_e = \text{Av. on } \{v \,:\, e(v) = e\}$$

(This formula holds for $\mathbf{Z}^d$, for $\mathbf{R}^d$ it is more complicated)

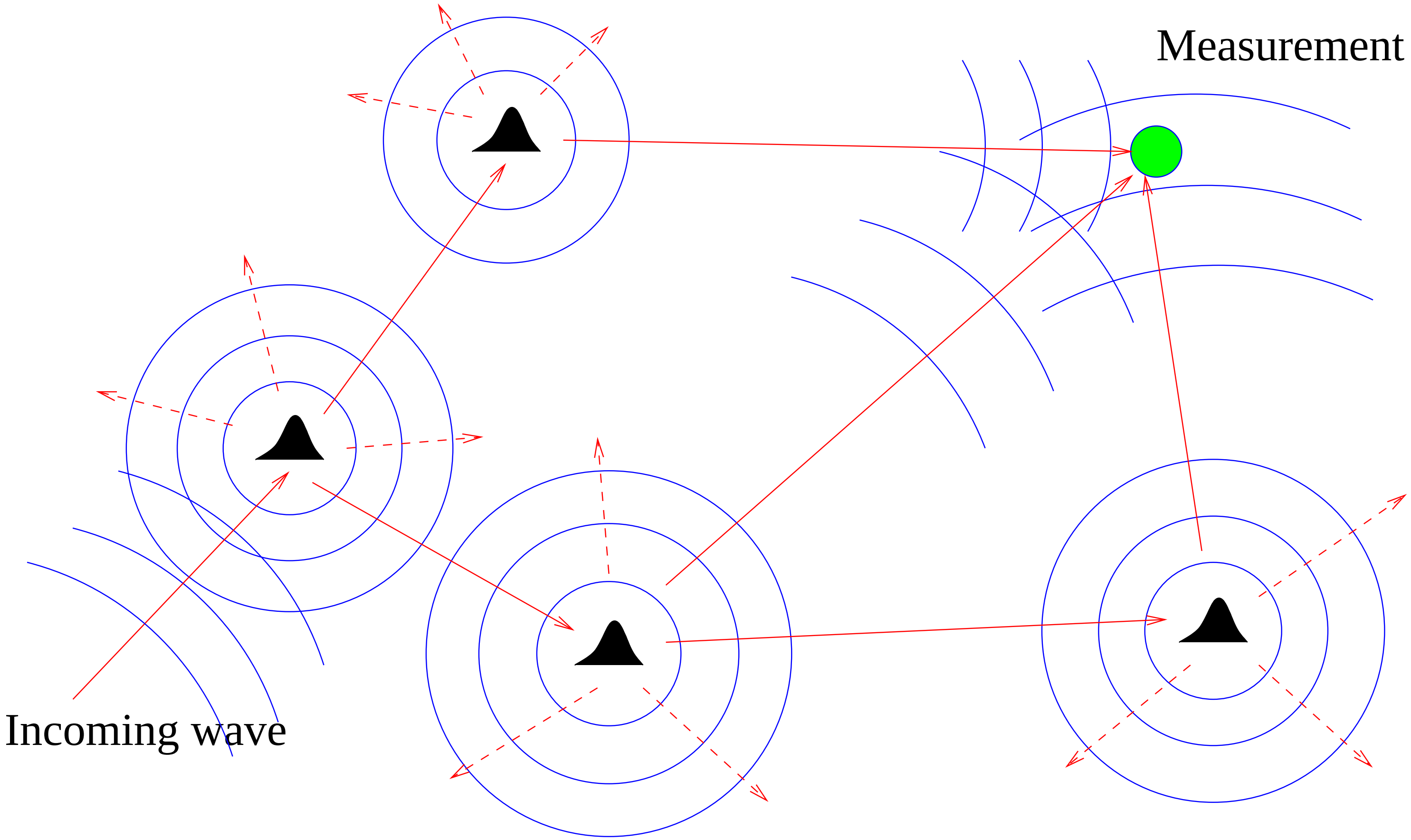
Quantum particle in a weak random environment converges to Brownian motion.

This is expected to hold for any κ in $d = 3$, but *not expected* to hold for $d = 2$ [localization].


Diffusive scale: X, T
Kinetic scale
Atomic scale: x, t
Length: λ^{−2−κ/2} Å
λ^{−2} Å
1 Å
Time: λ^{−2−κ}
λ^{−2}
1
HEAT EQ.
BOLTZMANN EQ.
SCHRODINGER EQ.
Scaling limit of RW [Wiener]
Quantum kinetic limit [Spohn, E−Yau]
Quantum diffusive limit [E−Salmhofer−Yau]

WHY IS IT DIFFICULT?

It is a multiple scattering process with n collisions, $n \to \infty$.



# of elementary waves to sum up $|\Lambda|^n \sim \lambda^{-\lambda^{-\kappa}}$.

SOME RELATED WORKS ON BOLTZMANN SCALE

- Phonon Boltzmann eq. (closer to Einstein's model) [E]
  Dehorence via phonons [Adami-E, '08]

- Lower bound on the eigenfunction localization length ℓ:

  [Schlag-Shubin-Wolff] $\ell \geq \lambda^{-2+\delta}$ in $d = 2$

  [Chen] $\ell \geq \lambda^{-2}$ (modulo logs) and for any d

  Main idea of [Chen]: Apply the kinetic limit for the eigenfn.

  $$\psi = (\text{phase})e^{itH}\psi \quad \text{that spreads on scale } \lambda^{-2}$$

- $\mathbf{E}(W - \mathbf{E}W)^2$ can also be controlled [Chen]

- Lattice wave eq. with random masses and cubic NL evolution near equilibrium [Lukkarian-Spohn]

All these can be extended to the diffusive regime ($t \sim \lambda^{-2-\kappa}$)

RELATED WORKS ON DIFFUSIVE SCALE

• Disertori-Spencer-Zirnbauer ['09]: Diffusion for a nonlinear sigma model (saddle point approximation of random Schrödinger)

• De Roeck–Fröhlich ['09]: Diffusion for **all** times with weakly coupled phonons and large mass in $d \geq 4$ with an additional fast internal degree of freedom to enhance memory loss.

• E-Knowles ['10]: Diffusion and eigenfunction delocalization for band matrices (interpolate between random Schrödinger and mean field Wigner matrices)

MAIN STEPS OF THE PROOF

1) Expansion into Feynman graphs with a stopping rule
[Monitor each elementary wave and stop if $\exists$ suff. recollision]

2) Renormalize the self-energy (2-legged subdiagram renorm.)

3) Identify the ladder graph as the main term – get limit eq.

4) Introduce the degree of a F. graph to measure the deviation from ladder (degree := # of non-ladder vertices).

5) Estimate the value of each F. graph as $(\lambda^{const})^{\text{deg}}$, i.e. gain a λ-power per each non-ladder vertex.

6) Estimate the # of graphs with a given degree.

7) Conclude that all non-ladders are negligible by comparing 5) and 6)

FEYNMAN GRAPHS (WITHOUT REPETITION)

$$\psi_t = e^{-itH}\psi_0 = e^{-it\Delta}\psi_0 + \int_0^t e^{-i(t-s)H} V e^{-is\Delta}\psi_0 \, \mathrm{d}s$$

$$= e^{-it\Delta}\psi_0 + \int_0^t e^{-i(t-s)\Delta} V e^{-is\Delta}\psi_0 \, \mathrm{d}s$$

$$+ \iint_{\sum s_j = t} e^{-is_1\Delta} V e^{-is_2\Delta} V e^{-is_3\Delta}\psi_0 + \ldots + \iint e^{-is_1 H} V e^{-is_2\Delta} \ldots$$

$$H = -\Delta + V, \qquad V = \sum_{\alpha \in \mathbb{Z}^d} V_\alpha, \quad \mathbf{E} V_\alpha = 0$$

Duhamel formula: $\quad \psi_t \sim \sum_A \psi_A, \quad A := (\alpha_1, \alpha_2, \cdots, \alpha_n)$

$$\psi_A = (-i)^n \int_{\sum s_j = t} e^{-is_0\Delta}\, V_{\alpha_1} \cdots V_{\alpha_n} e^{-is_n\Delta}\, \psi_0 \, \mathrm{d}s_0 \mathrm{d}s_1 \cdots \mathrm{d}s_n$$

Assume there is no repetition in A

$\psi_A =$ 

with propagator: ——— $= e^{-is\Delta}$

$$\widehat{\psi}_{A,t}(p) = \int dp_j \int_0^t \delta\Big(t - \sum_{j=0}^n s_j\Big) e^{is_0 e(p)} \widehat{V}_{\alpha_1}(p-p_1) e^{is_1 e(p_1)} \widehat{V}_{\alpha_2}(p_1-p_2)\ldots$$

$$= \underbrace{e^{\eta t}}_{\eta:=1/t} \int \prod_j dp_j \int_{-\infty}^{\infty} d\alpha\, e^{-i\alpha t} \prod_j \frac{1}{\alpha - e(p_j) + i\eta} \widehat{V}_{\alpha_j}(p_{j-1} - p_j)$$

$$\mathbf{E}\|\psi_t^{nr}\|^2 = \mathbf{E}\Big\| \sum_{A\,:\,\text{nonrep}} \psi_A \Big\|^2 = \sum_{A,B} \mathbf{E}\langle \psi_A, \psi_B\rangle$$

To compute the expectation, we need Wick's theorem:

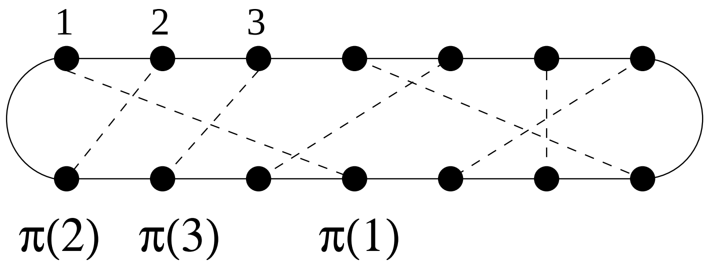
$$\mathbf{E}\overline{\widehat{V}_{\alpha_1}\widehat{V}_{\alpha_2}\ldots\widehat{V}_{\alpha_n}}\widehat{V}_{\beta_1}\widehat{V}_{\beta_2}\ldots\widehat{V}_{\beta_n} = \sum_\pi \prod_{i=1}^n \mathbf{E}\overline{\widehat{V}_{\alpha_i}}\widehat{V}_{\beta_{\pi(i)}}$$

and

$$\sum_{\alpha,\beta} \mathbf{E}\overline{\widehat{V}_\alpha(p)}\widehat{V}_\beta(q) = \sum_\alpha e^{i\alpha(p-q)} = \delta(p-q)$$

In particular, $B = \pi(A)$ (complete pairing – if no recollision)

$$E\,\|\Psi_t\|^2 = \sum_\pi$$



$$= \sum_\pi \mathrm{Val}(\pi)$$

$$\mathsf{Val}(\pi) := e^{2\eta t} \int d\mathbf{p} d\mathbf{q} d\alpha d\beta \; e^{it(\alpha-\beta)}$$

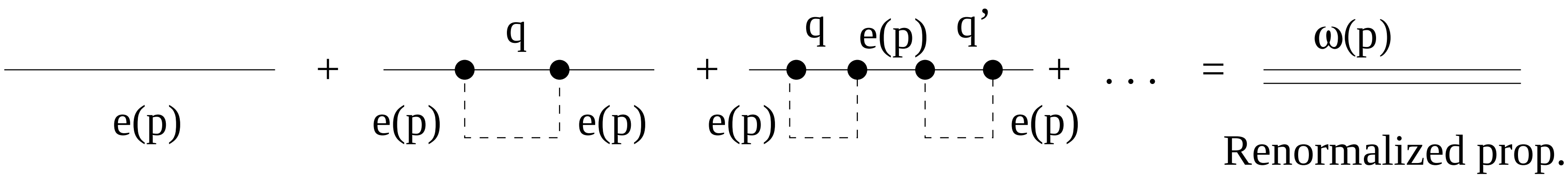
$$\times \prod_j \frac{\lambda}{\alpha - e(p_j) - i\eta} \, \frac{\lambda}{\beta - e(q_j) + i\eta} \, \Delta_\pi(\mathbf{p},\mathbf{q}) \; ,$$

$$\Delta_\pi(\mathbf{p},\mathbf{q}) := \prod_j \delta\Big(p_j - p_{j-1} = q_{\pi(j)} - q_{\pi(j)-1} \Big)$$

Structure: Multiple singular integral concentrated on a lower dimensional hypersurface with $\eta = t^{-1}$ regularization.

Real time propagators with hypersurface singularities.

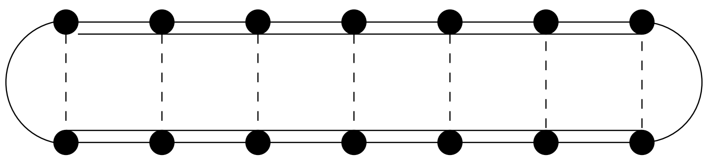
Special Recollision: Self-energy Renormalization



$$H = \underbrace{e(p) + \lambda^2\theta(p)}_{\omega(p)} + \lambda V - \lambda^2\theta(p)$$

with $$\theta(p) = \int \frac{\mathrm{d}q}{\omega(p) - \omega(q) + i0}, \qquad \sigma = \mathrm{Im}\theta$$

After renormalization: only the ladder has classical contribution and gives the limiting equation.



$$= \frac{(\sigma\lambda^2 t)^n}{n!} e^{-\mathrm{Im}\,\theta\,\lambda^2 t} \longrightarrow n \sim \lambda^2 t$$

$$\mathrm{Val}(id_n) = \lambda^{2n} e^{2\eta t} \int_{\mathbf{R}} \mathrm{d}\alpha \mathrm{d}\beta e^{it(\alpha-\beta)}$$

$$\times \prod_{j=0}^{n} \left(\int \frac{\mathrm{d}p_j}{(\alpha - \bar{\omega}(p_j) - i\eta)(\beta - \omega(p_j) + i\eta)} \right) |\widehat{\psi}_0(p_n)|^2$$

$$\approx \lambda^{2n} \int_{\mathbf{R}} \mathrm{d}\alpha \mathrm{d}\beta e^{it(\alpha-\beta)} \left(\frac{2\mathrm{Im}\theta}{\alpha - \beta + 2i\lambda^2 \mathrm{Im}\theta} \right)^{n+1}$$

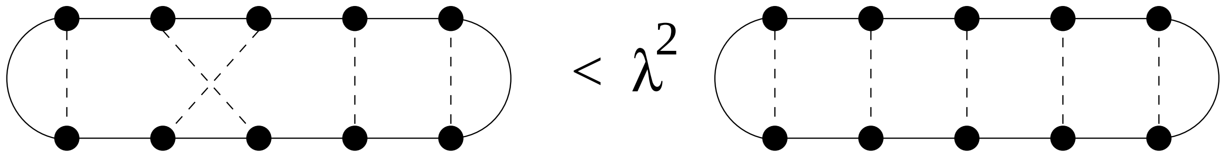
$$\approx \frac{(2\lambda^2 t \mathrm{Im}\theta)^n}{n!} e^{-2\lambda^2 t \mathrm{Im}\theta}$$

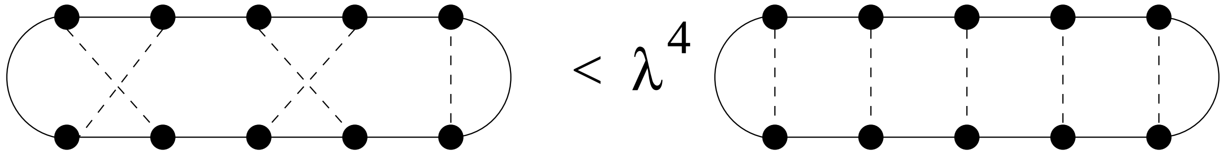
where we used residue calculus in the $\mathrm{d}(\alpha - \beta)$. In summary:

$$\sum_n \mathrm{Val}(id_n) = \sum_n \frac{(C\lambda^2 t)^n}{n!} e^{-C\lambda^2 t} = 1$$

and the main term is from $n \sim \lambda^2 t$.

All other graphs are small, but there are many ($n!$) of them.





$$\sum_n \frac{(\sigma\lambda^2 t)^n}{n!}\Big(\underbrace{1}_{A=B} + \underbrace{(n!-1)(\text{small})}_{A\neq B} \Big)$$

Converges for short kinetic time ($\lambda^2 t = T \leq T_0$) [Spohn, 1979]

Notice: "Most" graphs have many "crosses", hence they are much smaller due to phase decoherence

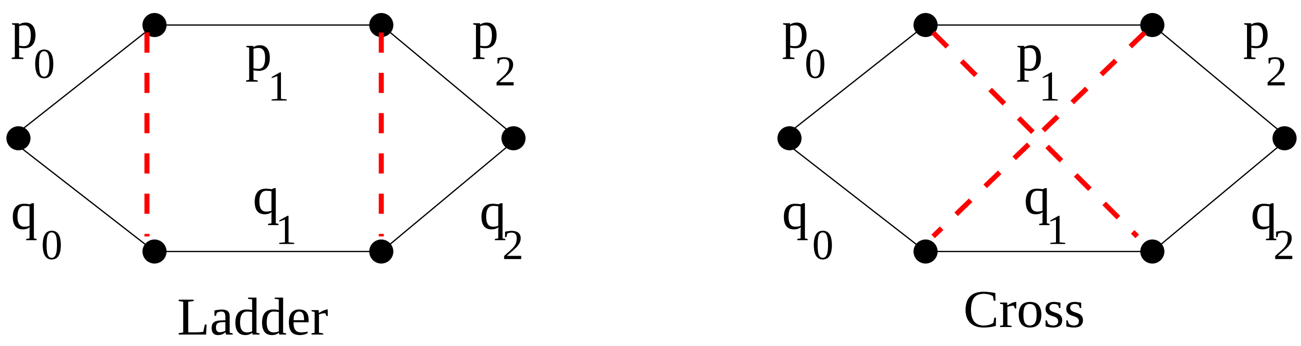
CROSSES ARE SMALLER THAN LADDERS

$$\text{Val}(\pi) = \int d\mathbf{p}d\mathbf{q}d\alpha d\beta\, e^{it(\alpha-\beta)} \prod_j \frac{\lambda}{\alpha - e(p_j) - i\eta} \frac{\lambda}{\beta - e(q_j) + i\eta} \Delta_\pi(\mathbf{p},\mathbf{q}) \,,$$

$\Delta_\pi(\mathbf{p},\mathbf{q})$ ($\equiv$ Kirchoff Law), may enhance singularity overlaps.

Power counting does not reveal this enhancement.

Example



Ladder delta functions: $p_j = q_j$

Crossing delta functions: $q_0 = p_0, \quad q_1 = p_0 - p_1 + p_2, \quad q_2 = p_2$

Ladder: ($\eta = \lambda^2 = t^{-1}$; kinetic scaling for simplicity)

$$\lambda^4 \int d\alpha d\beta\, e^{i(\alpha-\beta)t} \int \frac{1}{\alpha - e(p_0) - i\eta} \frac{1}{\alpha - e(p_1) - i\eta} \frac{1}{\alpha - e(p_2) - i\eta}$$

$$\times \frac{1}{\beta - e(p_0) + i\eta} \frac{1}{\beta - e(p_1) + i\eta} \frac{1}{\beta - e(p_2) + i\eta} dp_0 dp_1 dp_2$$

Main term: $|\alpha - \beta| \leq t^{-1} = \lambda^2 \Longrightarrow$ All sing.overlap, effectively

$$\int \frac{dp}{|\alpha - e(p) + i\eta|^2} \sim \frac{1}{|\eta|} = t \sim \lambda^{-2}$$

$$\text{Val(Ladder)} \sim \lambda^4 \lambda^2 (\lambda^{-2})^3 \sim O(1)$$

For the **cross**, $q_1 = p_0 - p_1 + p_2$, so the dp_1 integral is

$$\int dp_1 \frac{1}{|\alpha - e(p_1) + i\eta|} \frac{1}{|\alpha - e(p_0 - p_1 + p_2) + i\eta|} \sim \frac{1}{|p_0 + p_2| + \eta}$$

Frustration of singularities $\Longrightarrow$ Val(Cross)$\leq \lambda^2$ Val(Ladder)

PROOF FOR LONG KINETIC TIMES

Truncation of Duhamel + Unitarity of e^{itH} (No $\sum_0^\infty$).

$$\left\| \int_0^t e^{i(t-s)H} \underbrace{\int\int V e^{is_1\Delta} V e^{is_2\Delta} \ldots \mathrm{d}s}_{\psi_s} \right\| \le \underbrace{t}_{price} \sup_s \|\psi_s\|$$

On kinetic scale:

$$\sum_{n=0}^{N-1} \frac{(\sigma\lambda^2 t)^n}{n!} \Big(\underbrace{1}_{\text{ladder}} + \underbrace{n\lambda^2}_{\text{one cross}} + \underbrace{n!\lambda^4}_{\text{rest}} \Big)$$

$$+ \underbrace{t}_{unit.price} \frac{(\sigma\lambda^2 t)^N}{N!} \Big(\underbrace{1}_{\text{ladder}} + \underbrace{N\lambda^2}_{\text{one cross}} + \underbrace{N^2\lambda^4}_{\text{two cross}} + \underbrace{N!\lambda^6}_{\text{rest}} \Big)$$

Optimize $N = N(\lambda) \sim (\log \lambda)/(\log\log \lambda)$ to get convergence.
Gives the kinetic (Boltzmann) limit for all fixed T ($t = T\lambda^{-2}$).

For diffusion: one needs to classify **all** diagrams

Suppose $t = \lambda^{-2-\kappa}$, then the typical number of collisions is $n = \lambda^2 t = \lambda^{-\kappa}$, so we need to expand at least up to $N = \lambda^{-\kappa}$.

Compute up to p-crosses precisely:

$$\sum_{n=0}^{N-1} \frac{(\sigma\lambda^2 t)^n}{n!} \Big(\underbrace{1}_{\text{ladder}} + \underbrace{n\lambda^2}_{\text{one cross}} + \cdot + \underbrace{n^p \lambda^{2p}}_{p\text{-cross}} + \underbrace{n!\lambda^{2(p+1)}}_{\text{rest}} \Big)$$

$$\text{Last term} = (\sigma\lambda^2 t)^n \lambda^{2p} = \lambda^{-\kappa n}\lambda^{2p}$$

i.e. we need estimates on $p \geq \frac{\kappa}{2} n$ crosses.

Reality is worse: $1/n!$ prefactor is only for ladders. Without it:

$$n!(\sigma\lambda^2 t)^n \lambda^{2(p+1)} = \lambda^{-2\kappa n}\lambda^{2(p+1)}$$

i.e. $p \geq \kappa n$. In any case, we need

$$\text{Val}\Big(\text{Messy diagram of order } p\Big) \leq \lambda^{(const)p}$$

What is a good measure of "Mess"? Number of crossing is not exactly; the antiladder is not small. Instead: introduce a degree

KEY IDEAS IN THE PROOF OF QUANTUM DIFFUSION

Set $K := (\lambda^2 t)\lambda^{-\delta}$ ($\gg$ typical number of collisions), expand and stop:

$$\psi_t = \sum_{n=0}^{K-1} \psi_{n,t}^{nr} + \int_0^t e^{-i(t-s)H} \underbrace{\Big(\psi_{K,s}^{nr} + \sum_{n=0}^{K} \psi_{n,s}^{rep} \Big)}_{=:\psi_s^{err}} \mathrm{d}s$$

with $$\psi_{n,t}^{nr} := \sum_{A\,:\,\text{nonrep}} \psi_{A,t}, \qquad \psi_{n,s}^{rep} := \sum_{\substack{|A|=n \\ \text{one rep at } \alpha_n}} \psi_{A,s}$$

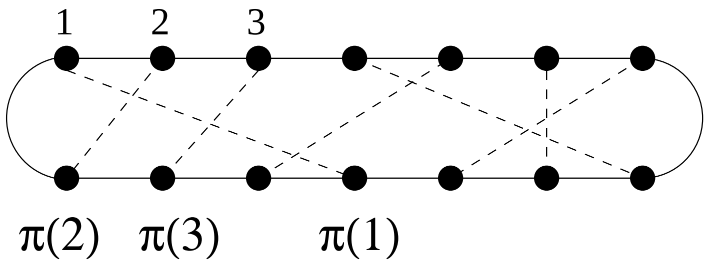
(i.e. the second sum is over $A = (\alpha_1, \dots \alpha_n)$ where $\alpha_n = \alpha_j$ for some $j < n$ and this is the first repetition).

Theorem 1. [Error terms are negligible]

$$\mathbf{E}\|\psi_s^{err}\| = o(t^{-2}) \quad \Longrightarrow \quad \mathbf{E}\Big\| \int_0^t e^{-i(t-s)H} \psi_s^{err} \mathrm{d}s \Big\|^2 = o(1)$$

We will focus on the non-repetition terms with $n < K$.

$$E\,\|\Psi_t\|^2 = \sum_\pi$$



$$= \sum_\pi \mathrm{Val}(\pi)$$

$$\mathsf{Val}(\pi) = \int d\mathbf{p}d\mathbf{q}d\alpha d\beta \; e^{it(\alpha-\beta)} \prod_j \frac{\lambda}{\alpha - e(p_j) - i\eta} \, \frac{\lambda}{\beta - e(q_j) + i\eta} \, \Delta_\pi(\mathbf{p},\mathbf{q}) \; ,$$

Theorem 2. [Only the ladder matters]

$$\mathbf{E}\|\psi_{n,t}^{nr}\|^2 = \mathsf{Val}(\mathsf{id}) + o(1)$$

$$\mathbf{E}W_{\psi_{n,t}^{nr}} = \mathsf{Val}_{\mathsf{Wig}}(\mathsf{id}) + o(1)$$

Theorem 3. [Wigner transform of the main term]

$$\mathsf{Val}_{\mathsf{Wig}}(\mathsf{id})$$

satisfies the heat equation.

For the rest, we focus on Theorem 2, $\mathbf{E}\|\psi_{n,t}^{nr}\|^2 = \mathsf{Val}(\mathsf{id}) + o(1)$

INTEGRATION OF GENERAL FEYNMAN GRAPH

$$\Delta_\pi(\mathbf{p},\mathbf{q}) = \prod_j \delta\Big((p_{j-1}-p_j)-(q_{\pi(j)-1}-q_{\pi(j)})\Big) = \prod_i \delta\Big(q_i - \sum_j M_{ij}p_j\Big)$$

The matrix M in $q_i = \sum_j M_{ij}p_j$ is totally unimodular (all subdeterminants are 0, ± 1), so change of variables is controlled.

TASK: Successively integrate out p_j's in

$$Q(M) := \lambda^{2n} \int d\alpha d\beta \int d\mathbf{p} \prod_{i=1}^{n} \frac{1}{|\alpha - \omega(p_i) - i\eta|} \, \frac{1}{\Big|\beta - \omega\Big(\sum_{j=1}^n M_{ij}p_j\Big) + i\eta\Big|}$$

and keep track of the change of M, $\mathcal{E}$.

Problem: Each p_j may appear in many denominators. One can integrate out one or two or three denom. but not any number.

$$|Val(\pi)| \Longrightarrow Q(M) := \lambda^{2n} \int d\alpha d\beta \int d\mathbf{p}$$

$$\prod_{i=1}^{n} \frac{1}{|\alpha - \omega(p_i) - i\eta|} \frac{1}{\left|\beta - \omega\Big(\sum_{j=1}^{n} M_{ij} p_j\Big) + i\eta\right|}$$

Triv. est $\left|\frac{1}{\beta - \omega(\ldots) + i\eta}\right| \leq \frac{1}{\lambda^2 |\mathrm{Im}\omega|} \sim \lambda^{-2} \quad \Longrightarrow \quad |Val(\pi)| \sim O(1)$

Number of tree momenta = number of loop momenta = n

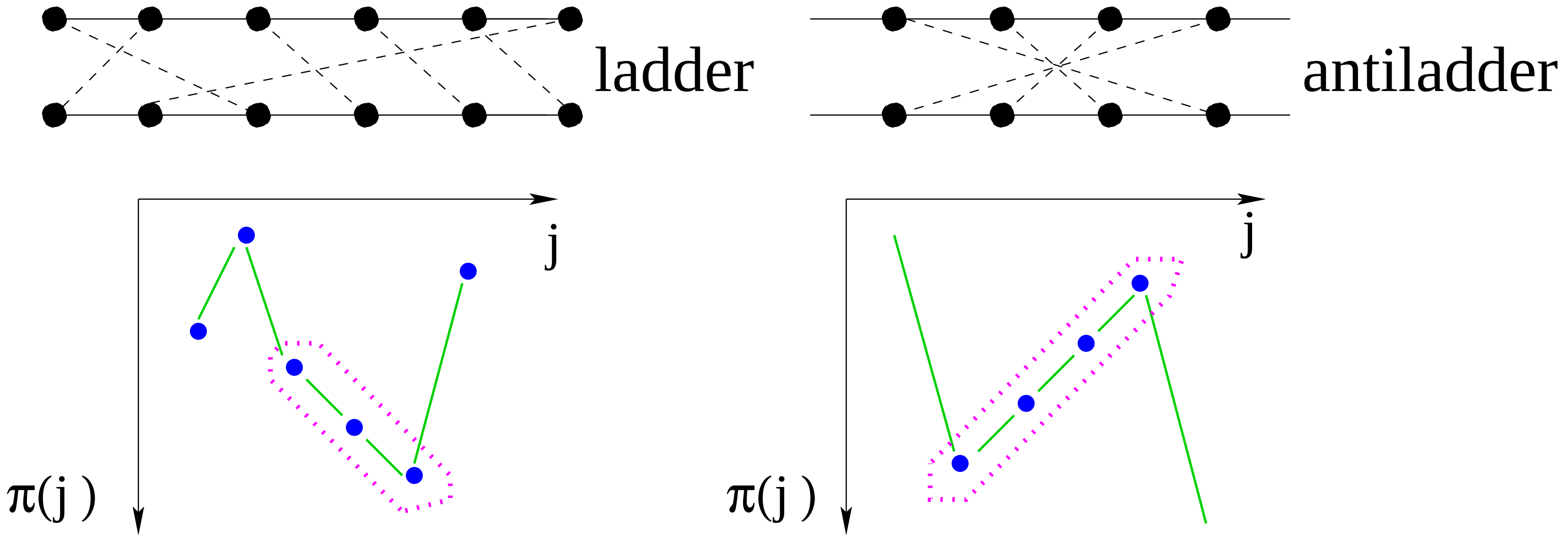
L^∞-bound on "tree"-denominators, L^1-bound on "loop"-denom.
$\Longrightarrow \lambda^{2n}\lambda^{-2n} = O(1)$. No gain. (Actually log factors)

We need to do better.

CONTROL OF CROSSING TERMS: THE DEGREE

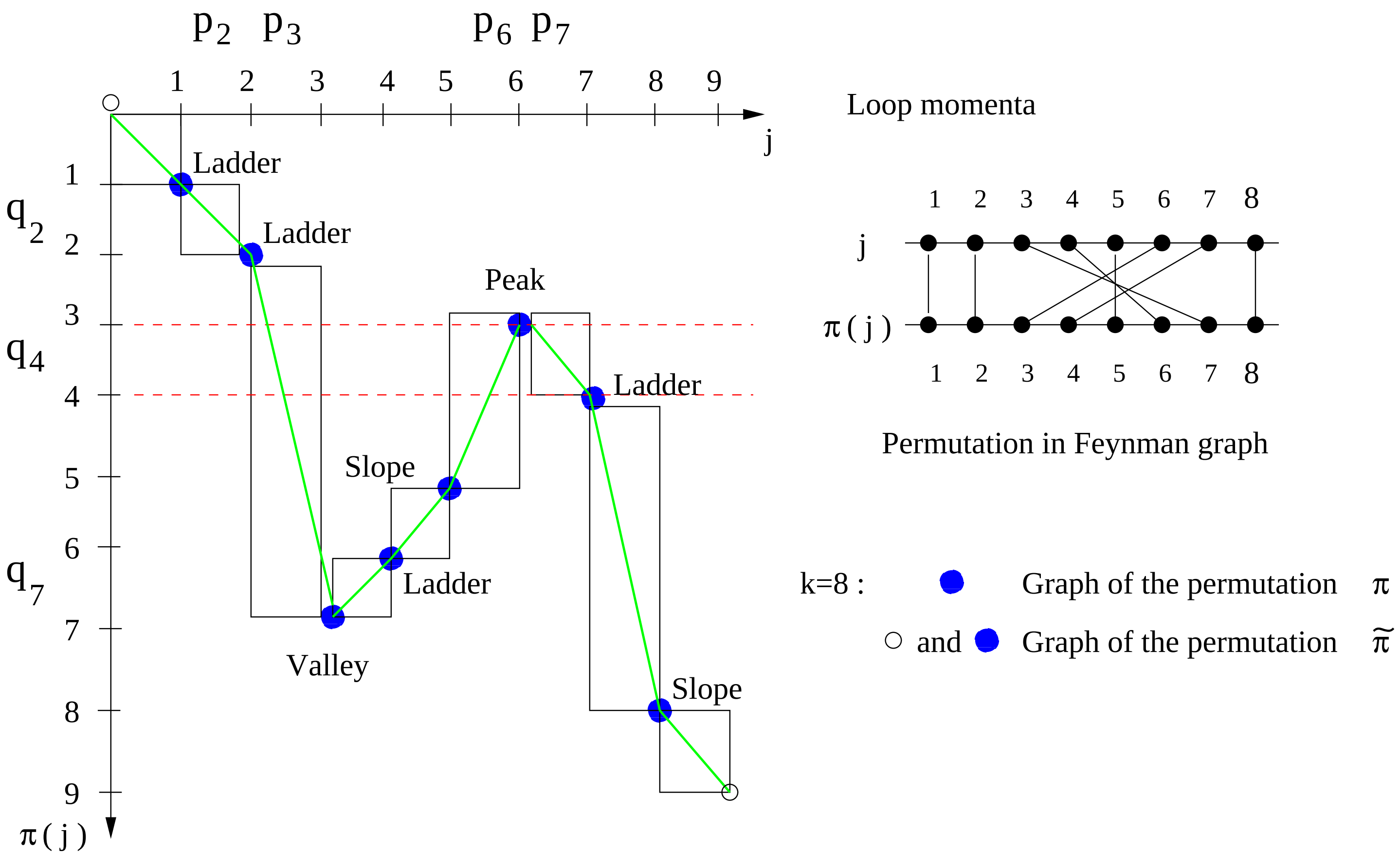
Classification of all Feynman diagrams based upon the complexity of the permutation $\pi : A \to B$ expressed by a degree $d(\pi)$

Def: $d(\pi)$ = deg of permutation = # non-ladder indices.



Philosophy: Most permutations have high degree (complexity)

MORE PRECISE DEFINITIONS



The boxes indicate the dependency structure of the tree momenta on the loop momenta: e.g.: $q_2 = p_2$, $q_4 = p_3 - p_6 + p_7$, and p_3 appears (with $+$ sign) in $q_3, q_4, \ldots, q_7$.

KEY ESTIMATES

Lemma 1. [Easy]

$$\#\{\pi \ : \ d(\pi) = d\} \sim n^d$$

Lemma 2. [Hard]

$$\text{Val}(\pi) \le \lambda^{\kappa d(\pi)} \qquad (*)$$

$$\text{Lemmas} \Longrightarrow \sum_{\pi} \text{Val}(\pi) = \sum_{d} \sum_{\pi : d(\pi) = d} \text{Val}(\pi) = \sum_{d} n^d \lambda^{\kappa d} < \infty$$

if $n \le \lambda^{-\kappa}$. Since $n \sim \lambda^2 t$, get convergence $t \le \lambda^{-2-\kappa}$.

Key: Gain a λ factor per each non-ladder vertex.

(*) should be valid for $\kappa = 2$ but not beyond.

Best bound: $\lambda^{2n} n! = (\lambda^2 n)^n$ and recall $n = \lambda^2 t$

Going beyond $t \sim \lambda^{-4}$ requires to resum 4-legged diagrams.

NEW ALGORITHM TO INTEGRATE OUT TREE MOMENTA

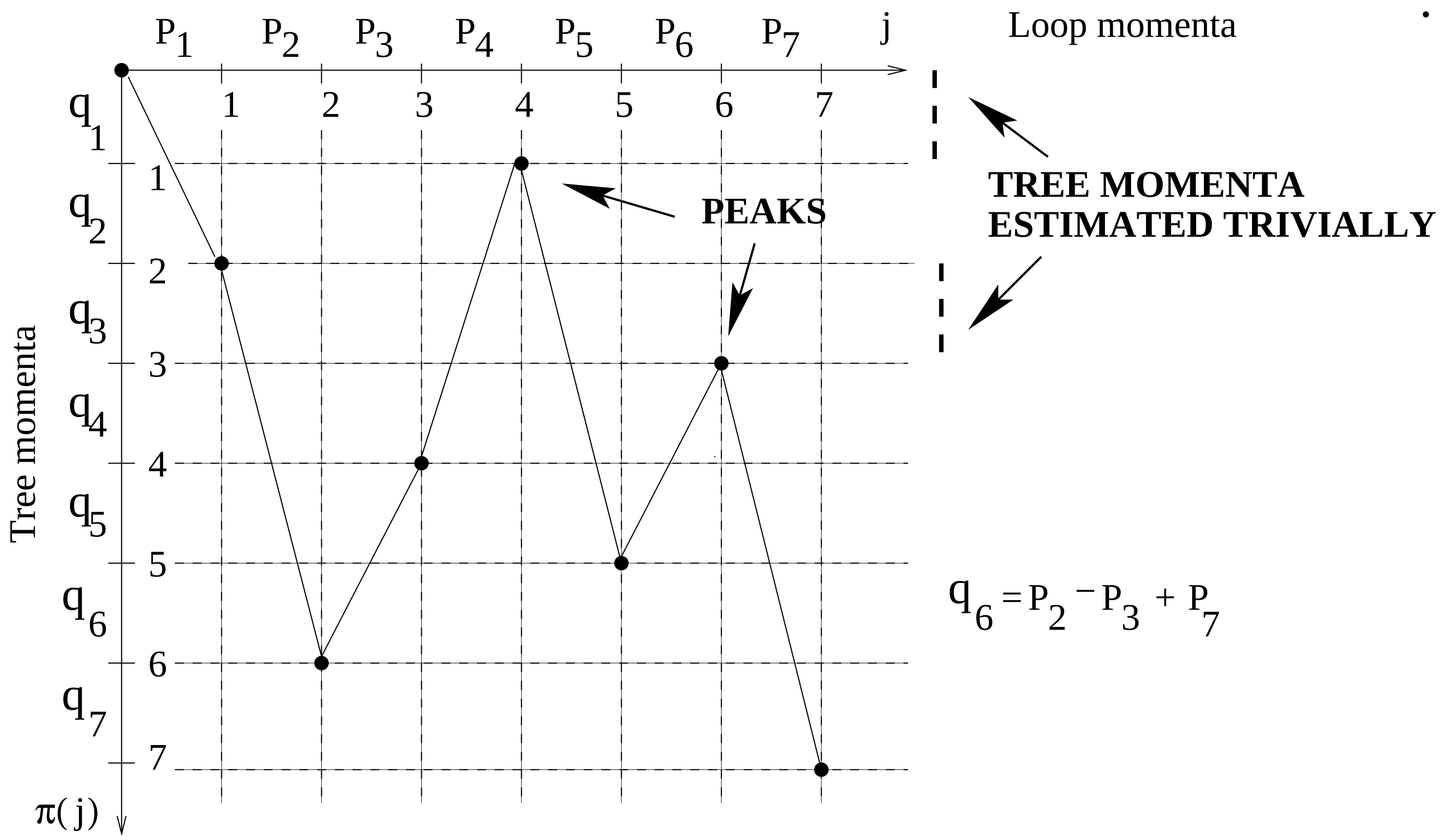
Use L^∞ bound

$$\frac{1}{|\beta - \omega(q) + i\eta|} \leq \eta^{-1} = t$$

on all tree momenta q that lies "above a peak", then the rest can be integrated out without further negative λ-power (modulo logs and point singularities).

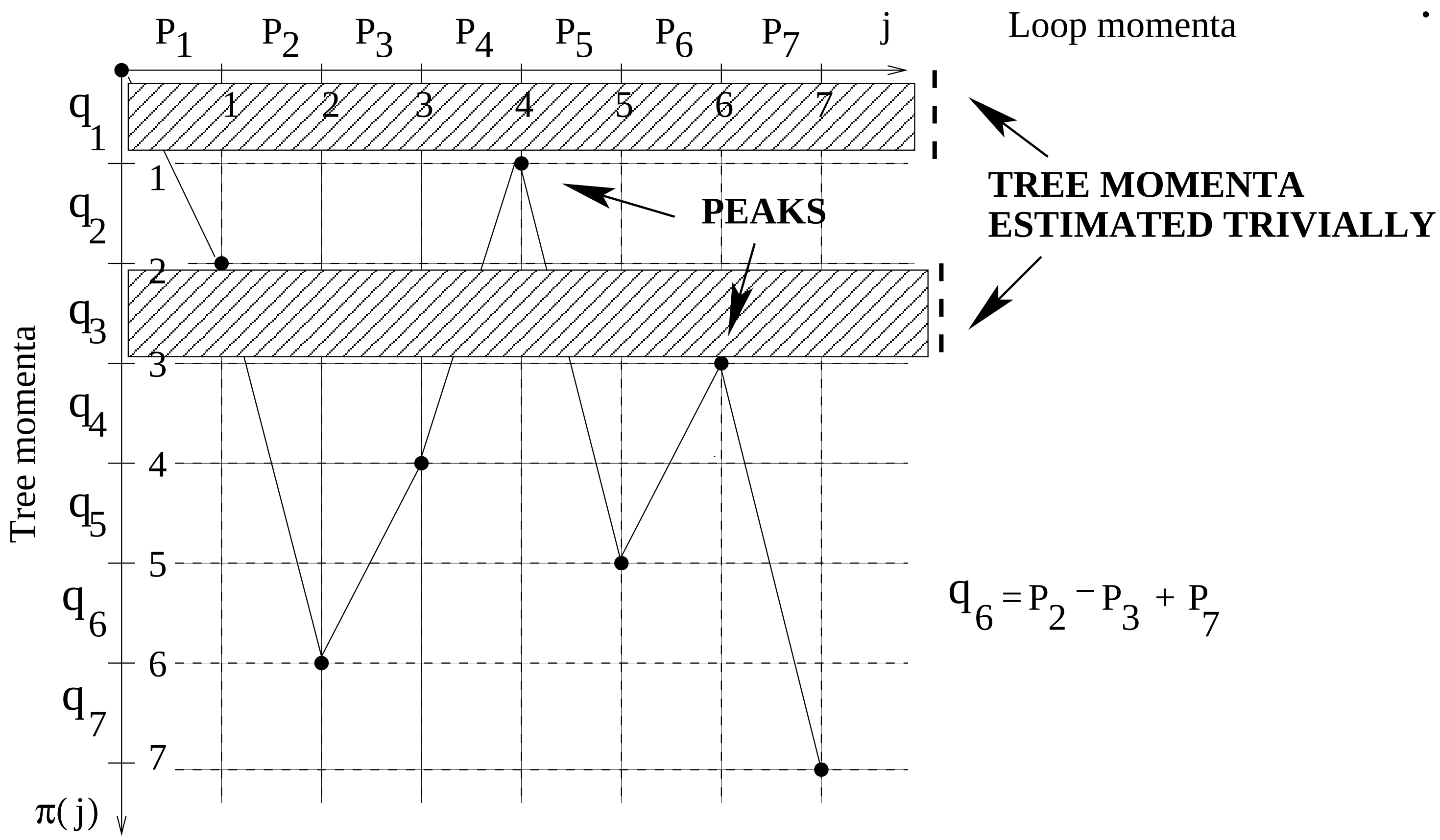
WHY?

Look at the remaning tree momenta successively. For each of them, there is a loop momenta (integration variable) that appears only in this tree propagator, so, together with the corresponding loop propagator, we need to do only a 2-denominator integration, that is doable (may lead to point singularity).



Estimate the tree denominators above the peaks trivially:

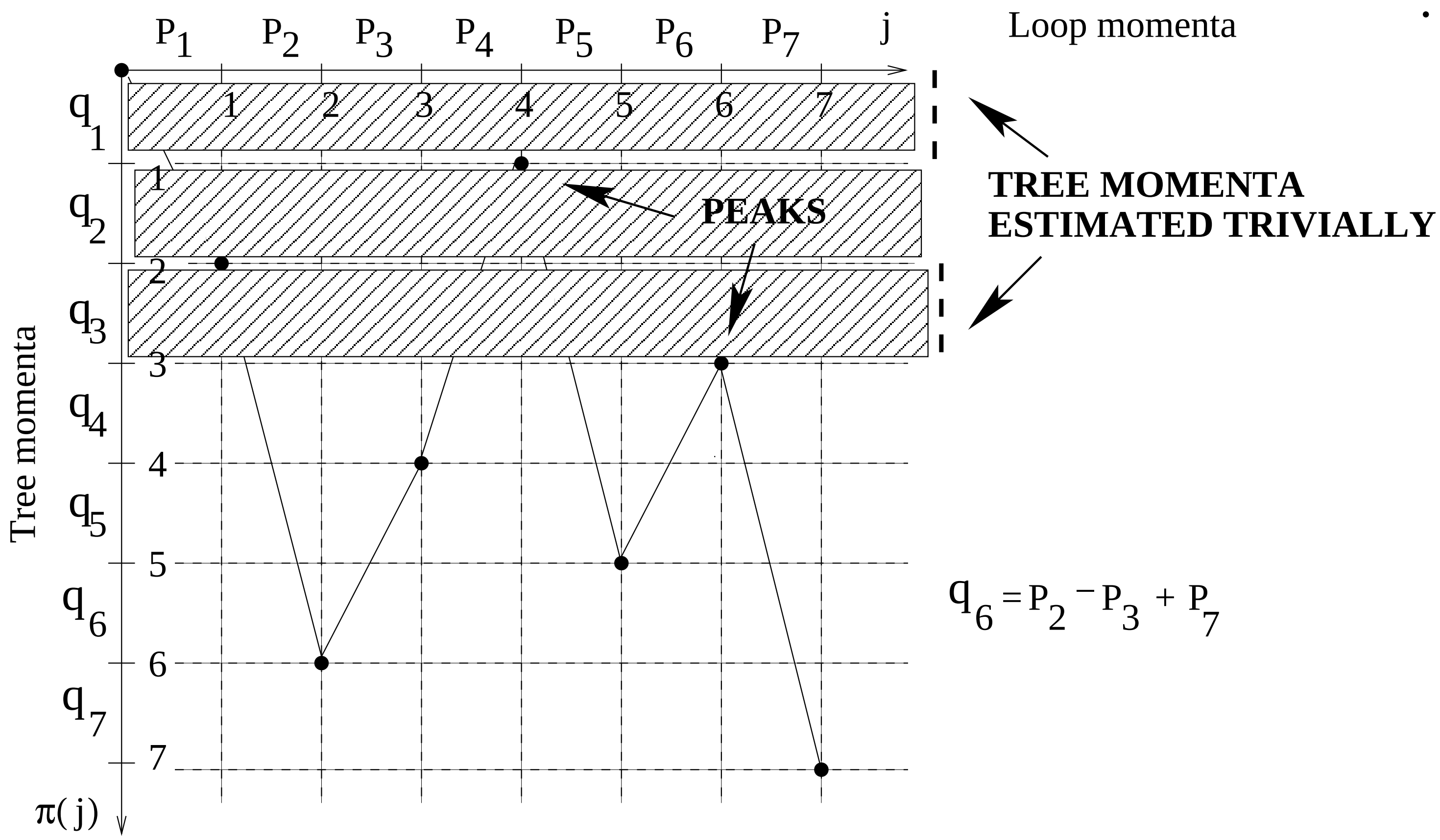
$$\frac{1}{|\beta - \omega(q_1) + i\eta|} \frac{1}{|\beta - \omega(q_3) + i\eta|} \leq \eta^{-2}$$



Consider the next tree momentum, q_2. Note that p_1 ends at q_2, so p_1 appears only in q_2, as $q_2 = p_1 - p_4 + p_5$. Integrate dp_1.

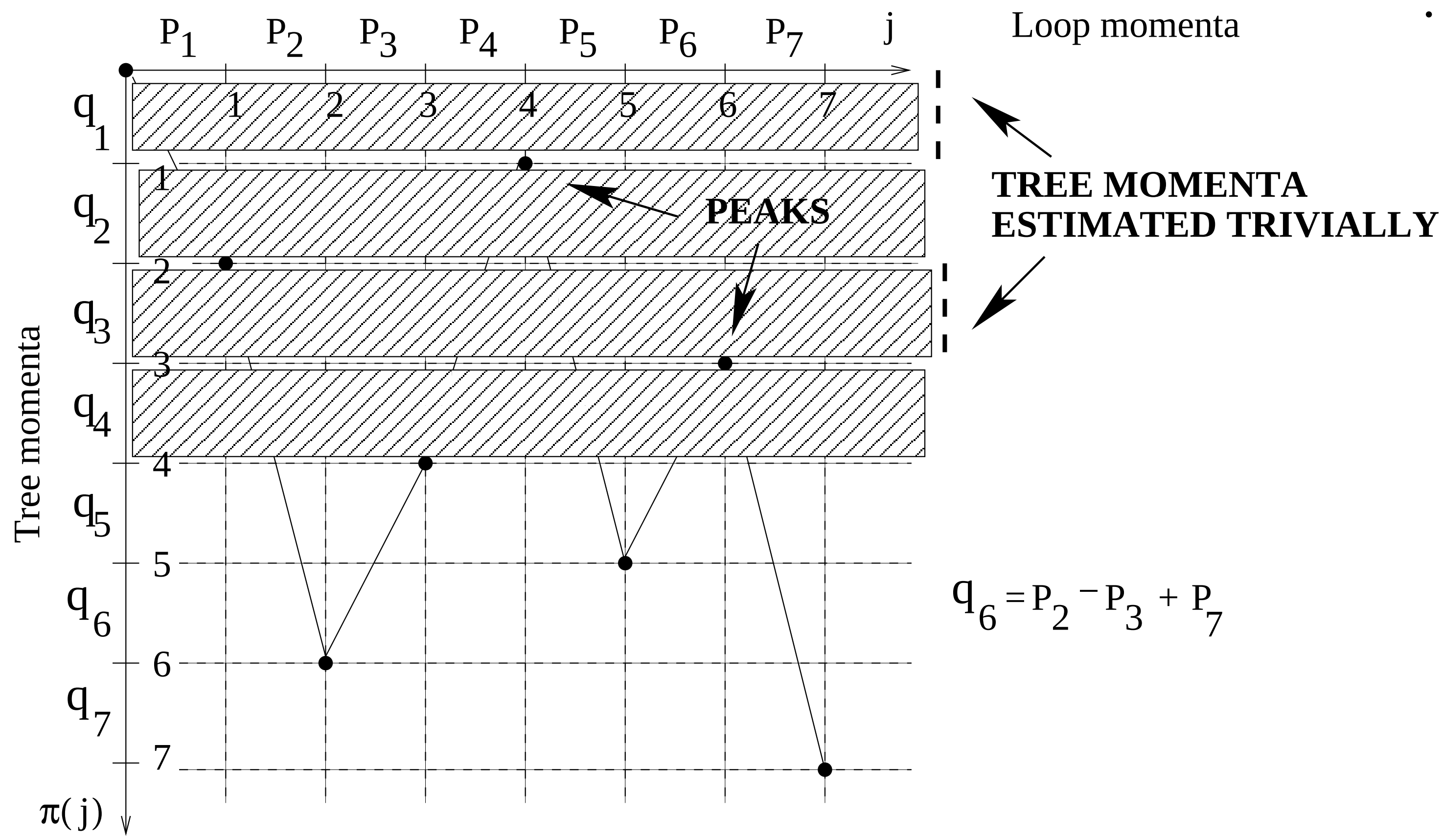
$$\int \frac{1}{|\alpha - \omega(p_1) + i\eta|} \frac{\mathrm{d}p_1}{|\beta - \omega(p_1 - p_4 + p_5) + i\eta|} \leq \frac{C|\log \eta|}{|p_4 - p_5| + \eta}$$

Neglect (for this discussion) the point singularity.



Next tree momentum is q_4. Note that p_4 ends at q_4, so p_4 appears only in q_4 as $q_4 = p_2 - p_4 + p_5 - p_6 + p_7$. Integrate d$p_4$

$$\int \frac{1}{|\alpha - \omega(p_4) + i\eta|} \frac{\mathrm{d}p_4}{|\beta - \omega(p_2 - p_4 + p_5 - p_6 + p_7) + i\eta|} \leq \frac{C|\log\eta|}{|p_2 + p_5 - p_6 + p_7| + \eta}$$



Next tree momentum is q_5. There are now two loop momenta, p_5 and p_6 that ends here, choose one, say p_5, and integrate.

$$\int \frac{1}{|\alpha - \omega(p_5) + i\eta|} \frac{\mathrm{d}p_5}{|\beta - \omega(p_2 - p_3 + p_5 - p_6 + p_7) + i\eta|} \leq \frac{C|\log \eta|}{|p_2 - p_3 - p_6 + p_7| + \eta}$$

Etc. One gets logarithmic factors but not worse.

The rest is just power-counting:

Let p be the number of peaks.

Assume first that there are no ladder indices, $\ell = 0$.

$$|\mathsf{Val}(\pi)| \leq \lambda^{2n}\eta^{-p} = \lambda^{2n-(2+\kappa)p}$$

Here

$$2n - 2p = 2n - p - v \geq n = d \qquad \text{and} \qquad p \leq \frac{n}{2} = \frac{d}{2}$$

Thus

$$|\mathsf{Val}(\pi)| \leq \lambda^{\left(1-\frac{\kappa}{2}\right)d}$$

which gives $\leq \lambda^{\kappa d}$ if $\kappa \leq 2/3$.

If there are ladder indices, $\ell \neq 0$, then first integrate them out:

$$\int \frac{\mathrm{d}p}{|\alpha - \omega(p) + i\eta|^2} = \lambda^{-2}$$

and then apply the previous argument for the rest:

$$|\mathrm{Val}(\pi)| \leq \lambda^{2n}\Big(\lambda^{-2}\Big)^{\ell}\eta^{-p} = \lambda^{2n-2\ell-(2+\kappa)p}$$

Use that

$$2n - 2p - 2\ell = (n - p - v - \ell) + (n - \ell) \geq n - \ell = d$$

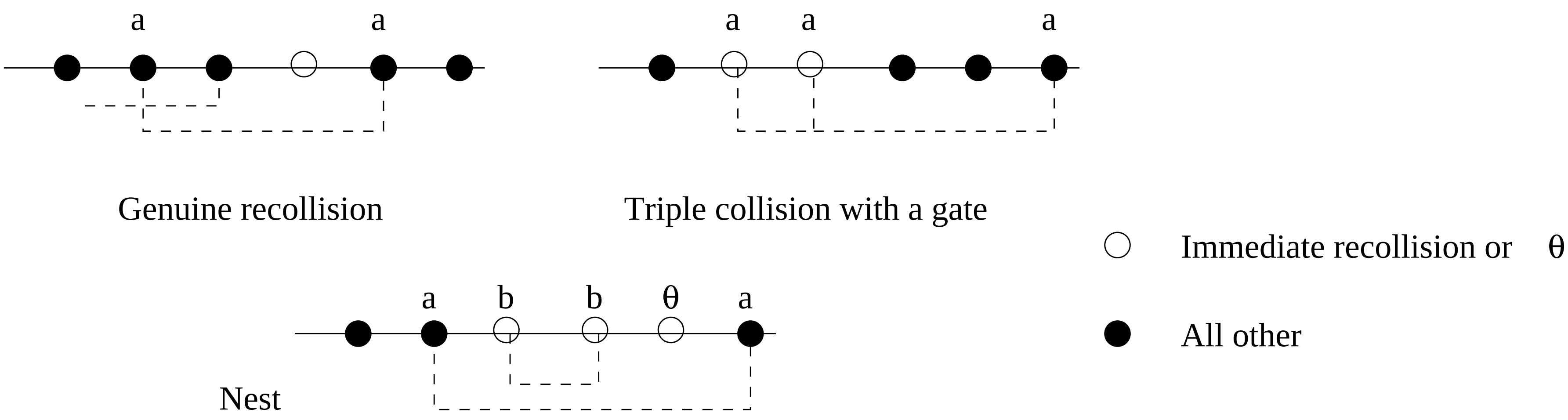
and

$$p \leq \frac{1}{2}(n - \ell) = \frac{d}{2}$$

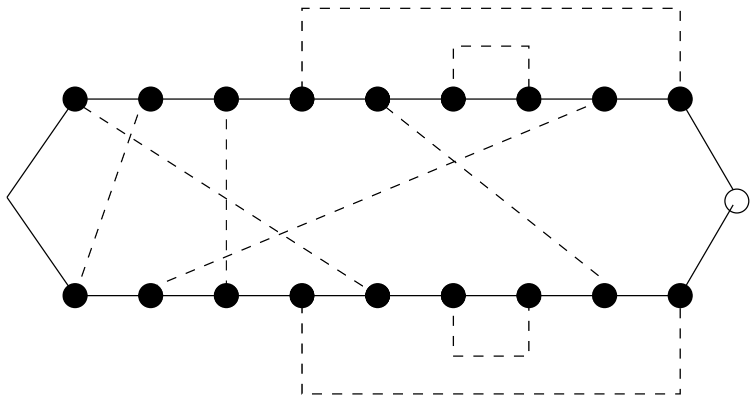
we again get

$$|\mathrm{Val}(\pi)| \leq \lambda^{\left(1-\frac{\kappa}{2}\right)d} \leq \lambda^{\kappa d}$$

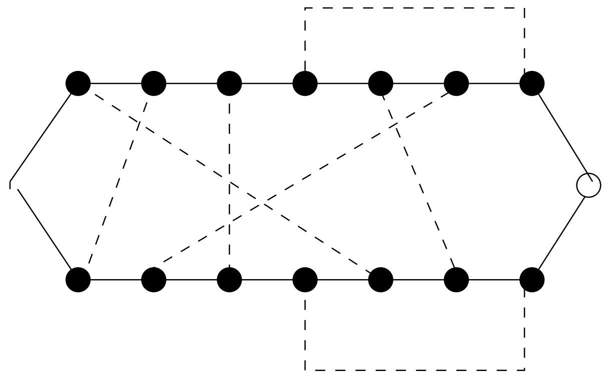
FEYNMAN GRAPHS WITH REPETITION



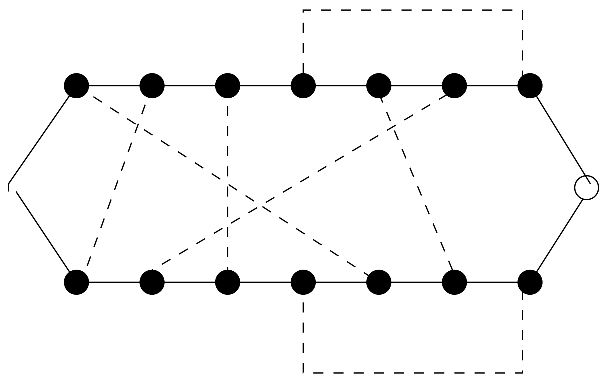
When one of these shows up $\Longrightarrow$ stop the expansion to reduce complications. Compute the term by "bare hand", E.g



$< \; C \, \lambda^{-2\kappa}$



Removal of the gates from a recollision



$< \; C \, \lambda^{6-3\kappa}$

Estimate of two−sided recollision graph

There is a delicate stopping rule that stops the expansion if the collision history has collected "enough" repetition (e.g. 3 gate, or 1 nest or 1 recollision).

This procedure has a side effect:

$$\sum_{\alpha_\ell \neq \alpha_k} \mathbf{E} \prod_j \overline{\widehat{V}_{\alpha_j}(p_{j+1} - p_j)} \widehat{V}_{\alpha_j}(q_{j+1} - q_j) = \sum_{\alpha_\ell \neq \alpha_k} \prod_j e^{i\alpha_j [p_{j+1} - p_j - (q_{j+1} - q_j)]}$$

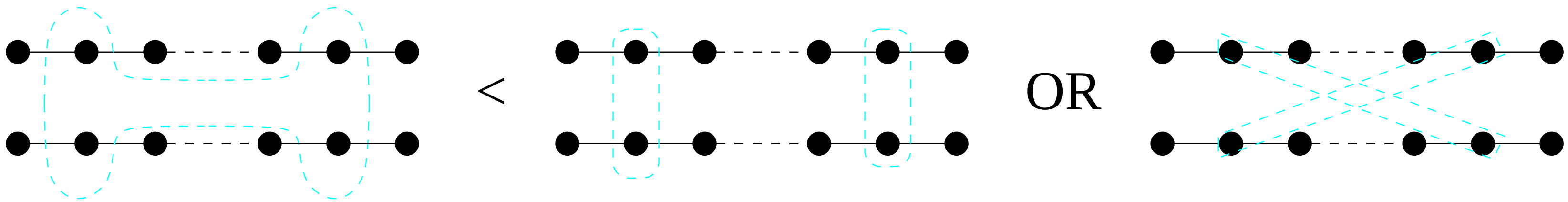
Restriction $\alpha_\ell \neq \alpha_k$ destroys the precise delta fn. $\sum_\alpha e^{i\alpha p} = \delta(p)$

Connected Graph Formula: $\mathcal{A}_n$ set of partitions of $\{1, \ldots, n\}$

$$\sum_{\alpha_\ell \neq \alpha_k} e^{iq_j \alpha_j} = \sum_{\mathbf{A} \in \mathcal{A}_n} \prod_\nu c(|A_\nu|) \delta\Big(\sum_{\ell \in A_\nu} q_\ell \Big) \qquad \mathbf{A} = (A_1, A_2, \ldots)$$

Instead of pairing $\Longrightarrow$ "hyperpairs" = lumps of size > 2.

Thm: Lumps can be broken into permutations.



Idea: Chose the break-up that gives the biggest $d(\pi)$

Let $\mathbf{B} = \{B_1, B_2, \ldots\}$ be a partition of vertices.

$$s(\mathbf{B}) := \frac{1}{2} \sum \{ \, |B_j| \; : \; |B_j| \geq 4\}$$

Prop. $\exists$ a perm. π, compatible with $\mathbf{B}$ such that $d(\pi) \geq \frac{1}{2} s(\mathbf{B})$.
Reduce to previous case.

CONCLUSIONS

• We rigorously proved Brownian motion from Hamiltonian quantum dynamics with a time-independent random scatterer environment.

• We controlled the interferences of random waves in a multiple scattering process with infinitely many collisions.

• We classified and estimated Feynman graphs up to all orders and gained an extra λ-power per each non-ladder vertex compared to the usual “power-counting” bound.

$\mathbf{Z}^d$ *IS HARDER THAN* $\mathbf{R}^d$

Typically $\mathbf{R}^d$ is harder because of the nuisance to control the UV regime that has no physical relevance in this problem.

We have to deal with this nuisance. But $\mathbf{Z}^d$ has an even more serious problem: the innocently looking crossing estimate is wrong.

$$\int \frac{\mathrm{d}p}{|\alpha - e(p) + i\eta|} \frac{1}{|\alpha - e(p+q) + i\eta|} \sim \frac{1}{|q| + \eta}$$

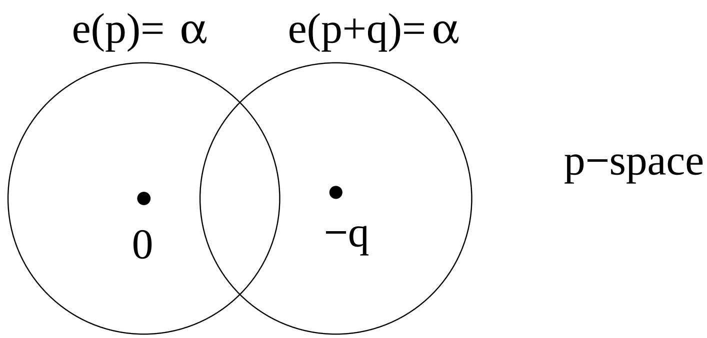
holds for $e(p) = p^2$, but it does not hold for $e(p) = \sum_1^3 (1 - \cos p_j)$!

The bound

$$\int \frac{\mathrm{d}p}{|\alpha - e(p) + i\eta|} \frac{1}{|\alpha - e(p+q) + i\eta|} \sim \frac{1}{|q| + \eta}$$

is about the overlap of a small nbhd. of two level sets

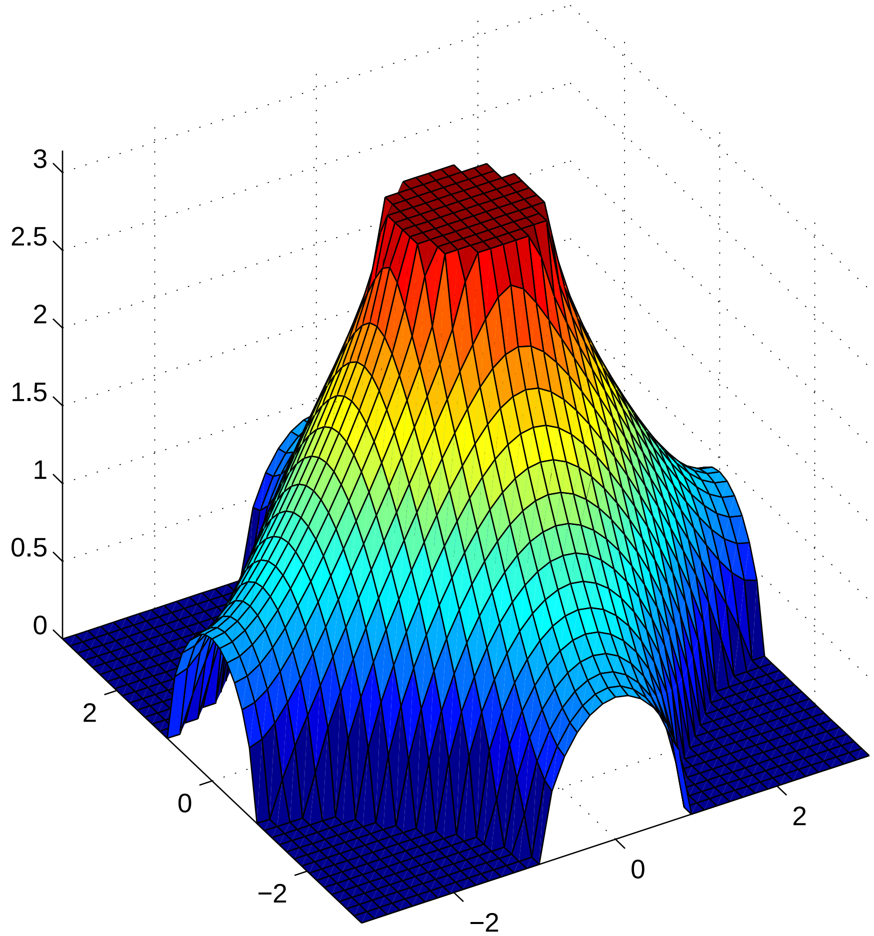
$$\{p \ : \ \alpha = e(p)\}, \qquad \{p \ : \ \alpha = e(p+q)\}$$



Convex level sets intersect transversally or touch at a point – small overlap.

The level sets of $e(p) = \sum_j (1 - \cos p^{(j)})$ for $2 < \alpha < 4$ are not convex (they even contain straight lines).

Level set for $\alpha = \sum_j (1 - \cos p^{(j)})$, $2 < \alpha < 4$



This problem is ultimately reduced to the decay of the Fourier transform of the α-level set, $\Sigma_\alpha = \{e(p) = \alpha\}$:

$$I(x) = I_\alpha(x) := \int_{\Sigma_\alpha} e^{ipx} \mathrm{d}p$$

For strictly convex level sets $I(x) \sim |x|^{-1}$ $(|x| \gg 1)$

Otherwise: no general result when K [Gauss curv.] vanishes.

Theorem [E-Salmhofer, '06] Let $\nu \in S^2$, $r > 0$

$$I(r\nu) \lesssim \frac{1}{r} + \frac{1}{r^{3/4}|D(\nu)|^{1/2} + 1}$$

with $D(\nu) = \min_j |\nu(p_j) - \nu|$, $\nu(p)$ is the normal at $p \in \Sigma$ and p_j's are finitely many points where the neutral direction of the Gauss map on $\Gamma = \{K = 0\} \cap \{e(p) = \alpha\}$ is parallel with Γ.

To complete the proof of diffusion in $\mathbf{Z}^d$, one needs a very precise information on the geometry of the surface.

The result

$$\int |I(x)|^4 \mathrm{d}x \le C|\log \eta|^{10}$$

depends on very detailed geometric properties of the level set Σ_α (that has to be proved for $e(p)$), namely:

• Let $\kappa_1(p), \kappa_2(p)$ be the two principal curvatures at $p \in \Sigma_\alpha$ and $v_1(p)$, $v_2(p)$ the principal curvature directions.

• Let γ_p be the integral curve of $v_1(p)$ through p (where $|\kappa_1| \le |\kappa_2|$)

• Let $K = \kappa_1\kappa_2$ (Gauss curv.)

• The surface $\{K = 0\}$ intersects Σ_α transversally

• The level curves of K on Σ_α, form a regular foliation, call Γ.

• The foliations γ and Γ are (unfortunately) not transversal but where they are parallel, their curvature differ, i.e. they touch each other at least quadratically at a few points.

All these guarantee that the normal vector $\nu(p)$ moves at least quadratically in one direction and linearly in the other direction away from a few exceptional points. These speeds degenerate with the first power of the distance to the closest exceptional point.

+ lots of dyadic decomposition and stationary phase estimates.

$$I(r\nu) \lesssim \frac{1}{r} + \sum_{p:\nu(p)=\nu} \left[\frac{1}{r^{3/4}|K(p)|^{1/2}+1} + \frac{1}{r^{3/4}|d(p)|^{1/2}+1} \right]$$

$d(p) = \min|p - p_j|$, p_j's are finitely many poins where the neutral direction of the Gauss map on $\Gamma = \{K = 0\} \cap \{e(p) = \alpha\}$ is parallel with Γ.

COMPUTATION OF THE MAIN TERM

Fourier Transform of the rescaled Wigner transform $W(X/\varepsilon, V)$

$$\widehat{W}_t(\varepsilon\xi, v) = \overline{\widehat{\psi}_t}\Big(v + \frac{\varepsilon\xi}{2}\Big)\widehat{\psi}_t\Big(v - \frac{\varepsilon\xi}{2}\Big)$$

Here $\varepsilon = \lambda^{2+\kappa/2}$ space rescaling, $t = \lambda^{-2-\kappa}T$.

Test against a macroscopic observable, compute

$$\langle \mathcal{O}, \mathbf{E}\widehat{W}_t\rangle = \langle \mathcal{O}(\xi, v), \mathbf{E}\widehat{W}_t(\varepsilon\xi, v)\rangle = \int \mathrm{d}v\mathrm{d}\xi\; \mathcal{O}(\xi, v)\, \mathbf{E}\widehat{W}_t(\varepsilon\xi, v)$$

Continuity property

$$\langle \mathcal{O}, \widehat{W}_\psi\rangle - \langle \mathcal{O}, \widehat{W}_\phi\rangle \le \|\mathcal{O}\|_\infty \|\psi - \phi\|\Big(\|\psi\| + \|\phi\|\Big)$$

Renormalize, expand ψ_t, keep the ladder term. The rest is $o(1)$.

$$
\begin{aligned}
\langle \mathcal{O}, \mathbf{E}\widehat{W}_t \rangle \approx \sum_{k \le K} \lambda^{2k} \int_{\mathbf{R}} \mathrm{d}\alpha \mathrm{d}\beta \; e^{it(\alpha-\beta)+2t\eta} & \\
\times \int \mathrm{d}\xi \mathrm{d}v \mathcal{O}(\xi, v) \overline{R_\eta\Big(\alpha, v + \frac{\varepsilon\xi}{2}\Big)} R_\eta\Big(\beta, v - \frac{\varepsilon\xi}{2}\Big) & \\
\times \prod_{j=2}^{k} \Big[\int \mathrm{d}v_j \overline{R_\eta\Big(\alpha, v_j + \frac{\varepsilon\xi}{2}\Big)} R_\eta\Big(\beta, v_j - \frac{\varepsilon\xi}{2}\Big) \Big] & \\
\times \int \mathrm{d}v_1 \overline{R_\eta\Big(\alpha, v_1 + \frac{\varepsilon\xi}{2}\Big)} R_\eta\Big(\beta, v_1 - \frac{\varepsilon\xi}{2}\Big) \overline{\widehat{W}_0}(\varepsilon\xi, v_1) \;, &
\end{aligned}
$$

with

$$
R_\eta(\alpha, v) := \frac{1}{\alpha - e(v) - \lambda^2 \theta(v) + i\eta} \;,
$$

We perform each $\mathrm{d}v_j$ integral.

KEY LEMMA

$f(p) \in C^1$, $a := (\alpha + \beta)/2$, $\lambda^{2+4\kappa} \le \eta \le \lambda^{2+\kappa}$. and $|r| \le \lambda^{2+\kappa/4}$.

$$\int \frac{\lambda^2 f(v)}{\Big(\alpha - e(v-r) - \lambda^2\overline{\theta}(v-r) - i\eta\Big)\Big(\beta - e(v+r) - \lambda^2\theta(v+r) + i\eta\Big)} \mathrm{d}v$$

$$= -2\pi i \int \frac{\lambda^2 f(v)\, \delta(e(v) - a)}{(\alpha - \beta) + 2(\nabla e)(v)\cdot r - 2i\lambda^2 \mathcal{I}(a)} \mathrm{d}v + o(\lambda^{1/4})$$

where

$$\mathcal{I}(a) := \mathrm{Im} \int \frac{\mathrm{d}v}{a - e(v) - i0} = \int \delta(e(v) - a)\, \mathrm{d}v\,, \qquad \mathcal{I}(e(p)) = \mathrm{Im}\theta(p)$$

IDEA:

$$\frac{\mathrm{d}v}{(\alpha - g(v-r) - i0)(\beta - g(v+r) + i0)} \approx \frac{1}{\alpha - \beta + g(v+r) - g(v-r)}$$

$$\times \Big[\frac{1}{\beta - g(v) + i0} - \frac{1}{\alpha - g(v) - i0}\Big]$$

Change variables $a = (\alpha + \beta)/2$, $b = (\alpha - \beta)/\lambda^2$, choose $\eta \ll t^{-1}$

$$\langle \mathcal{O}, \mathbf{E}\widehat{W}_t \rangle \approx \sum_{k \le K} \int \mathrm{d}\xi \mathrm{d}a \mathrm{d}b \; e^{it\lambda^2 b} \left(\prod_{j=1}^{k+1} \int \frac{-2\pi i F^{(j)}(\xi, v_j)\, \delta(e(v_j) - a)}{b + \lambda^{-2}\varepsilon(\nabla e)(v_j) \cdot \xi - 2i\mathcal{I}(a)} \mathrm{d}v_j \right)$$

$F^{(1)} = \widehat{W}_0$, $F^{(k+1)} = \mathcal{O}$, rest is $F^{(j)} \equiv 1$.

Let $\mathrm{d}\mu_a(v)$ be the probability measure the level surface $\{e(v) = a\}$

$$\int h(v) \mathrm{d}\mu_a(v) := \langle h \rangle_a = \frac{\pi}{\mathcal{I}(a)} \int h(v) \delta(e(v) - a) \mathrm{d}v$$

Let $H(v) := \frac{\nabla e(v)}{2\mathcal{I}(a)}$

$$2\mathcal{I}(a) \sum_{k \le K} \int \mathrm{d}\xi \int_{\mathbf{R}} \mathrm{d}a \mathrm{d}b \; e^{i2t\lambda^2 \mathcal{I}(a) b} \left(\prod_{j=1}^{k+1} \int \frac{-iF^{(j)}(\xi, v_j)}{b + \lambda^{-2}\varepsilon H(v_j) \cdot \xi - i} \mathrm{d}\mu_a(v_j) \right)$$

Expand the denominator up to second order

$$\int \frac{-i}{b + \varepsilon\lambda^{-2} H(v)\cdot\xi - i}\, \mathsf{d}\mu_a(v)$$
$$= \frac{-i}{b-i}\int \left[1 - \frac{\varepsilon\lambda^{-2}H(v)\cdot\xi}{b-i} + \frac{\varepsilon^2\lambda^{-4}[H(v)\cdot\xi]^2}{(b-i)^2} + O\Big((\varepsilon\lambda^{-2}|\xi|)^3\Big)\right]\mathsf{d}\mu_a(v)$$

After summation the error is $K(\varepsilon\lambda^{-2})^3 = \lambda^{-\kappa}(\lambda^{\kappa/2})^3 = o(1)$.

Linear term cancels by symmetry: $H(v) = -H(-v)$. Define

$$D(a) := 4\mathcal{I}(a)\int \mathsf{d}\mu_a(v)\; H(v)\otimes H(v)$$

$$\sum_{k\le K}\int \mathsf{d}\xi \int_{\mathbf{R}} \mathsf{d}a\; 2\mathcal{I}(a)\int \widehat{W}_0(\varepsilon\xi, v_1)\mathsf{d}\mu_a(v_1)\int \mathcal{O}(\xi, v)\mathsf{d}\mu_a(v)$$
$$\times \int_{\mathbf{R}} \mathsf{d}b\; e^{2i\lambda^2\mathcal{I}(a)tb}\left(\frac{-i}{b-i}\right)^{k+1}\left[1 + \underbrace{\frac{\varepsilon^2\lambda^{-4}\langle\xi, D(a)\xi\rangle}{4\mathcal{I}(a)}}_{=:B^2}\frac{1}{(b-i)^2}\right]^{k-1}$$

Note the geometric series.

$$\sum_{k=0}^{\infty}\left(\frac{-i}{b-i}\right)^{k+1}\left[1+\frac{B^2}{(b-i)^2}\right]^{k+1} = (-i)\frac{(b-i)^2+B^2}{(b-i)^3+i(b-i)^2+iB^2}$$

Perform the db integration analytically with $A := 2\lambda^2\mathcal{I}(a)$.

$$(-i)\int_{\mathbf{R}} \mathrm{d}b\, e^{itAb}\,\frac{(b-i)^2+B^2}{(b-i)^3+i(b-i)^2+iB^2} = 2\pi e^{-tAB^2} + o(1)$$

from the dominant residue $b = iB^2$. Compute

$$tAB^2 = \varepsilon^2\lambda^{-4-\kappa}\frac{T}{2}\langle\xi, D(a)\xi\rangle = \frac{T}{2}\langle\xi, D(a)\xi\rangle,$$

To get a nontrivial limit: $\varepsilon = O(\lambda^{-2-\kappa/2})$ – Diffusive scaling.

$$\langle\mathcal{O}, \mathbf{E}\widehat{W}\rangle \approx \int \mathrm{d}\xi \int_{\mathbf{R}} \mathrm{d}a\, \mathcal{I}(a)\Big(\int \mathcal{O}(\xi, v)\mathrm{d}\mu_a(v)\Big)\underbrace{\langle\widehat{W}_0\rangle_a \exp\Big(-\frac{T}{2}\langle\xi, D(a)\xi\rangle\Big)}_{=:f(T,\xi,a)}$$

where $f(T, \xi, a)$ is the sol. of the heat eq. in F-space. $\square$