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A BRIEF INTRODUCTION TO SINGULARITY THEORY

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1 Lecture 1. Some useful facts from Singularity Theory.

Throughout this course all definitions and statements are local, that is, we always operate with «sufficiently small» neighborhoods of the considered point. In other words, we deal with germs of functioms, maps, vector fields, etc.

1.1 Multiplicity of smooth functions and maps

One of the key notions of Singularity Theory is «multiplicity».

Definition 1:

Let $f(x) : \mathbb{R} \to \mathbb{R}$ be a smooth («smooth» means C^∞) function, and x^* is its critical point, i.e., $f'(x^*) = 0$. Multiplicity of the function $f(x)$ at the critical point x^* is the order of tangency of the graphs $y = f(x)$ and $y = f(x^*)$ at x^*, i.e., the natural number μ is defined by the condition

$$\frac{df}{dx}(x^*) = 0, \ \ldots, \ \frac{d^\mu f}{dx^\mu}(x^*) = 0, \quad \frac{d^{\mu+1} f}{dx^{\mu+1}}(x^*) \neq 0. \tag{1.1}$$

If such natural number μ does not exist, then we put $\mu = \infty$ and the function $f(x) - f(x^*)$ is called «∞-flat» or simply «flat» at the point x^*. We also put $\mu = 0$ for non-critical points.

Critical points with infinite multiplicity can occur, but we will deal only with finite multiplicities. If $\mu = 1$, then the critical point x^* is called «non-degenerated».

Exercise 1.

Prove that any function f with $\mu < \infty$ can be simplified to the form $f(x) = f(x^*) \pm (x - x^*)^{\mu+1}$ by means of a smooth change of the variable x (the sign $\pm$ coincides with the sign of the non-zero derivative in (1.1)). In particular, for non-degenerated critical point we get one-dimensional case of the Morse lemma.

Hint: Use Hadamard's lemma.[1]

Multiplicity is also defined for functions of several variables $x_1, \ldots, x_p$.

Recall that «formal series» of the variables $x_1, \ldots, x_p$ are power series

$$\sum_{n_1+\cdots+n_p=0}^{\infty} c_{n_1 \ldots n_p} \, x_1^{n_1} \cdots x_p^{n_p}, \tag{1.2}$$

where $n_1, \ldots, n_p$ are integer non-negative numbers, $c_{n_1 \ldots n_p}$ are «numbers», i.e., elements of some algebraic field (usually $\mathbb{R}$ or $\mathbb{C}$), in this course we will always deal with $\mathbb{R}$. The word «formal» means that the series (1.2) may be nonconvergent at all points (of course, except for the origin $x_1 = \cdots = x_p = 0$).

Clearly, it is possible to define the addition and multiplication of formal series similarly as for convergent series (it's even simpler, since we don't need to think about the convergence). Then we get the commutative ring and even algebra over the field $\mathbb{R}$ of formal series, it is usually denoted by $\mathbb{R}[[x_1, \ldots, x_p]]$.

In particular, for $p = 1$ the series (1.2) have the form

$$\sum_{n=0}^{\infty} c_n \, x^n, \tag{1.3}$$

[1] Hadamard's lemma: in a neighborhood of 0 any smooth function $f(x) : \mathbb{R}^p \to \mathbb{R}$ can be presented in the form

$$f(x) = f(0) + x_1 g_1(x) + \cdots + x_p g_p(x), \quad x = (x_1, \ldots, x_p),$$

with appropriate smooth functions $g_i(x)$. The proof can be found in many books, e.g. V.A. Zorich «Mathematical Analysis» (it is one of the best textbooks in Calculus).

and the corresponding algebra is denoted by $\mathbb{R}[[x]]$.

Why do we need to consider formal series? At first sight, it seems that this notion does not make sense: if the series (1.2) or (1.3) divergent they don't correspond to any real object. However formal series are very useful tool in many different fields of mathematics, including algebraic geometry.[2] and differential equations[3] Several examples will be given in this course.

Let $f(x_1, \ldots, x_p) : \mathbb{R}^p \to \mathbb{R}$ be a smooth function of p variables, and x^* is its critical point, i.e., $f_{x_i}(x^*) = 0$ for all indexes $i = 1, \ldots, p$. Without loss of generality assume $x^* = 0$ (origin in the space $\mathbb{R}^p$). The function $f(x_1, \ldots, x_p)$ is connected with so-called «gradient map» $\nabla f : \mathbb{R}^p \to \mathbb{R}^p$ defined by the formula

$$\nabla f : \quad (x_1, \ldots, x_p) \to (f_{x_1}, \ldots, f_{x_p}). \tag{1.4}$$

Let $A_p = \mathbb{R}[[x_1, \ldots, x_p]]$ be the algebra of the formal series of $x_1, \ldots, x_p$. By $I_{\nabla f}$ denote the ideal in the algebra of smooth functions of p variables generated by the functions $f_{x_1}, \ldots, f_{x_p}$. The ideal $I_{\nabla f}$ can be embedded in the algebra A_p of the formal series, since each smooth function of p variables has the Taylor series at 0, which is element of A_p.

It is not hard to see that the ideal $I_{\nabla f}$ is also a vector subspace of A_p. Then it is possible to define[4] the factor-algebra $A_p/I_{\nabla f}$, which is called the «local algebra» of the gradient map (1.4) and the initial function $f(x_1, \ldots, x_p)$ at the point 0. Notice that the definition of a factor-algebra includes some nuances (see below).

Remark. A factor-algebra can be constructed as follows.

Let A be an arbitrary algebra (over a field $\mathbb{K}$) and $I \subset A$ be an ideal of A that is also a vector subspace of A. Firstly, let's consider A as a ring and define the factor-ring A/I by the standard way: A/I consists of the residue classes of all elements of A modulo the additive subgroup I:

$$a + I = \{a + b \mid \forall\, b \in I\}.$$

Secondly, we can turn the factor-ring A/I into an algebra (over the same field $\mathbb{K}$) if we define the multiplication of the elements of A/I by the elements $k \in \mathbb{K}$ with the rule:

$$k(a + I) = ka + I, \quad \forall\, a \in A,$$

here we used the property $kb \in I$ for each $b \in I$ and $k \in \mathbb{K}$, since $I \subset A$ is a vector subspace. It is not hard to see that all axoims of a vector space hold true. $\square$

Definition 2:

The number $\mu = \dim A_p/I_{\nabla f}$ is called the multiplicity of the function

$$f(x_1, \ldots, x_p)$$

at the critical point 0 (here «dim» means the dimension of the local algebra $A_p/I_{\nabla f}$ as a vector space). If $\mu = 1$, then the critical point 0 is called «non-degenerated».

This definition of μ is invariant (does not depend on changes of the variables $x_1, \ldots, x_p$). Indeed, after such change of the variables we get a new function f' with new gradient map, new ideal $I_{\nabla f'}$ and new local algebra $A_p/I_{\nabla f'}$. However it is not hard to see that $A_p/I_{\nabla f'}$ is isomorphic to $A_p/I_{\nabla f}$. Hence the dimensions of both vector spaces coincide: $\mu' = \mu$.

Exercise 2.

Prove that in the case $p = 1$ the definitions 1 and 2 coincide.

[2] R.J. Walker. Algebraic curves. – Princeton, 1950.
[3] V.I. Arnold. Geometrical methods in the theory of ordinary differential equations. – Springer-Verlag 1988.
[4] E.B. Vinberg. A Course in Algebra (Graduate Studies in Mathematics, vol. 56).

Solution: Let's define the number μ by the formula (1.1). Suppose that $\mu < \infty$ (the extremal case $\mu = \infty$ we left to the reader). Then the Taylor series of the gradient map $\nabla f : x \to f'(x)$ starts with x^μ. Hence any element $g \in A_1 = \mathbb{R}[[x]]$ can be represented in the form $g = P_{\mu-1} + \alpha \cdot \nabla f$, where $P_{\mu-1}$ is a polynomial of degree $\mu - 1$ and $\alpha \in A_1$ is appropriate formal series. This representation implies that the vector space $A_1/I_{\nabla f}$ is isomorphic to the vector space of polynomials of degree $\mu - 1$. Since each polynomial of degree $\mu - 1$ is determined by μ coefficients, we get $\dim(A_1/I_{\nabla f}) = \mu$.

Exercise 3.

Prove that the given definition of non-degenerated critical point coincides with well-knows definition from Calculus: the Hessian of f is not equal to zero.

Solution: The condition (Hessian of f) $\neq 0$ is equivalent to the condition that the generators $f_{x_1}, \dots, f_{x_p}$ of the ideal $I_{\nabla f}$ have linearly independent gradients at 0. Then any element $g \in A_p = \mathbb{R}[[x_1, \dots, x_p]]$ can be written in the form

$$g(x_1, \dots, x_p) = c + \alpha_1 f_{x_1} + \cdots + \alpha_p f_{x_p},$$

where $\alpha_i \in A_p$ are appropriate formal series and $c = g(0)$. This implies $\dim(A_p/I_{\nabla f}) = 1$. The proof of the converse statement is similar.

Exercise 4.

Calculate the multiplicity at 0 of the functions $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$ given by the following formulas: $f = x^2 + y^2$, $f = \exp(x^2 + y^2)$, $f = x^2 + xy + y^2$, $f = x^2$, $f = x^2 + 2xy + y^2$, $f = x^2 y$, $f = x^3 + y^2$, $f = x^4 + y^2$, $f = x^3 + y^3$.

Answer: 1, 1, 1, ∞, ∞, ∞, 2, 3, 4.

In the case $\mu < \infty$ we have the following statement (a far generalization of the Morse lemma, which describes the partial case $\mu = 1$).

Tougeron's Theorem: in a neighborhood of the critical point 0 with $\mu < \infty$ there exit smooth local coordinates that the function $f(x_1, \dots, x_p)$ is a polynomial $P(x_1, \dots, x_p)$ of degree $\mu + 1$. Namely, the statement holds true if we put $P(x_1, \dots, x_p)$ = the Taylor polynomial of $f(x_1, \dots, x_p)$ at 0 in the initial coordinates.

Further we consider functions of several variables, but all of them (except for only one) play a role of parameters, and the multiplicity of a function is defined by this exceptional variable, i.e., using the formula similar to (1.1).

1.2 The Division Theorem

Let $F(x,y) : \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}$ is a smooth function. We separate the arguments of F such that $x \in \mathbb{R}$ plays a role of the «main variable», and $y \in \mathbb{R}^p$ play a role of parameters. Fixing the variables $y = (y_1, \dots, y_p)$, we get the function $f(x)$ of the variable x and consider the critical points of this function.

Let $T_0 = (x_0, y_0) \in \mathbb{R} \times \mathbb{R}^n$ be the critical point, i.e., $F_x(T_0) = 0$. Then the multiplicity by the variable x is defined by the following condition:

$$\frac{\partial F}{\partial x}(T_0) = 0, \ \dots, \ \frac{\partial^\mu F}{\partial x^\mu}(T_0) = 0, \quad \frac{\partial^{\mu+1} F}{\partial x^{\mu+1}}(T_0) \neq 0. \tag{1.5}$$

As before, we put $\mu = 0$ in the case of non-critical point.

For simplicity, we formulate the Division Theorem for the point $x_0 = 0$. The general case can be easily obtained using the change of variable $x \to x + x_0$.

Division Theorem. *In a neighborhood of the critical point $T_0 = (0, y_0)$ with $0 \leq \mu < \infty$ the function $F(x,y)$ has the form*

$$F(x,y) = F(0,y_0) + \varphi(x,y) \cdot \left(x^{\mu+1} + \sum_{i=0}^{\mu} a_i(y) x^{\mu-i} \right), \tag{1.6}$$

where $a_i(y), \varphi(x,y)$ are smooth functions, $\varphi(x,y) \neq 0$, and $a_i(y_0) = 0$.

Firstly this theorem was established by K. Weierstrass for analytic functions of complex variables.[5] The complex analytic and real analytic variants of this theorem are often called «Weierstrass division theorem». The smooth variant of this theorem (given above) is also called «Malgrange division theorem» or «Mather division theorem».

I do not give the proof of this theorem, because it can be found in many books, see for instanse:

- M. Golubitsky, V. Guillemin. Stable maps and their singularities. – Graduate Texts in Mathematics, vol. 14, Springer-Verlag, 1973.
- Th. Bröcker, L. Lander. Differentialble Germs and Catastrophes. – Cambridge University Press, 1975.

The Division Theorem allows to get a lot of useful facts, which we will use in the future lectures. In the following statement T_0 is not necessarily critical point of F:

Lemma 1. *Suppose that $F(x,y)$ is a smooth function vanishing on the smooth regular hypersurface $x = \gamma(y)$ in a neighborhood of the point $T_0 = (x_0, y_0)$. Then*

$$F(x,y) = (x - \gamma(y)) \cdot \varphi(x,y) \tag{1.7}$$

where φ is a smooth function.

Proof. Remark that the change of variable $x \to x - \gamma(y)$ turns the point T_0 to $(0, y_0)$ and turns the hypersurface $x = \gamma(y)$ to the hyperplane $x = 0$. Hence it is sufficient to prove this statement for $\gamma(y) \equiv 0$. Also remark that it is sufficient to prove this statement for the case when T_0 is the critical point of $F(x,y)$ by the variable x with $\mu < \infty$. Indeed, if it is not the case, we can add to the function $F(x,y)$ the term x^μ, $\mu > 0$, and establish the equality $F(x,y) + x^\mu = x\varphi \Leftrightarrow F(x,y) = x(\varphi - x^{\mu-1})$.

Thus assume that F satisfies the condition (1.5) with $\mu < \infty$. Then from the Division Theorem it follows that in a neighborhood of T_0 the function $F(x,y)$ has the form (1.6), where $F(0,y_0) = 0$ and the expression in the big brackets vanishes on the hyperplane $x = 0$. Hence $a_\mu(y) \equiv 0$, and we get the representation $F(x,y) = x\varphi(x,y)$ in the form:

$$F(x,y) = x \cdot \varphi(x,y) \left(x^\mu + \sum_{i=0}^{\mu-1} a_i(y) x^{\mu-1-i} \right).$$

□

Exercise 5.

Prove that for any natural n in a neighborhood of any point $T_0 = (0, y_0)$ any smooth function $F(x,y)$ has the form

$$F(x,y) = f_0(y) + x f_1(y) + \cdots + x^{n-1} f_{n-1}(y) + x^n f_n(x,y), \tag{1.8}$$

[5] Weierstrass K. Einige auf die Theorie der analytischen Functionen mehrerer Veränderlichen sich beziehende Sätze. – Mathematische Werke, V. II, Mayer und Müller, Berlin, 1895, 135–188.

where f_i are smooth functions.

Solution: Put $f_0(y) = F(0, y)$ and $g(x, y) = F(x, y) - f_0(y)$, then the function $g(x, y)$ vanishes on the hyperplane $x = 0$. Hence $g(x, y) = x\varphi(x, y)$ with a smooth function φ, and we get $F(x, y) = f_0(y) + x\varphi(x, y) \Rightarrow$ (1.8) with $n = 1$. Applying the same reasoning to the function $\varphi(x, y)$, we get representation (1.8) with $n = 2$, etc.

Exercise 6.

Prove that if in a neighborhood of some point $T_0 = (x_0, y_0)$ the derivative $F_x(x, y)$ is smooth and vanishes on the smooth regular hypersurface $x = \gamma(y)$, then the function $F(x, y)$ has the form

$$F(x, y) = f_0(y) + (x - \gamma(y))^2 \cdot \varphi(x, y) \tag{1.9}$$

with some smooth functions $f_0(y)$ and $\varphi(x, y)$.

Solution: Using the change of variable $x \to x - \gamma(y)$, we reduce the problem to the case $\gamma(y) \equiv 0$, i.e., hypersurface is the hyperplane $x = 0$. Using representation (1.8) with $n = 2$, we get

$$F(x, y) = f_0(y) + xf_1(y) + x^2 f_2(x, y).$$

Differentiation the above identity by x and substituting the value $x = 0$, we see that the condition $F_x(0, y) \equiv 0$ is equivalent to $f_1(y) \equiv 0$. Finally, after the inverse change of variable $x \to x + \gamma(y)$ we get formula (1.9).

Exercise 7.

Prove the Morse lemma:

If the smooth function $F(x_1, \ldots, x_m) : \mathbb{R}^m \to \mathbb{R}$ has non-degenerated critical point at the origin (0), then in a neighborhood of 0 there exit local coordinates such that

$$F(x_1, \ldots, x_m) = F(0) + \alpha_1 x_1^2 + \cdots + \alpha_m x_m^2, \quad \alpha_i = \pm 1. \tag{1.10}$$

Solution: For $m = 1$ the Morse lemma is trivial: applying two times Hadamard's lemma to the function $F(x)$, we get $F(x) = F(0) + x^2 g(x)$, where $g(0) \neq 0$. The change of variables $x \to x\sqrt{|g(x)|}$ gives $F(x) = F(0) + \alpha x^2$, where $\alpha = \operatorname{sgn} g(0)$.

Let $m > 1$. From Linear Algebra we know that it is possible to bring the quadratic part of the function $F(x_1, \ldots, x_m)$ to the canonical (diagonal) form using a linear change of the variables. Since the critical point 0 is non-degenerated, all diagonal elements are non-zero:

$$\begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_m \end{pmatrix}, \qquad a_i = \frac{\partial^2 F}{\partial x_i^2}(0) \neq 0.$$

Put $x := x_1$ и $y := (x_2, \ldots, x_m)$, then the derivative $F_x(x, y)$ is a smooth function and vanishes on a smooth regular hypersurface $x = \gamma(y)$. Hence the function $F(x, y)$ has the form (1.9), where $\varphi(0, 0) \neq 0$. Using the change of variable $x \to (x - \gamma(y))\sqrt{|\varphi(x, y)|}$, we get

$$F(x, y) = f_0(y) + \alpha_1 x^2, \quad \alpha_1 = \operatorname{sgn} \varphi(0),$$

where the smooth function $f_0(y)$ depends on $m - 1$ variables $y = (x_2, \ldots, x_m)$ and 0 is non-degenerated critical point of $f_0(y)$. Using induction on m, we get (1.10). □

1.3 Formal series and smooth functions.

Taking the Taylor series of a smooth (recall that «smooth» means C^∞) but not analytic function $f(x_1, \dots, x_p)$, we get a formal series, since the convergence does not follow from smoothness. In particular, this series may converge to some analytic function which is not f (see e.g. Exercises 8, 9).

Lemma 2. *Given formal series* (1.2) *there exists a smooth function*

$$f(x_1, \dots, x_p): \ \mathbb{R}^p \to \mathbb{R} \tag{1.11}$$

that the Taylor series at the origin 0 *coincides with the given formal series.*

To prove this Lemma, we need to construct smooth functions with special properties.

Exercise 8.

Construct a smooth monotone function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(x) = 0$ for all $x \le 0$ and $\varphi(x) = 1$ for all $x \ge 1$ (see the graph of φ on the fig. 1 a).

Solution. Define the function

$$f(x) = \begin{cases} 0, & x \le 0, \\ e^{-1/x}, & x > 0, \end{cases}$$

see the graph on the fig. 1 b. Then the required function φ is given by the formula

$$\varphi(x) = \frac{f(x)}{f(x) + f(1-x)}.$$

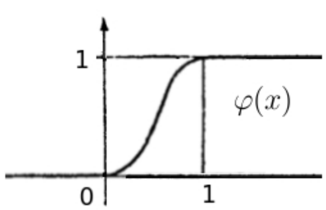


fig. 1 a

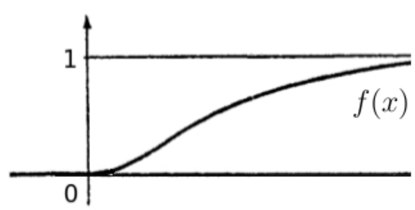


fig. 1 b

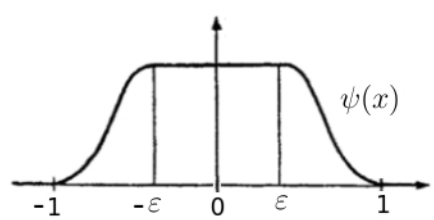


fig. 1 c

Exercise 9.

For any given $0 < \varepsilon < 1$ construct a smooth function $\psi : \mathbb{R} \to \mathbb{R}$ such that $\psi(x) = 1$ for $|x| \le \varepsilon$, $\psi(x) = 0$ for $|x| \ge 1$, and $\psi(x)$ is monotone on each connected interval from $\varepsilon < |x| < 1$ (see the graph of ψ on the fig. 1 c).

Solution. Define the function ψ by the formula

$$\psi(x) = \varphi\Big(\frac{1+x}{1-\varepsilon}\Big) \cdot \varphi\Big(\frac{1-x}{1-\varepsilon}\Big),$$

where φ is the function form Exercise 8.

Question: is it possible to construct analytic functions φ and ψ with above properties?

Using the functions constructed above, it is possible to obtain some non-obvious and useful results.

Exercise 10.

Prove the following statement (which is often called «Whitney Theorem»):

Any closed (in the standard Euclidean metric) subset of the space $\mathbb{R}^p$ is the set of points defined by the equation $f(x_1, \ldots, x_p) = 0$ for some smooth function $f : \mathbb{R}^p \to \mathbb{R}$.

Hint: the proof is based on the using of multi-dimensional analog of the function ψ from the Exercise 9:

$$\Psi(x) = \psi(\|x\|), \quad \text{where} \quad x = (x_1, \ldots, x_p), \quad \|x\| = \sqrt{x_1^2 + \cdots + x_p^2}. \tag{1.12}$$

Spoiler. The proof can be found in the book by Th. Bröcker and L. Lander cited above. By the way, Exercises 8, 9 are also taken from this book.

Proof of Lemma 2. Here I give the proof for the case $p = 1$, since the general case $p > 1$ is similar, but needs a bit longer formulas. Given formal series

$$\sum_{n=0}^{\infty} c_n\, x^n \tag{1.13}$$

construct the required function $f(x)$ in the form

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \psi(x/r_n), \tag{1.14}$$

where ψ is the function from Exercise 9, and r_n are positive numbers (depending on c_n) such that series from the right hanf side of (1.14) and all series obtained after k times differentiation (for any natural k) uniformly and absolutely convegre on the whole space $\mathbb{R}$. To satisfy these conditions, we can put

$$r_n = \frac{1}{n!(1 + |c_n|)}.$$

Then we have the following estimation:

$$\sum_{n=1}^{\infty} |c_n x^n \psi(x/r_n)| \leq \sum_{n=1}^{\infty} |c_n| r_n^n = \sum_{n=1}^{\infty} \frac{|c_n|}{(n!(1 + |c_n|))^n} = \sum_{n=1}^{\infty} \Big(\frac{1}{n!}\Big)^n \frac{|c_n|}{(1 + |c_n|)^n} \leq \sum_{n=1}^{\infty} \frac{1}{n!} < \infty.$$

In the above formula we firstly used the fact that $|x^n \psi(x/r_n)| \leq |x|^n$ if $|x| \leq r_n$, and $|x^n \psi(x/r_n)| = 0$ if $|x| > r_n$ (both statements follow from the definition of ψ). Secondly, we used the inequality $(1 + |c_n|)^n \geq |c_n|$, which holds true for $n \geq 1$ (that's why we started the series from 1 instead of 0).

Similarly, for the series obtained after one differentiation, we have the following estimation:

$$\sum_{n=2}^{\infty} |c_n (x^n \psi(x/r_n))'| = \sum_{n=2}^{\infty} |c_n (n x^{n-1} \psi(x/r_n) + x^n / r_n \psi'(x/r_n))| \leq$$

$$\leq \sum_{n=2}^{\infty} |c_n| r_n^{n-1} (n + M_1) = \sum_{n=2}^{\infty} \frac{|c_n|(n + M_1)}{(n!(1 + |c_n|))^{n-1}} \leq \sum_{n=2}^{\infty} \frac{(n + M_1)}{n!} < \infty,$$

where $M_1 = \max\limits_{[-1,1]} |\psi'(x)|$.

The reader can prove that the similar estimations hold true for all series obtained after k times differentiation. Hence the series from the right hand side of (1.14) converges to a smooth function on the whole space $\mathbb{R}$. Thus we need only to show that the Taylor series of this function $f(x)$ at the origin coincides with (1.13), that is, $f^{(n)}(0) = n!c_n$. The proof of this fact is trivial and left to the reader. $\square$

Exercise 11.

Give the proof of Lemma 2 for $p > 1$.

Lemma 3. *Any smooth function* $f(x, y_1, \dots, y_m) : \mathbb{R}^{m+1} \to \mathbb{R}$ *can be represented in the form*

$$f(x, y) = f_1(x^2, y) + x f_2(x^2, y), \quad y = (y_1, \dots, y_m), \tag{1.15}$$

where f_1 *and* f_2 *are smooth functions.*

Notice that the representation (1.15) is obvious for polynomials, formal series and analytic functions. Indeed, it is sufficient to write the Taylor series by the variable x (this means that we consider $y_1, \dots, y_m$ as parameters and x as the unique variable) and collect the monomilas x^n with even n and odd n. The validity of this operation (in the case of analytic functions) follows from the absolute convergence of the corresponding power series.

Proof. I give the proof for $m = 0$, the case $m > 0$ is similar (some differences are remarked below). Clearly, any smooth function $f(x) = f_{\text{ev}}(x) + f_{\text{od}}(x)$, where $f_{\text{ev}}(x)$ and $f_{\text{od}}(x)$ are even and odd smooth functions, respectively. Indeed, put

$$f_{\text{ev}}(x) = \frac{f(x) + f(-x)}{2}, \qquad f_{\text{od}}(x) = \frac{f(x) - f(-x)}{2}.$$

Clearly, any odd smooth function $f_{\text{od}}(x)$ has the form $f_{\text{od}}(x) = xg(x)$, where $g(x)$ is an even smooth function. To prove this claim, we can use Hadamard's lemma. Thus it is sufficient establish the representation

$$f(x) = f_1(x^2) \tag{1.16}$$

for even smooth functions.

Since $f(x) = f(|x|)$, equality (1.16) holds if and only if $f_1(\xi) = f(\sqrt{\xi})$ for all $\xi \geq 0$. Thus

$$f_1(\xi) = \begin{cases} f(\sqrt{\xi}), & \xi \geq 0, \\ \text{arbitrary}, & \xi < 0, \end{cases}$$

The only condition is required: the function $f_1(\xi)$ must be smooth on the whole $\mathbb{R}$. If such function $f_1 : \mathbb{R} \to \mathbb{R}$ exists, the Lemma is proved.

The function $f_1(\xi) = f(\sqrt{\xi})$ defined on the semiaxis $\xi \geq 0$ is smooth at all points $\xi > 0$. Let us prove that this function has right-hand side derivatives of any order n at $\xi = 0$. From Hadamard's lemma we get

$$f(x) = c_0 + c_1 x^2 + \dots + c_n x^{2n} + x^{2n+2} g(x), \tag{1.17}$$

where $g(x)$ is a smooth even function. The expression (1.17) does not contain monomials of odd degrees, since the function $f(x)$ is even. Substituting $x = \sqrt{\xi}$ in (1.17), we get

$$f_1(\xi) = c_0 + c_1 \xi + \dots + c_n \xi^n + \xi^{n+1} g(\sqrt{\xi}). \tag{1.18}$$

It is not hard to show (we left it to the reader) that function (1.18) has the right-hand side derivatives ${f_1}_+^{(n)}$ of any order $\leq n$ at the point $\xi = 0$.

Thus it is sufficient to define the function $f_1(\xi)$ on the semiaxis $\xi < 0$ such that it is smooth and its left-hand side derivatives at $\xi = 0$ exist and coincide with ${f_1}_+^{(n)}$. In other words, we have the formal series

$$\sum_{n=0}^{\infty} c_n \xi^n, \quad \text{where} \quad c_n = \frac{{f_1}_+^{(n)}(0)}{n!}, \tag{1.19}$$

and need to find a smooth function $\overline{f}_1(\xi)$ that its Taylor series at $\xi = 0$ coincides with (1.19). From Lemma 2 it follows that such function exists.

In the case $m > 0$ we separate the arguments of $f(x, y_1, \ldots, y_m)$ such that x plays a role of the «main variable», and $y = (y_1, \ldots, y_m)$ play a role of parameters. The words «even» and «odd» mean even and odd by the variable x. To establish the representations $f_{\mathrm{od}}(x, y) = xg(x, y)$ and (1.17), we can use representation (1.8) from Exercise 5.

Using the same reasonings as for $m = 0$, we get formal series (1.19), where the coefficients c_n smoothly depend on y. Hence we need to find a smooth function $\overline{f}_1(\xi, y)$ that its Taylor series by the variable ξ at $\xi = 0$ coincides with given formal series. To construct this function, we can apply Lemma 2 with $p = 1$ (here x is the unique variable, y is a parameter). □

For the reader familiar with Fourier series, it is possible to give another proof of Lemma 3.

Second Proof of Lemma 3. According to the previous reasonings, it is sufficient to prove the representation (1.16) for smooth even functions and $x \in [-\pi, \pi]$.

Expanding a smooth even function $f(x)$ in Fourier series, we get

$$f(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos nx, \quad \text{where} \quad c_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx. \tag{1.20}$$

Fourier series (1.20) absolutely and uniformly converges to f on the segment $[-\pi, \pi]$. Moreover, the series obtained from (1.20) after k times differentiation, absolutely and uniformly converge to the function $f^{(k)}(x)$ on the segment $[-\pi, \pi]$. Using the Taylor seires of cos, we can replace the functions $\cos nx$ in (1.20) by the corresponding power series.

Finally, we get the power series, which contains the monomials only of even degrees and absolutely and uniformly converges to f on the segment $[-\pi, \pi]$:

$$f(x) = \sum_{n=0}^{\infty} \alpha_n x^{2n}, \quad -\pi \le x \le \pi.$$

The above series gives the required representation $f(x) = f_1(x^2)$, where the function f_1 is defined on the segment $[-\pi^2, \pi^2]$ as a sum of absolutely and uniformly convergent series

$$f_1(\xi) = \sum_{n=0}^{\infty} \alpha_n \xi^n, \quad -\pi^2 \le \xi \le \pi^2.$$

□

2 Lecture 2. Implicit Differential Equations and Singularities of Plane Curves.

In this lecture we consider Implicit Differential Equations from the Singularity Theory viewpoint. The basic step is «lifting» of the equation from the phase plane to a sufrace in 3-dimensional projectivized tangent bundle (this construction was invited by Poincaré and similar to the introduction of the Riemann surface for a multivalued function of a complex variable on which this function becomes single-valued).

Singularities of plane curves appear when after the projection of the integral curves of the field defined on this sufrace to the phase plane.

2.1 Implicit Differential Equations: Lifting

An Implicit Differential Equation is a differential equation which is not solvable for the derivative, that is,

$$F(x, y, p) = 0, \quad \text{where} \quad p = \frac{dy}{dx}. \tag{2.1}$$

Suppose that F is a smooth function («smooth» means C^∞) and the equation $F = 0$ defines a regular surface $\mathscr{F}$ in the 3-dimensional space J^1 with coordinates (x, y, p), that is, 1-jet space of functions $y = y(x)$.

Let γ be the integral curve of equation (2.1), then its 1-graph $\overline{\gamma} \subset \mathscr{F}$. This means that the curve $\overline{\gamma}$ is tangent to the plane $F_x\,dx + F_y\,dy + F_p\,dp = 0$ at each point. From the relation $p = dy/dx$ it follows that $\overline{\gamma}$ is also tangent to the plane $p\,dx - dy = 0$, which is called the contact plane.

Exercise 1. If the field on the contact planes $p\,dx - dy = 0$ (defined in the whole space J^1) was integrable, then there was a family of smooth surfaces filling the space J^1 such that the contact planes are tangent to these surfaces. Then 1-graphs of the integral curves of (2.1) were the intersections of $\mathscr{F}$ with the surfaces of this family. But it is not the case: prove that the field of the contact planes is not integrable.

Hint: use the Frobenius Theorem.[6]

The curve $\overline{\gamma}$ is the 1-graph of some solution of equation (2.1) if and only if at each point it is tangent to both planes: $F_x\,dx + F_y\,dy + F_p\,dp = 0$ and $p\,dx - dy = 0$, i.e., it is tangent to their intersection. An intersection of two planes in 3-dimensional space is a line or a plane (if they coincide). The last case has codimension 2, since it is defined by two equations:

$$F_p = 0, \quad F_x + pF_y = 0. \tag{2.2}$$

Generically, equations $F_p = 0$, $F_x + pF_y = 0$, $F = 0$ are independent (this means that the gradients of the functions F_p, $F_x + pF_y$, F at the given point are linearly independent), hence the points of $\mathscr{F}$ where the tangent and contact planes coincide, are isolated on $\mathscr{F}$.

Exercise 2. Find the set of the points of $\mathscr{F}$ where the tangent and contact planes coincide, for the following functions: $F = p^2 - x$, $F = f(p) - xp + y$ and $F = (p + \alpha x)^2 - y$, where α is a real parameter.

Remark 1. The notions «genericity», «generic property», «generic case» must be used careful. The reader who solved the previous exercise, saw that there exist equations that such points on $\mathscr{F}$ are not isolated. The formal definition of «genericity» needs «Whitney» (or «fine») topology in the space C^∞.

[6] See e.g.: V.I. Arnold «Geometrical methods in the theory of ordinary differential equations» or Ph. Hartman «Ordinary Differential Equations».

We say that the objects from some fixed class C have the property P in «generic case» if the following two conditions hold. Firstly, if some object $\vartheta_1 \in C$ satisfies the property P then all objects $\vartheta \in C$ sufficiently close to ϑ_1 also satisfy P. Secondly, if some object $\vartheta_2 \in C$ does not satisfy P then there exists $\vartheta \in C$ arbitrarily close to ϑ_2 which satisfies P. In other words, the property P is «stable» (holds true under any sufficiently small perturbations) and each object $\vartheta \in C$ can be embedded into the subset $C_P = \{\vartheta \in C \,|\, P \text{ holds true}\}$ by means of an arbitrarily small perturbation. This means that the subset C_P is open and dense in C with some topology.

In our case we deal with class C^∞ of smooth functions, and «Whitney» (or «fine») topology is what we need. Roughly speaking, in Whitney topology two functions are «close» to each other if their values and the values of their derivatives are «close». The reader who needs the exact definition, is referred to the books:

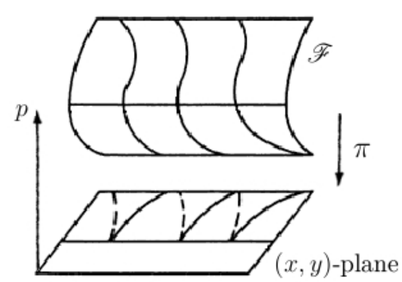
- M. Golubitsky, V. Guillemin. Stable mappings and their singularities. – Graduate Texts in Mathematics, vol. 14, Springer-Verlag, 1973.
- Th. Bröcker, L. Lander. Differentialble Germs and Catastrophes. – Cambridge University Press, 1975.

□

Thus at all the points of the surface $\mathscr{F}$ except for the points of the set (2.2) the intersection of the tangent and contact planes defines a direction field on $\mathscr{F}$. This direction field is called «lifted field» and this construction[7] is called «lifting» of equation (2.1) to the surface $\mathscr{F}$.

The relationship between the lifted field and initial eqiation (2.1) is obvious: 1-graphs of the solutions of (2.1) are the integral curves of the lifted field. Conversely, the solutions of equation (2.1) are projections of the integral curves of the lifted field onto the (x, y)-plane along the p-axis.

The direction field defined by equation (2.1) on the (x, y)-plane is multivalued, the lifted fields on the surface $\mathscr{F}$ is single-valued. This is the sense of lifting. Of course, we pay for this simplification: the phase space becomes a surface insted of a plane, and we have to consider the projection of the surface on the plane. The last circumstance explains appearance of sungularities of the solutions of equation (2.1).



To analize the integral curves of the lifted direction field, it is convenient to express it through a vector field. This means that we define the line l (at each point) through the basis vector of l. Of course, this correspondence is not one-to-one: it is possible to choose infinite number of basis vectors on a line. Formally, one can say that a direction field is a class of

[7]It was invited by Poincaré in his third «Mémoire sur les courbes définies par les équations différentielles». This approach allowed him to create the qualitative theory of Implicit Differential Equations that is essentially different from the previous analytic works of Lagrange, Cauchy, Briot, Bouquet, and others.

vector fields modulo a scalar-valued factor (i.e., each vector field of this class differs from each other by multiplying by a scalar-valued function). The vector field corresponding to the lifted direction field is also called «lifted».

Intersection of the tangent and contact planes corresponds to the system of equations

$$\begin{cases} F_x\,dx + F_y\,dy + F_p\,dp = 0 \\ p\,dx - dy = 0. \end{cases} \tag{2.3}$$

These equations are considered with respect to the variables dx, dy, dp, which are components of the required vector field. This system is linear and homogeneous with respect to these variables.[8]

System (2.3) defines the differentials dx, dy, dp up to a common factor. This is algebraic description of the fact that the vector field corresponding to the given direction field is defined up to a scalar factor. Substituting the expression $dy = p\,dx$ in the first eqiation of (2.3), we get

$$(F_x + pF_y)\,dx + F_p\,dp = 0.$$

Thie yields the formula for the lifted vector field:

$$\dot{x} = F_p, \quad \dot{y} = pF_p, \quad \dot{p} = -(F_x + pF_y). \tag{2.4}$$

Remark 2. For the reader unfamiliar with writing vector fields in such form (as systems of differential equations) I have to explain that the dot over a symbol here means differentiation by some additional parameter τ, which plays a role of time. Since the right hand sides of the equations do not contain τ, this system is autonomous. Any autonomous system defines a vector field in the phase space (the components of this vector field are right hand sides of the equations). Conversely, any vector field defines an autonomous system.

Exercise 3.

How the freedom of choice of the lifted vector fields is connected with the parameter τ?

Answer: Multiplication of vector field (2.4) by the scalar function $\mu(x, y, p)$ corresponds to the change of variable $\tau \to t$ defined by the equation $\dot{t} = \mu^{-1}(x, y, p)$, where the dot over a symbol means differentiation by τ.

Using the lifting, we passed from equation (2.1) to the lifted direction field on $\mathscr{F}$, which can be defined through vector field (2.4). The integral curves of (2.4) are 1-graphs of the solutions of (2.1). To get the graphs of the solutions, we need to make projection on the (x, y)-plane along the p-axis. The lifted direction field is defined at all points of the surface $\mathscr{F}$ except for the point (2.2), which are singular points of the vector field (2.4).[9] Geometrically, such points are defined by intersection of two sets:

$$\mathscr{K} = \{F = F_p = 0\} \quad \text{and} \quad \mathscr{L} = \{F = F_x + pF_y = 0\},$$

which are called the «criminant» and the «inflection curve» of equation (2.1), respectively. In the case of generic equation (2.1) the sets $\mathscr{K}$ and $\mathscr{L}$ are smooth curves on the surface $\mathscr{F}$, which intersect only at isolated points. However in some special cases they even may coincide (see e.g. Exercise 2).

[8] Systems of linear homogeneous equations on differentials of variables are named «Pfaffian systems» after Johann Friedrich Pfaff. Pfaffian system is a representation of a standard differential equations in symmetrical form: all variables in Pfaffian systems have «equal righs». When we use the notation dy/dx, we suggest that the variable x is distinguished (it is called «independent»). From geometrical viewpoint, the difference between the Pfaffian system and corresponding system of ordinary differential equations is difference between homogeneous and non-homogeneous coordinates in the tangent space = «space of differentials». Pfaffian systems define fields of subspaces, which are called «distributions».

[9] Recall that singular points of a vector field are the point where this field is equal to zero.

Exercise 4.

Explain the geometrical sense of the curves $\mathscr{K}$ and $\mathscr{L}$ (which explains their names).

Hint: For the criminant $\mathscr{K}$ consider the projection of the surface $\mathscr{F}$ on the (x, y)-plane along the p-axis. For the inflection curve $\mathscr{L}$ consider the second-order derivative

$$\frac{d^2y}{dx^2} = \frac{dp}{dx}$$

of the solution of equation (2.1) taking into account formula (2.4).

Exercise 5.

Draw the surface $\mathscr{F}$, the integral curves of the lifted field and their projection on the (x, y)-plane for the following equations: $p^2 - 1 = 0$, $p^2 + x = 0$, $p^3 + x = 0$, $p^2 - 4y = 0$, $p^2 - xp + y = 0$.

Exercise 6.

Suppose that the surface $\mathscr{F}$ is homeomorphic to a sphere. It is possible that $\mathscr{K} = \emptyset$ or $\mathscr{L} = \emptyset$ or $\mathscr{K} \cap \mathscr{L} = \emptyset$?

Hint: Use Poincaré–Hopf index theorem (the reader unfamiliar with this theorem can found the statement below).

The projection of the criminant $\mathscr{K}$ on the (x, y)-plane along the p-axis is calles the «discriminant curve» of the equation. (In the case when $F(x, y, p)$ is a polynomial of p, the discriminant curve is given by the equation $D = 0$, where D is the discriminant of F by p). Remark that a point of the discriminant curve can correspond to several different points of the surface $\mathscr{F}$ which belong to the preimage of the projection. Moreover, some of these points can not belong to the criminant $\mathscr{K}$.

Remark 3. A close geometric interpretation of differential equations was suggested by A. Clebsch, it is based on his «theory of connexes».

Using the modern terminology, we can say that Clebsch considered the tangent bundle of the (x, y)-plane with homogeneous coordinates and treated a differential equation as a connection between the points of the (x, y)-plane and the points of the projectivized tangent bundle. By the «connex» of equation (2.1) he meant the field of tangent planes to the surface $\mathscr{F}$. He also defined the «principal connex», which is the field of contact planes common for all such equations.

Clebsch put in correspondence to each differential equation its «principal concindence»: by his own definition, it is *the common things which has the connex of a given equation with the principal connex*. In fact, this concept corresponds to the intersection of the fields of tangent and contact planes, i.e., to the lifted field. However, this terminology turned out to be not very convenient and was forgotten some time after. □

Remark 4. The method of lifting invited by Poincaré, led him to studying differential equations (vector fields) on manifolds.

In particular, Poincaré considered the problem on the distribution of singular points of a vector field on a 2-manifold and proved a remarkable topological result: if a smooth vector field on a smooth compact orientable 2-manifold has only isolated singular points (for exmaple, n nodes, s saddles, and f focuses), then the sum of their indexes (in our example: $n + f - s$) for any such a field is the same and is equal to the Euler characteristic of this manifold. In 1926 this result was generalized by Hopf for manifolds of any finite dimension (Poincaré–Hopf index theorem). □

2.2 Legendre transformation

The «Legendre transformation» (or «Legendre transform») is the automorphism $(x, y, p) \to (X, Y, P)$ of the space J^1 given by the formula

$$x = P, \quad p = X, \quad y + Y = xp = XP. \tag{2.5}$$

Applying the Legendre transformation to equation (2.1) (this means that we substitute the expressions for the variables x, y, p through X, Y, P), we get another equation:

$$F^*(X, Y, P) := F(P, XP - Y, X) = 0, \tag{2.6}$$

which is called «dual equation» for (2.1).

Exercise 7. Prove that:

The Legendre transform is a global diffeomorphism of the space J^1.

The Legendre transform is an involution, i.e., it is its own inverse.

The Legendre transform turns the differential 1-form $p\,dx - dy$ to $-(P\,dX - dY)$.

The Legendre transform turns the integral curves of equation (2.1) to the integral curves of dual equation (2.6), and back.

The last property (correspondence of the integral curves) sometimes can be used for solving Implicit Differential Equations.[10] Indeed, if the dual equation (2.6) can be integrated easier than (2.1) and its general solution is $Y = \Phi(X, c)$, where c is the constant of integration, then the general solution of initial equation (2.1) can be written in the parametric form

$$x = \Phi_X(p, c), \quad y = p\Phi_X(p, c) - \Phi(p, c),$$

where p is the parameter.

Exercise 8.

Using the Legendre transform, solve the equations: $p^2 = x$, $e^p = x$, $\cos p = x$, $pe^p = x$, $p^3 + yp + x = 0$.

The reader who solved these equations saw that the criminants of the dual equations are empty (and the inflection curves of the initial equations are also empty). This is not an accidental coincidence. Indeed, the criminant and the inflection curve are connected by the dual relation:

Exercise 9.

Prove that the criminant of the initial equation is the inflection curve of the dual equation, and vice versa.

The Legendre transformation can be applied not only to integral curves of differential equations, but to all planar curves which have 1-graphs.

Indeed, let γ be an integarl curve on the (x, y)-plane and $\overline{\gamma}$ is its 1-graph. Since the curve $\overline{\gamma}$ belongs to the space J^1, the Legendre transform turns it to another curve $\overline{\gamma}^*$ in J^1. The projection of $\overline{\gamma}^*$ on the (X, Y)-plane along the P-axis is a planar curve γ^*, which is the image of γ under the Legendre transform. This construction can be illustrated by the following diagramm:

$$\begin{array}{ccc} (x, y, p) & \xrightarrow{\text{Legendre}} & (X, Y, P) \\ \uparrow & & \downarrow \\ (x, y) & \longrightarrow & (X, Y) \end{array}$$

[10] This method (as any concrete method of integration) is useful only for some class of such equations. Applications of this method for solving equations are described in the well-known handbook by Kamke on ordinary differential equations, some examples are also in: J.W. Bruce, P.J. Giblin. Curves and Singularities (Cambridge University Press, 1984).

The curve γ^* is called «dual» to the curve γ.

Exercise 10.

Find the dual curves to a straight line, parabola, cubic parabola and circle.

Exercise 11.

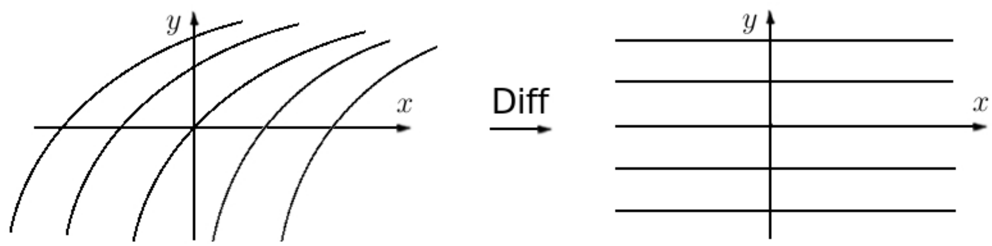
Suppose that γ is a smooth regular curve on a plane («regular» means that there exists a smooth parametrization of γ that the velocity never vanishes). Prove that the dual curve γ^* is regular at the points corresponding to the points of γ with non-zero curvature.

Remark 5.

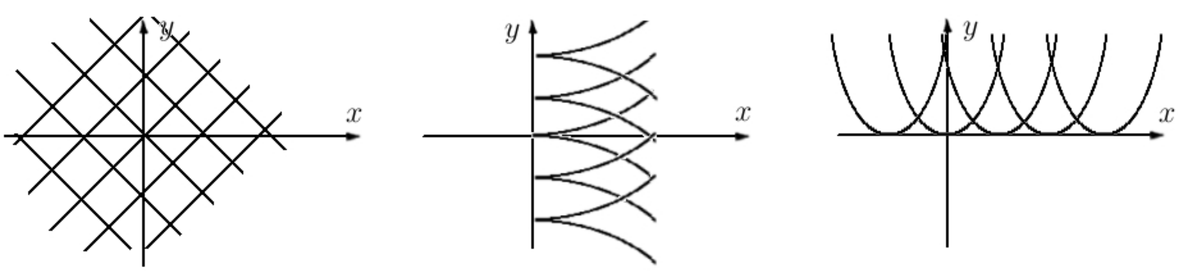
The Legendre transformation can be defined in multi-dimensional case and applied to partial differential equations.[11]

2.3 Implicit Differential Equations: Regular and Singular points

The local phase portrait of the ordinary differential equation $p = f(x, y)$ with a smooth function f at any point (x_0, y_0) is very simple: the family of the integtal curves is locally diffeomorphic to a family of straight lines. For instance, in a neighborhood of any point (x_0, y_0) the equation $p = f(x, y)$ can be brought to the form $p = 0$ with help of a smooth diffeomorphism of the (x, y)-plane:



The main question which we want to study in this mini-course is: how the local phase portraits of equation (2.1) look like? The reader who solved Exercise 5, saw that the local phase portraits of Implicit Differential Equations are different, some of them are not smoothly equivalent (and even not topologicaly equivalent). Indeed, it is sufficient to compare the phase portraits of the equations $p^2 - 1 = 0$, $p^2 - x = 0$, $p^2 - 4y = 0$ in a neighborhood of the origin:



Moreover, the equation $p^2 - x = 0$ shows that even individual integral curve of Implicit Differential Equations may have singularities (such phenomena never occurs for the ordinary differential equation $p = f(x, y)$). The question we study in this lecture is: what sort of singularities the integral curves of equation (2.1) may have? Thus in this lecture we will deal only with individual integral curves, not the phase portraits.

[11]R. Courant, D. Hilbert. Methods Of Mathematical Physics. – New York-London, 1962.

The appearance of the singularities of the integral curves is connected with so-called «singular points» of equation (2.1).

The point $T_0 = (x_0, y_0, p_0)$ of the surface $\mathscr{F}$ is called «singular point» of equation (2.1) if $F_p(T_0) = 0$. Otherwise T_0 is called «regular point» of this equation. Using the previous notations, we can say that T_0 is singular if $T_0 \in \mathscr{K}$, and T_0 is regular if $T_0 \in \mathscr{F} \setminus \mathscr{K}$. The geometric sense of this definition is clear: the singular points of equation (2.1) are critical points of the projection $\pi : \mathscr{F} \to (x, y)$ along the p-axis.

By Implicit Function Theorem, in a neighborhood of any regular point T_0 equation (2.1) is equivalent to equation $p = f(x, y)$ with a smooth function f. Hence we can apply to this equation all standard theorems, which guarantee existence, uniqueness and smoothness of solution.[12]

Now let T_0 be the singular point of equation (2.1), hence the Implicit Function Theorem is not applicable. What can we say about the lifted vector field (2.4) at the point T_0? Clearly, there are two different cases: $T_0 \in \mathscr{K} \setminus \mathscr{L}$ and $T_0 \in \mathscr{K} \cap \mathscr{L}$. The singular points of the first kind we call «proper», and the singular points of the second kind we call «improper».

At the proper points lifted vector field (2.4) in non-zero and vertical,[13] since $F_p(T_0) = 0$ and $F_x + pF_y(T_0) \neq 0$. At the improper points lifted vector field (2.4) in zero, that is, they are also the singular point of the lifted vector field.

2.4 Singularities of the integral curves

In the rest part of the lecture we deal only with proper singular points.

By the standard Existence and Uniqueness Theorem, for each proper singular point T_0 there exists a unique smooth integral curve $\overline{\gamma}$ of field (2.4) that passes through T_0. Hence the projection $\gamma = \pi(\overline{\gamma})$ on the (x, y)-plane is a unique solution of equation (2.1) with the initial condition $y(x_0) = y_0$, $y'(x_0) = p_0$. Since the projection $\pi : \mathscr{F} \to (x, t)$ at the point T_0 is not regular, the image $\pi(\overline{\gamma})$ is not a regular curve. What type of singularity do it have?

To answer this question, write the smooth curve $\overline{\gamma}$ in the parametric form:

$$\overline{\gamma}(\tau) : \quad x = \varphi(\tau), \quad y = \psi(\tau), \quad p = \vartheta(\tau), \tag{2.7}$$

where $\varphi(\tau), \psi(\tau), \vartheta(\tau)$ are smooth functions, $\overline{\gamma}(0) = T_0$.

It is convenient to use the notation $G := -(F_x + pF_y)$. Then

$$\dot{\varphi}(0) = F_p(T_0) = 0, \quad \dot{\psi}(0) = pF_p(T_0) = 0, \quad \dot{\vartheta}(0) = G(T_0) \neq 0. \tag{2.8}$$

This shows that curve (2.7) is regular (we can take the variable p as the parameter on this curve) and the point $\tau = 0$ is critical for the functions $\varphi(\tau), \psi(\tau)$. Suppose that the multiplicity μ of $\varphi(\tau)$ at this critical point is finite. Let $n = \mu + 1$, then

$$\dot{\varphi}(0) = 0, \ \ldots, \ \varphi^{(n-1)}(0) = 0, \ \varphi^{(n)}(0) \neq 0. \tag{2.9}$$

By $L_{\vec{V}}$ denote the differential operator

$$L_{\vec{V}} = F_p \frac{\partial}{\partial x} + pF_p \frac{\partial}{\partial y} + G \frac{\partial}{\partial p}$$

[12]Remark that we always consider C^1-smooth solutions, otherwise our reasonings are not correct. For example, for the differentiable but not continuously differentiable solutions the Implicit Function Theorem is not applicable. Fortunately, in the case of one-dimensional Implicit Differential Equations such solutions do not exist; see the textbook: I.G. Petrovskii. Lectures on the theory of ordinary differential equations. However it is necessary to remember that such solutions may exist in the case of multidimensional Implicit Differential Equations; see the paper: A.F. Filippov. Uniqueness of the solution of a system of differential equations not solved with respect to the derivatives (Differ. Equ. 41 (2005), no. 1, 90–95).

[13]By the «vertical» direction in the (x, y, p)-space we shall mean the direction of the p-axis.

corresponding to vector field (2.4). Then for any natural k we have $\varphi^{(k+1)}(0) = L^k_{\vec{V}}(F_p)\big|_{T_0}$. Then

$$\dot{\varphi}(0) = F_p\big|_{T_0} = 0, \quad \ddot{\varphi}(0) = L_{\vec{V}}(F_p)\big|_{T_0} = (F_pF_{xp} + pF_pF_{yp} + GF_{pp})\big|_{T_0} = GF_{pp}\big|_{T_0}. \tag{2.10}$$

Since $G(T_0) \neq 0$, we have $F_{pp}(T_0) \neq 0 \Leftrightarrow n = 2$. If $F_{pp}(T_0) = 0$, then continue differentiation:

$$\varphi^{(3)}(0) = L^2_{\vec{V}}(F_p)\big|_{T_0} = G\frac{\partial}{\partial p}(F_pF_{xp} + pF_pF_{yp} + GF_{pp})\big|_{T_0} = G^2F_{ppp}\big|_{T_0}.$$

Exercise 12. Prove that if $\varphi^{(k+1)}(0) = L^k_{\vec{V}}(F_p)\big|_{T_0} = 0$ for all $k < l$, then

$$\varphi^{(l+1)}(0) = L^l_{\vec{V}}(F_p)\big|_{T_0} = G^l\frac{\partial^{l+1}F}{\partial p^{l+1}}\Big|_{T_0}. \tag{2.11}$$

Hint: Use the induction on l with the basis given by (2.10).

Thus we have $n = \mu + 1$, where μ is the multiplicity of the function $F(x, y, p)$ by the variable p at the critical point T_0 (see formula (1.5) from the previous lecture, where x is replaced by p). The number n is the main factor which defines the type of singularity of the integral curve γ at the point (x_0, y_0).

To simplify the further reasonings, choose the coordinates on the (x, y)-plane such that T_0 is the origin of J^1. For example, we can make the translation bringing (x_0, y_0) to the origin of the plane and the linear transformation $y \to y - p_0x$. Thus from now we always assume $T_0 = 0$.

Exercise 13. Prove that the multiplicity of the function $\psi(\tau)$ at the critical point $\tau = 0$ is equal to n, i.e.,

$$\dot{\psi}(0) = 0,\ \ldots,\ \psi^{(n)}(0) = 0,\ \psi^{(n+1)}(0) \neq 0. \tag{2.12}$$

Using formulas (2.9) and (2.12) and Hadamard's lemma, we can present our curve in the form

$$\gamma(\tau): \quad x = \tau^n\overline{\varphi}(\tau), \quad y = \tau^{n+1}\overline{\psi}(\tau), \tag{2.13}$$

where $\overline{\varphi}(\tau)$ and $\overline{\psi}(\tau)$ are smooth fucntions non-vanishing at $\tau = 0$.

Formula (2.13) allows to understand how the curve γ looks like in a neighborhood of the origin, but it can be further simplified. The first step is as follows.

Exercise 14.

Prove that using a regular change of the parameter τ and linear change of the variables x and y in (2.13) we can obtain $\overline{\varphi}(\tau) = 1$ and $\overline{\psi}(\tau) = 1 + \chi(\tau)$ or $\overline{\varphi}(\tau) = 1 + \chi(\tau)$ and $\overline{\psi}(\tau) = 1$, where $\chi(\tau)$ is a smooth function and $\chi(0) = 0$.

Remark 6. Formula (2.13) with $\overline{\varphi}(\tau) = 1$ and $\overline{\psi}(\tau) = 1 + \chi(\tau)$ allows to write the curve γ is the form $y = y(x)$. Indeed, we have $\tau = x^{\frac{1}{n}}$ and can substitute this expression in $y = \tau^{n+1}(1+\chi(\tau))$. If the function $F(x, y, p)$ in equation (2.1) is analytic, then the functions $\varphi(\tau)$ and $\psi(\tau)$ are also analytic, and this procedure gives $y = h(x^{\frac{1}{n}})$, where h is an analytic function, i.e., we have a convergent power series of x, where the power takes not only integer, but also rational values: $(n + i)/n$, $i = 0, 1, 2, \ldots$ Such series are called «Puiseux series» or «Newton–Puiseux series».[14]

Let's forget (for several minutes) about singular points of equation (2.1) and consider the regular points lying on the inflection curve of this equation, but not on the criminant. Let

[14] Power series with fractional powers were used by Newton (1671) for the presentation of algebraic curves with branches. More detailed studying of such series were given by Franch mathematician Victor Alexandre Puiseux (1850).

$T_1 = (x_1, y_1, p_1)$ be such a point: $T_1 \in \mathscr{L} \setminus \mathscr{K}$. What can we say about the integral curve passing through T_1?

Obviously, there exists a unique integral curve of the lifted field (2.4) passing through T_1. The projection of this curve on the (x, y)-plane is the smooth solution of (2.1), which can be written is the form

$$y(x) = y_1 + p_1(x - x_1) + f(x - x_1), \quad f(0) = f'(0) = f''(0) = 0, \tag{2.14}$$

where f is a smooth function and $f''(0) = 0$, since the second-order derivative $y''(x) = G/F_p$ vanishes at T_1.

Let $m \geq 2$ be the multiplicity of the function $f(x)$ at the origin:

$$f'(0) = 0, \ \ldots, \ f^{(m)}(0) = 0, \ f^{(m+1)}(0) \neq 0. \tag{2.15}$$

If m is even, then (x_1, y_1) is the inflection point of the curve (2.14), this explains the name «inflection curve».

The reader should notice that the number n of the solution of the initial equation at the proper singular point T_0 is connected with the number m of the solution of the dual equation at the corresponding regular point T_1, since the criminant and the inflection curve of dual equations are connected with the Legendre transformation (Exercise 9).

Exercise 15.

Prove that $n = m$. In particular, the cusp of the semicubic parabola corresponds to the cubic inflection point on the dual curve.

Exercise 16.

Draw the integral curves of the equations from Exercise 8 and the dual equations.

Exercise 17.

We saw that any intergal curve of equation (2.1) passing through the proper singular point 0 with the number $n = \mu + 1$, where μ is the multiplicity of the function $F(x, y, p)$ by the variable p, can be presented in the form (2.13). Prove the inverse statement: for any curve γ given by formula (2.13) with any smooth functions $\overline{\varphi}(\tau)$ and $\overline{\psi}(\tau)$ there exists differential equation (2.1) such that γ is its integral curve.

Solution: Apply the Legendre transformation to the curve γ, let γ^* be the dual curve on the (X, Y)-plane, see formula (2.5). From the results obtained above it follows that γ^* has the form $Y = f(X)$, where f is a smooth function that satisfies (2.15) with $m = n$. Clearly, γ^* is the integral curve of the differential equation $P = f'(X)$, where $P = dY/dX$. Now apply the Legendre transformation to this differential equation $P = f'(X)$. Then we get the dual equation $f'(p) - x = 0$, and γ is the integral curve of $f'(p) - x = 0$.

The next natural question is: how to bring the curve given by formula (2.13) in a neighborhood of the origin to a more simple from using smooth changes of the coordinates x, y? For example, is it possible to bring (2.13) to the simplest form $x = \tau^n, \ y = \tau^{n+1}$? In other words, we want to establish smooth local normal forms of such curves. By $\#n$ define the number of equivalence classes for given n.

Exercise 18.

Prove that $\#2 = 1$, i.e., for any curve given by formula (2.13) with $n = 2$ there exists a smooth local change of the variables x, y bringing this curve to the form

$$x = \tau^2, \ y = \tau^3 \tag{2.16}$$

in a neighborhood of the origin.

Solution: Take formula (2.13) with $n = 2$ and the functions $\overline{\varphi}(\tau) = 1$, $\overline{\psi}(\tau) = 1 + \chi(\tau)$, where $\chi(0) = 0$ (see Exercise 14). Using Lemma 3 from the Lecture 1, we can present the function $\chi(\tau)$ in the form $\chi(\tau) = \chi_1(\tau^2) + \tau\chi_2(\tau^2)$, where χ_1, χ_2 are smooth functions. Then $y = \tau^3(1 + \chi_1(\tau^2)) + \tau^4\chi_2(\tau^2) = \tau^3(1 + \chi_1(x)) + x^2\chi_2(x)$, and the change of variable

$$y \to \frac{y - x^2\chi_2(x)}{1 + \chi_1(x)}$$

brings the curve to the required form (2.16) in a neighborhood of the origin. □

Exercise 19.

Prove that $\#3 = 1$, i.e., for any curve given by formula (2.13) with $n = 3$ there exists a smooth local change of the variables x, y bringing this curve to the form

$$x = \tau^3, \;\; y = \tau^4 \tag{2.17}$$

in a neighborhood of the origin.

Solution: Take formula (2.13) with $n = 3$ and the functions $\overline{\varphi}(\tau) = 1$, $\overline{\psi}(\tau) = 1 + \chi(\tau)$, where $\chi(0) = 0$. Using Hadamard's lemma $m + 1$ times, we get

$$x = \tau^3, \quad y = \tau^4(1 + \chi_1\tau + \cdots + \chi_m\tau^m + O(\tau^{m+1})). \tag{2.18}$$

Step One: kill the monomial τ^5. Clearly, it is sufficient to do this in the case $\chi_1 = 1$, because any other $\chi_1 \neq 0$ can be simplified to 1 by means of a linear change of the parameter τ and scalings of x and y. Notice that the monomials τ^i with $i > 5$ do not influence on this procedure. Hence it is sufficient to kill the monomial τ^5 for the curve

$$\gamma_1(\tau): \quad x = \tau^3, \quad y = \tau^4 + \tau^5.$$

The change of variable $x \to x + \alpha y$ with $\alpha = \frac{3}{4}$ brings the curve γ_1 to the form

$$\gamma_1(\tau): \quad x = \tau^3 + \alpha(\tau^4 + \tau^5), \quad y = \tau^4 + \tau^5.$$

After that make the change of the parameter $\tau \to t$ by the formula $t^3 = \tau^3 + \alpha(\tau^4 + \tau^5)$. Then we obtain

$$\gamma_1(t): \quad x = t^3, \quad y = \vartheta^4(t) + \vartheta^5(t),$$

where $\tau = \vartheta(t)$ is the inverse function to $t = \tau(1 + \alpha(\tau + \tau^2))^{\frac{1}{3}}$. It is not hard to show that in a neighborhood of the origin the function ϑ is smooth and $\vartheta(t) = t - t^2/4 + O(t^3)$. Substituting this expression in the equality $y = \vartheta^4(t) + \vartheta^5(t)$ and collecting similar terms, we get $y = t^4 + O(t^6)$, i.e., the monomial τ^5 is killed.

Step Two. Consider curve (2.18) with $\chi_1 = 0$:

$$x = \tau^3, \quad y = \tau^4(1 + \chi_2\tau^2 + \cdots + \chi_m\tau^m + O(\tau^{m+1})), \tag{2.19}$$

and show that we can kill all the monomials τ^i with $i > 5$. Remark that such monomials can be represented in the form $\tau^i = x^{i_1}y^{i_2} + O(\tau^{i+1})$ with integer i_1, i_2. For instance,

$$\tau^6 = x^2, \;\; \tau^7 = xy + O(\tau^8), \;\; \tau^8 = y^2 + O(\tau^9), \;\; \tau^9 = x^3, \;\; \tau^{10} = x^2y + O(\tau^{11}),$$
$$\tau^{11} = xy^2 + O(\tau^{12}), \;\; \tau^{12} = x^4, \;\; \tau^{13} = x^3y + O(\tau^{14}), \;\; \text{etc.}$$

Hence using the substitution

$$y \to y + \alpha_6x^2 + \alpha_7xy + \alpha_8y^2 + \alpha_9x^3 + \alpha_{10}x^2y + \alpha_{11}xy^2 + \alpha_{12}x^4 + \alpha_{13}x^3y + \cdots \tag{2.20}$$

with the appropriate coefficients α_i, we can kill all the monomials τ^i, $i = 6, 7, 8, \ldots$ Namely, each monomial $\alpha_i x^{i_1} y^{i_2}$, $3i_1 + 4i_2 = i$, $i > 5$, in (2.20) kills the monomial τ^i in (2.18) and does not change the monomials of degrees $< i$.

Since we need to kill all the monomials τ^i with $i > 5$, the right hand side of (2.20) contains an infinite number of terms, i.e., we get a formal series of x, y. If this series converges in a neighborhood of the origin, then formula (2.20) defines an analytic change of variable, which brings curve (2.19) to the required form (2.17). But in general, this is not the case: the obtained formal series may diverge at all points (except for the origin). To overcome this problem, we can use Lemma 2 form the Lecture 1.

Step Three. Let $f(x, y)$ be the smooth function such that its Taylor series at the origin coincides with formal series (2.20). Then the change of variables $y \to y + f(x, y)$ brings curve (2.19) to the form

$$x = \tau^3, \; y = \tau^4 + \omega(\tau),$$

where $\omega(\tau)$ is ∞-flat at the origin. It is clear that using the change of variable $y \to y - \omega(x^{\frac{1}{3}})$, we kill the «tail» $\omega(\tau)$ and obtain required form (2.17). But it is necessary to check that the function $\omega(x^{\frac{1}{3}})$ is smooth at 0. More exactly, we need to prove that all the derivatives of $\omega(x^{\frac{1}{3}})$ tend to zero as $x \to 0$. The proof of this fact is trivial and left to the reader (hint: use Hadamard's lemma). $\square$

The obtained results can suggest that $\#n = 1$ also for $n > 3$, but it is wrong.

Exercise 20.

Prove that $\#4 = 2$ and for any curve (2.13) with $n = 4$ there exists a smooth local change of the variables x, y bringing this curve to one of the following normal forms:

$$x = \tau^4, \; y = \tau^5; \qquad x = \tau^4, \; y = \tau^5 + \tau^7 \tag{2.21}$$

in a neighborhood of the origin.

Hint: This statement can be proved using the similar arguments as the previous.

We always used above C^∞-smooth changes of coordinates. It is reasonable to ask: what's happened if we weaken this condition to a finite smoothness?

Exercise 21.

Prove that two normal forms from (2.21) are C^1-smooth equivalent, but not C^2-smooth equivalent.

Solution: Establish C^1-smooth equivalence in more general case: for any n there exists a smooth local change of the variables x, y bringing curve (2.13) to the form $x = \tau^n, \; y = \tau^{n+1}$. To prove this statement for odd n it is convenient to use formula (2.13) with $\overline{\varphi}(\tau) = 1$ and $\overline{\psi}(\tau) = 1 + \chi(\tau)$, $\chi(0) = 0$. Using Hadamard's lemma n times, we get

$$\chi(\tau) = \chi_1 \tau + \cdots + \chi_{n-1} \tau^{n-1} + \tau^n \widetilde{\chi}(\tau),$$

where $\widetilde{\chi}(\tau)$ is a smooth function. Hence

$$y = \tau^{n+1}(1 + \chi(\tau)) = \tau^{n+1}(1 + \tau^n \widetilde{\chi}(\tau)) + \tau^n(\chi_1 \tau^2 + \cdots + \chi_{n-1} \tau^n),$$

and the required C^1-smooth change of coordinates is

$$y \to \frac{y - x(\chi_1 x^{\frac{2}{n}} + \cdots + \chi_{n-1} x^{\frac{n}{n}})}{1 + x \widetilde{\chi}(x^{\frac{1}{n}})}.$$

The proof for even n is similar and left to the reader. Hint: use formula (2.13) with the functions $\overline{\varphi}(\tau) = 1 + \chi(\tau)$ and $\overline{\psi}(\tau) = 1$.

Suppose that there exists a C^2-smooth local change of variables $\xi = f(x,y)$, $\eta = g(x,y)$ that turns the second curve from formula (2.21) to the first one. Applying Hadamard's lemma to the functions f and g two times, we can represent this change of variables in the form

$$\xi = a_1 x + a_2 y + (a_3 + \varphi_3(x,y))x^2 + (a_4 + \varphi_4(x,y))xy + (a_5 + \varphi_5(x,y))y^2,$$
$$\eta = b_1 x + b_2 y + (b_3 + \psi_3(x,y))x^2 + (b_4 + \psi_4(x,y))xy + (b_5 + \psi_5(x,y))y^2,$$

where a_i, b_i are constants, and $\varphi_i(x,y), \psi_i(x,y)$ are continuous functions vanishing at the origin. Substituting the above expressions for ξ and η in the equality $\xi^5 = \eta^4$, which defines the curve $\xi = \tau^4$, $\eta = \tau^5$, and substituting $x = \tau^4$, $y = \tau^5 + \tau^7$, we get the identity

$$(a_1\tau^4 + a_2(\tau^5 + \tau^7) + a_3\tau^8 + o(\tau^8))^5 = (b_1\tau^4 + b_2(\tau^5 + \tau^7) + b_3\tau^8 + o(\tau^8))^4.$$

Comparing the coefficients of the monomials of degrees ≤ 2 in the left and right hand sides of the identity, we see that $a_1 = b_1 = 0$. Whence the Jacobian of the map $(x,y) \to (\xi,\eta)$ is equal to zero, and the required C^2-smooth change of coordinates does not exist. $\square$

We will not further investigate the classification of the singularities of plane curves. There exist a lot of papers and books concerning this subject, for instance:

- V.I. Arnol'd. Simple Singularities of Curves. Proc. Steklov Institute of Mathematics, vol. 226 (1999), pp. 20-28.
- J.W. Bruce, T. Gaffney. Simple singularities of maps $(\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$. J. London Math. Soc. (2), 26 (1982), pp. 464–474.
- J.W. Bruce, P.J. Giblin. Curves and Singularities. Cambridge University Press, 1984.
- C.T.C. Wall. Singular Points of Plane Curves. London Math. Soc. Student Texts 63.
- M. Zhitomirskii, Fully simple singularities of plane and space curves. Proc. London Math. Soc. 96:3 (2008), pp. 792–812.

True, in this papers and books the analytic (but not smooth) classification was considered.

To my knowledge, C^k-smooth (with finite $k > 1$) classification of the singularities of plane curves is not studied yet. According to the cited paper by Bruce and Gaffney, in the analytic classification $\#n = \infty$ for all $n > 4$. On the other hand, C^1-smooth changes of the variables allow to bring the germ of each curve (2.13) to the from $x = \tau^n$, $y = \tau^{n+1}$ and even conjugate curves with different n. The difference between the C^k-smooth classifications with different $k > 1$ and $n > 4$ is not evident a priori Probably, this question is an interesting and not trivial problem waiting for the solution.

3 Lecture 3. Singularities of Maps.

In the previous lecture we studied singularities of the integral curves of implicit differential equations (this studying is based on the elementary facts form the Lecture 1). The next aim is to study phase portraits (i.e., the families of the integral curves) of implicit differential equations in neighborhoods of their singular points. This problem is considered in the next lecture, now we give some necessary facts for the Singularity Theory.

3.1 Malgrange Preparation Theorem.

Let $f(x) : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map (as before, smooth means C^∞) defined by n smooth functions:

$$f(x) = (f^1(x), \dots, f^n(x)), \quad x = (x_1, \dots, x_n), \quad f(0) = 0. \tag{3.1}$$

Our first aim is to define «multiplicity» of the germ $f(x)$ at 0. Recall that in the Lecture 1 we defined multiplicity of germs $f(x) : \mathbb{R}^n \to \mathbb{R}$ as the dimension of the local algebra of the gradient map (1.4). The similar construction is used for map (3.1).

Let $A_n = \mathbb{R}[[x_1, \dots, x_n]]$ = the algebra of formal series of the variables $x_1, \dots, x_n$. Consider the ideal $I_f \subset A_n$ generated by the functions $f^1(x), \dots, f^n(x)$ from (3.1). More exactly, the ideal $I_f \subset A_n$ is generated by the formal series corresponding to the functions $f^1(x), \dots, f^n(x)$.

Definition 1:

The factor-algebra $Q_f := A_n/I_f$ is called the «local algebra» of the map $f(x)$ at the point 0. The number $\mu = \dim(A_n/I_f)$ is called the multiplicity of the map $f(x)$ at 0 (here «dim» means the dimension of the local algebra as a vector field over $\mathbb{R}$).

In the case $n = 1$ we have two definitions of multiplicity of the map $f(x) : \mathbb{R}^1 \to \mathbb{R}^1$. Namely, the «old» multiplicity defined in the Lecture 1 (now denote it by μ'), and the «new» multiplicity μ defined above. It is not hard to see that $\mu = \mu' + 1$. The reason is clear: μ' is defined through the gradient map of f, that is, the map $f'(x) : \mathbb{R}^1 \to \mathbb{R}^1$. The differentiation reduces the multiplicity by 1. That's why we have $\mu' = \mu - 1$.

Definition 2:

Suppose $\mu < \infty$, that is, $Q_f = A_n/I_f$ is a finite-dimensional local algebra. A system of generators of Q_f is a set of elements $e_1(x), \dots, e_\mu(x)$ of the algebra A_n which goes back into a system of generators of the vector space Q_f on factoring by the ideal I_f.

This means that any element $g(x) \in A_n$ can be represented in the form

$$g(x) = \alpha_1 e_1(x) + \dots + \alpha_\mu e_\mu(x) \; + \; \beta_1(x) f^1(x) + \dots + \beta_n(x) f^n(x) \tag{3.2}$$

with appropriate constants $\alpha_i \in \mathbb{R}$ and elements $\beta_i(x) \in A_n$.

By $C^\infty[x_1, \dots, x_n]$ denote the algebra of smooth germs of the variables $x_1, \dots, x_n$ at the point 0. Put $A_n = C^\infty[x_1, \dots, x_n]$ and repeat all reasonings which we used above for definition of the local algebra $Q_f := A_n/I_f$, the multiplicity μ and the generators. Then we get the similar definitions in two categories: formal series and smooth germs.

The pleasant and useful fact is that for $\mu < \infty$ these notions (multiplicity and generators) coincide. Further we deal with the case $\mu < \infty$, hence we can operate simultaneously in two categories: $A_n = \mathbb{R}[[x_1, \dots, x_n]]$ and $A_n = C^\infty[x_1, \dots, x_n]$. The obtained results will be similar.

Exercise 1.

Calculate the multiplicity at 0 of the maps $f(x, y) : \mathbb{R}^2 \to \mathbb{R}^2$ given by the following formulas:

$f^1 = x,\ f^2 = x^2 + y^2;$
$f^1 = x,\ f^2 = xy;$
$f^1 = xy,\ f^2 = x^2 + y^2.$

Exercise 2.

Prove that if 0 is the regular point of the map $f(x)$, that is, the Jacobian $J_f(0) \neq 0$, then $\mu = 1$.

Solution: From the condition $J_f(0) \neq 0$ it follows that the functions $f^1(x), \dots f^n(x)$ are local coordinates in a neighborhood of 0. Thus it is sufficient to prove this statement for the identical map: $f^i(x) = x_i$. Using Hadamard's lemma, we can represent any function $g \in A_n$ in the form $g = g(0) + x_1 g_1(x) + \cdots + x_n g_n(x)$. Hence A_n/I_f (as a vector space) is isomorphic to $\mathbb{R}$, and $\mu = 1$. $\square$

Now we can formulate the main result:

Malgrange Preparation Theorem. *Let $A_n = \mathbb{R}[[x_1, \dots, x_n]]$ or $A_n = C^\infty[x_1, \dots, x_n]$, $\mu < \infty$ and $e_1(x), \dots, e_\mu(x)$ be generators of the corresponding local algebra Q_f. Then any element $h(x) \in A_n$ can be represented in the form*

$$h(x) = a_1(f(x))e_1(x) + \cdots + a_\mu(f(x))e_\mu(x), \quad a_i(\cdot) \in A_n. \tag{3.3}$$

This theorem is also called «Malgrange–Weierstrass Preparation Theorem». The proof in the category of formal series ($A_n = \mathbb{R}[[x_1, \dots, x_n]]$) is rather simple, it can be found in:

- V.I. Arnold, S.M. Gusein-Zade, A.N. Varchenko. Singularities of Differentiable Maps. – Birkhauser, Boston, 1985, Vol. 1, Monogr. Math. 82.

However in the smooth category ($A_n = C^\infty[x_1, \dots, x_n]$) the proof is much more difficult, it requires some advanced techniques, for instance, work with finitely generated modules and Nakayama's lemma. The proof in the smooth category can be found in:

- B. Malgrange. Ideals of differentiable functions. – Oxford University Press, Oxford, 1966.
- M. Golubitsky, V. Guillemin. Stable maps and their singularities. – Graduate Texts in Mathematics, vol. 14, Springer-Verlag, 1973.
- Th. Bröcker, L. Lander. Differentialble Germs and Catastrophes. – Cambridge University Press, 1975.

Exercise 3.

What does Malgrange Preparation Theorem (MPT) give in the case $\mu = 1$ (that is, f is a local diffeomorphism)?

Solution: In the case $\mu = 1$ the functions $f^1(x), \dots f^n(x)$ are local coordinates in a neighborhood of 0, and using Hadamard's lemma, we can represent any function $g \in A_n$ in the form $g = g(0) + \beta_1(x) f^1(x) + \cdots + \beta_n(x) f^n(x)$ with appropriate $\beta_i(x) \in A_n$. Comparing this representation with (3.2), we see that the function $e_1(x) = 1$ is the generator of the local algebra Q_f (we can put $e_1(x)$ equal to any constant function except for zero). Hence representation (3.3) reads $h(x) = a_1(f(x))e_1(x) = a_1(f(x))$. This equality does not contain any information: since f is a local diffeomorphism, we can take the new variable $y = f(x)$, then for any $h \in A_n$ the equality $h(x) = a_1(f(x))$ holds true with the function $a_1(y) = h(f^{-1}(y))$. $\square$

However MPT is a very powerful instrument when we deal with the germ of functions at critical points ($1 < \mu < \infty$). It allows to get a lot of important results, some of them will be described below.

Exercise 4.

Prove the Lemma 3 from the Lecture 1: any smooth function $h(x, y_1, \dots, y_{n-1}) : \mathbb{R}^n \to \mathbb{R}$ can be represented in the form

$$h(x, y_1, \dots, y_{n-1}) = a_1(x^2, y_1, \dots, y_{n-1}) + xa_2(x^2, y_1, \dots, y_{n-1}), \tag{3.4}$$

where a_1 and a_2 are smooth functions.

Solution: Denote $y = (y_1, \dots, y_{n-1})$ and consider the smooth map $f(x, y) : \mathbb{R}^n \to \mathbb{R}^n$ defined by the following n functions:

$$f^1(x, y) = y_1, \ \dots, \ f^{n-1}(x, y) = y_{n-1}, \ f^n(x, y) = x^2. \tag{3.5}$$

Applying representation (1.8) with $n = 2$ to the function $g(x, y)$ from (3.2) and using Hadamard's lemma, we get

$$\begin{aligned} g(x, y) &= g_0(y) + xg_1(y) + x^2 g_2(x, y) = \\ &= g_0(0) + \sum_{i=1}^{n-1} g_{0i}(x, y) y_i + x\left(g_1(0) + \sum_{i=1}^{n-1} g_{1i}(x, y) y_i\right) + x^2 g_2(x, y) = \\ &= g_0(0) + xg_1(0) + \sum_{i=1}^{n-1} \beta_i(x, y) f^i(x, y) + g_2(x, y) f^n(x, y), \end{aligned}$$

that is, representation (3.2) with $\mu = 2$ and $e_1(x, y) = 1$, $e_2(x, y) = x$. Hence the local algebra Q_f of map (3.5) is generated by the germs 1 and x. Then representation (3.3) reads $h(x, y) = a_1(f(x, y)) + xa_2(f(x, y))$. This is required equality (3.4). □

Exercise 5.

Prove the generalization of formula (3.4): for any integer number $p \geq 2$ any smooth function $h(x, y_1, \dots, y_{n-1}) : \mathbb{R}^n \to \mathbb{R}$ can be represented in the form

$$h(x, y) = a_1(x^p, y) + xa_2(x^p, y) + \dots + x^{p-1} a_{p-1}(x^p, y), \tag{3.6}$$

where a_i are smooth functions, $y = (y_1, \dots, y_{n-1})$.

Hint: consider the map $f(x, y) : \mathbb{R}^n \to \mathbb{R}^n$ defined by (3.5) where x^2 is replaced with x^p.

Exercise 6.

Apply MPT to the smooth map $f(x, y) : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the functions:

$$f^1(x, y) = x^3 + xy, \quad f^2(x, y) = y. \tag{3.7}$$

Solution: Prove that $\mu = 3$ and the local algebra Q_f of map (3.7) is generated by the functions $e_1(x, y) = 1$, $e_2(x, y) = x$, $e_3(x, y) = x^2$. Indeed, using representation (1.8) and Hadamard's lemma, we can represent any function $g(x, y) \in A_2$ in the form

$$g(x, y) = g(0) + \alpha_1 x + \alpha_2 x^2 + y\beta_1(x, y) + x^3\beta_2(x, y), \quad \beta_{1,2}(x, y) \in A_2, \quad \alpha_{1,2} \in \mathbb{R}.$$

Clearly, $g(x, y) = g(0) + \alpha_1 x + \alpha_2 x^2 + y\overline{\beta}_1(x, y) + (x^3 + xy)\beta_2(x, y)$, where $\overline{\beta}_1 = \beta_1 - x\beta_2$. Thus we have representation (3.2) with $\mu = 3$ and $e_1 = 1$, $e_2 = x$, $e_3 = x^2$. Then (3.3) gives

$$h(x, y) = a_1(f^1, y) + xa_2(f^1, y) + x^2 a_3(f^1, y), \tag{3.8}$$

where $f^1(x, y)$ is taken from (3.7). □

Exercise 7.

Establish formula (3.8) for the map $f(x, y) : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the functions:

$$f^1(x, y) = \varphi(x, y)(x^3 + xy), \quad f^2(x, y) = y, \tag{3.9}$$

where $\varphi(x, y)$ is a smooth non-vanishing function.

Hint: The proof is similar to the previous one.

3.2 Generic singularities of mappings $\mathbb{R}^2 \to \mathbb{R}^2$.

In this part of the lecture we study generic singularities of mappings $\mathbb{R}^2 \to \mathbb{R}^2$. Since we work with germs, i.e., do everything in sufficiently small neighborhoods of singular points, this is equivalent to studying of generic singularities of mappings $M^2 \to N^2$, where M^2 and N^2 are real smooth 2-manifolds.[15]

Consider the map $f(x, y) : \mathbb{R}^2 \to \mathbb{R}^2$ given by the smooth functions $f^1(x, y)$ and $f^2(x, y)$. Our aim is to simplify the functions f^1, f^2 using smooth changes of variables in the preimage (M^2) and the image (N^2). This means that if (x, y) are local coordinates on M^2 and (z, w) are local coordinates on N^2, we can use two independent changes of the variables:

$$\varphi : (x, y) \to (\widehat{x}, \widehat{y}) \quad \text{and} \quad \psi : (z, w) \to (\widehat{z}, \widehat{w}). \tag{3.10}$$

The map has the form $z = f^1(x, y)$, $w = f^2(x, y)$ in the old coordinates and the form $\widehat{z} = g^1(\widehat{x}, \widehat{y})$, $\widehat{w} = g^2(\widehat{x}, \widehat{y})$ in the new coordinates.

It is possible to write this in the following commutative diagram:

$$\begin{array}{ccc} M^2_{(x,y)} & \xrightarrow{f=(f^1,f^2)} & N^2_{(z,w)} \\ \Big\downarrow \varphi & & \Big\downarrow \psi \\ M^2_{(\widehat{x},\widehat{y})} & \xrightarrow{g=(g^1,g^2)} & N^2_{(\widehat{z},\widehat{w})} \end{array} \tag{3.11}$$

The word «commutative» means that composition of the left and lower arrows coincides with composition of the upper and rights arrows, i.e., $g \circ \varphi = \psi \circ f$, or equivalently,

$$g = \psi \circ f \circ \varphi^{-1}. \tag{3.12}$$

[15] I hope the reader see the relationship with implicit differential equations: the projection π from the surface $\mathscr{F}$ to the plane.

The change of coordinates $\varphi : (x, y) \to (\widehat{x}, \widehat{y})$ is called «right», and the change of coordinates $\psi : (z, w) \to (\widehat{z}, \widehat{w})$ is called «left» according to formula (3.12) in spite in diagram (3.11) it is vice versa.

Definition 3: The functions f and g at the point 0 are called «right-left equivalent» or shortly «RL-equivalent» if they are connected with relation (3.12) with diffeomorphisms φ and ψ. If we use only right change of variables φ (i.e., $\psi = id$), we get «R-equivalence»; if we use only left change of variables ψ (i.e., $\varphi = id$), we get «L-equivalence».

Clearly, we can define all these types of equivalence for maps $f : M^m \to N^n$ with any dimensions m and n. Besides, we already have dealt with some partial cases of these notions.

Example 1:

The Morse Lemma states that each smooth function $f : M^m \to N^1$ at any non-degenerated critical point is R-equivalent to its own quadratic part. Tougeron's Theorem states that each smooth function $f : M^m \to N^1$ at any critical point with $\mu < \infty$ (here multiplicity μ is defined as in the Lecture 1) is R-equivalent to its own Taylor polynomial of degree $\mu + 1$.

Example 2:

In the Exercises 18 – 20 from Lecture 2 we deal with RL-equivalence of maps $f : \mathbb{R}^1 \to \mathbb{R}^2$ (we change the coordinate τ in the preimage and the coordinates x, y in the image).

Example 3:

Any smooth map $f : M^m \to N^n$ which has the constant rang k (rang of a map is the rang of its derivative, i.e., Jacobi matrix) in a neighborhood of 0 is RL-equivalent to the map

$$f^i(x_1, \ldots, x_m) = x_i \ (i \leq k), \quad f^i(x_1, \ldots, x_m) = 0 \ (i > k).$$

The proof of this fact is left to the reader or can be found in: V.A. Zorich. Mathematical Analysis. In particular, in the case $n = m = k$ the diffeomorphism f is RL-equivalent to id. Moreover, any diffeomorphism is R-equivalent to id and also L-equivalent to id.

Consider the map $f(x, y) : M^2 \to N^2$ given by the functions

$$z = f^1(x, y), \quad w = f^2(x, y), \quad f^1(0) = f^2(0) = 0, \tag{3.13}$$

in a neighborhood of the origin. The Jacobi matrix of this map is

$$A_f = \begin{pmatrix} f^1_x & f^1_y \\ f^2_x & f^2_y \end{pmatrix} \tag{3.14}$$

The singular points of this map are defined by the condition $|A_f| = 0$, i.e., $\operatorname{rg} A_f = 1$ or $\operatorname{rg} A_f = 0$. The set $\{(x, y) : \operatorname{rg} A_f(x, y) = 1\}$ has codimension 1 and generically is a curve on the (x, y)-plane. The set $\{(x, y) : \operatorname{rg} A_f(x, y) = 0\}$ has codimension 4, hence in generic case it is empty.

Suppose the set of the singular points is $\{(x, y) : \operatorname{rg} A_f(x, y) = 1\}$. Then map (3.13) is R-equivalent to

$$z = F(x, y), \quad w = y, \quad F(0) = 0, \tag{3.15}$$

where $F(x, y)$ is a smooth function (the proof is trivial and left to the reader). The singular point of map (3.15) are given by the equation

$$|A_f| = \begin{vmatrix} F_x & F_y \\ 0 & 1 \end{vmatrix} = 0 \quad \Longleftrightarrow \quad F_x(x, y) = 0,$$

which defines a regular curve if $|F_{xx}| + |F_{xy}| \neq 0$.

Since the both sets $\{(x,y) : F_x = F_{xx} = F_{xy} = 0\}$ and $\{(x,y) : F_x = F_{xx} = F_{xxx} = 0\}$ have codimension 3, there are only two generic cases:

- $F_x(0) = 0, \;\; F_{xx}(0) \neq 0.$
- $F_x(0) = 0, \;\; F_{xx}(0) = 0, \;\; F_{xy}(0) \neq 0, \;\; F_{xxx}(0) \neq 0.$

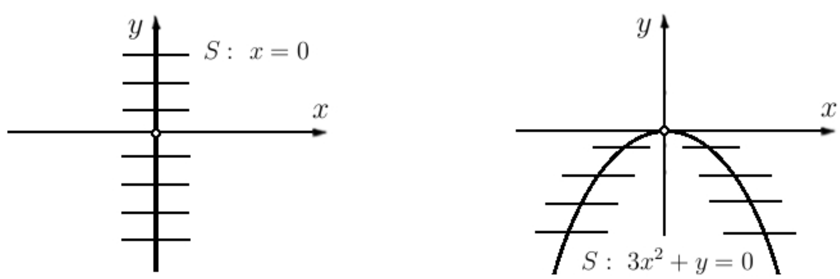
These singularities are called «fold» and «pleat», respectively. Sometimes they are called «Whitney fold» and «Whitney pleat».

Notice that to define fold and pleat of map (3.13), we used the special coordinates (such that the second function $f^2(x,y)$ coincides with one of the coordinates x, y in the preimage M^2). However it is possible to define fold and pleat using only invariant notions (which do not depend on the choose of coordinates).

There are two main invariant objects connected with the singularity of map f satisfying the condition $\operatorname{rg} df(0) = 1$. The first object is the curve S defined by the equation $|A_f| = 0$, this curves that consists of the singular points of the map. The second object is the direction field $\chi = \ker\, df$ defined at the points of S. The field χ is defined only on the curve S: at all points $\notin S$ the differential df is non-degenerated, and $\ker\, dx = 0$. Choosing the coordinates in M^2 and N^2, we can express the map $df : TM^2 \to TN^2$ through the matrix A_f, then the field $\chi = \ker\, A_f$.

Example 4:

The curve S and the field χ for maps (3.15) with $F = x^2$ and $F = x^3 + xy$ are depicted on the following picture:



Exercise 8.

Prove that fold and pleat can be defined by the following conditions:

- S is regular ($\nabla S \neq 0$) and χ is transversal to S at 0.
- S is regular, χ is tangent to S at 0, and the tangency has the first order.

Hint: Since the conditions of the second definition are geometrically invariant, it is sufficient to prove the equivalence of the first and second definitions for map (3.15)

Now we can prove the main result of this lecture: Whitney Theorem about the normal forms of maps in neighborhoods of fold and pleat.

Whitney Theorem. *In neighborhoods of fold and pleat any map $f(x,y) : M^2 \to N^2$ is RL-equivalent to the following normal forms:*

- $z = x^2, \quad w = y,$
- $z = x^3 + xy, \quad w = y,$

respectively.

It is sufficient to prove this statement for map (3.15).

Proof for Fold. By the condition $F_{xx}(0) \neq 0$, the curve $S : F_x(x, y) = 0$ has the form $x = \varphi(y)$ with a smooth function φ. The change of variable $x \to x - \varphi(y)$ turns this curve to the axis $x = 0$, and in the new coordinates the function $F_x(x, y)$ vanishes on the line $x = 0$. According to Exercise 6 form Lecture 1, we have the representation

$$F(x, y) = F_0(y) + x^2\varphi(x, y), \quad \text{where} \quad F_0(y) = F(0, y),$$

with a smooth function $\varphi(x, y)$. The condition $F_{xx}(0) \neq 0$ implies $\varphi(0) \neq 0$. Moreover, we may assume that $\varphi(0) > 0$, since the change of variable $z \to -z$ changes the sign of φ.

Then the change of variable $x \to x\sqrt{\varphi(x, y)}$ gives $F(x, y) = F_0(y) + x^2$, i.e., the map has the form

$$z = F_0(y) + x^2, \quad w = y.$$

Finally, using the change of variable $z \to z - F_0(w)$, we get the formula

$$\boxed{\text{Fold}: \quad z = x^2, \quad w = y} \tag{3.16}$$

Proof for Pleat.

Step One. Prove that map (3.15) with the function $F(x, y)$ satisfying the conditions

$$F_x(0) = 0, \quad F_{xx}(0) = 0, \quad F_{xy}(0) \neq 0, \quad F_{xxx}(0) \neq 0,$$

is RL-equivalent to the map (3.15) with $F = \varphi(x, y)(x^3 + xy)$, where φ is a smooth function, $\varphi(0) \neq 0$. According to Exercise 5 form Lecture 1, we can write the function F using representation (1.8) with $n = 1$:

$$F(x, y) = F_0(y) + xg(x, y), \quad \text{where} \quad F_0(y) = F(0, y),$$

where

$$g(0) = 0, \quad g_x(0) = 0, \quad g_y(0) \neq 0, \quad g_{xx}(0) \neq 0.$$

Then after the change of variable $z \to z - F_0(w)$ we get $F_0(y) \equiv 0$, i.e., $F = xg(x, y)$.

Apply the Division Theorem to the function $g(x, y)$. Since the multiplicity of $g(x, y)$ by the variable x is $\mu = 1$, we have

$$g(x, y) = \varphi(x, y)\,(x^2 + 2a(y)x + b(y)), \quad \varphi(0) \neq 0. \tag{3.17}$$

The change of variable $x \to x + a(y)$ kills the middle term in (3.17), i.e., $a(y) \equiv 0$. The conditions $g(0) = 0$, $g_y(0) \neq 0$ imply $b(0) = 0$, $b'(0) \neq 0$. Hence we can make the «right» change of variables $y \to b(y)$ and the «left» change of variables $w \to b(w)$. This yields

$$z = F(x, y), \quad w = y, \tag{3.18}$$

where $F(x, y) = x\varphi(x, y)(x^2 + y) = \varphi(x, y)(x^3 + xy)$, $\varphi(0) \neq 0$.

Step Two: consider map (3.18). According to the results obtained in Exercises 6 and 7 (from Lecture 3), any smooth function $h(x, y)$ can be represented in the form

$$h(x, y) = a_1(F, y) + xa_2(F, y) + x^2a_3(F, y), \quad \text{where} \quad F = \varphi(x, y)(x^3 + xy) \tag{3.19}$$

with appropriate smooth functions a_i of two variables. Applying representation (3.19) to the function $h(x, y) = x^3$ and changing $a_2 \to -a_2$, $a_3 \to 3a_3$, we get

$$x^3 = a_1(F, y) - xa_2(F, y) + 3x^2a_3(F, y), \quad \text{where} \quad F = \varphi(x, y)(x^3 + xy). \tag{3.20}$$

Representation (3.20) has the form

$$(x - a(F, y))^3 + b(F, y)(x - a(F, y)) = c(F, y), \quad \text{where} \quad F = \varphi(x, y)(x^3 + xy),$$

and $a = a_3$, $b = a_2 - 3a^2$, $c = a_1 - ab - a^3$. It is not hard to check that the following pair of right and left changes of variables

$$\begin{array}{lcll} (x, y) \to (\widehat{x}, \widehat{y}) & : & \widehat{x} = x - a(F(x, y), y), & \widehat{y} = b(F(x, y), y) \\ (z, w) \to (\widehat{z}, \widehat{w}) & : & \widehat{x} = c(z, w), & \widehat{w} = b(z, w) \end{array}$$

brings the map to the form

$$\boxed{\text{Pleat}: \quad z = x^3 + xy, \quad w = y} \tag{3.21}$$

□

Remark. The above theorem was initially proved in the paper:

- H. Whitney. On singularities of mappings of Euclidean spaces. I. Mappings of the plane into the plane. – Ann. of Math., 62 (1955), pp. 374–410.

In his proof Whitney used more elementary techniques (Malgrange Preparation Theorem was proved later on), but his proof is much longer.

Finally, it is very useful to understand how do maps (3.16) and (3.21) look like. Since in the both cases one coordinate in the image (w) coincides with one coordinate in the preimage (y), both maps can be realized as the projection π from a smooth surface $\mathscr{F}$ in the (x, y, z)-space to the (y, z)-plane along the x-axis. The set of the singular (critical) points $S \subset \mathscr{F}$ is said to be the criminant, and its projection $\pi(S)$ is said to be the discriminant curve.

In the case (3.16) this surface is the parabolic cylinder $z = x^2$. The projection π is a double covering of the semi-plane $(y, z > 0)$, with branching at $z = 0$. The criminant is the y-axis in the (x, y, z)-space, the discriminant curve is the y-axis on the (y, z)-plane.

In the case (3.21) this surface is the graph of the function $z = x^3 + xy$. The criminant is the parabola $3x^2 + y = 0$ on the surface $\mathscr{F}$. The discriminant curve is the semi-cubic parabola

$$y = -3x^2, \quad z = -2x^3.$$

on the (y, z)-plane. The cusp occurs at the point where the curve S is tangent to the direction field $\chi = x$-direction.

Exercise 9.

Draw the surface $\mathscr{F}$, the criminant and the discriminant curve for map (3.21). How many points does the preimage $\pi^{-1}(y, z)$ contain? How is this picture connected with the number of real roots of the cubic polynomial $p(x) = x^3 + ax + b$ with real coefficients a, b?

Exercise 10. Why the singularities of the maps:

$$\begin{array}{ll} 1) & z = x^3, \quad w = y, \\ 2) & z = x^4, \quad w = y, \end{array}$$

are not generic?

Hint: for the first map consider the perturbation $z = x^3 + \varepsilon x$, $w = y$.

4 Lecture 4. Implicit Differential Equations: Normal Forms and Phase Portraits.

In this lecture, we continue to study Implicit Differential Equations from the Singularity Theory viewpoint. We have already studied the singularities of the individual integral curves (lecture 2), now the aim is to get the normal forms of such equations and/or their phase portraits in neighborhoods of the singular points.

Recall that the singular points of the Implicit Differential Equation

$$F(x,y,p)=0, \quad \text{where} \quad p=\frac{dy}{dx}, \tag{4.1}$$

are the points of the surface $\mathscr{F}: F=0$ such that $F_p=0$, i.e., critical points of the projection $\pi:\mathscr{F}\to(x,y)$. Let $\mu<\infty$ be the multiplicity of F by the variable p at the singular point T_0:

$$\frac{\partial F}{\partial p}(T_0)=0,\ \ldots,\ \frac{\partial^\mu F}{\partial p^\mu}(T_0)=0, \quad \frac{\partial^{\mu+1} F}{\partial p^{\mu+1}}(T_0)\neq 0.$$

As we saw in the previous lecture, in generic case the projection π has only two types of singularities (= critical points). The first type is the Fold, it is defined by the condition

$$F_{pp}(T_0)\neq 0 \ \Leftrightarrow\ \mu=1. \tag{4.2}$$

The second type is the Pleat, it is defined by two conditions:

$$\Big(F_{pp}(T_0)=0,\ F_{ppp}(T_0)\neq 0 \ \Leftrightarrow\ \mu=2\Big) \quad \text{and} \quad \Big(F_{xp}(T_0)\neq 0 \ \text{or}\ F_{yp}(T_0)\neq 0\Big). \tag{4.3}$$

For generic equation (4.1) the criminant $\mathscr{K}$ consists of the points satisfying condition (4.2) except for the isolated points satisfying condition (4.3).

Moreover, for generic equation (4.1) the intersection of the criminant $\mathscr{K}$ and the inflection curve $\mathscr{L}$ consists of the isolated points that satisfy condition (4.2). The subset of the surface $\mathscr{F}$ defined by the equations $F_p=0$, $F_{pp}=0$, $G=0$ has codimension 3; hence in generic case it is empty.

Thus we will consider three types of the singular points:

- $F_{pp}(T_0)\neq 0$ and $G(T_0)\neq 0$.
- $F_{pp}(T_0)\neq 0$ and $G(T_0)=0$.
- condition (4.3) and $G(T_0)\neq 0$.

In the first and second cases $\pi:\mathscr{F}\to(x,y)$ is fold, and in the third case π is pleat.

4.1 «Regular» singular points

The singular points that satisfy the conditions

$$F_{pp}(T_0)\neq 0, \quad G(T_0)\neq 0$$

are called «regular».[16] There exist another definition: the singular points are said to be «regular» if the map $(x,y,p)\to(F,F_p)$ has maximal rang (=2) and the criminant is not tangent to the contact plane The second definition is used in the books:

[16]The combination of the words «regular singular points» seems strange. However it can be explained. This terminology belongs to V.I. Arnold and initially was in Russian; English translation is a calque from Russian. In Russian there are several words that can be translated as «singular» in English, that's why the initial Russian term does not sound like «hot ice».

- V.I. Arnold. Geometrical methods in the theory of ordinary differential equations. – Springer-Verlag 1988.
- V.I. Arnold, Yu.S. Il'yashenko. Ordinary differential equations. In Dynamical systems I, Encyclopaedia Math. Sci., vol. 1.
- A.A. Davydov. Qualitative Theory Of Control Systems. – Transl. Math. Monogr. 141, AMS, Providence, Rhode Island, 1994.

Exercise 1.
Check the equivalence on these definitions.

Regular singular points are «most typical» singular points in the following sense. The criminant of a generic Implicit Differential Equation consists of the regular singular points except for isolated points.

Example 1. The equation $p^2 + x = 0$ describes the characteristic curves of the partial differential equation $u_{xx} + xu_{yy} = 0$, which is called «Tricomi equation» (sometimes «Euler–Tricomi equation»). Tricomi equation is the simplest example of so-called «mixed» PDEs, i.e., linear PDEs of the second order that have different types at different points.[17]

Clearly, all singular points of the equation $p^2 + x = 0$ are regular. The discriminant curve $x = 0$ separates the (x, y)-plane into two parts. Tricomi equation is hyperbolic in the half plane $x < 0$ and elliptic in the half plane $x > 0$. The characteristic curves are semi-cubic parabolas, which belong to the hyperbolic semi-plane and have cusps at $x = 0$.

Definition: Two Implicit Differential Equations are called «equivalent» if there exists a change of variables (x, y) that transforms the family of the integral curves of the first IDE into the family of the integral curves of the second IDE. More exactly, IDEs are called «topologically equivalent» if shuch change of variables $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism, «smoothly equivalent» if f is a smooth diffeomorphism, «analytically equivalent» if f is an analytical diffeomorphism. In both cases the diffeomorphism f is called «conjugating diffeomorphism».

Example 2. The Implicit Differential Equation $p^4 - 1 = 0$ is smoothly (and analytically) equivalent to $p^2 - 1 = 0$. The conjugating diffeomorphism f is identical. Indeed, $p^4 - 1 = (p^2 - 1)(p^2 + 1)$. Hence the equations $p^4 - 1 = 0$ and $p^2 - 1 = 0$ have the same family of the integral curves, and not necessary to use any change of varibles!

Example 3. The Implicit Differential Equation $p^2 y = 1$ is smoothly (and analytically) equivalent to $p^2 = x$. The conjugating diffeomorphism f is interchanging x and y. Indeed, the interchanging x and y turns $p^2 = x$ to $p^{-2} = y$, that is, $p^2 y = 1$.

Theorem 1. *In a neighborhood of any regular singular point T_0, equation* (4.1) *is smoothly equivalent to $p^2 = x$.*

The normal form $p^2 = x$ is called «Cibrario normal form» (this name was suggested by V.I. Arnold) after Italian female mathematician Maria Cibrario (1905 – 1992). She has established this normal form when she studied the characteristics of linear second-order mixed PDEs with analytic coefficients:

- M. Cibrario. Sulla reduzione a forma canonica delle equazioni lineari alle derivative parzialy di secondo ordine di tipo misto. – Rend. Lombardo 65 (1932), pp. 889–906.

[17]Mixed PDEs became important in describing an object moving at supersonic speed (in the middle of XX century). Of course there were no supersonic aircraft in the time when Francesco Tricomi studied his famous equation (1923), but this equation was to play a major role in later studies of supersonic flight (another important mixed PDE is Chaplygin's equation).

Recall that IDEs which describe characteristics of linear second-order mixed PDEs have the form (4.1), where the function F is a second-order polynomial in p with coefficients depending on x, y. Later on, this normal form was obtained in general case and in the smooth and analytic categories simultaneously. This proof can be found in the famous textbook:

- V.I. Arnold. Geometrical methods in the theory of ordinary differential equations. – Springer-Verlag, 1988.

We follow the proof from this book, with some unsignificant modifications.

Proof.

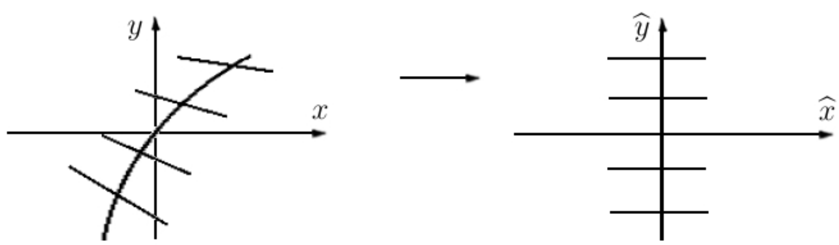
Step One. Without loss of generality we may assume that the point T_0 is the origin of J^1. Choosing appropriate local coordinates in a neighborhood of the origin on the (x, y)-plane, one can bring the criminant of equation (4.1) to the line $x = p = 0$ (the y-axis).

Indeed, equation (4.1) defines the direction field χ on the discriminant curve $\mathscr{D}$ that at each point $(x, y) \in \mathscr{D}$ the direction $dy : dx$ corresponds to the unique root $p = p(x, y)$ of the equation $F(x, y, p) = 0$ with respect to p. Let's show that the direction field χ is transversal to the discriminant curve $\mathscr{D}$. Clearly, it is sufficient to prove that the criminant $\mathscr{K} : F = F_p = 0$ is transversal to the contact plane $pdx - dy = 0$. This is equivalent to

$$\Delta = \begin{vmatrix} F_x & F_y & F_p \\ F_{xp} & F_{yp} & F_{pp} \\ p & -1 & 0 \end{vmatrix} \neq 0.$$

At point of the criminant we have $F_p = 0$, hence $\Delta = (F_x + pF_y)F_{pp} = F_{pp}G \neq 0$.

Since the direction field χ is transversal to the discriminant curve $\mathscr{D}$, there exist local coordinates $(\widehat{x}, \widehat{y})$ on the plane such that the discriminant curve $\mathscr{D}$ is $\widehat{x} = 0$ and the coordinate lines $\widehat{y} = \text{const}$ intersect $\widehat{x} = 0$ with tangent directions χ, see the figure:



Further we will use the same letters x, y for the new coordinates $\widehat{x}, \widehat{y}$, i.e., suppose that $\mathscr{K}$ is the y-axis in the initial coordinates.

Step Two. Using the Division Theorem, we can represent the function $F(x, y, p)$ in the form $F(x, y, p) = \varphi(x, y, p)(p^2 + 2a(x, y)p + b(x, y))$, where a, b, φ are smooth fucntions, $a(0) = b(0) = 0$ and $\varphi(0) \neq 0$. Hence in a neighborhood of the origin equation (4.1) is equivalent to

$$p^2 - 2a(x, y)p + b(x, y) = 0, \quad p = \frac{dy}{dx}. \tag{4.4}$$

The criminant of equation (4.4) is the y-axis, hence $a(0, y) = b(0, y) = 0$ $(\forall y)$. By the Lemma 1 (from Lecture 1) we have the representations $a(x, y) = x\alpha(x, y)$ and $b(x, y) = x\beta(x, y)$ with smooth functions α, β. The condition $G \neq 0$ for equation (4.4) implies $\beta(0) \neq 0$. Without loss of generality we may assume $\beta(0) < 0$, otherwise $x \to -x$.

Consider equation (4.4) as a square equation by p, then we get the solution

$$p = a \pm \sqrt{a^2 - b} = x\alpha \pm \sqrt{x(x\alpha^2 - \beta)} = x\alpha \pm \sqrt{x\gamma}, \quad \text{where} \quad \gamma = x\alpha^2 - \beta.$$

In a neighborhood of the origin, the function $\gamma(x, y) > 0$, and the above solution p is real in the semi-plane $x \geq 0$. Using the change of variable $x = \xi^2$, we get

$$p = \xi^2\alpha(\xi^2, y) \pm \xi\sqrt{\gamma(\xi^2, y)}. \tag{4.5}$$

Of course, the change of variables $(x, t) \to (\xi, y)$ is not a diffeomorphism and even not one-to-one map of a neighborhood of the origin of the (x, y)-plane. However it is one-to-one map of the half plane $x \geq 0$, which contains all solutions of equation (4.4). This justifies using of the change of variable $x = \xi^2$.

Step Three. The variable ξ takes both positive and negative values, hence we can put in formula (4.5) the sign «+» instead of «±». Since $p = dy/dx = dy/(2\xi d\xi)$, equality (4.5) gives the following differential equation for the variables ξ, y:

$$\frac{dy}{d\xi} = 2\xi\left(\xi^2\alpha(\xi^2, y) + \xi\sqrt{\gamma(\xi^2, y)}\right) = \xi^2\omega(\xi^2, y), \tag{4.6}$$

where $\omega = 2(\xi\alpha + \sqrt{\gamma})$ is a smooth function and $\omega(0) > 0$.

The integral curves of equation (4.6) intersect the y-axis (on the (ξ, y)-plane) with the horizontal tangent directions (= parallel to the ξ-axis). Moreover, form $\omega(0) \neq 0$ it follows that at the points with $\xi = 0$ the integral curves have the second order tangency with corresponding coordinate lines $y = \text{const}$ (like cubic parabolas). Hence equation (4.6) has the first integral

$$I(\xi, y) = y - \xi^3 f(\xi, y), \quad f(0) \neq 0,$$

where f is a smooth function. Using Lemma 3 (from Lecture 1), we can represent the function $f(\xi, y) = f_1(\xi^2, y) + \xi f_2(\xi^2, y)$ with smooth sunctions $f_{1,2}$. Then

$$I(\xi, y) = y - \xi^3 f_1(\xi^2, y) - \xi^4 f_2(\xi^2, y), \quad f_1(0) \neq 0. \tag{4.7}$$

Step Four. The change of variables $(\xi, y) \to (\Xi, Y)$ by the formula

$$\Xi = \xi\, f_1^{1/3}(\xi^2, y), \quad Y = y - \xi^4 f_2(\xi^2, y),$$

transforms the above first integral to $I = Y - \Xi^3$. Recall that $x = \xi^2$ and introduce the variable $X = \Xi^2$, that is $\Xi = \pm\sqrt{X}$. The change of variables $(x, y) \to (X, Y)$ by the formula

$$X = x\, f_1^{2/3}(x, y), \quad Y = y - x^2 f_2(x, y), \tag{4.8}$$

transforms the equation $Y - \Xi^3 = c$ into $Y \pm X^{3/2} = c$.

Hence the change of variables (4.8) transforms equation (4.1) into an implicit differential equation posessing the family of the integral curves $Y \pm X^{3/2} = c$. Using an appropriate scaling of X or Y, it can be brought to the form $Y = \pm\frac{2}{3}X^{3/2} + c$. The equation possessing the last family is

$$P^2 = X, \quad \text{where} \quad P = \frac{dY}{dX}.$$

This completes the proof. □

Comments.

Remark that Cibrario normal form coincides with the normal form (Whitney fold) of the projection $\pi : \mathscr{F} \to (x, y)$. Theorem 1 claims that there exist local coordinates on the (x, y)-plane such that given IDE is equivalent to $p^2 = x$. The variables (p, y) are local coordinates on the surface $\mathscr{F}$ of IDE $p^2 = x$, and the projection $\pi : (p, y) \to (x, y)$ has the normal form of the Whitney fold. However Theorem 1 and Whitney Theorem are different statements.

Indeed, the Whitney Theorem deals with maps $f : M \to N$, where M and N are two different manifolds with independent local coordinates: say, (x, y) on M and (z, w) on N. The map f is RL-equivalent to g if there exists a pair of local diffeomorphisms $\varphi : (x, y) \to (\widehat{x}, \widehat{y})$ and $\psi : (z, w) \to (\widehat{z}, \widehat{w})$ that in the new coordinates (with hats) the map f coincides with the map g. In the case of Theorem 1 we also deal with map $\pi : M \to N$, where $M = \mathscr{F}$ and $N = (x, y)$-plane, but the local coordinates on the manifolds M and N are not independent. Indeed, any change of the variables x, y denerates a change of the variable p, since $p = dy/dx$.[18]

4.2 «Pleated» singular points

Now consider the singular points that satisfy the conditions (4.3) and $G(T_0) \neq 0$. Without loss of generality assume that $T_0 = 0$.

Exercise 2. Prove that in a neighborhood of the pleated singular point 0, equation (4.1) is smoothly equivalent to

$$p\, h(y, p) = x, \quad \text{where} \quad p = \frac{dy}{dx}, \tag{4.9}$$

where $h(y, p)$ is a smooth function such that $h(0) = h_p(0) = 0$, $h_{pp}(0) \neq 0$, $h_y(0) \neq 0$.

Hint: From $G(0) \neq 0$ it follows that $F_x(0) \neq 0$. Hence equation (4.1) is locally equivalent to $x = f(y, p)$ (Implicit Functions Theorem). The smooth function $f(y, p)$ can be represented in the form $f(y, p) = g(y) + ph(y, p)$ with smooth g and h (Exercise 5 from Lecture 1). To kill the term $g(y)$, consider the change of variable $x \to x - g(y)$. □

As it was remarked above, in the case when singularity of the projection π is a fold, Theorem 1 cannot be derived from the Whitney Theorem, but the normal forms in both theorems are the same (and very simple). However in the case when singularity of π is a pleat, the generic IDE is not smoothly (and even topologically) equivalent to $p^3 + py = x$. Moreover, the smooth classification of pleated singular points has functional invariants, see:

- A.A. Davydov. Normal form of a differential equation, not solvable for the derivative, in a neighborhood of a singular point. – Functional Anal. Appl. 19:2 (1985), pp. 81–89.

J.W. Bruce suggested a construction[19] that gives a clear geometric description of the phase portrait in a neighborhood of pleated singular points. This construction is based on the special surface which is called «swallow tail».

This surface can be defined as follows. Consider the polynomials $P(t) = t^4 + at^2 + bt + c$ of the real variable t with the real coefficients a, b, c. Then «swallow tail» is a surface in the (a, b, c)-space defined by the condition that $P(t)$ has a multiple root, see the figure below.[20]

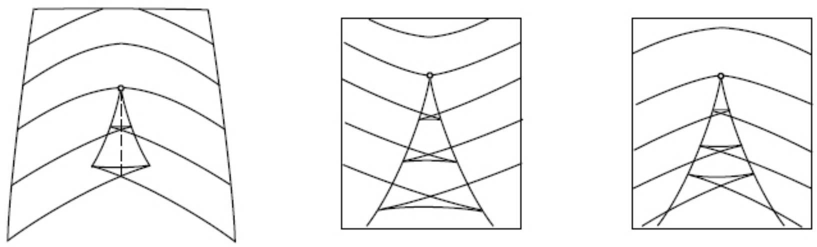
[18] It is similar to the well-known situation in Linear Algebra. Let $A : L_1 \to L_2$ be an invertible linear operator, L_1 and L_2 are vector spaces of the same dimension. It is possible to bring A to id using appropriate linear changes of variables in L_1 and L_2, that is, $\exists C_1$, $\exists C_2$: $C_1 A C_2^{-1} = id$. However if we use the same linear change of variables in both spaces (for example, if $C_1 = C_2$), one can bring A only to the Jordan normal form, that is, $\exists C$: $CAC^{-1} = J$, where $J \neq id$ if $A \neq id$.

[19] J.W. Bruce. A note on first-order differential equations of degree greater than one and wavefront evolution. – Bull. London Math. Soc., 16 (1984).

[20] In 1983, spanish painter Salvador Dali created the picture which is called «Swallow Tail» (in fact, this painting contains not the whole surface, but its section). This work became his last painting.

Exercise 3. Prove that the swallow tail divides the (a,b,c)-space into 3 regions which correspond to the numbers of the real roots of $P(t)$. Find the number of the roots at different points of the space. Write the swallow tail in the parametric form, i.e., as the image of a smooth map $\mathbb{R}^2 \to \mathbb{R}^3$.

Exercise 4. Consider the similar construction for the polynomials $P(t) = t^3 + at^2 + bt + c$ of the real variable t with the real coefficients a, b, c. Describe the surface in the (a,b,c)-space defined by the condition that $P(t)$ has a multiple root.



(a) Swallow tail and its sections (b) Elliptic pleated point (c) Hyperbolic pleated point

J.W. Bruce showed that the phase portrait of IDE (4.9) near the origin can be obtained from the family of sections of the swallow tail constructed above in the (a,b,c)-space by planes $a =$ const using an appropriate submersion $(a,b,c) \to (x,y)$. He showed that the phase portraits of such IDEs are essentially different for two types of pleated points, which are called «elliptic» and «hyperbolic» (see the figure above) and correspond to the cases $h_y(0) > 0$ and $h_y(0) < 0$, respectively. Bruce conjectured that the smooth classification of IDEs at pleated singular points has functional invariants. Later on, Davydov proved it even for the topological classification, see the paper cited above.

4.3 «Folded» singular points

Finally, consider the singular points that satisfy the conditions $F_{pp}(T_0) \neq 0$ and $G(T_0) = 0$. At such points the projection $\pi : \mathscr{F} \to (x,y)$ is fold, and the lifted vector field $\vec{V}$ vanishes. There are three generic types of singular point of a vector field on a two-dimensional surface: saddle, node and focus.

The point T_0 is said to be «well-folded» if three additional genericity conditions hold true: T_0 is saddle or node or focus of the lifted field $\vec{V}$, ratio of the eigenvalues $\lambda_{1,2}$ of the linear part of $\vec{V}$ at T_0 is different from ± 1, and the eigenvectors are not tangent to the criminant and to the p-direction.

Remark. Recall that the lifted field $\vec{V}$ is given by the formula

$$\dot{x} = F_p, \quad \dot{y} = pF_p, \quad \dot{p} = G, \quad \text{where} \quad G := -(F_x + pF_y). \tag{4.10}$$

Notice that the above formula defines a vector field not only on the surface $\mathscr{F}$, but in whole (x,y,p)-space,[21] i.e., for the vector field on $\mathscr{F}$ one of the coordinates x, y, p is superfluous. Computing eigenvalues of the matrix

$$A = \begin{pmatrix} F_{xp} & F_{yp} & F_{pp} \\ pF_{xp} & pF_{yp} & F_p + pF_{pp} \\ G_x & G_y & G_p \end{pmatrix} (T_0) = \begin{pmatrix} F_{xp} & F_{yp} & F_{pp} \\ pF_{xp} & pF_{yp} & pF_{pp} \\ G_x & G_y & G_p \end{pmatrix} (T_0),$$

[21] This circumstance allows to use the same construction (lifting) for studying IDEs with non-regular surface $\mathscr{F}$ (for example, $\mathscr{F}$ can be a cone).

we get three values: $\lambda_1, \lambda_2, 0$. The linear part of the vector field $\vec{V}$ obtained by restriction of (4.10) to $\mathscr{F}$, has the eigenvalues $\lambda_{1,2}$ (generally speaking, one or even both of them can be also equal to zero, but it is not the generic case). This statement follows from the fact that F is the first integral of field (4.10).

Exercise 5. Let $\vec{V}$ be an arbitrary smooth vector field in n-dimensional phase space with the coordinates $x = (x^1, \dots, x^n)$, and $\vec{V}(x_*) = 0$. By $(\lambda_1, \dots \lambda_n)$ denote the eigenvalues of the linear part of $\vec{V}$ at x_* (of course, this set may contain the equal values). Briefly, we say that $(\lambda_1, \dots \lambda_n)$ is the spectrum of $\vec{V}$ at x_*.

Suppose that $\mathscr{F}$ is a smooth invariant hypersurface of the field $\vec{V}$, and consider the restriction $\vec{V}\big|_{\mathscr{F}}$. Prove that the spectrum of $\vec{V}\big|_{\mathscr{F}}$ at x_* can be obtained from $(\lambda_1, \dots \lambda_n)$ after deleting one eigenvalue λ_i. Prove that this eigenvalue $\lambda_i = 0$ if the hypersurface $\mathscr{F}$ has the form $F = 0$, where F is a regular first integral of $\vec{V}$ («regular» means $\nabla F \neq 0$). $\square$

A. A. Davydov obtained a list of smooth normal forms of IDE (4.1) at well-folded singular points that satisfy the additional «linearizability condition»: in a neighborhood of given point, the lifted vector field $\vec{V}$ on $\mathscr{F}$ is smoothly equivalent to its linear part.[22] I stress that here (and below) the lifted vector field $\vec{V}$ means the field in the surface $\mathscr{F}$.

The sufficient condition for linearizability can be expressed through the eigenvalues $\lambda_{1,2}$ of the lifted vector field $\vec{V}$ at T_0. Namely, linearizability condition holds true if there are no relations

$$\lambda_i = n_1\lambda_1 + n_2\lambda_2, \quad n_{1,2} \in \mathbb{Z}_+, \quad n_1 + n_2 \geq 2, \quad i \in \{1, 2\}. \tag{4.11}$$

The relations (4.11) are called «resonances» between the eigenvalues $\lambda_{1,2}$. In general, presence of the resonance is an obstacle for the linearizability of a vector field. There is the following result (which is a partial case of more general Sternberg–Chen Theorem): if there are no resonances (4.11) then the vector field $\vec{V}$ is locally smoothly equivalent to its linear part. This statement is valid for vector fields in the phase space of any finite dimension.

Exercise 6.

Find the resonances in the following cases:

1. $\lambda_1 : \lambda_2 \in \mathbb{N}$,
2. $n\lambda_1 + m\lambda_2 = 0$, where $n, m \in \mathbb{N}$,
3. one of the eigenvalues $\lambda_{1,2}$ is zero.

A. Davydov proved that in a neighborhood of any well-folded singular point T_0 safisfying the linearizability condition, equation (4.1) is smoothly equivalent to

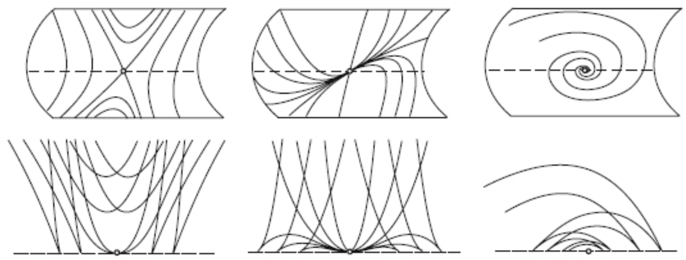
$$(p + \alpha x)^2 = y, \quad \text{where} \quad \alpha < 0 \ \text{ or } \ 0 < \alpha < 1/8 \ \text{ or } \ \alpha > 1/8. \tag{4.12}$$

Three intervals for the parameter α in normal form (4.12) correspond to three types of the lifted field $\vec{V}$ on the surface $\mathscr{F}$: saddle, node, focus. The proof can be found in:

- A.A. Davydov. Normal form of a differential equation, not solvable for the derivative, in a neighborhood of a singular point. – Functional Anal. Appl. 19:2 (1985), pp. 81–89.
- A.A. Davydov. Qualitative Theory of Control Systems, Math. Monogr., 141, Amer. Math. Soc., Providence, Rhode Island (1994).

The phase portraits of equation (4.1) at such singular points are obtained after projection of the saddle, node, focus of the lifted vector field from the surface $\mathscr{F}$ to the (x, y)-plane. They are called «folded saddle», «folded node», «folded focus», respectively:

[22] Two vector fields are called «smoothly equivalent» if there exists a smooth diffeomorphism of the phase space (= change of the variables) that turns the first vector field into the second one.



In fact, the phase portraits of folded saddle, folded node, folded focus were firstly obtained by Poincaré in his third «Mémoire sur les courbes définies par les équations différentielles» (1885). Later on, such singularities were considered by several authors before A. A. Davydov. It is interesting to remark that folded saddle, folded node, folded focus occurred in a physical work (1971) devoted to studying the conversion of electromagnetic waves into plasma waves in an anisotropic plasma with two-dimensional inhomogeneity.[23]

In 1975 L. Dara formulated the conjecture about normal form (4.12) of equation (4.1) in a neighborhoods of folded saddles, folded nodes, folded foci, but he didn't prove this statement.[24] The proof was obtained by A. Davydov in 1985.

Later on, A. Davydov obtained normal forms (more complex that (4.12)) for well-folded singular points without the linearizability condition (with resonances) and even in much more degenerated case when one of the eigenvalues $\lambda_{1,2}$ is zero.

[23]The interested reader is referred to the original paper: A.D. Piliya, V.I. Fedorov. «Singularities of electromagnetic wave field in a cold plasma with two-dimensional inhomogeneity». – ZhETF, 60, N 1 (1971). The similar text can be found in the book: A.G. Litvak (ed.). «High-frequency plasma heating». – American Institute of Physics (1992), Chapter 6.

[24]More exactly, L. Dara suggested another normal form, which is equivalent to (4.12); see: Lak Dara. Singularités générique des équations différentielles multiformes. – Bol. Soc. Bras. Math., 1975, v. 6, N 2, pp. 95–128.

In this paper, L. Dara also suggested suggested simple normal forms for the pleated singular points, however this hypothesis is proved wrong.

5 Appendix.

5.1 Legendre transformation

In Lecture 2 we defined the Legendre transformation of planar curves. Recall this definition. Let γ be such a curve given by the formula

$$\gamma: \quad x = \varphi(\tau), \quad y = \psi(\tau), \tag{5.1}$$

and $\overline{\gamma}$ be the 1-graph of γ:

$$\overline{\gamma}: \quad x = \varphi(\tau), \quad y = \psi(\tau), \quad p = \frac{\psi'(\tau)}{\varphi'(\tau)}. \tag{5.2}$$

Let $\mathscr{L}$ be the diffeomorphism $(x, y, p) \to (X, Y, P)$ given by the formula

$$x = P, \quad p = X, \quad y + Y = xp = XP. \tag{5.3}$$

Then the diffeomorphism $\mathscr{L}$ turns the curve $\overline{\gamma}$ in the (x, y, p)-space to the curve $\overline{\gamma}^*$ in the (X, Y, P)-space. Consider the projection of the curve $\overline{\gamma}^*$ on the (X, Y)-plane along the P-axis:

$$\begin{array}{ccc} \overline{\gamma} \subset (x, y, p) & \xrightarrow{\mathscr{L}} & \overline{\gamma}^* \subset (X, Y, P) \\ \uparrow & & \downarrow \\ \gamma \subset (x, y) & \longrightarrow & \gamma^* \subset (X, Y) \end{array} \tag{5.4}$$

The curve γ^* is called «dual» to the curve γ or «Legendre transformation» of γ.

Consider the important particular case when the curve γ is a graph of a smooth function: $y = f(x)$, that is, in formula (5.1) we can put $\varphi(\tau) = \tau$ and $\psi(\tau) = f(\tau)$. Then formula (5.2) reads

$$\overline{\gamma}: \quad x = \tau, \quad y = f(\tau), \quad p = f'(\tau).$$

and using the above construction, we get the dual curve

$$\gamma^*: \quad X = f'(\tau), \quad Y = \tau f'(\tau) - f(\tau). \tag{5.5}$$

If the curve (5.5) can be written in the form $Y = f^*(X)$ with some function f^*, it is natural to say that the function f^* is the «Legendre transformation» of the function f. For example, if the function $f' : \mathbb{R} \to \mathbb{R}$ is surjective, and f'' is not vanishing on $\mathbb{R}$, then f^* is defined and smooth on $\mathbb{R}$. However if one of these conditions does not hold, the function f^* can be not defined or not regular at some points of $\mathbb{R}$. (Consider the example: $f(x) = x^n$ with $n \in \mathbb{N}$).

Let $U \subset \mathbb{R}$ be an open domain. Let $\mathfrak{C}(U)$ be the class of C^2-smooth functions $f(x) : U \to \mathbb{R}$ satisfying the following condition: $f''(x) > 0$ for all $x \in U$. This implies that the derivative map $f'(x) : U \to U^*$, where $U^* := f'(U)$, is surjective.

Then for any function $f \in \mathfrak{C}(U)$ the Legendre transformation $f^*(X) : U^* \to \mathbb{R}$ is defined by the following formula

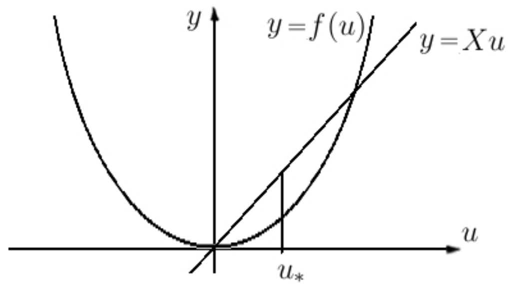
$$f^*(X) = \sup_{u \in U} (Xu - f(u)). \tag{5.6}$$

To prove this claim, consider the function $F(u) = Xu - f(u)$ of the variable $u \in U$ depending on the parameter $X \in U^*$. Since $F'(u) = X - f'(u)$ and the map $f'(x) : U \to U^*$ is surjective, for each value $X \in U^*$ the equation $F'(u) = 0 \Leftrightarrow f'(u) = X$ has a unique solution $u_* \in U$.

From the condition $f''(x) > 0$ it follows $F''(u) < 0$. Hence the fucntion $F(u)$ has a global maximum on U at the point u_*. Therefore

$$f^*(X) = \sup_{u \in U} (Xu - f(u)) = Xu_* - f(u_*), \quad \text{where} \quad f'(u_*) = X, \tag{5.7}$$

see the following picture. Comparing this formula with (5.5) and (5.6), we see that for any function $f(x) \in \mathfrak{C}(U)$ formulas (5.4) – (5.5) and (5.6) give the same function $f^*(X)$.



Exercise 1.
Prove that $f \in \mathfrak{C}(U) \Rightarrow f^* \in \mathfrak{C}(U^*)$ and $(f^*)^* = f$.
Exercise 2.
Prove the Young inequality: $f(x) + f^*(X) \geq xX$ for $\forall\, f \in \mathfrak{C}(U)$, $x \in U$, $X \in U^*$.
Exercise 3.
Calculate the Legendre transformation of the function

$$f(x) = \sqrt{1 + x^2}, \quad x \in \mathbb{R},$$

using both definitions: (5.4) – (5.5) and (5.6).

Notice that for functions $f \notin \mathfrak{C}(U)$ the definitions of f^* by formulas (5.4) – (5.5) and (5.6) can be non-equivalent.
Exercise 4.
Prove that the Legendre transformations of the functions $f(x)$ and $g(x) = -f(x)$ defined by formulas (5.4) – (5.5) are connected by the following relation: $g^*(X) = -f^*(-X)$. However this is not true in the case of definition (5.6). For example, consider the function $f(x) = -x^2/2$ on $U = \mathbb{R}$. Clearly, for any X the right hand side of (5.6) is equal to $+\infty$.
Exercise 5.
Define the similar Legendre transformation in the multidimensional case, i.e., for functions $f(x_1, \ldots, x_n) : \mathbb{R}^n \to \mathbb{R}$.
Hint: In formula (5.3) put $x = (x_1, \ldots, x_n)$, $p = (p_1, \ldots, p_n)$ and $xp = x_1p_1 + \cdots + x_np_n$, and similarly for X and P. In formulas (5.6) and (5.7) also put $Xu = X_1u_1 + \cdots + X_nu_n$.

5.2 Lagrangian and Hamiltonian equations

Consider the space J^1 with coordinates (x, y, y') and a smooth function $L(x, y, y')$ of three independent real variables.[25] Suppose that for each fixed pair (x, y) the function $L(x, y, y')$ as a function of the variable y' belongs to $\mathfrak{C}(\mathbb{R})$.

[25]Later on we put $y' = dy/dx$, that's why we use this special notation for the third variable.

Then the Legendre transformation of the function L with respect to y' (here the variables x, y play the role of parameters) is defined and belongs to $\mathfrak{C}(\mathbb{R})$. According to formulas (5.6) and (5.7), the Legendre transformation of L with respect to y' is defined by the relations

$$H(x,y,p) = py' - L(x,y,y'), \quad \text{where} \quad y' = y'(p) \; : \; p = L_{y'}(x,y,y'), \tag{5.8}$$

here we used H instead of L^*, p instead of X, y' instead of u_*.

Formula (5.8) suggests doing the following change of variables in the space J^1:

$$(x,y,y') \to (x,y,p), \quad \text{where} \quad p = L_{y'}(x,y,y'). \tag{5.9}$$

By the above supposition, (5.9) defines a smooth diffeomorphism of J^1, and the variables (x,y,p) are global coordinates in J^1.

Consider the Euler–Lagrange equation with the Lagrangian $L(x,y,y')$, that is,

$$\frac{d}{dx}L_{y'} - L_y = 0, \quad \text{where} \quad y' = \frac{dy}{dx}. \tag{5.10}$$

The differential of the function $H(x,y,p)$ from (5.8) is

$$dH = \boxed{pdy'} + y'dp - L_x dx - L_y dy - \boxed{L_{y'}dy'} = y'dp - L_x dx - L_y dy,$$

since $p = L_{y'}$. Hence we get: $H_x = -L_x$, $H_y = -L_y$, $H_p = y'$. Finally, using equation (5.10), the equalities $p = L_{y'}$ and $y' = dy/dx$, we obtain

$$H_y = -L_y = -\frac{d}{dx}L_{y'} = -\frac{dp}{dx} \quad \text{and} \quad H_p = y' = \frac{dy}{dx}.$$

Thus the Euler–Lagrange equation (5.10) in the variables (x,y,p) gives the Hamiltonian system

$$\frac{dy}{dx} = H_p, \quad \frac{dp}{dx} = -H_y, \tag{5.11}$$

where the function $H(x,y,p)$ defined by (5.8) is the Hamiltonian corresponding to the Lagrangian $L(x,y,y')$, and the correspondence being the Legendre transformation with respect to the variable y' («velocity»). The variable p defined by (5.9), is called «impulse», and (x,y,p) are called «canonical variables».

Exercise 6.

Prove the equivalence of the Lagrangian and Hamiltonian systems in the multidimensional case: $y = (y_1,\dots,y_n)$, $y' = (y'_1,\dots,y'_n)$, $p = (p_1,\dots,p_n)$, where $p_i = L_{y'_i}$. Notice that in the multidimensional case the change of variables $y' \to p$ is a local (not necessary global) diffeomorphism of the space.

Exercise 7.

Prove that if the Lagrangian L does not depend on the variable x, i.e., $L = L(y,y')$, then the function $I(y,y') = y'L_{y'} - L$ is a first integral of the Euler–Lagrange equation (5.10), which is called the «energy integral», and the Hamiltonian $H(y,p)$ is a first integral of the Hamiltonian equation (5.11). This claim is valid in the multidimensional case, if we put $I(y,y') = y'L_{y'} - L$, where $y'L_{y'} := y'_1 L_{y'_1} + \dots + y'_n L_{y'_n}$.

Exercise 8.

Prove that if the Lagrangian L does not depend on the variable y_i for some $i \in \{1,\dots,n\}$, then the function $I(x,y,y') = L_{y'_i}$ is a first integral of the Euler–Lagrange equation (5.10), which is called the «impulse integral». What is the first integral of the corresponding Hamiltonian equation?

Example 1.

Consider the motion of a material point in n-dimensional space with the Cartesian coordinates $y = (y_1, \dots, y_n)$. Let the variable x mean the time, and $y' = dy/dx$ the velocity. The kinetic energy of this mechanical system is $T(y') = \frac{m}{2}\sum(y_i')^2$, where m is the mass of the point. Assume that the potential energy of the point is a smooth function of the coordinates: $U(y)$, and there is no friction.

According to the «minimum action principle», the motion of the point is realized along extremals of the action functional, i.e., it is described by the Euler–Lagrange equation (5.10) with the Lagrangian

$$L(y, y') = T(y') - U(y) = \frac{m}{2}\sum_{i=1}^{n}(y_i')^2 - U(y).$$

To get the corresponding Hamiltonian system, introduce the variables $p_i = L_{y_i'} = my_i'$. Hence in this case the change of variables $y' \to p$ is a scaling. According to formula (5.8) and Exercise 5, the Hamiltonian is defined by the relation

$$H(y,p) = \sum_{i=1}^{n} p_i y_i' - L(y, y'), \quad \text{where} \quad p_i = my_i' \Leftrightarrow y_i' = \frac{p_i}{m}.$$

Therefore

$$H(y,p) = \sum_{i=1}^{n} p_i \frac{p_i}{m} - L\Big(y, \frac{p}{m}\Big) = \sum_{i=1}^{n}\frac{p_i^2}{m} - \frac{m}{2}\sum_{i=1}^{n}\frac{p_i^2}{m^2} + U(y) = \frac{1}{2m}\sum_{i=1}^{n} p_i^2 + U(y), \quad (5.12)$$

and the corresponding Hamiltonian system reads

$$\frac{dy_i}{dx} = H_{p_i} = \frac{p_i}{m}, \qquad \frac{dp_i}{dx} = -H_{y_i} = -U_{y_i}(y). \quad (5.13)$$

□

Exercise 9.

What is the mechanical sense of the Hamiltonian?

Answer: the Hamiltonian is the total energy.

Indeed, since $y_i' = \frac{p_i}{m}$, the kinetic energy is $T = \frac{m}{2}\sum(y_i')^2 = \frac{1}{2m}\sum p_i^2$, and from formula (5.12) we obtain $H = T + U(y)$. Since $H(y,p)$ is a first integral of equation (5.13), the function H has a constant value on each solution. The mechanical sense of this fact is energy conservation law: the total energy of the mechanical system is constant.

Exercise 10.

Write the Hamiltonian equation for the mechanical system which consists of N material points with masses m_i, $i = 1, \dots, N$, in 3-dimensional space assuming that the potential energy of the mechanical system is a smooth function of the coordinates of the points, and there is no friction.

Hint: The total number of coordinates is $n = 3N$. Put $y = (y_1, y_2, y_3; \dots; y_{n-2}, y_{n-1}, y_n)$, where each triple $(y_{3i-2}, y_{3i-1}, y_{3i})$ is the Cartesian coordinates of the i-th point. Then the Lagrangian of this mechanical system is $L(y, y') = T(y') - U(y)$, where the kinetic energy is

$$T(y') = \frac{1}{2}\sum_{i=1}^{N} m_i((y_{3i-2}')^2 + (y_{3i-1}')^2 + (y_{3i}')^2).$$

Example 2.

The equation of geodesic lines on a 2-dimensional smooth manifold with coordinates x, y and the metric

$$ds^2 = a\,dy^2 + 2b\,dxdy + c\,dx^2 \tag{5.14}$$

is the Euler–Lagrange equation (5.10) with the Lagrangian

$$L(x, y, y') = \frac{ds}{dx} = \sqrt{a(y')^2 + 2by' + c}. \tag{5.15}$$

Here the coefficients a, b, c are smooth functions of the coordinates x, y.

For instance, in the case of the standard Euclidean metric $ds^2 = dy^2 + dx^2$ the Lagrangian $L = \sqrt{(y')^2 + 1}$ does not depend on the variables x and y, and the corresponding equation (5.10) has both energy and impulse integrals. Let's use the impulse integral:

$$L_{y'} = \frac{y'}{\sqrt{(y')^2 + 1}} = \text{const} \quad \Leftrightarrow \quad y' = \text{const}.$$

This yields that the geodesics are straight lines $y = \alpha x + \beta$, where α, β are arbitrary real coefficients.[26] Let's write the corresponding Hamiltonian equation. The canonical variable is $p = L_{y'} = y'/\sqrt{(y')^2 + 1}$. Notice that

$$-1 < p = \frac{y'}{\sqrt{(y')^2 + 1}} < 1 \quad \text{and} \quad L_{y'y'} = \frac{1}{((y')^2 + 1)^{\frac{3}{2}}} > 0 \qquad \forall\, y' \in \mathbb{R}.$$

Thus the function $L(y')$ belongs to the class $\mathfrak{C}(U)$ with $U = \mathbb{R}$, and $U^* = (-1, +1)$. Hence the Legendre transformation $H(p) : U^* \to \mathbb{R}$ is defined by the relation

$$H(p) = py' - L(y'), \quad \text{where} \quad p = \frac{y'}{\sqrt{(y')^2 + 1}} \ \Leftrightarrow\ y' = \frac{p}{\sqrt{1 - p^2}},$$

i.e., $H(p) = -\sqrt{1 - p^2}$, where $p \in U^*$. Substituting this expression into (5.11), we get the corresponding Hamiltonian equation. □

Exercise 11.

Find the geodesic lines in the metric

$$ds^2 = \frac{dy^2 + dx^2}{y^2} \tag{5.16}$$

on the upper half-plane $y > 0$, using the energy integral. Write the Hamiltonian equation for the geodesics in this metric. □

Remark. This metric is called the «Klein metric», it is used in the Klein–Poincaré model of the Lobachevsky plane ($\mathbb{L}^2$). In this model the «lines» on $\mathbb{L}^2$ are geodesics in the metric (5.16). By $\mathrm{Iso}(\mathbb{L}^2)$ denote the group of the isometries of $\mathbb{L}^2$. This group can be discribed as follows.

Each point (x, y) of the upper half-plane can be interpreted as the complex number $z = x + iy$. Consider the group of fractional linear transformations (FLTs):

$$z \to \frac{az + b}{cz + d}, \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}). \tag{5.17}$$

[26] The attentive reader must see that here we lost the line $x = 0$, because we used the derivative $y' = dy/dx$. Generally speaking, we must to consider also the Lagrangian obtained by interchanging x and y.

Recall that the group $\mathrm{SL}(2,\mathbb{R})$ consists of matrices $A \in \mathrm{GL}(2,\mathbb{R})$ such that $|A| = 1$.

It is not hard to prove that FLTs (5.17) define real maps $(x, y) \to (x, y)$ which preserve the upper half-plane and the Klein metric, that is, they define isometries of $\mathbb{L}^2$. More exactly, FLTs (5.17) define a subgroup of $\mathrm{Iso}(\mathbb{L}^2)$. The whole group $\mathrm{Iso}(\mathbb{L}^2)$ consists of FLTs (5.17) and their compositions with the transformation $z \to -\overline{z}$, i.e., $(x, y) \to (-x, y)$. $\square$

Hamiltonian equations (as well as Lagrangian equations) are not «generic» second-order ODEs, they posesse a lot of special properties.

For instance, the divergence of Hamiltonian equations is identically zero. Indeed, let $\vec{V}_H$ be the vector field defined in the (y, p)-space by Hamiltonian equation (5.11). Then

$$\operatorname{div} \vec{V}_H = \frac{\partial (H_p)}{\partial y} + \frac{\partial (-H_y)}{\partial p} = H_{py} - H_{py} = 0.$$

Clearly, this statement is valid in the multidimensional case as well.

The vector fields $\vec{V}$ satisfying the condition $\operatorname{div} \vec{V} = 0$ are called «divergence free». The phase flow of any divergence free vector field $\vec{V}$ preserves the volume in the phase space. In particular, the phase flow of the Hamiltonian vector field $\vec{V}_H$ preserves the volume in the (y, p)-space. Using the correspondence between Lagrangian and Hamiltonian equations, one can establish the same property for Euler–Lagrange equations.

In the Examples 1 and 2 we dealt with smooth Lagrangians satisfying the condition $L_{y'y'} \neq 0$. In the multidimensional case the analogous condition is

$$|L_{y'y'}| := \left| \frac{\partial^2 L}{\partial y'_i \, \partial y'_j} \right| \neq 0, \tag{5.18}$$

that is, the Hessian matrix of the function L with respect to $y' = (y'_1, \ldots, y'_n)$ is non-zero. If condition (5.18) holds true, the Legendre transformation of L by y' is defined (at least, locally), and the passage to the canonical variables $y' \to p = L_{y'}$ is a local diffeomorphism.

In most «reasonable problems» Lagrangians are smooth and satisfy condition (5.18). For instance, in classical mechanics most of Lagrangians have the form $L = T - U$, where the kinetic energy T is a non-degenerated quadratic form of $y' = (y'_1, \ldots, y'_n)$ and the potential energy U does not depend on y'. However there are some exceptions.

The Lagrangian $L(x, y, y')$ is called «singular» at the point (x, y, y'), if at this point condition (5.18) does not hold: $|L_{y'y'}| = 0$ or $|L_{y'y'}|$ is the indeterminacy $0/0$. The simplest example of singular Lagrangians can be obtained if we consider geodesic lines in non-regular metrics (here the Lagrangian is the square root of the metric).

For example, let S be a surface in a Euclidean space. Then the geodesic lines on S are extremals of the length functional, i.e., solutions of the Euler–Lagrange equation, where L is the square root of the metric ds^2 induced on S by the metric of the ambient Euclidean space. The Lagrangian $L = \sqrt{ds^2}$ is singular at the points where the surface S in not regular (cone, cuspidal edge, swallow tail, etc.). Another example: let S be a regular surface in a pseudo-Euclidean space E. Then the metric ds^2 induced on S by the metric of the ambient space E, is indefinite, and ds^2 is degenerated at the points where S in tangent to the isotropic cone of E. At such points the Lagrangian $L = \sqrt{ds^2}$ is singular.[27]

[27] See, for example:

A.O. Remizov. Geodesics on 2-surfaces with pseudo-Riemannian metric: singularities of changes of signature. Mat. Sb., 200:3 (2009), pp. 75–94.

A.O. Remizov. Singularities of a geodesic flow on surfaces with a cuspidal edge. Proceed. Steklov Institute of Math. 268 (2010), pp. 258–267.

Singular Lagrangians also occur in theoretical physics in describing relativistic effects. For the first time they were studied in 1950 by Dirac, Anderson, and Bergman, and later on, by many other authors.[28] In particular, an important problem is the Hamiltonian formalism for the Euler–Lagrange equation in a neighborhood of the singular point, since the standard passage to the canonical coordinates $y' \to p = L_{y'}$ at such points is not a local diffeomorphism.

The reader may ask a natural question: why do we need to pass from the Euler–Lagrange equation to the Hamiltonian one? In the examples 1 and 2 considered above, the Hamiltonian equation does not give us any advantages, and the passage to the canonical coordinates looks like a waste of time. However in many cases the Hamiltonian approch gives a lot of advantages.[29]

5.3 Hamiltonian equations and integrating factor

Consider the Hamiltonian vector field

$$\frac{dx}{dt} = H_y(x,y), \quad \frac{dy}{dt} = -H_x(x,y), \tag{5.19}$$

where $x, y \in \mathbb{R}$ are the independent (phase) variables, t plays the role of time, and the Hamiltonian $H(x, y)$ is a smooth function. (Formula (5.19) concides with the previous formula (5.11) up to a renaming the variables. The only difference is that now we deal with the case when the Hamiltonian H does not depend on the time.)

The vector field (5.19) is divergence free. The inverse statement is also true:

Exercise 12.

Prove that any smooth divergence free vector field $\vec{V}$:

$$\frac{dx}{dt} = v(x,y), \quad \frac{dy}{dt} = -w(x,y) \tag{5.20}$$

in a simply connected open domain D of the (x, y)-plane is a Hamiltonian one (recall that an open domain D is called «simply connected» if any continuous loop in D is homotopic to a point).

Solution: To prove the statement it is necessary and sufficient to show that there exists a smooth function $H(x, y)$ such that $v = H_y$ and $w = H_x$. It is well-known[30] that in a simply connected open domain such a function H exists $\Leftrightarrow v_x = w_y \Leftrightarrow \operatorname{div} \vec{V} = v_x - w_y = 0$. $\square$

Give the following definition: the vector fields $\vec{V}$ and $\vec{V}_1$ are «collinear» if $\vec{V}_1 = \varphi \vec{V}$, where φ is a smooth non-vanishing scalar function of the same variables as the fields. This means that $\vec{V}$ and $\vec{V}_1$ have the same integral curves, but the velocities of motion on these curves are different. Using the notion of a direction field (Lecture 2), we can say that vector fields $\vec{V}$ and $\vec{V}_1$ are collinear iff they define the same direction field.

Exercise 13.

Prove that in a neighborhood of any non-singular point any smooth vector field is locally collinear to a divergence free vector field (hence, to a Hamiltonian one).

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[28] See, for example:
[1] P.A.M. Dirac. Lectures on Quantum Mechanics. – Yeshiva University, New York (1964).
[2] J.F. Cariñena. Theory of singular Lagrangians. – Fortschr. Phys., 38 (1990), No. 9, pp. 641–679.

[29] See the book [1] cited above. Hamiltonian format is also native for Pontryagin's maximum principle.

[30] V.A. Zorich «Mathematical Analysis», vol. II, chapter «Elements of Vector Analysis and Field Theory». In this book the reader can also find the explaination why the condition that D is simply connected is important.

Solution: Let $\vec{V}$ be a smooth vector field written in the form (5.20) in a neighborhood of 0 such that $\vec{V}(0) \neq 0$. We have to establish the local existence of a smooth function $\varphi(x, y) \neq 0$ such that the vector field $\varphi\vec{V}$ is divergence free: $\operatorname{div}(\varphi\vec{V}) = 0$. This gives the first-order PDE

$$v\varphi_x - w\varphi_y + \varphi(v_x - w_y) = 0 \tag{5.21}$$

with known functions $v(x, y)$, $w(x, y)$ and the unknown function $\varphi(x, y)$.

The condition $|v(0)| + |w(0)| \neq 0$ provides the local existence (and uniqueness) of the solution of PDE (5.21) with any initial condition $\varphi\big|_\Gamma = \varphi_\Gamma$, where Γ is a regular curve on the (x, y)-plane passing through the origin transversal to $\vec{V}$ (one can put Γ = one of the coordinate axis: $x = 0$ or $y = 0$), and φ_Γ is a smooth function on Γ. Choosing the initial function φ_Γ such that $\varphi_\Gamma(0) \neq 0$, we get the required function φ. □

In other words, this means that any smooth vector field (5.20) can be transformed into a Hamiltonian one in a neighborhood of a non-singular point, using appropriate regular change of the time t.

The function φ found above is called an «integrating factor» or «Jacobi multiplier» of the vector field (5.20) and the corresponding differential equation $w\,dx + v\,dy = 0$. The facts proved in Exercises 12, 13 can be used for the exact solution of this equation. Indeed, after multiplication the both sides of the equation $w\,dx + v\,dy = 0$ by the function $\varphi(x, y) \neq 0$ satisfying equation (5.21), we get the equivalent equation

$$\varphi w\,dx + \varphi v\,dy = 0 \;\Leftrightarrow\; \exists H \,:\, H_x\,dx + H_y\,dy = 0 \;\Leftrightarrow\; d\,H(x, y) = 0,$$

which has the general solution $H(x, y) = \text{const}$.

At first sight, this trick does not give any advantages: we just reduce the solution of equation $w\,dx + v\,dy = 0$ to the solution of another differential equation (5.21). However it is sufficient to find only one partial solution of (5.21). There exist a lot of concrete types of ODEs with known recipes for integrating factors. «The masters of integrating differential equations (Jacobi, for example) attained great success in the solution of specific applied problems using this technique.» (V.I. Arnold). For example:

Exercise 14.

Prove that the integrating factor of the equation $w\,dx + v\,dy = 0$ satisfies the PDE

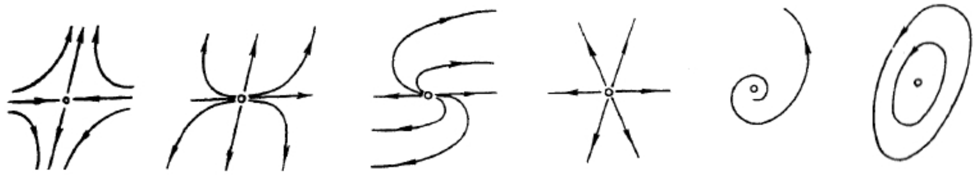
$$v\,\frac{\partial \ln\varphi}{\partial x} - w\,\frac{\partial \ln\varphi}{\partial y} = w_y - v_x,$$

and if the function $(w_y - v_x)/v = a$ does not depend on y, the integrating factor is $\varphi = \exp\Big(\int a(x)\,dx\Big)$; if the function $(w_y - v_x)/w = b$ does not depend on x, the integrating factor is $\varphi = \exp\Big(-\int b(y)\,dy\Big)$.

Remark that we used the condition $\vec{V} \neq 0$ for prooving the existence of an integrating factor. Does an integrating factor exist in neighborhoods of singular points of vector fields? Further we will always assume that 0 be a non-degenerated singular point of vector field $\vec{V}$ given by (5.20). The term «non-degenerated» means that both eigenvalues $\lambda_{1,2}$ of the linear part of $\vec{V}$ at 0 are non-zero, that is, the Jacobi matrix

$$A = \frac{\partial\vec{V}}{\partial(x, y)}(0) = \frac{\partial(v, -w)}{\partial(x, y)}(0)$$

is non-degenerated (= invertible). Then there are 4 possible phase portraits of $\vec{V}$ in a neighborhood of 0:



Stress that if $v(x, y)$ and $w(x, y)$ are linear functions, then the case of center $\Leftrightarrow \lambda_{1,2} = \pm i\omega$, where i is the imaginary unit and $\omega > 0$. However if $v(x, y)$ and $w(x, y)$ are non-linear, there is only implication: center $\Rightarrow \lambda_{1,2} = \pm i\omega$, $\omega > 0$. In this case the difference between center (the integral are loops) and focus (the integral are spirals) is determined by the non-linear terms.

Exercise 15.

Prove that in the case of node (Jordan node and dicritical node as well) and in the case of focus the vector field $\vec{V}$ does not have an integrating factor in a neighborhood of 0.

Solution: If a function $I(x, y)$ is a first integral of the vector field $\vec{V}$, then I is a first integral of any collinear vector field $\varphi\vec{V}$. Suppose that the field $\varphi\vec{V}$ is divergence free, then it is Hamiltonian vector field with some Hamiltonian function $H(x, y)$; hence both fields $\varphi\vec{V}$ and $\vec{V}$ possess the continuous first integral $I = H$.

Suppose that in the case of node or focus there exists a continuous function $I(x, y)$ such that $I(\gamma(t)) \equiv \text{const}$ (the constant depends on $\gamma(\cdot)$) for each integral curve γ. Since all integral curves tend to 0 as $t \to \pm\infty$ (the sign «$\pm$» = «$+$»/«$-$» corresponds to stable/unstable equilibrium point 0), we have $I(\gamma(t)) \to I(0)$ as $t \to \pm\infty \Rightarrow I(\gamma(t)) \equiv I(0)$ for all integral curves $\gamma(\cdot)$. Since for each point (x, y) there exists the integral curve γ passing through this point, we get $I(x, y) \equiv I(0)$, i.e., I is a constant function. Thus we proved that $\vec{V}$ does not have a continuous first integral in a neighborhood of $0 \Rightarrow \vec{V}$ does not have an integrating factor in a neighborhood of 0. □

Remark. However discountinuous first integrals exist. For example, the dicritical node $dy/dx = y$, $dp/dx = p$ has the first integral $I = y/p$ discountinuous at the origin.

Thus we proved that node and focus do not have integrating factor. What about saddle and center? First of all, we can try to apply the reasoning from Exercise 15 to saddle and center. But unlike node and focus, it does not give the indentity $I(x, y) \equiv I(0)$. In the case of saddle we only conclude that $I(\gamma(t)) \equiv I(0)$ for two separatrices, but not for others integral curves.

Exercise 16.

In the case of saddle prove the following necessary condition: if the field $\vec{V}$ has an integrating factor in a neighborhood of 0 then $\lambda_1 + \lambda_2 = 0$ (recall that $\lambda_{1,2}$ are the eigenvalues of the linear part of $\vec{V}$ at 0, i.e., of the matrix A defined above).

Solution: From the identity

$$\operatorname{div}(\varphi\vec{V}) = \varphi \operatorname{div}\vec{V} + \langle \vec{V}, \nabla\varphi \rangle$$

it follows that at the singular point ($\vec{V}(0) = 0$) we have the equality $\operatorname{div}(\varphi\vec{V})(0) = \varphi \operatorname{div} \vec{V}(0)$. Hence $\operatorname{div}(\varphi\vec{V})(0) = 0 \Leftrightarrow \operatorname{div} \vec{V}(0) = 0$. Thus if the field $\vec{V}$ is collinear to a divergence free field $\varphi\vec{V}$, then $\operatorname{div} \vec{V}(0) = 0$. Taking into account the relation $\operatorname{div} \vec{V}(0) = \operatorname{tr} A(0)$, we get the equality $\lambda_1 + \lambda_2 = 0$. □

Remark. The attentive reader may say the last proof can be also used for proving the previous Exercise 15. Indeed, among the above types of non-generated singular points (node, saddle, focus, center) only saddle and center satisfy the necessary condition $\lambda_1 + \lambda_2 = 0$. This proofs the statement from Exercise 15.

However I prefer to give both proofs simultaneously, because they are based on absolutely different ideas. The first one uses the topological (more «rough») properties of the phase portrait, but the second proof uses the eigenvalues of the linear part, which are smooth, but not topological invariants of vector fields.[31] Moreover, the first proof can be also used for some types of degenerated singular points such that $\lambda_{1,2} = 0$. □

Thus we proved that a vector field may have an integrating factor in a neighborhood of a non-degenerated singular point 0 only if 0 is a center or saddle with the eigenvalues $\lambda_{1,2} = \pm\alpha$. This condition is necessary, but is it sufficient? In fact, no:

Exercise 17.

Prove that the vector field $\vec{V}$ has an integrating factor in a neighborhood of a non-degenerated singular point 0 if and only if there exist smooth local coordinates such that $\vec{V}$ is collinear to the linear field

$$\frac{dx}{dt} = \varepsilon y, \quad \frac{dy}{dt} = x, \quad \text{where} \quad \varepsilon = \pm 1. \tag{5.22}$$

<u>Solution:</u> Consider the vector field $\vec{V}$ given by formula (5.20).

[⇒] Suppose that φ is the integrating factor of $\vec{V}$, that is, the field $\varphi\vec{V}$ is Hamiltonian $\Rightarrow \varphi v = H_y$ and $\varphi w = H_x$ for a smooth function $H(x, y)$. By the Morse Lemma, we can choose smooth local coordinates such that $H = (-x^2 + \varepsilon y^2)/2 \Rightarrow$ the field $\varphi\vec{V}$ has the form (5.22). Thus in these («canonical») coordinates the vector field $\vec{V}$ is collinear to (5.22).

[⇐] Notice that any diffeomorphism (smooth change of variables) turns any non-degenerated singular point of $\vec{V}$ into a non-degenerated singular point (moreover, it preserves the eigenvalues of the linear part of $\vec{V}$). Suppose that $\vec{V}$ is collinear to (5.22) in some coordinates (x, y), i.e., $\vec{V}$ is given by the formula

$$\frac{dx}{dt} = \varphi\, \varepsilon y, \quad \frac{dy}{dt} = \varphi\, x, \quad \text{where} \quad \varepsilon = \pm 1, \tag{5.23}$$

with a smooth non-vanishing function φ.

The vector field (5.23) has the first integral $(-x^2 + \varepsilon y^2)/2 \Rightarrow$ in any another coordinates the field $\vec{V}$ has the first integral $I(x, y)$ obtained from the previous one by the corresponding change of variables. Clearly, the function $I(x, y)$ also has a non-degenerated critical point at 0. Thus we proved that in the initial coordinates the vector field (5.20) has the first integral $I(x, y) \Rightarrow vI_x - wI_y = 0 \Rightarrow v = \psi I_y$ and $w = \psi I_x$ for some function $\psi(x, y)$, that is,

$$\frac{dx}{dt} = \psi\, I_y, \quad \frac{dy}{dt} = -\psi\, I_x.$$

[31] Homeomorphisms do not preserve the eigenvalues. In particular, node (including Jordan and dicritical ones) and focus are topologically equivalent, see e.g.:

- V.I. Arnold, Yu.S. Il'yashenko. Ordinary differential equations. – Dynamical systems I, Encyclopaedia Math. Sci., vol. 1.
- Ph. Hartman. Ordinary Differential Equations.

From the condition that 0 is a non-degenerated singular point, it follows that $\psi(0) \neq 0$. Thus the vector field (5.20) is collinear to the Hamiltonian vector field with $H = I$. $\square$

Example 3. Consider the vector field $\vec{V}$ given by formula (5.20) with the functions $v(x, y) = x$ and $w(x, y) = y$. The equation $dy/dx = -x/y$ has the first integral $I = xy$, which can be brought to the required form $(-x^2 + y^2)/2$ using a linear change of the variables.

The condition that 0 is a non-degenerated singular point provides the condition $\varphi \neq 0$ for the integrating factor. Indeed, consider the following example.

Example 4. The vector field

$$\frac{dx}{dt} = x^2 + y^2, \quad \frac{dy}{dt} = -(x^2 + y^2) \tag{5.24}$$

is obtained from the Hamiltonian vector field

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = -1$$

(the Hamiltonian function $H = x + y$) after multiplication by the function $x^2 + y^2$, which vanishes at 0. It is not hard to see that field (5.24) does not have an integrating factor, i.e., it cannot be brought to a Hamiltonian vector field by means of multiplication by a smooth non-vanishing function $\varphi(x, y)$. Indeed, such smooth function φ satisfies the equation

$$\frac{\partial}{\partial x}\Big((x^2 + y^2)\varphi\Big) - \frac{\partial}{\partial y}\Big((x^2 + y^2)\varphi\Big) = (x^2 + y^2)(\varphi_x - \varphi_y) + 2(x - y)\varphi = 0,$$

which implies that $\varphi(0) = 0$.

Remark. The statements of both Exercises 15 and 16 can be derived from Exercise 17. However it seems reasonable to give them separately, because the proofs in Exercises 15, 16 can be used in more general situation than the last one. In fact, the proof of Exercise 17 is based on the Morse lemma, which is applicable only for non-degenerated singular points of $\vec{V}$. On the other hand, it is not hard to see that the statements in Exercises 15, 16 can be applied to degenerated singular points.

Finally, consider the multidimensional Hamiltonian vector field

$$\frac{dx_i}{dt} = H_{y_i}(x, y), \quad \frac{dy_i}{dt} = -H_{x_i}(x, y), \quad i = 1, \ldots, n, \tag{5.25}$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are the independent (phase) variables, t plays the role of time, and the Hamiltonian $H(x, y)$ is a smooth function. The vector field (5.25) is divergence free.[32]

However the inverse implication: «divergence free $\Rightarrow$ Hamiltonian field» in the case $n > 1$ is false (give an example!).

Exercise 18. Let $\vec{V}$ be the Hamiltonian vector field given by formula (5.25) and S be a smooth invariant manifold of $\vec{V}$. Is the restriction $\vec{V}\big|_S$ a Hamiltonian vector field? Is the restriction $\vec{V}\big|_S$ a divergence free vector field?

[32]From the correspondence between Lagrangian and Hamiltonian equations it follows that vector fields generated by Euler–Lagrange equations, are also divergence free. Remark that this statement for Hamiltonian equations of any dimension is trivial, but for the multidimensional Euler–Lagrange equations it requires some calculations.

Exercise 19.

Prove the famous Poincaré Recurrence Theorem:

Let g^t be the phase flow of a Hamiltonian vector field $\vec{V}$ which maps a bounded region D of the phase space onto itself: $g^t(D) = D$ (for example, the field $\vec{V}$ is defined on a smooth compact $2n$-dimensional manifold D). Then in any neighborhood U of any point $(x, y) \in D$ there is a point $(x', y') \in U$ which returns to U, that is, $\exists n \in \mathbb{N}$: $g^n(x', y') \in U$.

Hint: $\vec{V}$ is Hamiltonian $\Rightarrow$ $\vec{V}$ is divergence free $\Rightarrow$ $\vec{V}$ preserves the volume of the phase space $\Rightarrow$ the series

$$\sum_{n=0}^{\infty} \mathrm{Vol}\,(g^n(U)) = \sum_{n=0}^{\infty} \mathrm{Vol}\,(U)$$

diverges $\Rightarrow \exists k, l \in \mathbb{N}$: $g^k(U) \cap g^k(U) \neq \emptyset$.

The reader interested in further studying of Hamiltonian systems, is referred to the book:

- V.I. Arnold. Mathematical Methods of Classical Mechanics.