Navier–Stokes and Euler equations: Cauchy problem and controllability

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Abstract

The course is devoted to studying the Navier–Stokes and Euler systems in a bounded domain. We begin with the investigation of the initial-boundary value problems for both equations. A complete proof of well-posedness is given in the 2D case and some results are announced in the 3D case. We next turn to the problem of controllability. To avoid technical difficulties, consideration is given to the 1D Burgers equation. We also discuss the situation for Navier–Stokes and Euler equations and formulate some open questions.

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1 Navier–Stokes Equations

Let us consider the Navier–Stokes system describing the motion of an incompressible fluid:

$$\partial_t u + \langle u, \nabla \rangle u - \Delta u + \nabla p = f(t, x), \quad \operatorname{div} u = 0, \tag{1.1}$$

$$u\big|_{\partial D} = 0. \tag{1.2}$$

Here $D \subset \mathbb{R}^d$ is a bounded domain with smooth boundary, $u = (u_1, \dots, u_d)$ and p are unknown velocity field and pressure, f is a given external force, and

$$\langle u, \nabla \rangle = \sum_{j=1}^{d} u_j(t, x) \frac{\partial}{\partial x_j}.$$

Our aim is to investigate the well-posedness of the Cauchy problem for (1.1), (1.2). This question was first studied by Leray [Ler34], whose results were developed later by many others; see the books [Lad63, Lio69, Tem79, CF88, Soh01] and the references therein.

An essential ingredient of the modern theory of the Navier–Stokes system is a detailed description of the functional spaces. They are introduced and studied in the first subsection. We next show that the problem in question can be reduced to a nonlocal evolution equation in a Hilbert space. The latter is studied in detail for the 2D case. We conclude this section by formulating two results concerning the 3D system.

1.1 Functional spaces and Leray projection

Let $D \subset \mathbb{R}^d$ be a connected bounded domain with smooth boundary ∂D and let $L^2(D, \mathbb{R}^d)$ be the space of vector functions $u = (u_1, \dots, u_d)$ on D whose components are square-integrable. We denote by $(\cdot, \cdot)$ the natural scalar product in $L^2(D, \mathbb{R}^d)$ and by $\|\cdot\|$ the corresponding norm. If $u \in L^2(D, \mathbb{R}^d)$ is such that

$$\operatorname{div} u = \sum_{j=1}^{d} \frac{\partial u}{\partial x_j} = 0 \quad \text{in } D, \tag{1.3}$$

where the equality is understood in the sense of distributions, then we say that u is *divergence free*.

Definition 1.1. We shall say that a divergence-free vector field $u \in L^2(D, \mathbb{R}^d)$ has *zero normal component on ∂D* if [1]

$$\int_D \langle u, \nabla \varphi \rangle \, dx = 0 \quad \text{for any } \varphi \in H^1(D). \tag{1.4}$$

In this case, we shall write $\langle u, \boldsymbol{n} \rangle\big|_{\partial D} = 0$, where $\boldsymbol{n}$ stands for the outward unit normal to ∂D.

[1] Note that if $u \in C^1(\overline{D}, \mathbb{R}^d)$ is divergence free and satisfies (1.4), then the scalar product of u with the normal vector to ∂D vanishes.

For any integer $s \geq 0$, let $H^s(D)$ be the Sobolev space of order s, let $H_0^1(D)$ be the space of functions in $H^1(D)$ vanishing on ∂D, and let $\dot{H}^s(D)$ be the space of functions $u \in H^s(D)$ with zero mean value:

$$\langle u \rangle := \frac{1}{\mathrm{vol}(D)} \int_D u(x)\,dx = 0, \tag{1.5}$$

where $\mathrm{vol}(D)$ stands for the volume of D. Similar spaces of $\mathbb{R}^d$-valued functions will be denoted by $H^s(D,\mathbb{R}^d)$, etc.

Let us endow the space $H_0^1 = H_0^1(D,\mathbb{R}^d)$ with scalar product

$$(\nabla u, \nabla v) = \sum_{i,j=1}^{d} \int_D \partial_i u_j \partial_i v_j dx, \quad u, v \in H_0^1(D,\mathbb{R}^d).$$

The norm corresponding to this scalar product is given by $\|\nabla u\|$. Denote by $H^{-1} = H^{-1}(D,\mathbb{R}^d)$ the space of $\mathbb{R}^d$-valued distributions on D such that

$$|(u,\varphi)| \leq C\,\|\nabla \varphi\| \quad \text{for any } \varphi \in C_0^\infty(D,\mathbb{R}^d).$$

In other words, H^{-1} is the dual space of H_0^1. By the Riesz representation theorem, there is a natural isometry between H^{-1} and H_0^1, and it is easy to see that it is given by the Laplacian. Namely, for any $f \in H^{-1}$ there is a unique function $u_f \in H_0^1$ such that

$$f(v) = (\nabla u_f, \nabla v) \quad \text{for any } v \in H_0^1.$$

It follows from Proposition 4.4 that u_f is the unique solution of (4.9). In what follows, the space H^{-1} is endowed with the natural scalar product

$$(f,g)_{-1} = g(u_f) = f(u_g) = (\nabla u_f, \nabla u_g), \quad f, g \in H^{-1}(D,\mathbb{R}^d), \tag{1.6}$$

and the corresponding norm is denoted by $\|\cdot\|_{-1}$.

Let us introduce the functional spaces

$$H(D) = \{u \in L^2(D,\mathbb{R}^d) : \mathrm{div}\, u = 0 \text{ in } D,\ \langle u, \boldsymbol{n}\rangle\big|_{\partial D} = 0\},$$
$$Z(D) = \{u \in H^1(D) : \Delta u = 0 \text{ in } D\}.$$

We denote by $\nabla H_0^1(D)$ the space of vector fields $u \in L^2(D,\mathbb{R}^d)$ that are representable in the form $u = \nabla p$ with some $p \in H_0^1(D)$ and define $\nabla Z(D)$ in a similar way. In what follows, we shall often omit the domain D from the notation and write L^2, H, Z, etc. It follows from the Friedrichs and Poincaré inequalities (see the Appendix) that ∇H_0^1 and ∇Z are closed subspaces of L^2.

Theorem 1.2 (Hodge-Kodaira decomposition)**.** *Let $D \subset \mathbb{R}^d$ be a bounded domain with C^1-smooth boundary ∂D. Then the space of square-integrable vector fields in D is representable as the direct sum*

$$L^2(D,\mathbb{R}^d) = H \oplus \nabla H_0^1 \oplus \nabla Z. \tag{1.7}$$

Proof. Let us take an arbitrary function $u \in L^2(D,\mathbb{R}^d)$ and show that is representable in the form

$$u = v + \nabla w + \nabla z, \tag{1.8}$$

where $v \in H$, $w \in H_0^1$, and $z \in Z$. Consider the elliptic boundary value problems

$$\Delta w = \operatorname{div} u \quad \text{in } D, \qquad w\big|_{\partial D} = 0, \tag{1.9}$$

$$\Delta z = 0 \quad \text{in } D, \qquad \langle \nabla z - u + \nabla w, \boldsymbol{n}\rangle\big|_{\partial D} = 0. \tag{1.10}$$

By Proposition 4.4 and 4.5, problems (1.9) and (1.10) are uniquely solvable in the spaces $H_0^1(D)$ and $\dot{H}^1(D)$, respectively. Let us set $v = u - \nabla w - \nabla z$. We need to prove that $v \in H$. For any $\varphi \in C_0^\infty(D)$, we have

$$(v, \nabla\varphi) = (u - \nabla w - \nabla z, \nabla\varphi) = (\operatorname{div} u - \Delta w - \Delta z, \varphi) = 0,$$

and therefore $\operatorname{div} v = 0$ in D. Furthermore, the second relation in (1.10) implies that $\langle v, \boldsymbol{n}\rangle = 0$ on ∂D. We have thus proved (1.8).

To complete the proof of the theorem, it suffices to show that the spaces H, ∇H_0^1, and ∇Z are pairwise orthogonal. This fact is a straightforward consequence of the definition and the density of $C_0^\infty(D)$ in the space $H_0^1(D)$. □

Regularity of solutions for elliptic equations (see Prositions 4.4 and 4.5) and the explicit description of the functions on the right-hand side of (1.8) imply the following result:

Corollary 1.3. *Let $u \in H^s(D,\mathbb{R}^d)$ for some integer $s \geq 0$. Then the functions v, w, and z entering decomposition* (1.8) *satisfy the inclusions $v \in H^s$ and $w, z \in H^{s+1}$. Moreover, there is a constant $C_s > 0$ such that*

$$\|v\|_s + \|w\|_{s+1} + \|z\|_{s+1} \leq C_s \|u\|_s. \tag{1.11}$$

In what follows, we denote by $\Pi : L^2(D,\mathbb{R}^d) \to H$ the orthogonal projection in L^2 associated with the Hodge–Kodaira decomposition (1.7). It is called the *Leray projection*. Corollary 1.3 implies that Π is continuous in any Sobolev space $H^s(D,\mathbb{R}^d)$. Another useful consequence of Theorem 1.2 is the following necessary and sufficient condition for the representation of an L^2 vector field as the gradient of a function.

Corollary 1.4. *A vector function $u \in L^2(D,\mathbb{R}^d)$ can be represented in the form $u = \nabla p$ for some $p \in \dot{H}^1$ if and only if*

$$(u, \varphi) = 0 \quad \textit{for any } \varphi \in H. \tag{1.12}$$

In this case, the function p is unique, and if $u \in H^s(D,\mathbb{R}^d)$ for some integer $s \geq 0$, then $p \in H^{s+1}(D)$.

Let $V = V(D)$ be the space of divergence-free functions $u \in H_0^1(D,\mathbb{R}^d)$ and let $\mathcal{V} = \mathcal{V}(D) = C_0^\infty(D,\mathbb{R}^d) \cap V$. The latter space is not empty; for instance, for any $\psi \in C_0^\infty(D)$, the function $(-\partial_2\psi, \partial_1\psi, \dots)$ belongs to $\mathcal{V}$. The following result of fundamental importance implies, in particular, that $\mathcal{V}$ is much bigger.

Theorem 1.5. *A function $u \in H^{-1}(D, \mathbb{R}^d)$ is representable in the form $u = \nabla p$ for some $p \in L^2(D)$ if and only if*

$$(u, \varphi) = 0 \quad \text{for any } \varphi \in \mathcal{V}(D). \tag{1.13}$$

To prove this theorem, we shall need the following natural result, which is of independent interest. Its proof is given at the end of this subsection.

Proposition 1.6. *Let p be a distribution in D such that $\nabla p \in H^{-1}(D, \mathbb{R}^d)$. Then $p \in L^2(D)$. Moreover, there is a constant $C > 0$ depending only on D such that*

$$\|p - \langle p \rangle\| \le C \|\nabla p\|_{H^{-1}}. \tag{1.14}$$

Proof of Theorem 1.5. Step 1. Consider the operator $\nabla : L^2(D) \to H^{-1}(D, \mathbb{R}^d)$ taking a function p to its gradient ∇p. By Problem 1, the image $F(D)$ of ∇ is a closed subspace in $H^{-1}(D, \mathbb{R}^d)$. Furthermore, since the adjoint of ∇ coincides with the operator div : $H_0^1 \to L^2$ taking u to $\operatorname{div} u$, in view of Problem 2, we have

$$\begin{aligned} F(D) &= \big(\operatorname{Ker}(\operatorname{div})\big)^{\perp} = (V(D))^{\perp} \\ &= \{f \in H^{-1} : f(u) = 0 \text{ for any } u \in V(D)\}. \end{aligned} \tag{1.15}$$

Let us represent D as the union of an increasing sequence of connected domains D_k with smooth boundaries such that $\overline{D}_k \subset D_{k+1}$. Suppose we have proved that for any integer $k \ge 1$ the restriction of u to D_k (which we denote by u_k) belongs $F(D_k)$. In this case, we can find a function $p_k \in L^2(D_k)$ such that $u_k = \nabla p_k$. Arguing by induction, we can construct $p \in L^2_{\text{loc}}(D)$ such that $u = \nabla p$ in D. Applying Proposition 1.6, we conclude that $p \in L^2(D)$.

Step 2. We now prove that $u_k \in F(D_k)$. In view of relation (1.15) applied to D_k, it suffices to show that

$$u_k(\varphi) = 0 \quad \text{for any } \varphi \in V(D_k). \tag{1.16}$$

Let $\{\omega_\varepsilon, \varepsilon > 0\}$ be a family of mollifying kernels. Since $\overline{D}_k \subset D$, we see that $\omega_\varepsilon * \varphi \in \mathcal{V}(D)$ for $\varepsilon \ll 1$, and hence

$$u(\omega_\varepsilon * \varphi) = 0. \tag{1.17}$$

On the other hand, the family $\omega_\varepsilon * \varphi$ converges to φ in the space H^1 as $\varepsilon \to 0^+$. Passing to the limit in (1.17) as $\varepsilon \to 0^+$, we arrive at (1.16). □

Corollary 1.7. **(i)** *The closure of $\mathcal{V}(D)$ in $L^2(D, \mathbb{R}^d)$ coincides with $H(D)$.*

(ii) *The closure of $\mathcal{V}(D)$ in $H^1(D, \mathbb{R}^d)$ coincides with $V(D)$.*

Proof. **(i)** We need to show that if $u \in H$ satisfies (1.13), then $u = 0$. Indeed, by Theorem 1.5, the function u can be written as $u = \nabla p$ for some $p \in L^2(D)$. Since $u \in L^2$, we see that $p \in H^1(D)$. In view of Corollary 1.4, the function u must be orthogonal to H, and we conclude that $u = 0$.

(ii) Let $u \in H^{-1}(D,\mathbb{R}^d)$ be such that $u(\varphi) = 0$ for any $\varphi \in \mathcal{V}(D)$. The required assertion will be proved if we show that $u = 0$ on $V(D)$. By Theorem 1.5, we can write $u = \nabla p$ for some $p \in L^2(D)$. It follows that

$$\nabla p(\varphi) = \int_D p(x)\,\mathrm{div}\,\varphi(x)\,dx = 0 \quad \text{for any } \varphi \in V(D).$$

This completes the proof of the corollary. □

Proof of Proposition 1.6. We shall only outline the proof, leaving the details to the reader. Let us remark $\Delta p = \mathrm{div}\,u \in H^{-2}(D)$, and therefore, by elliptic regularity, we conclude that $p \in L^2_{\mathrm{loc}}(D)$. Let us denote by $\mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$ the d-dimensional torus. It is straightforward to verify, using the Fourier series, that for any $p \in L^2(\mathbb{T}^d)$ with zero mean value we have

$$\|p\| \le C\,\|\nabla p\|_{-1}. \tag{1.18}$$

Approximating any function $p \in L^2(D)$ by smooth functions with compact support, we can prove easily that (1.14) holds for any $p \in L^2(D)$. Thus, it suffices to show that any distribution p such that $\nabla p \in H^{-1}$ belongs to L^2. This is a local result, and by rectifying the boundary, we can assume without loss of generality that $D = (-1,1)^d$. In this case, for any $\alpha \in (0,1)$ the function $p_\alpha(x) = p(\alpha x)$ belongs to $L^2(D)$ and therefore

$$\|p_\alpha - \langle p_\alpha\rangle\| \le C\|\nabla p_\alpha\| \le C_1.$$

Since the unit ball in L^2 is weakly compact, we can find a sequence $\alpha_n \to 1$ and a function $\tilde p \in L^2$ such that $p_{\alpha_n} - \langle p_{\alpha_n}\rangle \to \tilde p$ weakly in L^2. On the other hand, $p_\alpha \to p$ in the sense of distributions. It follows that $\langle p_{\alpha_n}\rangle$ converges to a constant $c \in \mathbb{R}$, and $p = \tilde p + c \in L^2$. □

In what follows, we shall need also the dual space of V; it is denoted by V^*. Since V is a closed subspace of H^1_0, we can identify V^* with the quotient space of H^{-1} over the closed subspace

$$\{u \in H^{-1} : u(\varphi) = 0 \text{ for any } \varphi \in V\} = \{u \in H^{-1} : u = \nabla p \text{ for some } p \in L^2\}.$$

It is straightforward to verify that the Leray projection admits a continuous extension from H^{-1} to V^*; see Problem 5 for a hint.

1.2 Reduction to an evolution equation

In what follows, we assume that $d = 2$ or 3. Let us fix $T > 0$ and denote by J_T the time interval $[0,T]$. If there is no ambiguity, we shall omit the subscript T.

Definition 1.8. Let $f \in L^2(J,H^{-1})$. A pair of distributions (u,p) in the domain $(0,T)\times D$ is called a *weak solution of* (1.1), (1.2) if

$$u \in L^\infty(J,L^2) \cap L^2(J,H^1_0), \qquad p = \partial_t q \quad \text{for some } q \in L^\infty(J,L^2),$$

and Eqs. (1.1) are satisfied in the sense of distributions.

The regularity imposed on (u, p) by the above definition is dictated by a standard *energy estimate*. Namely, let (u, p) be a smooth solution of (1.1), (1.2). Multiplying the first equation of (1.1) by $2u$ and integrating over D, we derive

$$\partial_t \|u\|^2 + 2\|\nabla u\|^2 + 2(\langle u, \nabla \rangle u + \nabla p, u) = 2(f, u). \tag{1.19}$$

The third term on the left-hand side of (1.19) vanishes, and the right-hand side can be estimated by $C\|f\|_{-1}^2 + \|\nabla u\|^2$. Substituting this into (1.19), integrating in time, and taking the supremum over $t \in J$, we obtain

$$\sup_{t\in J} \left(\|u(t)\|^2 + \int_0^t \|\nabla u\|^2 ds \right) \le \|u(0)\|^2 + C\|f\|^2_{L^2(J,H^{-1})}.$$

This justifies the choice of the functional space for u. Furthermore, integrating the first equation of (1.1) in time, we derive

$$\nabla \left(\int_0^t p(s)\, ds \right) = u(0) - u(t) + \int_0^t \big(f + \Delta u - \langle u, \nabla \rangle u \big)\, ds.$$

The right-hand side of this relation belongs to $L^\infty(J, H^{-1})$, and therefore $\int_0^t p ds$ must be an element of $L^\infty(J, L^2)$.

The following proposition gives some additional information on weak solutions. It also suggests how to reduce the Navier–Stokes system to an evolution equation.

Proposition 1.9. *Let (u, p) be a weak solution of* (1.1), (1.2). *Then*

$$u \in \mathcal{X}_T := L^\infty(J, H) \cap L^2(J, V), \tag{1.20}$$

$$\langle u, \nabla \rangle u \in L^1(J, H^{-1}), \quad \Delta u \in L^2(J, H^{-1}), \tag{1.21}$$

$$\partial_t u \in L^1(J, V^*), \quad u \in C(J, V^*), \tag{1.22}$$

and we have the relation

$$\partial_t u + \langle u, \nabla \rangle u - \Delta u = f \quad \text{in } V^* \text{ for almost every } t \in J. \tag{1.23}$$

Proof. Inclusions (1.20) and (1.21) are easy to establish, and we confine ourselves to the proof of the first relation in (1.21). For any $u \in V$, we have

$$\langle u, \nabla \rangle u = \sum_{i=1}^d u_i \partial_i u = \sum_{i=1}^d \partial_i (u_i u). \tag{1.24}$$

Using the continuous embedding $H^1 \subset L^4$, which is valid for $d \le 4$, we derive

$$\|u_i u\| \le C_1 \|u\|^2_{L^4} \le C_2 \|\nabla u\|^2. \tag{1.25}$$

Since the derivation is continuous from L^2 to H^{-1}, relations (1.24) and (1.25) imply that

$$\|\langle u, \nabla \rangle u\|_{-1} \le C_3 \|\nabla u\|^2. \tag{1.26}$$

This estimate implies the required inclusion.

We now prove (1.22) and (1.23). It follows from the first equation of (1.1) that

$$u(t) + \nabla q(t) = \hat{u} + \int_0^t (f + \Delta u - \langle u, \nabla \rangle u)\, ds \quad \text{for almost every } t \in J, \tag{1.27}$$

where $\hat{u}$ is an element of H^{-1}, and the equality holds in H^{-1}. Applying both sides of (1.27) to a test function $\varphi \in \mathcal{V}$, we obtain

$$(u(t), \varphi) = (\hat{u}, \varphi) + \int_0^t (f + \Delta u - \langle u, \nabla \rangle u, \varphi)\, ds \quad \text{for almost every } t \in J.$$

Since $\mathcal{V}$ is dense in V, we conclude that

$$u(t) = \hat{u} + \int_0^t (f + \Delta u - \langle u, \nabla \rangle u)\, ds, \tag{1.28}$$

where the equality holds in V^* for almost every $t \in J$. Relation (1.28) implies inclusions (1.22), as well as equality (1.23). □

Proposition 1.9 suggests that projecting the first equation of (1.1) to the space V^*, we can reduce the Navier–Stokes system to an evolution equation. Guided by this idea, let us consider the equation

$$\partial_t u + Lu + B(u, u) = g(t), \tag{1.29}$$

where $g \in L^2(J, V^*)$, and we set

$$Lu = -\Pi\Delta, \quad B(u, v) = \Pi\left(\langle u, \nabla \rangle u\right). \tag{1.30}$$

Using Problem 5 and (1.24) – (1.26), it is straightforward to see that the linear operator $L : V \to V^*$ and the bilinear form $B(u, v) : V \times V \to V^*$ are continuous.

Definition 1.10. A distribution $u(t, x)$ in the domain $(0, T) \times D$ is called a *weak solution of* (1.29) if $u \in \mathcal{X}_T$, $\partial_t u \in L^1(J, V^*)$, and relation (1.29) holds in V^* for almost every $t \in J$.

Proposition 1.9 shows that if (u, p) is a weak solution of (1.1), (1.2), then u is a weak solution for (1.29) with $g = \Pi f$. The converse assertion is also true.

Proposition 1.11. *Let $f \in L^2(J, H^{-1})$ and let $u(t, x)$ be a weak solution of Eq.* (1.29) *with $g = \Pi f$. Then there is $q \in L^\infty(J, L^2)$ such that the pair (u, p) with $p = \partial_t q$ is a weak solution of* (1.1), (1.2).

Proof. Relation (1.29) implies that $u \in C(J, V^*)$, and

$$u(t) - \hat{u} + \int_0^t \big(Lu + B(u, u) - \Pi f(t)\big)\, ds = 0 \quad \text{for any } t \in J, \tag{1.31}$$

where $\hat{u} \in V^*$, and the equality holds in V^*. Let us set

$$h(t) = u(t) - \hat{u} + \int_0^t \big(\langle u, \nabla\rangle u - \Delta u - f(t)\big)\, ds.$$

Then $h \in L^\infty(J, H^{-1})$, and in view of (1.31), there is a subset $J_0 \subset J$ of full measure such that

$$(h(t), \varphi) = 0 \quad \text{for any } \varphi \in \mathcal{V} \text{ and } t \in J_0.$$

By Theorem 1.5 and Proposition 1.6, there is a function $q \in L^\infty(J, L^2)$ such that $h(t) = \nabla q(t)$. This relation is equivalent to the first equation of (1.1). The second equation follows from the inclusion $u \in L^2(J, V)$. □

From now on, we study the evolution equation (1.29).

1.3 Existence, uniqueness, and regularity in the 2D case

We begin with the investigation of the Cauchy problem for the Stokes equation. Namely, we consider the problem

$$\partial_t u + Lu = g(t), \tag{1.32}$$

$$u(0) = u_0, \tag{1.33}$$

where $u_0 \in H$ and $g \in L^2(J, V^*)$ are given functions. We shall show that L is a positive (self-adjoint) operator in H with discrete spectrum and then apply the Hille-Yosida theorem to construct a solution of (1.32), (1.33). We shall need the concept of Friedrichs extension for semi-bounded symmetric operators; it is briefly described in the Appendix.

We define an operator L_0 in H by $\mathcal{D}(L_0) = \mathcal{V}$ and $L_0 u = -\Pi\Delta u$ for $u \in \mathcal{V}$. It is straightforward to show that L_0 satisfies inequalites (4.15) and (4.16) with $M = 0$ and therefore, by Theorem 4.6, admits a self-adjoint extension L called the Friedrichs extension. The following proposition describes the operator L and some of its properties.

Proposition 1.12. *The domain $\mathcal{D}(L)$ of L coincides with the space $H^2(D, \mathbb{R}^d) \cap V$, and $Lu = -\Pi\Delta u$ for $u \in \mathcal{D}(L)$. Moreover, L has a compact resolvent, and there is an orthonormal basis of H consisting of the eigenfunctions of L.*

Proof. Step 1. We first show that $\mathcal{D}(L) = H^2(D, \mathbb{R}^d) \cap V$. Indeed, the norm $\|\cdot\|_q$ defined in the Appendix coincides with $\|\cdot\|_1$, and therefore, by Corollary 1.7, we have

$$\mathcal{D}(Q) = V, \quad Q(u, v) = (\nabla u, \nabla v).$$

Recall that

$$\mathcal{D}(L) = \{u \in V : \text{there is } f \in H \text{ such that } (\nabla u, \nabla v) = (f, v) \text{ for any } v \in V\}. \tag{1.34}$$

This implies immediately that $\mathcal{D}(L) \supset H^2(D, \mathbb{R}^d) \cap V$. We now prove the converse inclusion. To this end, we need the following lemma.

Lemma 1.13. *For any $f \in H^{-1}(D, \mathbb{R}^d)$ the problem*

$$-\Delta u + \nabla p = f, \quad \operatorname{div} u = 0, \tag{1.35}$$

has a unique solution $(u, p) \in V \times \dot{L}^2$. Moreover, if $f \in L^2$, then $u \in H^2$ and $p \in H^1$.

The existence and uniqueness of solution is a straightforward consequence of the Riesz representation theorem. The regularity of solution in the general case follows from the elliptic theory; see [ADN64]. However, in the 2D case, a simpler proof can be given by reducing (1.35) to a biharmonic equation. The corresponding argument is outlined in Problem 8.

Now let $u \in \mathcal{D}(L)$. Then, by (1.34), there is $f \in H$ such that

$$(\Delta u + f, \varphi) = 0 \quad \text{for any } \varphi \in \mathcal{V}. \tag{1.36}$$

By Theorem 1.5, there is $p \in \dot{L}^2$ such that $\Delta u + f = \nabla p$. Thus, (u, p) is the unique solution of problem (1.35). Since $f \in L^2$, in view of Lemma 1.13, the function $u \in V$ must belong to H^2. Furthermore, it follows from (1.36) that $f = Lu = -\Pi\Delta u$.

Step 2. The remaining assertions will be proved if we show that for any $f \in H$ the equation $Lu = f$ has a unique solution $u \in \mathcal{D}(L)$. Indeed, if this proved, then $\lambda = 0$ is in the resolvent set of L, and the inverse L^{-1} is compact. Thus, L is a self-adjoint operator with compact resolvent, and therefore it has a discrete spectrum; see Chapters VI–VIII of [RS80] for details.

To prove the unique solvability in $\mathcal{D}(L)$ of the equation $Lu = f$, it suffices to note that it is equivalent to problem (1.35), and therefore all the claims follow from Lemma 1.13. □

Proposition 1.12 and The Hille–Yosida theorem enable one to prove the existence and uniqueness of solution for the Stokes equation. Namely, we have the following result whose proof is left to the reader.

Proposition 1.14. *For any $u_0 \in H$ and $g \in L^2(J, V^*)$, problem* (1.32), (1.33) *has a unique solution $u \in \mathcal{Y}_T := C(J, H) \cap L^2(J, V)$, which satisfies the relation*

$$\|u(t)\|^2 + 2\int_0^t \|\nabla u(s)\|^2 ds = \|u(0)\|^2 + 2\int_0^t (g(s), u(s))\, ds, \quad 0 \le t \le T. \tag{1.37}$$

Moreover, if $u_0 \in V$ and $g \in L^2(J, H)$, then $u \in C(J, V) \cap L^2(J, \mathcal{D}(L))$.

From now on until the end of this subsection, we assume that $d = 2$. The next step in the construction of solution of the Navier–Stokes system is the investigation of the quasilinear equation

$$\partial_t u + Lu + B(v_1, u) + B(u, v_2) = g(t), \tag{1.38}$$

where v_1, v_2, and g are given functions.

Proposition 1.15. *For any $v_1, v_2 \in \mathcal{Y}_T$, $g \in L^2(J, V^*)$, and $u_0 \in H$, problem* (1.38), (1.33) *has a unique solution $u \in \mathcal{Y}_T$, which satisfies the inequality*

$$\|u\|_{\mathcal{Y}_T} \leq C\left(\|u_0\| + \|g\|_{L^2(J,V^*)}\right), \tag{1.39}$$

where $C > 0$ does not depend on v_1, g, and u_0.

Proof. Step 1: Uniqueness and a priori estimate. It suffices to establish inequality (1.39). If $u, v_1, v_2 \in \mathcal{Y}_T$, then the function $B(v_1, u) + B(u, v_2)$ belongs to $L^2(J, V^*)$; see Problem 9. Therefore, by (1.37), we can write

$$\|u(t)\|^2 + 2\int_0^t \|\nabla u(s)\|^2 ds = \|u_0\|^2 + 2\int_0^t (g - B(v_1, u) - B(u, v_2), u)\, ds. \tag{1.40}$$

Now note that

$$(B(v_1, u), u) = 0,$$
$$|(B(u, v_2), u)| \leq C_1 \|v_2\|_1\, \|u\|_{L^4}^2 \leq C_2 \|v_2\|_1\, \|u\|\, \|\nabla u\|.$$

Substituting this into (1.40) and using the Cauchy inequality, we derive

$$\mathcal{E}_u(t) \leq \|u_0\|^2 + C_3 \int_0^t \|g\|_{V^*}^2 ds + C_3 \int_0^t \|v_2\|_1^2 \mathcal{E}_u(s)\, ds, \tag{1.41}$$

where we set

$$\mathcal{E}_u(t) = \|u(t)\|^2 + \int_0^t \|\nabla u\|^2 ds. \tag{1.42}$$

Applying the Gronwall inequality (see (4.2)), we obtain

$$\mathcal{E}_u(t) \leq C_4 \exp\left(\int_0^T \|v_2\|_1^2 ds\right) \left(\|u_0\|^2 + \|g\|_{L^2(J,V^*)}^2\right), \quad t \in J.$$

This implies the required estimate (1.39).

Step 2: Existence. We now use the contraction mapping principle to construct a solution. We fix $\tau > 0$ and define the ball

$$B_R = \{u \in \mathcal{Y}_\tau : \|u\|_{\mathcal{Y}_\tau} \leq R\}.$$

Let $\mathcal{F} : B_R \to \mathcal{Y}_\tau$ be a mapping that takes $\tilde{u}$ to the solution $u \in \mathcal{Y}_\tau$ of the problem

$$\partial_t u + Lu = g - B(v_1, \tilde{u}) - B(\tilde{u}, v_2), \quad u(0) = u_0.$$

We claim that there are constants R and τ depending only on the norms of the functions v_1, v_2, g and u_0 such that $\mathcal{F}$ is a contraction of the ball B_R into itself. Indeed, it follows from (1.37) that

$$\|u(t)\|^2 + 2\int_0^t \|\nabla u\|^2 ds \leq \|u_0\|^2 + 2\int_0^t (g - B(v_1, \tilde{u}) - B(\tilde{u}, v_2), u)\, ds$$
$$\leq \|u_0\|^2 + \int_0^t \Big(C_5 \|g\|_{V^*}^2 + \|\nabla u\|^2 + C_6\big(\|v_1\|_{L^4}^2 + \|v_2\|_{L^4}^2\big)\|\tilde{u}\|_{L^4}^2\Big)\, ds.$$

By interpolation inequality, it follows that

$$\sup_{t\in J} \mathcal{E}_u(t) \le \|u_0\|^2 + C_5\|g\|^2_{L^2(J,V^*)} + \frac{1}{8}\int_0^t \|\nabla \tilde{u}\|^2 ds \\ + C_7\Big(\|v_1\|^4_{L^4(Q_\tau)} + \|v_2\|^4_{L^4(Q_\tau)}\Big) \sup_{s\in[0,t]} \|\tilde{u}(s)\|^2. \quad (1.43)$$

Let us choose

$$R = 3\Big(\|u_0\|^2 + C_5\|g\|^2_{L^2(J,V^*)}\Big)^{1/2}.$$

Then, for sufficiently small $\tau > 0$, the right-hand side of (1.43) is smaller than $R^2/4$, whence it follows that $\|u\|_{\mathcal{Y}_\tau} \le R$. This proves that $\mathcal{F}$ maps the ball B_R into itself. A similar argument shows that $\mathcal{F}$ is a contraction. □

Theorem 1.16. *For any* $u_0 \in H$ *and* $g \in L^2(J, V^*)$*, problem* (1.29), (1.33) *has a unique solution* $u \in \mathcal{Y}_T$*, which satisfies relation* (1.37).

We shall need the following lemma; we refer the reader to Problem 11 for some hints how to establish it.

Lemma 1.17. *Let* $u \in L^2(J,V)$ *be such that* $\dot{u} \in L^2(J,V^*)$*. Then* $u \in C(J,H)$ *and*

$$\|u\|^2_{C(J,H)} \le C\,\|u\|_{L^2(J,V)}\|\dot{u}\|_{L^2(J,V^*)}. \quad (1.44)$$

In particular, if $u \in \mathcal{X}_T$ *is a weak solution of* (1.29)*, then* $u \in \mathcal{Y}_T$*.*

Proof of Theorem 1.16. Step 1: Uniqueness and a priori estimate. Let $u \in \mathcal{Y}_T$ be a solution. Then, by Problem 9, we have $B(u,u) \in L^2(J,V^*)$. Hence, by (1.37), we derive

$$\|u(t)\|^2 + 2\int_0^t \|\nabla u(s)\|^2 ds = \|u(0)\|^2 + 2\int_0^t (g - B(u,u), u) ds.$$

Since $(B(u,u),u) = 0$, we conclude that (1.37) remains valid for solutions of (1.29). Furthermore, using the inequality

$$|(g,u)| \le \|g\|_{V^*}\|\nabla u\| \le \frac{1}{2}\big(\|g\|^2_{V^*} + \|\nabla u\|^2\big),$$

we see that

$$\mathcal{E}_u(t) \le \|u(0)\|^2 + \int_0^t \|g\|^2_{V^*} ds.$$

It follows that

$$\|u\|_{\mathcal{Y}_\tau} \le 2\left(\|u_0\| + \|g\|_{L^2(J_\tau,V^*)}\right) \quad \text{for any } \tau \in J_T. \quad (1.45)$$

Suppose now that $u_1, u_2 \in \mathcal{Y}_T$ are two solutions for (1.29), (1.33). Then their difference $u = u_1 - u_2$ vanishes at $t = 0$ and satisfies the equation

$$\partial_t u + Lu = B(u_2,u) + B(u,u_1).$$

Using (1.37), we derive

$$\|u(t)\|^2 + 2\int_0^t \|\nabla u(s)\|^2 ds = 2\int_0^t \big(B(u_2,u) + B(u,u_1),u\big)\,ds. \tag{1.46}$$

The first term under the integral on the right-hand side of (1.46) vanishes, while the second can be estimated by

$$|B(u,u_1),u)| \le C_1\|\nabla u_1\|\,\|u\|\,\|\nabla u\| \le \frac{1}{2}\big(\|\nabla u\|^2 + C_1^2\|\nabla u_1\|^2\,\|u\|^2\big).$$

Substituting this into (1.46), we obtain

$$\mathcal{E}_u(t) \le C_1^2\int_0^t \|\nabla u_1\|\,\mathcal{E}_u(t)\,ds. \tag{1.47}$$

Applying the Gronwall inequality, we conclude that $u \equiv 0$.

Step 2: Reduction to local existence. We now show that it suffices to establish the existence of solution on a small time interval $[0,\tau]$ depending on the initial function and the right-hand side. Indeed, let us set

$$T_* = \sup\{\tau > 0 : \text{there is a solution } u \in \mathcal{Y}_\tau \text{ for (1.29), (1.33)}\}.$$

Suppose that $T_* < T$. It follows from (1.45) that $u \in \mathcal{X}_{T_*}$ and therefore, by Lemma 1.17, we have $u \in \mathcal{Y}_{T_*}$. Let us set $u_* = u(T_*)$. By assumption, we can solve Eq. (1.29) with the initial condition $u(T_*) = u_*$ on a small interval $[T_*, T_*+\tau]$. We thus obtained a solution $u \in \mathcal{Y}_{T_*+\tau}$; this contradicts the definition of T_*. Thus, T_* must coincide with T, and repeating the above argument, we obtain a solution $u \in \mathcal{Y}_T$.

Step 3: Construction of a local solution. The solution is sought in the form $u = z + v$, where $z = \exp(-tL)u_0$, and v is an unknown function. Substituting this into (1.29), we obtain the following problem for v:

$$\partial_t v + Lv + B(v,v) + B(v,z) + B(z,v) = \tilde{g}, \quad v(0) = 0, \tag{1.48}$$

where $\tilde{g} = g - B(z,z) \in L^2(J_\tau, V^*)$. Let us fix positive constants $\tau \le 1$ and M and denote by $K(\tau, M)$ the set of functions $v \in \mathcal{Y}_\tau$ such that

$$\mathcal{E}_v(t) \le M\int_0^t \|\tilde{g}(s)\|_{V^*}^2 ds \quad \text{for } 0 \le t \le \tau. \tag{1.49}$$

It is clear that $K(\tau, M)$ is a closed subset in the space $\mathcal{Y}_\tau$. Let us define a mapping $\mathcal{F} : K(\tau, M) \to \mathcal{Y}_\tau$ that takes $\tilde{v}$ to the solution $v \in \mathcal{Y}_\tau$ of the problem (cf. (1.48))

$$\partial_t v + Lv + B(\tilde{v},v) + B(v,z) + B(z,v) = \tilde{g}, \quad v(0) = 0. \tag{1.50}$$

It is clear that a function $v \in \mathcal{Y}_\tau$ is the solution of (1.48) if and only if it is a fixed point for $\mathcal{F}$. We claim that for an appropriate choice of τ and M, the mapping $\mathcal{F}$ is a contraction of the set $K(\tau, M)$ into itself. Indeed, it follows from (1.37) and the relation $(B(w,v),v) = 0$ that

$$\|v(t)\|^2 + 2\int_0^t \|\nabla v(s)\|^2 ds = 2\int_0^t \big(\tilde{g} - B(v,z), v\big)\, ds. \tag{1.51}$$

Now note that

$$|(\tilde{g}, v)| \le \frac{1}{4}\|\nabla v\|^2 + \|\tilde{g}\|_{V^*}^2,$$
$$|(B(v,z),v)| \le C_2\|v\|_{L^4}^2\|\nabla z\| \le \frac{1}{4}\|\nabla v\|^2 + C_2^2\|v\|^2\|\nabla z\|^2.$$

Substituting these inequalities into (1.51) and carrying out some simple transformations, we derive

$$\mathcal{E}_v(t) \le 2\int_0^t \|\tilde{g}\|_{V^*}^2 ds + C_3\int_0^t \|\nabla z\|^2\mathcal{E}_v(s)\, ds. \tag{1.52}$$

Application of the Gronwall inequality shows that (1.49) holds with

$$M = 2\exp\bigg(C_3\int_0^1 \|\nabla z\|^2 ds\bigg). \tag{1.53}$$

Thus, the mapping $\mathcal{F}$ takes the set $K(\tau, M)$ into itself for any $\tau \in (0,1]$. Let us prove that it is a contraction for sufficiently small τ.

Take any $\tilde{v}_i \in K(\tau, M)$, $i = 1,2$, and set $v_i = \mathcal{F}(\tilde{v}_i)$. Then the difference $v = v_1 - v_2$ vanishes at $t = 0$ and satisfies the equation

$$\partial_t v + Lv + B(v,z) + B(z,v) = -B(\tilde{v}, v_2) - B(\tilde{v}_1, v),$$

where $\tilde{v} = \tilde{v}_1 - \tilde{v}_2$. Using again (1.37) and repeating the above arguments, we derive

$$\begin{aligned}\mathcal{E}_v(t) &\le C_4\int_0^t \big(\|\nabla z\|^2\|v\|^2 + \|v_2\|^2\|\nabla\tilde{v}\|^2 + \|\nabla v_2\|^2\|\tilde{v}\|^2\big)\, ds\\ &\le C_4\int_0^t \|\nabla z\|^2\mathcal{E}_v(s)\, ds + C_5\|\tilde{v}\|_{\mathcal{Y}_\tau}^2 \sup_{s\in J_t}\mathcal{E}_{v_2}(s).\end{aligned}$$

Applying the Gronwall inequality and using (1.49) with $v = v_2$, we derive

$$\mathcal{E}_v(t) \le C_6\|\tilde{v}\|_{\mathcal{Y}_\tau}^2\int_0^\tau \|\tilde{g}\|_{V^*}^2 ds \quad \text{for } 0 \le t \le \tau,$$

where $C_6 > 0$ does not depend on v_1 and v_2. The above inequality implies that $\mathcal{F}$ is a contraction for $\tau \ll 1$ and, hence, has a fixed point $v \in \mathcal{Y}_\tau$. This completes the proof of the theorem. □

We now turn to the problem of smoothness of solution. The following result shows that the weak solution corresponding to smooth data is smooth.

Theorem 1.18. *Let $u_0 \in V$ and $g \in L^2(J,H)$. Then the solution $u \in \mathcal{Y}_T$ of* (1.29), (1.33) *belongs to the space $C(J,V) \cap L^2(J,H^2)$.*

Proof. Using the methods applied for proving Theorem 1.16, it is not difficult to show that for any $R > 0$ there is $\tau > 0$ such that problem (1.29), (1.33) has a unique solution $u \in C(J_\tau, V) \cap L^2(J_\tau, H^2)$ for any $u_0 \in V$ and $g \in L^2(J_\tau, H)$ satisfying the inequality

$$\|u_0\| + \|g\|_{L^2(J_\tau,H)} \le R.$$

Therefore, it suffices to establish that the solution remains bounded in H^1. To this end, we shall need the lemma below, whose proof is a straightforward consequence of Hölder and interpolation inequalities.

Lemma 1.19. *There is a constant $C > 0$ such that*

$$|(B(u,u),Lu)| \le C\,\|u\|^{1/2}\|\nabla u\|\,\|Lu\|^{3/2} \quad \textit{for any } u \in H^2 \cap H_0^1. \tag{1.54}$$

We now establish an a priori estimate for smooth solutions. We shall confine ourselves to a formal derivation, leaving it to the reader to justify the calculations. Taking the scalar product in L^2 of (1.29) with $2tLu$, we obtain

$$\partial_t\big(t\|\nabla u\|^2\big) - \|\nabla u\|^2 + 2t\|Lu\|^2 + 2t(B(u,u),Lu) = 2t(g,Lu). \tag{1.55}$$

In view of the Cauchy inequality and Lemma 1.19, we have

$$|2t(g,Lu)| \le \frac{t}{2}\|Lu\|^2 + 2t\|g\|^2,$$

$$|2t(B(u,u),Lu)| \le C_1 t\|u\|^{1/2}\|\nabla u\|\,\|Lu\|^{3/2} \le \frac{t}{2}\|Lu\|^2 + C_2 t\|\nabla u\|^4\|u\|^2.$$

Substituting these inequalities into (1.55), we derive

$$\partial_t\big(t\|\nabla u\|^2\big) + t\|Lu\|^2 \le \|\nabla u\|^2 + 2t\|g\|^2 + C_2 t\|\nabla u\|^4\|u\|^2.$$

Integration in time results in

$$\varphi(t) \le \int_0^t h(s)\,ds + C_2\int_0^t \|\nabla u\|^2\|u\|^2\varphi(s)\,ds, \tag{1.56}$$

where we set

$$\varphi(t) = t\|\nabla u(t)\|^2 + \int_0^t s\|Lu(s)\|^2 ds, \quad h(t) = \|\nabla u(t)\|^2 + 2t\|g(t)\|^2.$$

Applying the Gronwall inequality to (1.56), we obtain

$$t\|\nabla u(t)\|^2 + \int_0^t s\|Lu(s)\|^2 ds \le C_3\big(T, \|u_0\|, \|g\|_{L^2(J,H)}\big).$$

This estimate shows, in particular, that the solution remains bounded in H^1 on any finite time inteval. □

1.4 Remarks on the 3D case

In the 3D case, the situation is much more complicated. Roughly speaking, one can prove existence of a global weak solution and existence and uniqueness of a local strong solution. However, it is an open question if there is a functional class in which (global) existence and uniqueness hold simultaneously.

In this section, we confine ourselves to the formulation of two results: local existence and uniqueness of smooth solutions and weak-strong uniqueness. As before, we consider Eq. (1.29).

Definition 1.20. Let $T > 0$. A function $u(t, x)$ defined in the domain $(0, T) \times D$ is called a *strong solution of* (1.29) if

$$u \in C(J, V) \cap L^2(J, H^2), \quad \partial_t u \in L^2(J, H),$$

and Eq. (1.29) holds for almost every $t \in J$.

The proofs of the following two results can be found in [Tay97, Soh01].

Theorem 1.21 (local well-posedness)**.** *For any $R > 0$ there is $T = T(R) > 0$ such that if $u_0 \in V$ and $g \in L^2(J_T, H)$ satisfy the inequality $\|u_0\| + \|g\|_{L^2(J_T,H)} \leq R$, then Eq.* (1.29) *has a unique strong solution $u(t, x)$ defined on J_T and satisfying the initial condition $u(0) = u_0$.*

Theorem 1.22 (weak-strong uniqueness)**.** *Let u be a strong solution of* (1.29) *defined on J_T and let v be a weak solution of* (1.29) *that is defined on the same interval and satisfies the energy inequality*

$$\|v(t)\|^2 + 2\int_0^t \|\nabla v(s)\|^2 ds \leq \|v(0)\|^2 + 2\int_0^t (g(s), v(s))\, ds$$

for almost every $0 \leq t \leq T$. In this case, if $u(0) = v(0)$, then $u \equiv v$.

2 Euler equations

In this section, we study the initial-boundary value problem for the incompressible Euler system

$$\partial_t u + \langle u, \nabla \rangle u + \nabla p = f(t, x), \quad \operatorname{div} u = 0, \quad x \in D, \tag{2.1}$$

where $D \subset \mathbb{R}^d$ is a bounded domain with smooth boundary, and $d = 2$ or 3. To simplify the presentation, we assume that D is simply-connected, i.e., any two continuous curves with the same endpoints are homotopic. Let us emphasise, however, that the main results remain valid in the general case. Equations (2.1) are supplemented with the boundary and initial conditions

$$\langle u, \boldsymbol{n} \rangle\big|_{\partial D} = 0, \tag{2.2}$$

$$u(0, x) = u_0(x), \tag{2.3}$$

where $\boldsymbol{n}$ stands for the outward unit normal to ∂D. As for the Navier–Stokes system, the problem in question is well posed in the 2D case and locally well posed in the 3D case. Our presentation follows essentially the classical works [Wol33, Kat67].

We begin with a study of some properties of smooth vector fields and corresponding flows. These results are used in the next two subsections to prove existence and uniqueness of a global smooth solution in the 2D case. In conclusion, we discuss briefly some results on the 3D system.

2.1 Smooth vector fields and flows

For an integer, $k \geq 0$, we denote by $C^k(\overline{D})$ the space of k time continuously differentiable functions $v : \overline{D} \to R$. If $s > 0$ is a non-integer, then we write $[s]$ for the integer part of s and denote by $C^s(D)$ the space of functions $v \in C^{[s]}(\overline{D})$ whose derivatives of order $[s]$ are Hölder continuous with exponent $\gamma = s - [s]$. This space is endowed with the natural norm

$$|v|_s = \max_{|\alpha| \leq [s]} \sup_{x \in D} |\partial^\alpha v(x)| + \max_{|\alpha|=[s]} \sup_{0<|x-y|\leq 1} \frac{|\partial^\alpha v(x) - \partial^\alpha v(y)|}{|x-y|^\gamma}.$$

We write $C^k = C^k(\overline{D}, \mathbb{R}^d)$ and $C^s = C^s(D, \mathbb{R}^d)$ for similar spaces of $\mathbb{R}^d$-valued functions on D.

Let us fix a time-dependent vector field $u \in L^\infty(J, C^s)$, where $J = [0, T]$ and $s > 1$ is a non-integer, and consider the ordinary differential equation

$$\dot{x} = u(t, x). \tag{2.4}$$

Proposition 2.1. *Suppose that $u \in L^\infty(J, C^s)$ satisfies* (2.2) *for almost every $t \in J$. Then, for any $y \in \overline{D}$, Eq.* (2.4) *has a unique solution $x \in W^{1,\infty}(J, \mathbb{R}^d)$ that satisfies the initial condition*

$$x(0) = y. \tag{2.5}$$

Moreover, the function $\varphi : J \times \overline{D} \to \mathbb{R}^d$ taking (t, y) to $x(t)$ belongs to $W^{1,\infty}(J, C^s)$.

Proof. The local existence, uniqueness, and regularity are standard; for instance, see [CL55, Har82]. The fact that solutions are global is implied by the following simple observation. Since the vector field is tangent to ∂D, the solution starting from a point $y \in \partial D$ remains on the boundary. It follows that if $y \in D$, then $\varphi(t, y) \in D$ for any $t \in J$, an therefore a blow-up cannot occur. □

Let us denote by $LL(D)$ the space of log-Lipschitz functions, that is, the space of $v \in C(\overline{D})$ such that

$$\|v\|_{LL} := \sup_{x \in D} |v(x)| + \sup_{0<|x-y|\leq 1} \frac{|v(x) - v(y)|}{\lambda(|x-y|)} < \infty,$$

where $\lambda(r) = r(|\ln r| + 1)$. We write $LL = LL(D, \mathbb{R}^d)$ for the corresponding space of vector functions.

Proposition 2.2. *Let $u \in L^\infty(J, LL)$ be such that* (2.2) *holds for almost every $t \in J$. Then for any $y \in \overline{D}$ problem* (2.4), (2.5) *has a unique solution $x \in W^{1,\infty}(J, \mathbb{R}^d)$. Moreover, there is $\gamma > 0$ depending only on $M := \|u\|_{L^\infty(J,LL)}$ such that the function $\varphi(t,y)$ defined in Proposition 2.1 belongs to the Hölder space $C^{1,\gamma}(Q, \mathbb{R}^d)$, where $Q = J \times D$. Finally, the norm of φ in $C^{1,\gamma}(Q, \mathbb{R}^d)$ is bounded by a constant depending only on M.*

Proof. The local existence is a version of the Peano theorem (see [CL55]), and if we have uniqueness, then the same argument as in the proof of Proposition 2.1 shows that all solution remain confined in D. Let us sketch the proof of the claims about uniqueness and Hölder continuity.

Let $y_1, y_2 \in D$ be two initial points and let $x_1, x_2 \in W^{1,\infty}(J, \mathbb{R}^d)$ be the corresponding solutions of (2.4). Let $z(t) = |x_1(t) - x_2(t)|^2$. Differentiating z and using the log-Lipschitz property of u, we derive

$$\dot{z} \le -Mz \ln\Big(\frac{z}{C}\Big),$$

where $C > 1$ is a constant not depending on y_1 and y_2, and the inequality holds on any interval $I \subset J$ such that $x_1(t) \neq x_2(t)$ for $t \in I$. Applying the comparison principle (see the Appendix), we obtain

$$z(t) \le C\, z(s)^{\exp(-M(t-s))} \quad \text{for } t, s \in I, s \le t. \tag{2.6}$$

This inequality implies the uniqueness and the Hölder continuity of the function $\varphi(t,y)$ in y with exponent $\gamma = \exp(-MT)$.

Let us show that φ is Lipschitz continuous in t. Since $\partial_t \varphi(t,y) = u(t, \varphi(t,y))$, integrating in time, we see that

$$|\varphi(t, y_1) - \varphi(t, y_2)| \le \|u\|_{L^\infty} |t_1 - t_2| \quad \text{for } t_1, t_2 \in I, y \in D.$$

This completes the proof of the proposition. □

From now on we assume that $d = 2$ and D is simply-connected, which means that $\overline{D}$ is homeomorphic to the closed unit disc in $\mathbb{R}^2$. For a vector field $u = (u_1, u_2)$ and a scalar function v, we set

$$\operatorname{curl} u = \partial_1 u_2 - \partial_2 u_1, \quad \operatorname{curl} v = (\partial_2 v, -\partial_1 v).$$

The following proposition gives a necessary and sufficient condition for a vector function to be representable as a gradient; cf. Theorem 1.5.

Proposition 2.3. *Let $D \subset \mathbb{R}^2$ be a simply-connected bounded domain with smooth boundary and let $s \ge 0$. Then a vector field $u \in C^s(\overline{D})$ can be written as $u = \nabla p$ for some $p \in C^{s+1}(\overline{D})$ if and only if* $\operatorname{curl} u = 0$ *in D. In this case, there is a constant $C_s > 0$ not depending on $u(x)$ such that*

$$|p - \langle p \rangle|_{s+1} \le C_s |u|_s. \tag{2.7}$$

Proof. It is clear that if $u = \nabla p$ for some $p \in C^{s+1}(\overline{D})$, then $\operatorname{curl} u = 0$. Let us prove the converse implication.

By Theorem 1.5, we know that if $u \in L^2$ satisfies (1.13), then $u = \nabla p$ for some $p \in H^1(D)$. In view of Problem 4, in this case we have $p \in C^1(\overline{D})$. Thus, it suffices to show that if $\operatorname{curl} u = 0$, then (1.13) holds. To this end, note that

$$0 = (\operatorname{curl} u, \psi) = (u, \operatorname{curl} \psi) \quad \text{for any } \psi \in C_0^\infty(D).$$

It follows that the required assertion will be proved if we show that any $\varphi \in \mathcal{V}$ can be written as $\varphi = \operatorname{curl} \psi$ for some $\psi \in C_0^\infty(D)$.

Let us fix any $x_0 \in \partial D$ and define

$$\psi(x) = \int_{\Gamma(x_0,x)} (-\varphi_2 dy_1 + \varphi_1 dy_2), \quad x \in \overline{D},$$

where $\Gamma(x_0, x) \subset \overline{D}$ is an arbitrary smooth curve with the endpoints x_0 and x. Since D is simply-connected, the Stokes formula (see [Tay97]) and the relation $\operatorname{div} \varphi = 0$ imply that ψ is well defined. It is straightforward to verify that $\psi \in C_0^\infty(D)$ and $\operatorname{curl} \psi = \varphi$. The proof of (2.7) is left to the reader. ☐

For any $\omega \in C^s(D)$, we denote by $\Delta_D^{-1}\omega$ the unique solution of the equation $\Delta u = \omega$ satisfying the Dirichlet boundary condition. It is well known that, for any non-integer $s > 0$, the operator Δ_D^{-1} is continuous from C^s to C^{s+2}, see [GT01].

Proposition 2.4. *Let $D \subset \mathbb{R}^2$ be a simply-connected bounded domain with smooth boundary and let $s > 0$ be a non-integer. Then for any $\omega \in C^s(D)$ the problem*

$$\operatorname{curl} u = \omega, \quad \operatorname{div} u = 0, \quad \langle u, \boldsymbol{n} \rangle\big|_{\partial D} = 0, \tag{2.8}$$

has a unique solution $u \in C^{s+1}(D, \mathbb{R}^2)$. This solution is given by

$$u = -\operatorname{curl}(\Delta_D^{-1}\omega). \tag{2.9}$$

Moreover, if $\omega \in C(\overline{D})$, then the function $u(x)$ defined by (2.9) *belongs to $LL(D, \mathbb{R}^2)$ and satisfies the inequality*

$$\|u\|_{LL} \le C\|\omega\|_{L^\infty}. \tag{2.10}$$

Proof. It is easy to see that the function u defined by (2.9) belongs to $C^{s+1}(D, \mathbb{R}^2)$ and satisfies (2.8). Moreover, if $v \in C^{s+1}$ is another solution, then $w = u - v$ belongs to $H \cap C^{s+1}$ and satisfies the relation $\operatorname{curl} w = 0$. By Proposition 2.3, there is $p \in C^{s+1}(D)$ such that $w = \nabla p$. Since ∇H^1 is orthogonal to H, we conclude that $w = 0$.

We now assume that $\omega \in C(\overline{D})$ and prove that $u \in LL(D, \mathbb{R}^2)$. Let us denote by $G(x, y)$ the Green function of the Dirichlet problem for the Laplace operator in the domain D. In other words, G is the kernel of the operator Δ_D^{-1}:

$$(\Delta_D^{-1}\omega)(x) = \int_D G(x, y)\omega(y)\, dy. \tag{2.11}$$

It is well known that for any bounded domain with smooth boundary there is a constant $C_1 > 0$ such that

$$|\partial_x^\alpha G(x,y)| \le C_1 \, |x-y|^{-|\alpha|} \quad \text{for } x,y \in D, \tag{2.12}$$

where $\alpha = (\alpha_1, \alpha_2)$ is an arbitrary non-zero multi-index such that $|\alpha| \le 2$. It follows from (2.9) and (2.11) that

$$u(x) = -\int_D (\mathrm{curl}_x\, G)(x,y)\omega(y)\, dy. \tag{2.13}$$

Combining this with (2.12), we obtain

$$|u(x)| \le C_2 \|\omega\|_{L^\infty} \quad \text{for } x \in D. \tag{2.14}$$

Furthermore, for $z \in D$ and $r > 0$, let us define the set $D_r(z) = \{y \in D : |y-z| < r\}$ and denote by $D_r^c(z)$ its complement in D. Let us fix any points $x_1, x_2 \in D$ and set $d = |x_1 - x_2|$. It follows from (2.13) and (2.12) that

$$\begin{aligned}
|u(x_1) - u(x_2)| &\le \int_D \big|\big((\mathrm{curl}_x\, G)(x_1,y) - (\mathrm{curl}_x\, G)(x_2,y)\big)\omega(y)\big|\, dy \\
&= \int_{D_{2d}^c(x_1)} + \int_{D_{2d}(x_1)} \\
&\le C_3 \|\omega\|_{L^\infty} \bigg(d \sup_{z\in D} \int_{D_d^c(z)} |y-z|^{-2} dy \\
&\qquad + \int_{D_{2d}(x_1)} \big(|y-x_1|^{-1} + |y-x_2|^{-1}\big)\, dy \bigg)
\end{aligned}$$

A simple computation shows that the integrals on the right-hand sides can be estimated by $C_4 d(|\ln d| + 1)$. Combining this with inequality (2.14), we see that $u \in LL(D, \mathbb{R}^2)$ and (2.10) holds. □

2.2 Reduction to an evolution equation in the 2D case

Let us fix a constant $T > 0$ and a non-integer $s > 0$. We denote y $\dot{C}^s$ the space of scalar functions $p \in C^s(D)$ with zero mean value. The following definition concerns both 2D and 3D cases.

Definition 2.5. Let $s > 1$ be a non-ineteger and let $f \in L^1(J, C^s)$. A pair of functions (u, p) is called a *classical solution of the Euler system* (2.1), (2.2) if

$$u \in L^\infty(J_T, C^s) \cap W^{1,1}(J_T, C^{s-1}), \quad p \in L^1(J_T, \dot{C}^s), \tag{2.15}$$

and Equations (2.1), (2.2) are satisfied in the sense of distributions.

We now show how to reduce formally the 2D Euler system to an evolution equation. Let us note that if $\mathrm{div}\, u = 0$, then

$$\mathrm{curl}\big(\langle u, \nabla\rangle u\big) = \langle u, \nabla \rangle\, \mathrm{curl}\, u. \tag{2.16}$$

Therefore, setting $\omega = \operatorname{curl} u$ and $g = \operatorname{curl} f$ and applying the operator curl to the first relation in (2.1), we obtain the transport equation

$$\partial_t \omega + \langle u, \nabla \rangle \omega = g(t,x). \tag{2.17}$$

Taking into account Proposition 2.4, we see that u andω must be connected by relation

$$u(t,x) = -\big(\operatorname{curl} \Delta_D^{-1} \omega\big)(t,x) \quad \text{for } t \in J. \tag{2.18}$$

The following proposition shows that the Euler system is essentially equivalent to the evolution problem (2.17), (2.18).

Proposition 2.6. *Let $s > 1$ be a non-integer, let $f \in L^1(J, C^s)$, and let (u,p) be a classical solution of the Euler system. Then the function $\omega = \operatorname{curl} u$ belongs to $L^\infty(J, C^{s-1})$ and satisfies Eq.* (2.17)*, in which $g = \operatorname{curl} f$, and $u(t,x)$ is given by* (2.18)*. Conversely, if $\omega \in L^\infty(J, C^{s-1})$ is a solution of Eqs.* (2.17), (2.18) *with $g = \operatorname{curl} f$, then there is a unique function $p \in L^1(J, \dot{C}^s)$ such that the pair (u,p) is a classical solution of* (2.1), (2.2).

Proof. The fist part of the proposition is a simple consequence of the above calculations. Let us show that to any solution $\omega \in L^\infty(J, C^{s-1})$ of (2.17), (2.18) there corresponds a unique classical solution of the Euler system.

Relations (2.17), (2.18) imply that

$$u \in L^\infty(J, C^s), \quad \partial_t u = -\big(\operatorname{curl} \Delta_D^{-1}\big)\big(g - \langle u, \nabla \rangle \omega\big) \in L^1(J, C^{s-1}). \tag{2.19}$$

Let us consider the function $h = f - \partial_t u - \langle u, \nabla \rangle u$. It follows from (2.19) that $h \in L^1(J, C^{s-1})$. Moreover, in view of (2.17), we have

$$\operatorname{curl} h = g - \partial_t \omega - \langle u, \nabla \rangle \omega = 0 \quad \text{in the sense of distributions.}$$

Hence, using Proposition 2.3, we conclude that $h = \nabla p$ for some $p \in L^1(J, \dot{C}^s)$, and the pair (u,p) is a classical solution of the Euler system. The uniqueness of p is a straightforward consequence of Proposition 2.3. □

Let us define the space

$$C^s_\sigma = \{u \in C^s : \operatorname{div} u = 0 \text{ in } D, \langle u, \boldsymbol{n} \rangle|_{\partial D} = 0\}.$$

Proposition 2.6 implies that if $u_0 \in C^s_\sigma$ for some $s > 1$ and $\omega \in L^\infty(J, C^{s-1})$ is a solution of (2.17), (2.18) such that

$$\omega(0,x) = \omega_0(x), \tag{2.20}$$

where $\omega_0 = \operatorname{curl} u_0$, then one can construct a unique function $p \in L^1(J, \dot{C}^s)$ such that (u,p) is a classical solution of (2.1) – (2.3). In what follows, when talking about solutions of the Euler system, we shall often omit the function p, because it is uniquely determined by the vector field u.

2.3 Existence and uniqueness in the 2D case

Theorem 2.7. *Let $D \subset \mathbb{R}^2$ be a simply-connected bounded domain with a smooth boundary and let $s > 1$ be a non-integer. Then for any $u_0 \in C^s_\sigma$ problem* (2.1) – (2.3) *has a unique classical solution (u, p).*

Proof. We first outline the scheme of the proof. To prove the uniqueness, we assume that there are two classical solutions u_1 and u_2. In this case, the difference $u = u_1 - u_2$ satisfies the equations

$$\partial_t u + \langle u_1, \nabla \rangle u + \langle u, \nabla \rangle u_2 + \nabla p = 0, \quad \operatorname{div} u = 0. \tag{2.21}$$

Multiplying the first equation by $2u$, integrating over D, and carrying out some simple transformations, we arrive at the differential inequality

$$\partial_t \|u(t, \cdot)\|^2 \le C \, \|u(t, \cdot)\|^2.$$

Application of the Gronwall inequality shows that $u \equiv 0$.

To prove the existence of a solution, we construct a solution of the transport equation (2.17). It is based on the Leray–Schauder fixed point theorem; see the Appendix. We define a mapping $\mathcal{F}$ that takes a continuous scalar function $\widetilde{\omega}(t, x)$ defined on the cylinder $Q = J \times \overline{D}$ to the solution of the problem

$$\partial_t \omega + \langle \tilde{u}, \nabla \rangle \omega = g(t, x), \quad \omega(0, x) = \omega_0(x), \tag{2.22}$$

where $g = \operatorname{curl} f$, $\omega_0 = \operatorname{curl} u_0$, and $\tilde{u} = -\operatorname{curl} \Delta_D^{-1} \widetilde{\omega}$. It turns out that $\mathcal{F}$ is a continuous mapping of an appropriately chosen compact subset of $C(Q)$ into itself. By the Leray–Schauder theorem, it must have a fixed point $\omega \in C(Q)$. Some further argument shows that ω has the needed regularity.

We now turn to the accurate proof. It is divided into three steps.

Step 1: Uniqueness. Suppose that (u_i, p_i), $i = 1, 2$, are two classical solutions of the Euler system. Setting $u = u_1 - u_2$ and $p = p_1 - p_2$, we easily verify that (2.21) holds. Let us set $r(t) = \|u(t, \cdot)\|^2$ and calculate $\dot{\varphi}(t)$. Using the first relation in (2.21), we derive

$$\dot{r}(t) = 2(\partial_t u, u) = -2(\langle u_1, \nabla \rangle u + \langle u, \nabla \rangle u_2 + \nabla p, u), \tag{2.23}$$

where the equality holds for almost every $t \in J$. Now note that

$$(\nabla p, u) = 0, \quad (\langle u_1, \nabla \rangle u, u) = 0,$$
$$|(\langle u, \nabla \rangle u_2, u)| \le C_1 \|\nabla u_2\|_{L^\infty} \|u\|^2.$$

Combining this with (2.23), we obtain

$$\dot{r}(t) \le C_2 r(t) \quad \text{for almost every } t \in J,$$

where $C_2 > 0$ is a constant depending only on u_2. Integrating in time, applying the Gronwall inequality, and recalling that $r(0) = 0$, we conclude that $r \equiv 0$.

Step 2: Fixed point argument. We now assume that $1 < s < 2$ and construct a solution $\omega \in L^\infty(J, C^{s-1})$ of problem (2.17), (2.18). We fix positive constants M, C, δ and a non-decreasing function $\rho(r) \geq 0$ defined for $r \geq 0$ and vanishing at $r = 0$. Denote by $\mathcal{K} = \mathcal{K}(M, C, \delta, \rho)$ the set of functions $\omega \in C(Q)$ such that

$$|\omega(t,x)| \leq M, \quad |\omega(t_1,x_1) - \omega(t_2,x_2)| \leq \rho(|t_1 - t_2|) + C\,|x_1 - x_2|^\delta$$

for all $(t_1, x_1), (t_2, x_2) \in Q$. By the Arzelà–Ascoli theorem, $\mathcal{K}$ is a compact convex set in $C(Q)$. Let us define an operator $\mathcal{F} : \mathcal{K} \to C(Q)$ taking a function $\widetilde{\omega} \in \mathcal{K}$ to the solution $\omega(t,x)$ of problem (2.22). We claim that, for an appropriate choice of M, C, δ, and ρ, the operator $\mathcal{F}$ is well defined, maps the set $\mathcal{K}$ into itself, and is continuous.

Indeed, let us denote by $\varphi(t,y)$ the flow associated with the time-dependent vector field $\tilde{u}(t,x)$ and by $\psi(t,x)$ the inverse of φ regarded as a function of y. Since $\widetilde{\omega} \in C(Q)$ and $\|\widetilde{\omega}\|_{L^\infty} \leq M$, Proposition 2.4 implies that

$$\|\tilde{u}\|_{L^\infty(J,LL)} \leq M_1,$$

where $M_1 > 0$ depends only on M. Proposition 2.2 now implies that

$$|\psi(t,x)| \leq C_1, \quad |\psi(t_1,x_1) - \psi(t_2,x_2)| \leq C_2(|t_1 - t_2| + |x_1 - x_2|^\gamma), \qquad (2.24)$$

and similar estimates holds for φ. Here C_1, C_2, and γ are some constants depending only on M.

Now recall that the solution of problem (2.22) is uniquely determined by the relation

$$\omega(t, \varphi(t,y)) = \omega_0(y) + \int_0^t g\big(\tau, \varphi(\tau,y)\big)\, d\tau, \quad (t,y) \in Q. \qquad (2.25)$$

Setting $x = \varphi(t,y)$, we can rewrite (2.25) in the form

$$\omega(t,x) = \omega_0(\psi(t,x)) + \int_0^t g\big(\tau, \varphi(\tau, \psi(t,x))\big)\, d\tau, \quad (t,x) \in Q. \qquad (2.26)$$

It follows from (2.24) and (2.26) that

$$|\omega(t,x)| \leq \|\omega_0\|_{L^\infty} + \int_0^t \|g(\tau,\cdot)\|_{L^\infty} d\tau,$$

$$|\omega(t,x_1) - \omega(t,x_2)| \leq C_3|x_1 - x_2|^{\gamma(s-1)} + C_4|x_1 - x_2|^{\gamma^2(s-1)},$$

$$|\omega(t_1,x) - \omega(t_2,x)| \leq C_3|t_1 - t_2|^{s-1} + C_4|t_1 - t_2|^{\gamma(s-1)} + \left|\int_{t_1}^{t_2} \|g(\tau,\cdot)\|_{L^\infty} d\tau\right|,$$

where $C_3 = C_3(\omega_0, M)$ and $C_4 = C_4(g, M)$ are some positive constants. Choosing

$$M = \|\omega_0\|_{L^\infty} + \int_0^T \|g(\tau,\cdot)\|_{L^\infty} d\tau, \qquad C = C_3 + C_4,$$

$$\rho(r) = C_3 r^{s-1} + C_4 r^{\gamma(s-1)} + \sup_{0 \leq t_1 - t_2 \leq r} \int_{t_1}^{t_2} \|g(\tau,\cdot)\|_{L^\infty} d\tau, \qquad \delta = \gamma^2(s-1),$$

we see that $\mathcal{F}(\mathcal{K}) \subset \mathcal{K}$.

We now show that $\mathcal{F}$ is continuous. Let $\{\widetilde{\omega}_n\} \subset \mathcal{K}$ be a sequence that converges to $\omega \in \mathcal{K}$. We set $\omega_n = \mathcal{F}(\widetilde{\omega}_n)$ and $\omega = \mathcal{F}(\widetilde{\omega})$. Since $\mathcal{K}$ is compact in $C(Q)$, there is $n_j \to \infty$ such that $\{\omega_{n_j}\}$ converges to a function $\overline{\omega} \in \mathcal{K}$. On the other hand, the continuous dependence of solutions to ODE's on the vector field implies that $\omega_n(t,x)$ must converge to $\omega(t,x)$ for any $(t,x) \in Q$. It follows that $\overline{\omega} \equiv \omega$, and the entire sequence $\{\omega_n\}$ converges to ω.

We have thus shown that $\mathcal{F} : \mathcal{K} \to \mathcal{K}$ is a continuous mapping. By the Leray–Schauder theorem, there is $\omega \in \mathcal{K} \subset L^\infty(J, C^\delta)$ such that $\mathcal{F}(\omega) = \omega$. It is clear that ω is a solution of (2.17), (2.18).

Step 3: Regularity. To complete the proof of the theorem, it remains to show that $\omega \in L^\infty(J, C^{s-1})$. Suppose that $1 < s < 2$. Since $\omega \in L^\infty(J, C^\delta)$, Proposition 2.4 implies that $u \in L^\infty(J, C^{1+\delta})$. By Proposition 2.1, we see that $\varphi, \psi \in W^{1,\infty}(J, C^{1+\delta})$. Combining this with the explicit formula (2.26), we conclude that $\omega \in L^\infty(J, C^{s-1})$.

We now assume that $s \in (k, k+1)$ for some integer $k \geq 2$ and it is already proved that $\omega \in L^\infty(J, C^{r-1})$ for any $r < k$. Repeating the above argument, we see that $u \in L^\infty(J, C^r)$ and $\varphi, \psi \in W^{1,\infty}(J, C^r)$. Using again representation (2.26) for ω, we obtain the required result. The proof is complete. ☐

2.4 The 3D case: Beale-Kato-Majda theorem

We now discuss some results concerning the 3D Euler system. For a smooth vector field $u = (u_1, u_2, u_3)$, we define the vorticity as

$$\operatorname{curl} u = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1).$$

Let us set $\omega = \operatorname{curl} u$ and $g = \operatorname{curl} f$. Applying the operator curl to the first equation in (2.1) and taking into account the relations

$$\operatorname{curl}(\nabla p) = 0, \quad \operatorname{curl}\big(\langle u, \nabla \rangle u\big) = \langle u, \nabla \rangle \omega - \langle \omega, \nabla \rangle u,$$

we obtain the following system for ω:

$$\partial_t \omega + \langle u, \nabla \rangle \omega - \langle \omega, \nabla \rangle u = g(t,x). \tag{2.27}$$

Compared to the 2D case, system (2.27) contains the additional nonlinear term $\langle \omega, \nabla \rangle u$. If we try to construct a solution with the help of a fixed point argument similar to that applied in the previous subsection, we shall encounter the following difficulty. The estimates that can be obtained for solutions of the associated transport equations are not sufficient to construct an invariant subset. This can by done only if T is sufficiently small. More precisely, we have the following result (see [EM70, Tem76]).

Theorem 2.8. *Let $D \subset \mathbb{R}^3$ be a bounded domain with smooth boundary and let $s > 1$ be a non-integer. For any $R > 0$ there is $T = T(R) > 0$ such that if $u_0 \in C^s_\sigma(D, \mathbb{R}^3)$, $f \in L^1(J_T, C^s)$, and*

$$|u_0|_s + \|f\|_{L^1(J_T, C^s)} \leq R,$$

then problem (2.1) – (2.3) *has a unique classical solution* (u, p) *on the interval* J_T.

It is still an open question to decide if a blow-up can happen for smooth initial data. It turns out, however, that if a classical solution cannot be continued beyond some point, then the L^∞ norm of the vorticity must blow up. The following result is established in [BKM84] for the case of the entire space or the torus and in [Fer93] for a general bounded domain.

Theorem 2.9. *Let* $T > 0$ *be a constant and let* $s > 1$ *be a non-integer. For any* $u_0 \in C_\sigma(D, \mathbb{R}^3)$ *and* $f \in L^1(J_T, C^s)$*, let us set*

$$T_* = \sup\{\tau \in J_T : \textit{problem } (2.1) – (2.3) \textit{ has a classical solution on } [0, \tau]\}.$$

In this case, if $T_* < T$*, then*

$$\int_0^{T_*} \|\omega(t)\|_{L^\infty} dt = \infty.$$

3 Controllability

In this section, we discuss the problem of controllability for some nonlinear PDE's. Our presentation is based on the approach introduced by Agrachev and Sarychev in [AS05]. For simplicity, we study in detail only the case of the 1D Burgers equation

$$\partial_t u - \nu \partial_x^2 u + u \partial_x u = f(t, x), \tag{3.1}$$

where $x \in (0, \pi)$, $t > 0$, $\nu > 0$ is a parameter, and f is a given function. Equation (3.1) is supplemented with the boundary and initial conditions

$$u(0, t) = u(\pi, t) = 0, \tag{3.2}$$

$$u(0, x) = u_0(x). \tag{3.3}$$

We first recall a theorem on well-posedness of problem (3.1) – (3.3). We next describe the concept of controllability we are interested in and formulate the main result. The principal ideas of the Agrachev–Sarychev approach are presented in Subsection 3.3, and the details are given in Subsection 3.4. We conclude this section by discussing some results on the Navier–Stokes and Euler systems and formulating two open questions.

3.1 Cauchy problem for Burgers equation

Let us fix a constant $T > 0$ and set $Q = J_T \times [0, \pi]$. To simplify the notation, we shall write

$$H = L^2(0, \pi), \quad H_0^1 = H_0^1(0, \pi), \quad \mathcal{Z}_T = C(J, H) \cap L^2(J, H_0^1).$$

The following result can be established by using the same methods as in the case of Navier–Stokes system. However, the corresponding arguments are much simpler.

Proposition 3.1. *For any $f \in L^2(Q)$ and $u_0 \in H$, problem* (3.1) – (3.3) *has a unique solution $u \in \mathcal{Z}_T$. Moreover, the operator $\mathcal{R} : H \times L^2(J, H) \to \mathcal{Z}_T$ taking a pair (u_0, f) to the solution $u \in \mathcal{Z}_T$ is uniformly Lipschitz continuous on bounded subsets.*

In what follows, we denote by $\mathcal{R}_t : H \times L^2(Q) \to H$ the restriction of $\mathcal{R}$ at time $t \in J$. Proposition 3.1 implies that $\mathcal{R}_t$ satisfies the inequality

$$\|\mathcal{R}_t(u_0, f) - \mathcal{R}_t(v_0, g)\| \leq C(R)\big(\|u_0 - v_0\| + \|f - g\|_{L^2(Q)}\big), \tag{3.4}$$

where u_0, v_0 and f, g belong to the balls of radius R centred at origin in the spaces H and $L^2(Q)$, respectively, and $C(R) > 0$ is a constant depending only on R.

3.2 Formulation of the main result

We now fix a finite-dimensional space $E \subset H$ and assume that

$$f(t, x) = h(t, x) + \eta(t, x), \tag{3.5}$$

where $h \in L^2(Q)$ is a given function and η is a control with range in E.

Definition 3.2. We shall say that problem (3.1), (3.2), (3.5) is *controllable at time T by an E-valued control* if for any constant $\varepsilon > 0$, any functions $u_0, \hat{u} \in H$, and any finite-dimensional subspace $F \subset H$ there is a control $\eta \in C^\infty(J, E)$ such that

$$\|\mathcal{R}_T(u_0, h + \eta) - \hat{u}\| < \varepsilon, \tag{3.6}$$

$$\mathsf{P}_F \mathcal{R}_T(u_0, h + \eta) = \mathsf{P}_F \hat{u}, \tag{3.7}$$

where $\mathsf{P}_F : H \to H$ stands for the orthogonal projection in H onto F.

The following theorem is the main result of this section. It was first established in [AS05] for a more complicated case of the Navier–Stokes equation.

Theorem 3.3. *Let E be the vector space spanned by the function* $\sin x$ *and* $\sin(2x)$*. Then, for any $\nu > 0$, $T > 0$, and $h \in L^2(Q)$, problem* (3.1), (3.2), (3.5) *is controllable at time T by an E-valued control.*

We now present the scheme of the proof of the above theorem; the details are given in Subsection 3.4.

3.3 Agrachev–Sarychev approach

We shall show that the required result is a consequence of the so-called *uniform approximate controllability*. We then outline the proof of the latter property.

Reduction to uniform approximate controllability

Definition 3.4. Let us fix a constant $\varepsilon > 0$, a function $u_0 \in H$, and a compact set $\mathcal{K} \subset H$. Problem (3.1), (3.2), (3.5) is said to be *$(\varepsilon, u_0, \mathcal{K})$-controllable at time T by an E-valued control* if there is a continuous mapping $\Psi : \mathcal{K} \to L^2(J, E)$ such that

$$\sup_{u \in \mathcal{K}} \|\mathcal{R}_T(u_0, h + \Psi(u)) - u\| < \varepsilon. \tag{3.8}$$

In what follows, the time T and the control space E are fixed, and we shall simply say that Eq. (3.1) is *$(\varepsilon, u_0, \mathcal{K})$-controllable*.

Definition 3.5. Problem (3.1), (3.2), (3.5) is said to be *uniformly approximately controllable* if it is $(\varepsilon, u_0, \mathcal{K})$-controllable for any $\varepsilon > 0$, $u_0 \in H$, and $\mathcal{K} \Subset H$.

Theorem 3.3 is a consequence of the following result. Its proof is sketched below.

Theorem 3.6. *Let E be the vector span of the functions* $\sin x$ *and* $\sin(2x)$. *Then for any* $\nu > 0$ *and* $h \in L^2(Q)$ *problem* (3.1), (3.2), (3.5) *is uniformly approximately controllable by an E-valued control.*

Proof of Theorem 3.3. Let us fix a constant $\varepsilon > 0$, functions $u_0, \hat{u} \in H$, and a finite-dimensional space $F \subset H$. Without loss of generality, we can assume that $\hat{u} \in F$; otherwise, we can replace F by the larger space spanned by F and $\hat{u}$.

Let us denote by $B_F(R)$ the ball in F of radius R centred at origin and define $\mathcal{K} = B_F(\|\hat{u}\| + \varepsilon)$. Since $\mathcal{K}$ is a compact subset of H, in view of Theorem 3.6, we can construct a continuous mapping $\Psi : \mathcal{K} \to L^2(J, E)$ satisfying inequality (3.8). Furthermore, since $\mathcal{K} \subset H$ is compact and $C^\infty(J, E)$ is dense in $L^2(J, E)$, we can assume that the range of Ψ is contained in $C^\infty(J, E)$; otherwise, we can replace the function Ψ by its convolution with a mollifying kernel. Let us consider the mapping

$$\Phi : \mathcal{K} \to F, \quad \Phi(u) = \mathsf{P}_F \mathcal{R}_T(u_0, h + \Psi(u)).$$

It follows from (3.8) that Φ is a continuous mapping satisfying the inequality

$$\sup_{u \in \mathcal{K}} \|\Phi(u) - u\| < \varepsilon.$$

Proposition 4.8 (see the Appendix) implies that the image of Φ contains the ball $B_F(\|\hat{u}\|)$. In particular, there is $\bar{u} \in \mathcal{K}$ such that $\Phi(\bar{u}) = \hat{u}$. Setting $\eta = \Psi(\bar{u})$, we see that

$$\mathsf{P}_F \mathcal{R}_T(u_0, h + \eta) = \hat{u}. \tag{3.9}$$

Furthermore, it follows from (3.8) and (3.9) that

$$\begin{aligned} \|\mathcal{R}_T(u_0, h + \eta) - \hat{u}\| &= \|\mathcal{R}_T(u_0, h + \eta) - \mathsf{P}_F \mathcal{R}_T(u_0, h + \eta)\| \\ &\le \|\mathcal{R}_T(u_0, h + \Psi(\bar{u})) - \bar{u}\| < \varepsilon, \end{aligned}$$

where we used the facts that $\bar{u} \in F$ and that P_F is an orthogonal projection. This completes the proof of Theorem 3.3. □

Scheme of the proof of Theorem 3.6

Let us fix a constant $\varepsilon > 0$, a function $u_0 \in H$, and a compact set $\mathcal{K} \Subset H$. We need to show that Eq. (3.1) is $(\varepsilon, u_0, \mathcal{K})$-controllable by an E-valued control.

Step 1: Extension principle. Let $G \subset H^2 \cap H_0^1$ be an arbitrary finite-dimensional subspace. Along with (3.1), consider the equation

$$\partial_t u - \nu\partial_x^2(u + \zeta(t,x)) + (u + \zeta(t,x))\partial_x(u + \zeta(t,x)) = h(t,x) + \eta(t,x), \quad (3.10)$$

where η and ζ are G-valued control functions. This is a Burgers-type equation, and using the same methods as for the Navier–Stokes system, one can prove that the Cauchy problem for (3.10) is well posed. We shall say that problem (3.10), (3.2) is $(\varepsilon, u_0, \mathcal{K})$-controllable by G-valued controls (η, ζ) if there is a continuous mapping $\widehat{\Psi} : \mathcal{K} \to L^2(J, G \times G)$ such that

$$\sup_{u \in \mathcal{K}} \|\widehat{\mathcal{R}}_T(u_0, \widehat{\Psi}(u)) - u\| < \varepsilon, \quad (3.11)$$

where $\widehat{\mathcal{R}}_t : H \times L^2(J, G \times G) \to H$ stands for the operator that takes the triple (u_0, η, ζ) to the solution $u(t,\cdot)$ of problem (3.10), (3.2), (3.3).

Even though Eq. (3.10) is "more controlled" than Eq. (3.1), it turns out that the property of uniform approximate controllability is equivalent for them. Namely, we have the following result.

Proposition 3.7. *Equation* (3.1) *is* $(\varepsilon, u_0, \mathcal{K})$*-controllable if and only if so is problem* (3.10), (3.2).

Step 2: Convexification principle. Now let $N \subset H^2 \cap H_0^1$ be another finite-dimensional subspace such that

$$N \subset G, \quad B(N) \subset G, \quad (3.12)$$

where $B(u) = u\partial_x u$. Denote by $\mathcal{F}(N, G)$ the intersection of $H^2 \cap H_0^1$ with the vector space spanned by the functions of the form[2]

$$\eta + \xi\partial_x\tilde{\xi} + \tilde{\xi}\partial_x\xi, \quad (3.13)$$

where $\eta, \xi \in G$ and $\tilde{\xi} \in N$. It is easy to see that $\mathcal{F}(N, G)$ is a well-defined finite-dimensional space. The following proposition, which is an infinite-dimensional analogue of the well-known convexification principle for controlled ODE's (e.g., see [AS04, Theorem 8.7]), is a key point of the proof of Theorem 3.6.

Proposition 3.8. *Let N and G be finite-dimensional subspaces in $H^2 \cap H_0^1$ that satisfy* (3.12)*. Then* (3.10), (3.2) *is* $(\varepsilon, u_0, \mathcal{K})$*-controllable by a $G \times G$-valued control if and only if* (3.1), (3.2), (3.5) *is* $(\varepsilon, u_0, \mathcal{K})$*-controllable by an $\mathcal{F}(N, G)$-valued control.*

[2]Note that a function of the form (3.13) does not necessarily belong to $H^2 \cap H_0^1$, and therefore the space $\mathcal{F}(N, G)$ may be not larger than G.

Step 3: Saturating property. Propositions 3.7 and 3.8 imply the following result, which is a kind of "relaxation property" for the controlled Navier–Stokes system.

Proposition 3.9. *Let N and G be finite-dimensional subspaces in $H^2 \cap H_0^1$ that satisfy* (3.12)*. Then Eq.* (3.1) *is $(\varepsilon, u_0, \mathcal{K})$-controllable by a G-valued control if and only if it is $(\varepsilon, u_0, \mathcal{K})$-controllable by an $\mathcal{F}(N, G)$-valued control.*

We now introduce the subspaces $E_k = \{\sin(jx), 1 \le j \le k\}$, so that the space E defined in Theorem 3.3 coincides with E_2. We wish to apply Proposition 3.9 to the subspaces $N = E_1$ and $G = E_k$.

Lemma 3.10. *For any integer $k \ge 2$, we have $\mathcal{F}(E_1, E_k) = E_{k+1}$.*

Proposition 3.9 and Lemma 3.10 imply that, for any integer $k \ge 2$, problem (3.1), (3.2), (3.5) is $(\varepsilon, u_0, \mathcal{K})$-controllable by an E_k-valued control if and only if it is $(\varepsilon, u_0, \mathcal{K})$-controllable by an E_{k+1}-valued control. Thus, Theorem 3.6 will be established if we find an integer $N \ge 2$ such that Eq. (3.1) is $(\varepsilon, u_0, \mathcal{K})$-controllable by an E_N-valued control. We shall be able to do that due to the *saturating property*

$$\bigcup_{k=2}^{\infty} E_k \text{ is dense in } H, \tag{3.14}$$

which is a straightforward consequence of the definition of E_k.

Step 4: Case of a large control space. It is easy to construct a continuous mapping $\Psi_0 : \mathcal{K} \to L^2(J, H)$ such that

$$\sup_{u \in \mathcal{K}} \|\mathcal{R}_T(u_0, h + \Psi_0(u)) - u\| < \varepsilon. \tag{3.15}$$

Since $\mathcal{K} \subset H$ is a compact set, the image $\Psi_0(\mathcal{K})$ is compact in $L^2(J, H)$. Using (3.14), it is not difficult to approximate Ψ_0, within any accuracy $\delta > 0$, by a continuous function $\Psi : \mathcal{K} \to L^2(J, H)$ with range in $L^2(J, E_N)$:

$$\sup_{u \in \mathcal{K}} \|\Psi_0(u) - \Psi(u)\| < \delta. \tag{3.16}$$

Since the function $\mathcal{R}_t(u_0, h + \eta)$ is Lipschitz continuous on bounded subsets, inequalities (3.15) and (3.16) with $\delta \ll 1$ imply (3.8). This completes the proof of Theorem 3.6.

3.4 Details of proof of Theorem 3.3

To simplify the presentation, we shall assume that $\mathcal{K}$ consists of a single point $\hat{u} \in H$. The proof in the general case can be carried out by similar arguments, following carefully the dependence of all the objects on the final point $\hat{u}$; cf. the papers [AS05, Shi07] and Problems 12, 13. In what follows, the constant ε, the function u_0, and the subset $\mathcal{K} = \{\hat{u}\}$ are fixed, and we shall say simply *ε-controllable* rather than *$(\varepsilon, u_0, \mathcal{K})$-controllable*.

3.4.1 Extension principle

In this subsection, we prove Proposition 3.7. It is clear that if problem (3.1), (3.2) is ε-controllable, then so is problem (3.10), (3.2), because it suffices to take $\zeta \equiv 0$. Let us establish the converse assertion.

Let $(\hat{\eta}, \hat{\zeta}) \in L^2(J, G)$ be an arbitrary control such that

$$\|\widehat{\mathcal{R}}_T(u_0, \hat{\eta}, \hat{\zeta}) - \hat{u}\| < \varepsilon. \tag{3.17}$$

In view of continuity of $\widehat{\mathcal{R}}_T(u_0, \eta, \zeta)$ with respect to $\zeta \in L^2(J, H)$, there is no loss of generality in assuming that

$$\hat{\zeta} \in C^\infty(J, G), \quad \hat{\zeta}(0) = \hat{\zeta}(T) = 0. \tag{3.18}$$

Consider the function $u(t, x) = \widehat{\mathcal{R}}_t(u_0, \hat{\eta}, \hat{\zeta}) + \hat{\zeta}(t, x)$. It is straightforward to see that it belongs to $\mathcal{Z}_T$ and satisfies Eqs. (3.1), (3.2) with $\eta = \hat{\eta} + \partial_t \hat{\zeta} \in L^2(J, G)$. Moreover, it follows from (3.17) and (3.18) that

$$u(0) = u_0, \quad \|u(T) - \hat{u}\| = \|\widehat{\mathcal{R}}_T(u_0, \hat{\eta}, \hat{\zeta}) - \hat{u}\| < \varepsilon.$$

Thus, problem (3.1), (3.2), (3.5) is ε-controllable.

3.4.2 Convexification principle

Let us prove Proposition 3.8. It follows from the extension principle that if problem (3.10), (3.2) is ε-controllable by a $G \times G$-valued control, then Eq. (3.1) is ε-controllable by a G-valued control and all the more by an $\mathcal{F}(N, G)$-valued control. The proof of the converse assertion is divided into several steps. We need to show that if $\eta_1 \in L^2(J, H)$ is an $\mathcal{F}(N, G)$-valued control such that

$$\|\mathcal{R}_T(u_0, h + \eta_1) - \hat{u}\| < \varepsilon, \tag{3.19}$$

then there are $\eta, \zeta \in L^2(J, G)$ such that

$$\|\widehat{\mathcal{R}}_T(u_0, \eta, \zeta) - \hat{u}\| < \varepsilon. \tag{3.20}$$

Step 1. We first show that it suffices to consider the case in which η_1 is a piecewise constant function. Indeed, suppose Proposition 3.8 is proved in that case and denote $G_1 = \mathcal{F}(N, G)$. For a given $\eta_1 \in L^2(J, G_1)$, we can find a sequence $\{\eta^m\}$ of piecewise constant G_1-valued functions such that

$$\|\eta_1 - \eta^m\|_{L^2(J, G_1)} \to 0 \quad \text{as } m \to \infty.$$

By continuity of $\mathcal{R}_t$, there is an integer $n \geq 1$ such that

$$\|\mathcal{R}_T(u_0, h + \eta^n) - \hat{u}\| < \varepsilon. \tag{3.21}$$

Since the result is true for piecewise constant controls, for any $\delta > 0$ there are $\eta, \zeta \in L^2(J, G)$ such that

$$\|\mathcal{R}_T(u_0, h + \eta^n) - \widehat{\mathcal{R}}_T(u_0, \eta, \zeta)\| < \delta. \tag{3.22}$$

Comparing (3.21) and (3.22), for a sufficiently small $\delta > 0$ we arrive at (3.20).

Step 2. We now consider the case of piecewise constant G_1-valued controls. A simple iteration argument combined with the continuity of $\mathcal{R}_t$ and $\widehat{\mathcal{R}}_t$ shows that it suffices to consider the case of one interval of constancy. Thus, we shall assume that $\eta_1(t) \equiv \eta_1 \in G_1$.

We shall need the lemma below, whose proof is given at the end of this subsection. Recall that $B(u) = u\partial_x u$.

Lemma 3.11. *For any $\eta_1 \in \mathcal{F}(N, G)$ and any $\delta > 0$ there is an integer $k \geq 1$, constants $\alpha_j > 0$, and vectors $\eta, \zeta^j \in G$, $j = 1, \dots, k$, such that*

$$\sum_{j=1}^{k} \alpha_j = 1, \tag{3.23}$$

$$\Big\|\eta_1 - B(u) - \Big(\eta - \sum_{j=1}^{k} \alpha_j \big(B(u + \zeta^j) - \nu\partial_x^2 \zeta^j\big)\Big)\Big\| \leq \delta \quad \textit{for any } u \in H^1. \tag{3.24}$$

We fix a small $\delta > 0$ and choose constants $\alpha_j > 0$ and vectors $\eta, \zeta^j \in G$ satisfying (3.23), (3.24). Let us consider the equation

$$\partial_t u - \nu\partial_x^2 u + \sum_{j=1}^{k} \alpha_j \big(B(u + \zeta^j(x)) - \nu\partial_x^2 \zeta^j(x)\big) = h(t, x) + \eta(x). \tag{3.25}$$

This is a Burgers-type equation, and using the same arguments as in the case of the Navier–Stokes system, it can be proved that problem (3.25), (3.2), (3.3) has a unique solution $\tilde{u} \in \mathcal{Z}_T$. On the other hand, we can rewrite (3.25) in the form

$$\partial_t u - \nu\partial_x^2 u + u\partial_x u = h(t, x) + \eta_1(x) - r_\delta(t, x), \tag{3.26}$$

where $r_\delta(t, x)$ stands for the function under sign of norm on the left-hand side of (3.24) in which $u = \tilde{u}(t, x)$. Since $\mathcal{R}_t$ is Lipschitz continuous on bounded subsets, there is a constant $C > 0$ depending only on the L^2 norm of η_1 such that

$$\begin{aligned}\|\mathcal{R}_T(u_0, h + \eta_1) - \tilde{u}(T)\| &= \|\mathcal{R}_T(u_0, h + \eta_1) - \mathcal{R}_T(u_0, h + \eta_1 - r_\delta)\| \\ &\leq C\|r_\delta\|_{L^2(J,H)} \leq C\sqrt{T}\delta,\end{aligned}$$

where we used inequality (3.24). Combining this with (3.19), we see that if $\delta > 0$ is sufficiently small, then

$$\|\tilde{u}(T) - \hat{u}\| < \varepsilon. \tag{3.27}$$

We shall show that there is a sequence $\zeta_m \in L^2(J, G)$ such that

$$\|\widehat{\mathcal{R}}_T(u_0, \eta, \zeta_m) - \tilde{u}(T)\| \to 0 \quad \text{as } m \to \infty. \tag{3.28}$$

In this case, inequalities (3.27) and (3.28) with $m \gg 1$ will imply the required estimate (3.20) in which $\zeta = \zeta_m$.

Step 3. Following a classical idea in the control theory, we define a sequence $\zeta_m \in L^2(J, G)$ by the relation $\zeta_m(t) = \zeta(mt/T)$, where $\zeta(t)$ is a 1-periodic G-valued function such that

$$\zeta(t) = \zeta^j \quad \text{for } 0 \le t - (\alpha_1 + \cdots + \alpha_{j-1}) \le \alpha_j,\ j = 1, \ldots, k.$$

Let us rewrite (3.25) in the form

$$\partial_t u - \nu \partial_x^2 (u + \zeta_m(t,x)) + B(u + \zeta_m(t,x)) = h(t,x) + \eta(x) + f_m(t,x),$$

where we set $f_m = f_{m1} + f_{m2}$,

$$f_{m1} = -\nu \partial_x^2 \zeta_m + \nu \sum_{j=1}^{k} \alpha_j \partial_x^2 \zeta^j, \tag{3.29}$$

$$f_{m2} = B(\tilde{u} + \zeta_m) - \sum_{j=1}^{k} \alpha_j B(\tilde{u} + \zeta^j). \tag{3.30}$$

We now define an operator $K : L^2(J, H) \to \mathcal{Z}_T$ that takes a function f to the solution $u(t,x)$ of the equation

$$\partial_t u - \nu \partial_x^2 u = f(t,x),$$

supplemented with initial and boundary conditions (3.2), (3.3) with $u_0 = 0$. In other words,

$$(Kf)(t) = \int_0^t e^{\nu(t-s)A} f(s)\, ds,$$

where A stands for the operator $\frac{d^2}{dx^2}$ with the domain $\mathcal{D}(A) = H^2 \cap H_0^1$. Setting $v_m = \tilde{u} - Kf_m$, we see that $v_m \in \mathcal{Z}_T$ satisfies the equation

$$\partial_t v - \nu \partial_x^2 (v + \zeta_m) + B(v + \zeta_m + Kf_m) = h + \eta. \tag{3.31}$$

Suppose we have shown that

$$\|Kf_m\|_{\mathcal{Z}_T} \to 0 \quad \text{as } m \to \infty. \tag{3.32}$$

Then, by the Lipschitz continuity of the resolving operator for (3.31) on bounded subsets, we have

$$\|\widehat{\mathcal{R}}_T(u_0, \eta, \zeta_m) - \tilde{u}(T)\| \le \|\widehat{\mathcal{R}}_T(u_0, \eta, \zeta_m) - v_m(T)\| + \|Kf_m(T)\| \to 0$$

as $m \to \infty$. Thus, it remains to prove (3.32).

Step 4. We first note that $\{f_m\}$ is a bounded sequence in $L^2(J, H)$. It follows that

$$\|Kf_m\|_{C(J,H^1)} + \|Kf_m\|_{L^2(J,H^2)} \le C_1, \tag{3.33}$$

where we denote by C_i, $i = 1, 2, \dots$, positive constants not depending on m. Furthermore, we have the interpolation inequalities

$$\|v\| \le C_2 \|v\|_1^{1/2} \|v\|_{-1}^{1/2}, \quad \|v\|_1 \le C_3 \|v\|_2^{2/3} \|v\|_{-1}^{1/3} \quad \text{for } v \in H^2 \cap H_0^1.$$

Combining this with (3.33), we obtain

$$\begin{aligned} \|Kf_m\|_{\mathcal{Z}_T} &\le \|Kf_m\|_{C(J,H)} + \|Kf_m\|_{L^2(J,H^1)} \\ &\le C_4 \Big(\|Kf_m\|_{C(J,H^{-1})}^{1/2} + \|Kf_m\|_{L^2(J,H^{-1})}^{1/3} \Big). \end{aligned}$$

Thus, convergence (3.32) will be established if we show that

$$\|Kf_m\|_{C(J,H^{-1})} \to 0 \quad \text{as } m \to \infty. \tag{3.34}$$

Step 5. To prove (3.34), we write

$$(Kf_m)(t) = F_m(t) + G_m(t), \tag{3.35}$$

where

$$F_m(t) = \int_0^t f_m(s)\, ds, \quad G_m(t) = \nu \int_0^t A e^{\nu(t-s)A} F_m(s)\, ds.$$

Since $\|Ae^{\tau A}\|_{\mathcal{L}(H,H^{-1})} \le C_5 \tau^{-1/2}$ for $\tau > 0$, where $\|\cdot\|_{\mathcal{L}(H,H^{-1})}$ stands for the usual norm of operators from H to H^{-1}, we have

$$\begin{aligned} \|G_m\|_{C(J,H^{-1})} &\le \nu \sup_{t\in[0,T]} \int_0^t \|Ae^{\nu(t-s)A}\|_{\mathcal{L}(H,H^{-1})} \|F_m(s)\|\, ds \\ &\le C_6 \, \|F_m\|_{C(J,H)}. \end{aligned}$$

Comparing this with (3.35), we see that (3.34) will be established if we show that

$$\|F_m\|_{C(J,H)} \to 0 \quad \text{as } m \to \infty. \tag{3.36}$$

This convergence is a straightforward consequence of relations (3.29) and (3.30); cf. [Shi06, Section 3.3]. The proof of Proposition 3.8 is complete.

Proof of Lemma 3.11. It suffices to find functions $\eta, \tilde{\zeta}^j \in G$, $j = 1, \dots, m$, such that

$$\Big\| \eta_1 - \eta + \sum_{j=1}^k B(\tilde{\zeta}^j) \Big\| \le \delta. \tag{3.37}$$

If such vectors are constructed, then we can set $k = 2m$,

$$\alpha_j = \alpha_{j+m} = \frac{1}{2}, \quad \zeta^j = -\zeta^{j+m} = \tilde{\zeta}^j \quad \text{for } j = 1, \dots, m.$$

To construct $\eta, \tilde{\zeta}^j \in G$ satisfying (3.37), note that if $\eta_1 \in \mathcal{F}(N, G)$, then there are functions $\tilde{\eta}_j, \xi_j \in G$ and $\tilde{\xi}_j \in N$ such that

$$\eta_1 = \sum_{j=1}^{k} \big(\tilde{\eta}_j - \xi_j \partial_x \tilde{\xi}_j - \tilde{\xi}_j \partial_x \xi_j\big). \tag{3.38}$$

Now note that, for any $\varepsilon > 0$,

$$\xi_j \partial_x \tilde{\xi}_j + \tilde{\xi}_j \partial_x \xi_j = B(\varepsilon \xi_j + \varepsilon^{-1} \tilde{\xi}_j) - \varepsilon^2 B(\xi_j) - \varepsilon^{-2} B(\tilde{\xi}_j).$$

Combining this with (3.38), we obtain

$$\eta_1 - \sum_{j=1}^{k} \big(\tilde{\eta}_j + \varepsilon^{-2} B(\tilde{\xi}_j)\big) + \sum_{j=1}^{k} B(\varepsilon \xi_j + \varepsilon^{-1} \tilde{\xi}_j) = \varepsilon^2 \sum_{j=1}^{k} B(\xi_j).$$

Choosing $\varepsilon > 0$ sufficiently small and setting

$$\eta = \sum_{j=1}^{k} \big(\tilde{\eta}_j + \varepsilon^{-2} B(\tilde{\xi}_j)\big), \quad \tilde{\zeta}^j = \varepsilon \xi_j + \varepsilon^{-1} \tilde{\xi}_j,$$

we arrive at (3.37). □

3.4.3 Saturating property

Let us prove Lemma 3.10 and the inclusion $B(E_1) \subset E_2$. For $\xi = \sin(jx)$ and $\tilde{\xi} = \sin x$, we have

$$\begin{aligned} \xi \partial_x \tilde{\xi} + \tilde{\xi} \partial_x \xi &= \sin(jx) \cos x + j \sin x \cos(jx) \\ &= \frac{1}{2}\big((j+1) \sin(j+1)x - (j-1) \sin(j-1)x\big). \end{aligned} \tag{3.39}$$

It follows that $B(E_1) \subset E_2$ and $\mathcal{F}(E_1, E_k) \subset E_{k+1}$. Furthermore, taking $j = k$ in (3.39), we write

$$\sin(k+1)x = \frac{k-1}{k+1} \sin(k-1)x + \frac{2}{k+1}\big(\sin(kx)\, \partial_x \sin x + \sin x\, \partial_x \sin(kx)\big).$$

This relation implies that the function $\sin(k+1)x$ belongs to $\mathcal{F}(E_1, E_k)$ and therefore $E_{k+1} \subset \mathcal{F}(E_1, E_k)$.

3.4.4 Case of a large control space

We wish to construct a control $\eta \in L^2(J, E_N)$ with a large integer $N \geq 2$ such that

$$\|\mathcal{R}_T(u_0, h+\eta) - \hat{u}\| < \varepsilon. \tag{3.40}$$

To this end, consider the function $u_\mu(t,x)$ defined as

$$u_\mu(t,x) = T^{-1}\big(te^{\mu A}\hat{u} + (T-t)e^{\nu t A}u_0\big), \tag{3.41}$$

where A denotes the operator $\frac{d^2}{dx^2}$ with the domain $\mathcal{D}(A) = H^2 \cap H_0^1$, and $\mu > 0$ is a small constant that will be chosen later. The function u_μ belongs to the space $\mathcal{Z}_T$ and satisfies Eqs. (3.1) – (3.3) and (3.5), in which

$$\eta = \eta_\mu := \partial_t u_\mu - \nu \partial_x^2 u_\mu + u_\mu \partial_x u_\mu - h. \tag{3.42}$$

This function belongs to $L^2(J, H)$. Furthermore,

$$\|u_\mu(T) - \hat{u}\| = \|e^{\mu A}\hat{u} - \hat{u}\| \to 0 \quad \text{as } \mu \to 0. \tag{3.43}$$

Choosing $\mu > 0$ sufficiently small in (3.43) and approaching $\eta_\mu \in L^2(J, H)$ by continuous H-valued functions, we can find $\tilde{\eta} \in C(J, H)$ such that

$$\|\mathcal{R}_T(u_0, h+\tilde{\eta}) - \hat{u}\| < \varepsilon. \tag{3.44}$$

Let us denote by $\mathsf{P}_k : H \to H$ the orthogonal projection in H onto the subspace E_k. In view of the saturating property (3.14), we have

$$\sup_{t\in[0,T]} \|\mathsf{P}_k\tilde{\eta}(t) - \tilde{\eta}(t)\| \to 0 \quad \text{as } k \to \infty.$$

By continuity of $\mathcal{R}_t$, we obtain

$$\|\mathcal{R}_T(u_0, h+\mathsf{P}_k\tilde{\eta}) - \mathcal{R}_T(u_0, h+\tilde{\eta})\| \to 0 \quad \text{as } k \to \infty.$$

Combining this with (3.44), we see that for a sufficiently large $N \geq 1$ the function $\eta = \mathsf{P}_N\tilde{\eta}$ satisfies (3.40). This completes the proof of Theorem 3.6 in the case $\mathcal{K} = \{\hat{u}\}$.

3.5 Remarks on the Navier–Stokes and Euler equations

Let us turn to the problem of controllability for the Navier–Stokes and Euler equations. It was established by Agrachev and Sarychev [AS05, AS06] that, for both problems considered on a 2D torus, the velocity field can be controlled in the sense of Definition 3.2 by a finite-dimensonal external force. Their results were extended to other boundary conditions by Rodrigues [Rod06] and to 3D Navier–Stokes equations on a torus by the author [Shi06, Shi07]. The case of a general Riemannian manifold satisfying some topological constraints was studied in [AS07, Rod07]. More recently, some progress has been towards controlling the finite-dimensional projections of the pressure [Ner08].

Let us formulate without proof a result on controllability of the Navier–Stokes system on the 2D torus $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$. Denote by $E \subset L^2(\mathbb{T}^2, \mathbb{R}^2)$ the six-dimensional vector space spanned by the functions

$$\begin{pmatrix}1\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\end{pmatrix}, \quad \begin{pmatrix}\sin x_2\\0\end{pmatrix}, \quad \begin{pmatrix}\cos x_2\\0\end{pmatrix}, \quad \begin{pmatrix}\sin(x_1+x_2)\\-\sin(x_1+x_2)\end{pmatrix}, \quad \begin{pmatrix}\cos(x_1+x_2)\\-\cos(x_1+x_2)\end{pmatrix}.$$

Consider the control problem

$$\partial_t u + \langle u, \nabla\rangle u - \Delta u + \nabla p = h(t,x) + \eta(t,x), \quad \operatorname{div} u = 0, \tag{3.45}$$

$$u(0,x) = u_0(x), \tag{3.46}$$

where $x \in \mathbb{T}^2$, $t \in J_T$, $h \in L^2(J_T \times \mathbb{T}^2)$ is a given function, and η is a control with range in E. Let H be the space of divergence-free functions that belong to $L^2(\mathbb{T}^2, \mathbb{R}^2)$. The following result is established in [AS06].

Theorem 3.12. *For any $\varepsilon > 0$, any functions $u_0, \hat{u} \in H$, and any finite-dimensional subspace $F \subset H$, there is a control $\eta \in C^\infty(J_T, E)$ such that the weak solution (u, p) of problem* (3.45), (3.46) *satisfies the relations*

$$\|u(T,\cdot) - \hat{u}\| < \varepsilon, \quad \mathsf{P}_F u(T,\cdot) = \mathsf{P}_F \hat{u},$$

where $P_F : H \to H$ denotes the orthogonal projection onto F.

The above-mentioned results on controllability of the Navier–Stokes and Euler systems have one common point: for all of them, the boundary conditions ensure that the Leray projection commutes with the Laplacian. It is a challenging problem to construct an example for which that property does not hold and still the controllability is true. For instance, the following question is completely open.

Problem 1. *Let us define the strip $D = \mathbb{R} \times (0,1)$ and consider the controlled Navier–Stokes system* (3.45) *in D supplemented with the periodicity condition in the horizontal direction and the Dirichlet condition in the vertical one:*

$$u(x_1, 0) = u(x_1, 1) = 0, \quad u(x_1+1, x_2) = u(x_1, x_2), \quad p(x_1+1, x_2) = p(x_1, x_2).$$

Is it possible to find a finite-dimensional vector space $E \subset L^2$ such that Eq. (3.45) *is controllable in the sense of Definition 3.2 by an E-valued control $\eta(t,x)$?*

Another question of interest concerns the problem of exact controllability. It is well known that Eq. (3.45) considered on the torus $\mathbb{T}^2$ is exactly controllable by an external force η supported by a given open set $\omega \subset \mathbb{T}^2$; see the references in [Fur00, Cor07]. Nothing is known for a similar problem in the case of a finite-dimensional control (see [Shi08] for the case of the Euler system).

Problem 2. *Suppose that $x \in \mathbb{T}^2$ and $h \in C^\infty(\mathbb{R}_+ \times \mathbb{T}^2)$. Let $(\hat{u}, \hat{p})$ be a weak solution of Eq.* (3.45) *with $\eta \equiv 0$. Is it possible to find a finite-dimensional space $E \subset L^2$ such that for any $u_0 \in H$ there is a time $T > 0$ and a control $\eta \in L^2(J_T, E)$ for which $u(T) = \hat{u}(T)$, where (u, p) stands for the weak solution of* (3.45), (3.46)*?*

4 Appendix

Gronwall inequality

Proposition 4.1. *Let $J \subset \mathbb{R}$ be a closed interval, let $\tau \in J$, let $b \geq 0$ be a constant, and let $\varphi \in C(J)$ and $a \in L^1(J)$ be non-negative functions such that*

$$\varphi(t) \leq \left| \int_\tau^t a(s)\varphi(s)\, ds \right| + b \quad \text{for } t \in J. \tag{4.1}$$

Then

$$\varphi(t) \leq b \exp\left| \int_\tau^t a(s)\, ds \right| \quad \text{for } t \in J. \tag{4.2}$$

Proof. It suffices to establish (4.1) for $t \geq \tau$; the general case can be reduced to the former by the change of variable $t = \tau - s$. Consider the function

$$\psi(t) = \exp(-A(t)) \left(\int_\tau^t a(s)\varphi(s)\, ds + b \right),$$

where $A(t) = \int_\tau^t a(s)\, ds$. Inequality (4.1) implies that

$$\dot{\psi}(t) = a(t) \exp(-A(t)) \left(\varphi(t) - \int_\tau^t a(s)\varphi(s)\, ds - b \right) \leq 0,$$

whence it follows that

$$\psi(t) \leq \psi(\tau) = b.$$

Recalling the definition of ψ, we obtain the inequality

$$\varphi(t) \leq \psi(t) \exp(A(t)) \leq b \exp(A(t)),$$

which coincides with (4.2) for $t \geq \tau$. □

Comparison principle

Proposition 4.2. *Let $\Lambda \subset \mathbb{R}$ be an open interval, let $V \in C^1(\Lambda)$, and let x and y be two absolutely continuous functions on $J = [0, T]$ such that*

$$x(t), y(t) \in \Lambda \quad \text{for } t \in J, \tag{4.3}$$

$$\dot{x}(t) \leq V(x(t)), \quad \dot{y}(t) = V(y(t)) \quad \text{for almost every } t \in J. \tag{4.4}$$

In this case, if $x(0) \leq y(0)$, then $x(t) \leq y(t)$ for all $t \in J$.

Proof. For any $\varepsilon > 0$ we denote by $y_\varepsilon(t)$ the solution of the problem

$$\dot{z} = V(z) + \varepsilon, \quad z(0) = y(0).$$

It follows from (4.3) that y_ε is defined on an interval $[0, T_\varepsilon]$, where $T_\varepsilon \le T$ and $T_\varepsilon \to T$ as $\varepsilon \to 0$. Moreover,

$$y_\varepsilon(t) \to y(t) \quad \text{as } \varepsilon \to 0 \text{ for any } t \in [0, T). \tag{4.5}$$

Let us set

$$\tau_\varepsilon = \max\{t \in [0, T_\varepsilon] : x(s) \le y_\varepsilon(s) \text{ for } 0 \le s \le t\}.$$

If $\tau_\varepsilon < T_\varepsilon$, then $x(\tau_\varepsilon) = y_\varepsilon(\tau_\varepsilon)$. Furthermore, it follows from (4.4) and the definition of y_ε that $\dot{x}(\tau_\varepsilon) < \dot{y}_\varepsilon(\tau_\varepsilon)$. We conclude that $x(t) < y_\varepsilon(t)$ on some interval $[\tau_\varepsilon, \tau_\varepsilon + \delta]$. This contradicts the definition of τ_ε, and therefore $\tau_\varepsilon = T_\varepsilon$.

We have thus shown that $x(t) \le y_\varepsilon(t)$ for $0 \le t \le T_\varepsilon$. Passing to the limit as $\varepsilon \to 0$ and taking into account (4.5), we arrive at the required result. □

Poincaré and Friedrichs inequalities

The following elegant result is taken from Konkov's lecture notes [Kon02].

Theorem 4.3. *Let $D \subset \mathbb{R}^d$ be a bounded domain with C^1 boundary and let p be a continuous functional on $H^1(D)$ that does not vanish on non-zero constants. Then there is $C > 0$ such that*

$$\|u\|_1 \le C\big(\|\nabla u\| + |p(u)|\big) \quad \text{for any } u \in H^1(D). \tag{4.6}$$

Applying the above theorem to the functionals

$$p_1(u) = \int_D u\,dx, \quad p_2(u) = \int_{\partial D} u\,d\sigma,$$

we obtain the Poincaré and Friedrich inequalities:

$$\|u - \langle u\rangle\| \le C\,\|\nabla u\| \quad \text{for any } u \in H^1(D) \tag{4.7}$$

$$\|u\| \le C\,\|\nabla u\| \quad \text{for any } u \in H_0^1(D). \tag{4.8}$$

Some boundary value problems for the Laplace operator

Let us consider the Dirichlet problem for the Laplacian:

$$-\Delta u = f, \quad u\big|_{\partial D} = 0, \tag{4.9}$$

where $f \in H^{-1}(D)$ is a given function. Recall that a function $u \in H_0^1(D)$ is called a *solution* for (4.9) if

$$\int_D \langle \nabla u, \nabla \varphi\rangle\,dx = f(\varphi) \quad \text{for any } \varphi \in C_0^\infty(D).$$

The following result is well known; its proof can be found in [GT01, Chapter 8].

Proposition 4.4. *For any function $f \in H^{-1}(D)$, problem* (4.9) *has a unique solution $u \in H_0^1(D)$, which satisfies the inequality*

$$\|u\|_1 \leq C\,\|f\|_{-1}, \tag{4.10}$$

where $C > 0$ does not depend on f. Moreover, if $f \in H^s(D)$ for an integer $s \geq 0$, then $u \in H^{s+2}(D)$, and there is a constant $C_s > 0$ such that

$$\|u\|_{s+2} \leq C_s\,\|f\|_s, \tag{4.11}$$

A similar result is true for the Neumann problem with inhomogeneous boundary condition. Namely, let us fix a divergence-free function $g \in L^2(D, \mathbb{R}^d)$ and consider the problem

$$\Delta u = 0, \quad \langle \nabla u - g, \boldsymbol{n}\rangle\big|_{\partial D} = 0. \tag{4.12}$$

A function $u \in H^1(D)$ is called a *solution* for (4.12) if

$$\int_D \langle \nabla u - g, \nabla \varphi\rangle\, dx = 0 \quad \text{for any } \varphi \in H^1(D).$$

Recall that $\dot{H}^1(D)$ stands for the space of function in $H^1(D)$ with zero mean value. We have the following existence, uniqueness, and regularity theorem.

Proposition 4.5. *For any divergence-free function $g \in L^2(D, \mathbb{R}^d)$, problem* (4.12) *has a unique solution $u \in \dot{H}^1(D)$, which satisfies the inequality*

$$\|u\|_1 \leq C\,\|g\|. \tag{4.13}$$

Moreover, if $g \in H^s(D)$ for an integer $s \geq 1$, then $u \in H^{s+1}(D)$, and

$$\|u\|_{s+1} \leq C_s\,\|g\|_s. \tag{4.14}$$

Proof. We confine ourselves to the proof of existence of solution, because the regularity and inequality (4.14) are standard; see [GT01] or [Tay97, Section 5.7]. Let us endow $\dot{H}^1(D)$ with the scalar product $(\nabla u, \nabla v)$ and consider a continuous functional $\ell : \dot{H}^1(D) \to \mathbb{R}$ defined by the relation

$$\ell(\varphi) = \int_D g \nabla \varphi\, dx, \quad \varphi \in \dot{H}^1(D).$$

By the Riesz representation theorem (see [Yos95, Section III.6]), there is a unique $u \in \dot{H}^1(D)$ such that $\ell(\varphi) = (\nabla u, \nabla \varphi)$ for any $\varphi \in \dot{H}^1(D)$. This relation coincides with (4.12). □

Friedrichs extension

Let H be a Hilbert space with scalat product $(\cdot,\cdot)$ and the norm $\|\cdot\|$ and let L_0 be a semi-bounded symmetric operator in H with domain $\mathcal{D}(L_0)$:

$$(L_0 u, v) = (u, L_0 v) \quad \text{for } u, v \in \mathcal{D}(L_0), \tag{4.15}$$

$$(L_0 u, u) \geq -M\|u\|^2 \quad \text{for } u \in \mathcal{D}(L_0), \tag{4.16}$$

where $M \geq 0$ does not depend on u. Let us define an operator L by the following rule:

- denote by $\mathcal{D}(Q) \subset H$ the closure of $\mathcal{D}(L_0)$ with respect to the norm $\|u\|_q = \big((L_0 u, u) + (M+1)\|u\|^2\big)^{1/2}$ and define a quadratic form on $\mathcal{D}(Q)$ by the relation
$$Q(u, v) = \lim_{n\to\infty} (L_0 u_n, v_n), \qquad (4.17)$$
where $\{u_n\}, \{v_n\} \subset \mathcal{D}(L_0)$ are arbitrary sequences converging to u and v, respectively, in the norm $\|\cdot\|_q$.
- denote by $\mathcal{D}(L) \subset H$ the space of vectors $u \in \mathcal{D}(Q)$ for which there is $f_u \in H$ such that
$$Q(u, v) = (f_u, v) \quad \text{for any } v \in \mathcal{D}(Q);$$
denote by $L : \mathcal{D}(L) \to H$ the operator taking u to f_u.

A proof of the following important result can be found in [Yos95, RS80].

Theorem 4.6. *The operator L is well defined. Moreover, it is self-adjoint and satisfies the inequality* $(Lu, u) \geq -M\|u\|^2$ *for* $u \in \mathcal{D}(L)$.

Brouwer and Leray–Schauder theorems

The well-known Brouwer theorem says that any continuous application of the unit ball in R^n into itself has a fixed point. The following result is a generalisation of the Brouwer theorem to the infinite-dimensional case.

Theorem 4.7 (Leray–Schauder)**.** *Let K be a compact convex subset of a Banach space X and let* $\mathcal{F} : \mathcal{K} \to \mathcal{K}$ *be a continuous mapping. Then* $\mathcal{F}$ *has at least one fixed point.*

A proof of the above theorem can be found in [Tay97]. The following proposition gives a sufficient condition for the image of a continuous mapping to contain a ball.

Proposition 4.8. *Let F be a finite-dimensional vector space, let* $R > \varepsilon > 0$ *be some constants, and let* $\Phi : B_F(R) \to F$ *be a continuous operator such that*
$$\|\Phi(u) - u\|_F \leq \varepsilon \quad \text{for any } u \in B_F(R). \qquad (4.18)$$
Then $\Phi(B_F(R)) \supset B_F(R - \varepsilon)$.

Proof. Let us fix any $\hat{u} \in B_F(R - \varepsilon)$ and consider the continuous function
$$\Psi : B_F(R) \to F, \quad \Psi(u) = \hat{u} - \Phi(u) + u.$$
It follows from (4.18) that
$$\|\Psi(u)\|_F \leq \|\hat{u}\|_F + \|\Phi(u) - u\|_F \leq R.$$
Thus, Ψ is a continuous function from the ball $B_F(R)$ into itself. By the Brouwer theorem, Ψ has a fixed point $v \in B_F(R)$. It is easy to see that $\Phi(v) = \hat{u}$. □

5 Problems

1. Let $D \subset \mathbb{R}^d$ be a bounded domain with smooth boundary.
   (a) Prove that the image of the operator $\nabla : H^{k+1}(D) \to H^k(D, \mathbb{R}^d)$ is closed for any integer $k \geq 0$. *Hint:* Use the Poincaré inequality.
   (b) Prove that the image of the operator $\nabla : L^2(D) \to H^{-1}(D, \mathbb{R}^d)$ is closed. *Hint:* Use Proposition 1.6.
2. Let X and Y be Banach spaces, let $A \in \mathcal{L}(X, Y)$, and let $A^* \in \mathcal{L}(Y^*, X^*)$ be the adjoint operator of A. Show that
$$(\mathrm{Im}\, A)^{\perp} = \mathrm{Ker}\, A^*, \tag{5.1}$$
   where for a vector space $F \subset Y$ we denote by $F^{\perp}$ the space of linear functionals $\ell \in Y^*$ such that $\ell(f) = 0$ for any $f \in F$. Furthermore, if Y is a reflexive space and F is a closed subspace of Y, then
$$\mathrm{Im}\, A = (\mathrm{Ker}\, A^*)^{\perp}, \tag{5.2}$$
   where for a vector space $L \subset Y^*$ we denote by $L^{\perp}$ the space of vectors $f \in Y$ such that $\ell(f) = 0$ for any $\ell \in L$.
3. Give the details of the proof of Proposition 1.6.
4. Let $p \in H^1(D)$ be such that $\nabla p \in C(\overline{D})$. Show that $p \in C^1(\overline{D})$.
5. Show that the Leray projection can be extended by continuity to an operator from H^{-1} to V^*. *Hint:* It suffices to show that
$$|(\Pi u, \varphi)| \leq \|u\|_{-1} \|\varphi\|_1 \quad \text{for any } \varphi \in V.$$
6. Show that any function $f \in H^{-1}(D, \mathbb{R}^d)$ can be written as $f = v + \nabla p$, where $v \in H^{-1}(D, \mathbb{R}^d)$, $\mathrm{div}\, v = 0$, and $p \in L^2(D)$. *Hint:* Use the Hodge–Kodaira decomposition for the function $u_f = -\Delta^{-1} f \in H_0^1$.
7. Show that the operators $L : V \to V^*$ and $B(u, v) : V \times V \to V^*$ defined by (1.30) are continuous.
8. The aim of this problem is to construct a solution of problem (1.35) in any dimension and to prove the regularity of solution for $d = 2$.
   (a) Show that (1.35) is equivalent to (1.36). Use then the Riesz representation theorem and Theorem 1.5 to construct a solution $(u, p) \in V \times \dot{L}^2$.
   (b) Suppose that $d = 2$. Show that for any $u \in V$ there is $\psi \in H^2 \cap H_0^1$ such that $u = \mathrm{curl}\, \psi = (-\partial_2 \psi, \partial_1 \psi)$.
   (c) Show that if $(u, p) \in V \times \dot{L}^2$ is a solution of problem (1.35) with $f \in L^2$, then $-\Delta^2 \psi = \partial_1 f_2 - \partial_2 f_1$. Use this and elliptic regularity for the biharmonic operator to prove that $\psi \in H^3$, $u \in H^2$, and $p \in H^1$.
9. Show that $\|B(u, v)\|_{-1} \leq C \|u\|_{L^4} \|v\|_{L^4}$ for $u, v \in L^4(D)$ with $\mathrm{div}\, u = 0$. Use this and an interpolation inequality to show that, in the 2D case, we have
$$\|B(u, v)\|_{L^2(J, V^*)} \leq C \|u\|_{\mathcal{X}_T} \|v\|_{\mathcal{X}_T} \quad \text{for } u, v \in \mathcal{X}_T. \tag{5.3}$$

10. Show that for any divergence-free vector field $u = (u_1, u_2, u_3)$ we have
$$\operatorname{curl}\big(\langle u, \nabla \rangle u\big) = \langle u, \nabla \rangle \omega - \langle \omega, \nabla \rangle u, \tag{5.4}$$
Show also that if $u = (u_1, u_2, 0)$, where u_1 and u_2 do not depend on x_3, then
$$\omega = (0, 0, \omega_3), \quad \operatorname{curl}\big(\langle u, \nabla \rangle u\big) = (0, 0, \langle u, \nabla \rangle \omega_3).$$

11. The aim of this problem is to prove Lemma 1.17.
   (a) Let $u \in C^1(\mathbb{R}, V)$. Show that
   $$\|u(t)\|^2 = \|u(s)\|^2 + 2\int_s^t (u(r), \dot{u}(r))\, dr \quad \text{for any } s \le t. \tag{5.5}$$
   (b) Let $u \in L^2(\mathbb{R}, V)$ be such that $u \in L^2(\mathbb{R}, V)$ and $u(t) = 0$ for $|t| \ge T$. Show that u satisfies (5.5).
   (c) Prove Lemma 1.17.

12. The aim of this problem is to establish the extension and convexification principles for an arbitrary compact set $\mathcal{K} \subset H$; see Subsection 3.4. Let $T > 0$ be a constant and let N and G be finite-dimensional subspaces in $H^2 \cap H_0^1$ satisfying (3.12).
   (a) Let $\widehat{\Psi} : \mathcal{K} \to L^2(J_T, G \times G)$ be a continuous mapping for which (3.11) holds, let $\omega_n \in C_0^\infty(\mathbb{R})$ be a family of mollifying kernels, and let $\chi_n \in C_0^\infty(J_T)$ be a sequence such that $0 \le \chi_n \le 1$ and $\chi_n(t) = 1$ for $\frac{1}{n} \le t \le T - \frac{1}{n}$. We write $\widehat{\Psi}(u) = (\hat{\eta}(t; u), \hat{\zeta}(t; u))$ and set
   $$\zeta_n(t; u) = \big(\omega_n * \zeta(\cdot; u)\big)(t), \quad \eta_n(t; u) = \hat{\eta}(t; u) + \partial_t\big(\chi_n(t)\zeta_n(t; u)\big),$$
   where we extended $\zeta(\cdot; u)$ by zero outside J_T. Show that
   $$\sup_{u \in \mathcal{K}} \|\mathcal{R}_T(u, h + \eta_n(\cdot; u)) - u\| < \varepsilon \quad \text{for } n \gg 1.$$
   (b) Let $\Psi : \mathcal{K} \to L^2(J_T, \mathcal{F}(N, G))$ be a continuous function satisfying (3.8). For integers $s \ge 1$ and $r \ge 0$, denote by $I_{r,s}(t)$ the indicator function of the interval $s^{-1}[rT, (r+1)T)$. Construct a finite set $A = \{\eta_1^l, l = 1, \dots, m\} \subset \mathcal{F}(N, G)$ and continuous functions $c_{lr} : \mathcal{K} \to \mathbb{R}$ such that the mapping
   $$\Psi_1(t; u) = \sum_{l=1}^{m} \sum_{r=0}^{s-1} c_{lr}(u) I_{r,s}(t) \eta_1^l$$
   satisfies the inequality
   $$\sup_{u \in \mathcal{K}} \|\mathcal{R}_T(u, h + \Psi_1(\cdot; u)) - u\| < \varepsilon.$$
   Combine this with Lemma 3.11 to show that there is a continuous mapping $\widehat{\Psi} : \mathcal{K} \to L^2(J_T, G \times G)$ for which (3.11) holds.

13. Let $E_k \subset H$ be an increasing sequence of subspaces for which (3.14) holds. Show that for any constant $\varepsilon > 0$ there is an integer $N \ge 1$ and a continuous function $\Psi : \mathcal{K} \to L^2(J_T, E_N)$ satisfying (3.8). *Hint:* Use relations (3.41) and (3.42) to construct first a continuous mapping $\Psi_0 : \mathcal{K} \to L^2(J_T, H)$ such that (3.15) holds. Show then that the mapping $\Psi(u) = \mathsf{P}_N \Psi_0(u)$, where P_k is the orthogonal projection to E_k, possesses the required property for $N \gg 1$.

14. Use the compactness of the embedding $H^1 \subset L^2$ to prove Theorem 4.3.

Notation

We denote by $\mathbb{R}^d$ the d-dimensional Euclidean space, by $\langle \cdot, \cdot \rangle$ the standard scalar product in $\mathbb{R}^d$, and by $|\cdot|$ the corresponding norm.

If f is a distribution and φ is a test function, then we write $f(\varphi)$ or (f, φ) for the value of f on φ.

For a bounded domain $D \subset \mathbb{R}^d$ and constants $p \in [1, \infty]$, $s \in \mathbb{Z}$, and $k \in \mathbb{Z}_+$, we denote by $L^p(D)$ the standard Lebesgue space, by $H^s(D)$ the Sobolev space of order s, and by $C^k(\overline{D})$ the space of k times continuously differentiable functions on $\overline{D}$. The Hölder space $C^s(D)$ for a non-integer $s > 0$ and the corresponding norm are defined in Section 2.1.

If X is a subspace in $L^1(D)$, then $\dot{X}$ stands for the space of functions $u \in X$ whose mean value is zero (see (1.5)).

For $\nu, \gamma \in (0, 1]$, $T > 0$, and a bounded domain $D \subset \mathbb{R}^d$, we write $Q = (0, T) \times D$ and denote by $C^{\nu,\gamma}(Q)$ the space of continuous functions u on Q such that

$$\|u\|_{C^{\nu,\gamma}} := \sup_{(t,x)\in Q} |u(t,x)| + \sup_{(t_1,x_1)\neq(t_2,x_2)} \frac{|u(t_1,x_1) - u(t_2,x_2)|}{|t_1 - t_2|^\nu + |x_1 - x_2|^\gamma} < \infty.$$

If $T > 0$ is a constant and X is a Banach space, then we write $J = [0, T]$ and denote by $C(J, X)$ the space of continuous functions $f : J \to X$ with the natural norm. For $1 \leq p < \infty$, let $L^p(J, X)$ be the completion of $C(J, X)$ with respect to the norm

$$\|f\|_{L^p(J,X)} = \left(\int_0^T \|f(t)\|_X^p dt \right)^{1/p}.$$

$L^\infty(J, X)$ is defined as the space of functions $f : J \to X$ for which there is a bounded sequence $\{f_n\} \subset C(J, X)$ such that $f_n(t) \to f(t)$ for almost every $t \in J$. This space is endowed with the norm

$$\|f\|_{L^\infty(J,X)} = \operatorname*{ess\,sup}_{t\in J} \|f(t)\|_X.$$

For $1 \leq p \leq \infty$, we define $W^{1,p}(J, X)$ as the space of functions $f \in L^p(J, X)$ that can be written in the form

$$f(t) = f_0 + \int_0^t g(s)\, ds, \quad 0 \leq s \leq T,$$

where $g \in L^p(J, X)$ and $f_0 \in X$. This space is endowed with the norm

$$\|f\|_{W^{1,p}(J,X)} = \|f\|_{L^p(J,X)} + \|g\|_{L^p(J,X)}.$$

The spaces $\mathcal{X}_T$ and $\mathcal{Y}_T$ are defined in Propositions 1.9 and 1.14, respectively.

We denote by $C, C_1, C_2, \ldots$ unessential positive constants.

References

[ADN64] S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II*, Comm. Pure Appl. Math. **17** (1964), 35–92.

[AS04] A. A. Agrachev and Yu. L. Sachkov, *Control Theory from Geometric Viewpoint*, Springer-Verlag, Berlin, 2004.

[AS05] A. A. Agrachev and A. V. Sarychev, *Navier–Stokes equations: controllability by means of low modes forcing*, J. Math. Fluid Mech. **7** (2005), 108–152.

[AS06] ______, *Controllability of 2D Euler and Navier–Stokes equations by degenerate forcing*, Commun. Math. Phys. **265** (2006), no. 3, 673–697.

[AS07] ______, *Solid controllability in fluid dynamics*, Instabilities in Models Connected with Fluid Flow. I (C. Bardos and A. Fursikov, eds.), Springer, 2007, pp. 1–35.

[BKM84] J. T. Beale, T. Kato, and A. Majda, *Remarks on the breakdown of smooth solutions for the 3-D Euler equations*, Comm. Math. Phys. **94** (1984), no. 1, 61–66.

[CF88] P. Constantin and C. Foias, *Navier-Stokes Equations*, University of Chicago Press, Chicago, 1988.

[CL55] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGill-Hill Book Company, New York–Toronto–London, 1955.

[Cor07] J.-M. Coron, *Control and Nonlinearity*, American Mathematical Society, Providence, RI, 2007.

[EM70] D. G. Ebin and J. Marsden, *Groups of diffeomorphisms and the notion of an incompressible fluid.*, Ann. of Math. (2) **92** (1970), 102–163.

[Fer93] A. B. Ferrari, *On the blow-up of solutions of the 3-D Euler equations in a bounded domain*, Comm. Math. Phys. **155** (1993), no. 2, 277–294.

[Fur00] A. V. Fursikov, *Optimal Control of Distributed Systems. Theory and Applications*, American Mathematical Society, Providence, RI, 2000.

[GT01] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 2001.

[Har82] P. Hartman, *Ordinary Differential Equations*, Birkhäuser, Boston–Basel–Stuttgart, 1982.

[Kat67] T. Kato, *On classical solutions of the two-dimensional nonstationary Euler equation*, Arch. Rational Mech. Anal. **25** (1967), 188–200.

[Kon02] A. A. Konkov, *Elliptic Equations*, lecture notes, unpublished, 2002.

[Lad63] O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York, 1963.

[Ler34] J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math. **63** (1934), no. 1, 193–248.

[Lio69] J.-L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Gauthier-Villars, Paris, 1969.

[Ner08] H. Nersisyan, *Personal communication*.

[Rod06] S. S. Rodrigues, *Navier-Stokes equation on the rectangle: controllability by means of low mode forcing*, J. Dyn. Control Syst. **12** (2006), no. 4, 517–562.

[Rod07] ______, *Controllability of nonlinear PDE's on compact Riemannian manifolds*, Workshop on Mathematical Control Theory and Finance, Lisbon, 10–14 April, 2007, pp. 462–493.

[RS80] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I. Functional Analysis*, Academic Press, New York, 1980.

[Shi06] A. Shirikyan, *Approximate controllability of three-dimensional Navier–Stokes equations*, Commun. Math. Phys. **266** (2006), no. 1, 123–151.

[Shi07] ______, *Exact controllability in projections for three-dimensional Navier-Stokes equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **24** (2007), no. 4, 521–537.

[Shi08] ______, *Euler equations are not exactly controllable by a finite-dimensional external force*, Physica D (2008), to appear.

[Soh01] H. Sohr, *The Navier-Stokes Equations*, Birkhäuser Verlag, Basel, 2001.

[Tay97] M. E. Taylor, *Partial Differential Equations. I–III*, Springer-Verlag, New York, 1996-97.

[Tem76] R. Temam, *Local existence of C^∞ solutions of the Euler equations of incompressible perfect fluids*, Turbulence and Navier-Stokes equations (Proc. Conf., Univ. Paris-Sud, Orsay, 1975), Springer, Berlin, 1976, pp. 184–194. Lecture Notes in Math., Vol. 565.

[Tem79] ______, *Navier–Stokes Equations*, North-Holland, Amsterdam, 1979.

[Wol33] W. Wolibner, *Un theorème sur l'existence du mouvement plan d'un fluide parfait, homogène, incompressible, pendant un temps infiniment long*, Math. Z. **37** (1933), no. 1, 698–726.

[Yos95] K. Yosida, *Functional Analysis*, Classics in Mathematics, Springer-Verlag, Berlin, 1995.