Thin films with many small cracks

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Abstract We show with an example that the limiting theory as thickness goes to zero of a thin film with many small cracks can be three-dimensional rather than two-dimensional.

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1 Introduction

There has been much recent interest in the rigourous derivation of limiting theories for thin films (see for example [1, 7, 2, 3, 4, 8]). These works start with a three-dimensional theory and pass to the limit without any a priori assumption or ansatz about the limiting deformation to obtain an effective theory which is two-dimensional: Consider a film $\Omega_\varepsilon = \omega \times (0, \varepsilon)$ (ω an open subset of $\mathbb{R}^2$). The behavior of this film is obtained by minimizing the elastic energy

$$\int_{\Omega_\varepsilon} W(x, Du)\, dx,$$

over all possible displacements $u \in \mathrm{W}^{1,p}(\Omega_\varepsilon; \mathbb{R}^3)$ satisfying suitable boundary conditions. It is convenient to divide this functional by ε, change variables and rewrite it as

$$\int_\Omega W\Big(x_1, x_2, \varepsilon x_3, D_1 u, D_2 u, \frac{1}{\varepsilon} D_3 u\Big)\, dx, \tag{1.1}$$

where $\Omega = \omega \times (0,1)$ is independent of ε and $u \in \mathrm{W}^{1,p}(\Omega; \mathbb{R}^3)$. To study the limiting behavior of this functional as $\varepsilon \to 0$, the natural mathematical setting is that of Γ-convergence, a notion of convergence of functionals which roughly speaking is equivalent to the requirement that minimizers (minimizing sequences) and minima of a sequence of functionals converge to a minimizer (minimizing sequence) and minimum of the limit functional. Using these methods, it is shown that if W

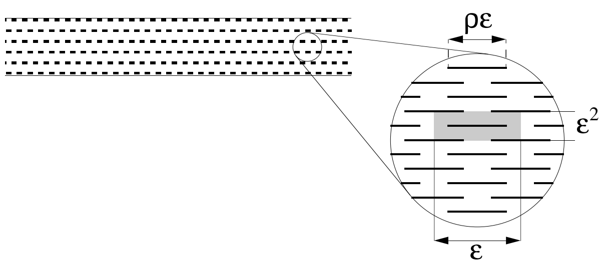

Figure 1: Thin-film with many small cracks

satisfies some growth conditions, then the asymptotic behavior as $\varepsilon \to 0$ of the functional (1.1) is described by the functional

$$\int_{\Omega} \widetilde{W}(x_\alpha, D_\alpha v)\, dx,$$

where D_α denotes differentiation with respect to the in-plane variable $x_\alpha := (x_1, x_2)$, $\widetilde{W}$ is a suitable energy density and v is subject to the constraint $D_3 v = 0$. Consequently, the integrand is independent of x_3 and the limiting functional is the two-dimensional 'membrane energy'

$$\int_{\omega} \widetilde{W}(x_\alpha, D_\alpha v)\, dx_\alpha.$$

The works cited above have established this result and have calculated $\widetilde{W}$ for various situations including finite elasticity, inhomogeneous materials, phase transforming materials and films with corrogated surfaces.

This paper deals with the situation where the set Ω_ε contains holes or cracks so that the energy density does *not* satisfy any growth condition. We show with an example that the limiting functional has the form

$$\int_{\Omega} \widetilde{W}(Dv)\, dx,$$

with $v \in \mathrm{W}^{1,p}(\Omega; \mathbb{R}^n)$, without the constraint $D_3 v = 0$. Thus, the limiting membrane theory is fully three-dimensional! As we want to highlight the effect of small fissures, which may be relevant regardless to the structure of W, we consider the simplest case of $W(F) = |F|^2$ and u scalar-valued. Moreover, we consider a 2D to 1D reduction problem, which can be trivially extended to the 3D to 2D case.

Consider the thin strip with holes $\Omega_\varepsilon \cap \tilde{E}_\varepsilon$ shown in Figure 1 where $\Omega_\varepsilon = I \times (0, \varepsilon)$, I is an interval of $\mathbb{R}$ and

$$\tilde{E}_\varepsilon = \mathbb{R}^2 \setminus \bigcup_{i,j \in \mathbb{Z}} \left(\left((\varepsilon i, \varepsilon^2 j) + \frac{\varepsilon}{2} e_1 + \left[-\frac{\rho}{2}, \frac{\rho}{2} \right] \varepsilon e_1 \right) \cup \left((\varepsilon i, \varepsilon^2 j) + \frac{\varepsilon^2}{2} e_2 + \left[-\frac{\rho}{2}, \frac{\rho}{2} \right] \varepsilon e_1 \right) \right),$$

for some $1/2 < \rho < 1$. The elastic energy of the strip is

$$\tilde{J}_\varepsilon(U) = \int_{\Omega_\varepsilon \cap \tilde{E}_\varepsilon} \Big(\Big|\frac{\partial U}{\partial X}\Big|^2 + \Big|\frac{\partial U}{\partial Y}\Big|^2 \Big) \, dX \, dY$$

where $U \in \mathrm{H}^1(\Omega_\varepsilon \cap \tilde{E}_\varepsilon; \mathbb{R})$. The main result of this paper (Theorem 3.1) shows that as $\varepsilon \to 0$, the asymptotic behavior of this strip is described by the functional

$$J(u) = \int_\Omega \Big(\Big|\frac{\partial u}{\partial x}\Big|^2 + \frac{1}{2\rho - 1}\Big|\frac{\partial u}{\partial y}\Big|^2 \Big) \, dx \, dy$$

where $\Omega = I \times (0,1)$ and $u \in \mathrm{H}^1(\Omega; \mathbb{R})$.

2 Notation and Preliminaries

In the sequel $\mathbb{M}^{m \times n}$ stands for the space of $m \times n$ matrices. We use standard notation for Lebesgue and Sobolev spaces. The scalar product is denoted by $\langle \cdot, \cdot \rangle$, and, for fixed $z \in \mathbb{R}^n$, $\langle z, x \rangle$ denotes the linear function given by this product as well. The letter c denotes a strictly positive constant whose value may vary from line to line and does not depend on the parameters of the problem.

2.1 Γ-convergence

We recall the definition of De Giorgi's Γ-convergence in L^p spaces, $1 \le p < +\infty$ (see [6], [5] or [3]). Given a family of functionals $J_\varepsilon : \mathrm{L}^p(\Omega; \mathbb{R}^m) \to [0, +\infty]$, $\varepsilon > 0$, for $u \in \mathrm{L}^p(\Omega; \mathbb{R}^m)$ we define

$$\Gamma\text{-}\liminf_{\varepsilon \to 0+} J_\varepsilon(u) := \inf\big\{ \liminf_{\varepsilon \to 0+} J_\varepsilon(u_\varepsilon) \ : \ u_\varepsilon \to u \text{ in } \mathrm{L}^p(\Omega; \mathbb{R}^m) \big\},$$

and

$$\Gamma\text{-}\limsup_{\varepsilon \to 0+} J_\varepsilon(u) := \inf\big\{ \limsup_{\varepsilon \to 0+} J_\varepsilon(u_\varepsilon) \ : \ u_\varepsilon \to u \text{ in } \mathrm{L}^p(\Omega; \mathbb{R}^m) \big\}.$$

If these two quantities coincide then their common value is called the Γ-*limit* of the sequence (J_ε) at u, and is denoted by $\Gamma\text{-}\lim_{\varepsilon \to 0+} J_\varepsilon(u)$. We recall that the Γ-upper and lower limits defined above are L^p-lower semicontinuous functions.

The importance of Γ-convergence in the calculus of variations lies in the following theorem about the convergence of minima, together with the stability properties of Γ-convergence by continuous perturbations. Its applications to thin films theory is described for example in [7].

Theorem 2.1 *Let $J = \Gamma\text{-}\lim_{\varepsilon \to 0} J_\varepsilon$, and suppose that there exists a family (u_ε) of J_ε which lies in a compact subset of $\mathrm{L}^p(\Omega; \mathbb{R}^m)$ such that $J_\varepsilon(u_\varepsilon) = \inf J_\varepsilon + o(1)$ as $\varepsilon \to 0$. Then the limit functional admits a minimum and*

$$\lim_{\varepsilon \to 0} \inf J_\varepsilon = \min J.$$

Moreover, if (ε_j) is a family of positive numbers converging to 0 such that (u_{ε_j}) converges to u in $\mathrm{L}^p(\Omega; \mathbb{R}^m)$ then u is a minimizer of J.

2.2 Homogenization

The following theorem can be found in [3] Theorem 14.8 and Example 14.11.

Theorem 2.2 *Let E be a 1-periodic set containing a unbounded connected component, let $W : \mathbb{M}^{m\times n} \to [0,+\infty)$ be a convex function satisfying a growth condition of the form*

$$c(-1+|F|^p) \le W(F) \le C(1+|F|^p).$$

If Ω is a bounded open set of $\mathbb{R}^n$ and we set for all $\varepsilon > 0$

$$J_\varepsilon(u) = \begin{cases} \displaystyle\int_{\Omega\cap\varepsilon E} W(Du(x))\,dx & \text{if } u \in \mathrm{W}^{1,p}(\Omega\cap\varepsilon E;\mathbb{R}^m) \\ +\infty & \text{otherwise in } \mathrm{L}^p(\Omega;\mathbb{R}^m), \end{cases} \tag{2.1}$$

then we have

$$\Gamma\text{-}\lim_{\varepsilon\to 0} J_\varepsilon(u) = \begin{cases} \displaystyle\int_{\Omega} W_\#(Du(x))\,dx & \text{if } u \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^m) \\ +\infty & \text{otherwise in } \mathrm{L}^p(\Omega;\mathbb{R}^m), \end{cases}$$

where $W_\# : \mathbb{M}^{m\times n} \to [0,+\infty)$ is a convex function satisfying the homogenization formula

$$W_\#(F) = \inf\Big\{\int_{(0,1)^n\cap E} W(F+Du(x))\,dx :\ u \in \mathrm{W}^{1,p}_\#((0,1)^n\cap E;\mathbb{R}^m)\Big\} \tag{2.2}$$

for all $F \in \mathbb{M}^{m\times n}$, where $\mathrm{W}^{1,p}_\#((0,1)^n\cap E;\mathbb{R}^m)$ is the set of 1-periodic functions belonging to $\mathrm{W}^{1,p}_{\mathrm{loc}}(\mathbb{R}^n\cap E;\mathbb{R}^m)$.

3 The example

Recall the strip $\Omega_\varepsilon \cap \tilde{E}_\varepsilon$ and the energy $\tilde{J}_\varepsilon(U)$ described earlier. It is convenient to change variables

$$x = X, \quad y = Y/\varepsilon, \quad u(x,y) = U(X(x),Y(y)), \quad J_\varepsilon(u) = \frac{1}{\varepsilon}\tilde{J}_\varepsilon(U)$$

and work on the fixed domain $\Omega = I \times (0,1)$. We obtain,

$$J_\varepsilon(u) = \int_{\Omega\cap\varepsilon E}\Big(\Big|\frac{\partial u}{\partial x}\Big|^2 + \frac{1}{\varepsilon^2}\Big|\frac{\partial u}{\partial y}\Big|^2\Big)\,dx\,dy,$$

on $\mathrm{H}^1(\Omega\cap\varepsilon E)$ where

$$E = \mathbb{R}^2 \setminus \bigcup_{i\in\mathbb{Z}^2}\Big(\Big(i+\frac{1}{2}e_1+\Big[-\frac{\rho}{2},\frac{\rho}{2}\Big]e_1\Big)\cup\Big(i+\frac{1}{2}e_2+\Big[-\frac{\rho}{2},\frac{\rho}{2}\Big]e_1\Big)\Big)$$

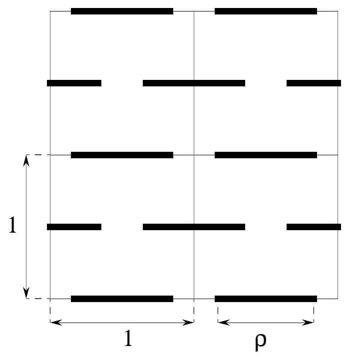

Figure 2: The set E

is shown in Figure 2. We extend $J_\varepsilon(u) = +\infty$ to $\mathrm{L}^2(\Omega) \setminus \mathrm{H}^1(\Omega \cap \varepsilon E)$. We will prove the following result.

Theorem 3.1 *The functionals J_ε defined above Γ-converge with respect to the $L^2(\Omega)$-convergence to the functional J defined on $\mathrm{L}^2(\Omega)$ by*

$$J(u) = \begin{cases} \displaystyle\int_\Omega \Big(\Big|\frac{\partial u}{\partial x}\Big|^2 + \frac{1}{2\rho - 1}\Big|\frac{\partial u}{\partial y}\Big|^2 \Big)\, dx\, dy & \text{if } u \in \mathrm{H}^1(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

3.1 Lower bound

Proposition 3.2 *Let $z \in \mathbb{R}^2$ and let*

$$f(z) = \min\Big\{ \int_{(0,1)^2 \cap E} \Big|\frac{\partial u}{\partial x}\Big|^2 dx\, dy : u - \langle z, x\rangle \in \mathrm{H}^1_\#((0,1)^2 \cap E),\ \frac{\partial u}{\partial y} = 0 \ a.e. \Big\}. \tag{3.1}$$

Then we have

$$f(z) = z_1^2 + \frac{1}{2\rho - 1} z_2^2.$$

Proof. Clearly f is a quadratic form, and by the symmetry of E we easily infer that $f(z_1, z_2) = \alpha z_1^2 + \beta z_2^2$.

To compute $\alpha = f(1,0)$ we remark that for all u with $u - \langle z, x\rangle \in \mathrm{H}^1_\#((0,1)^2 \cap E)$ we have

$$\int_{(0,1)^2 \cap E} \Big|\frac{\partial u}{\partial x}\Big|^2 dx\, dy \geq \int_0^1 \Big| \int_0^1 \frac{\partial u}{\partial x}\, dx \Big|^2 dy = 1,$$

so that $f(1,0) \geq 1$. Since this value is achieved trivially on $u(x,y) = x$ we obtain $\alpha = 1$.

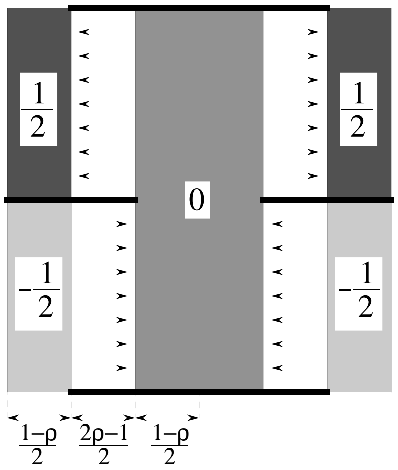

Figure 3: The minimizer for $\beta = f(0,1)$.

As for $\beta = f(0,1)$, note that $f(0,1) = \min_X \mathcal{J}$, where

$$\mathcal{J}(u) = \int_{(-1/2,1,2)^2 \cap E} \left|\frac{\partial u}{\partial x}\right|^2 dx\, dy,$$

and

$$\begin{aligned} X &= \Big\{u : u(x,y) - y \in \mathrm{H}^1_{\#}((-1/2,1/2)^2 \cap E), \\ &\quad u(\pm 1/2,-1/2) = -1/2, u(\pm 1/2, 1/2) = 1/2, \frac{\partial u}{\partial y} = 0 \text{ a.e.}\Big\}. \end{aligned}$$

As $\mathcal{J}$ is lower semicontinuous and coercive on X there exists a minimum point u, which is unique up to constants since $\mathcal{J}$ is strictly convex in $\frac{\partial u}{\partial x}$. As also $u(-x,y)$ and $-u(x,y)$ are minimizers, we obtain that

$$u(x,y) = u(-x,y) = -u(x,-y).$$

¿From these conditions, the periodicity assumption on u, and the condition $\frac{\partial u}{\partial y} = 0$ we deduce that

$$u(x,y) = 0 \text{ if } (x,y) \in \Big(-\frac{1-\rho}{2}, \frac{1-\rho}{2}\Big) \times \Big(-\frac{1}{2}, \frac{1}{2}\Big),$$

$$u(x,y) = -\frac{1}{2} \text{ if } (x,y) \in \Big(\Big(-\frac{1}{2}, -\frac{\rho}{2}\Big) \cup \Big(-\frac{\rho}{2}, -\frac{1}{2}\Big)\Big) \times \Big(-\frac{1}{2}, 0\Big),$$

and

$$u(x,y) = \frac{1}{2} \text{ if } (x,y) \in \Big(\Big(-\frac{1}{2}, -\frac{\rho}{2}\Big) \cup \Big(-\frac{\rho}{2}, -\frac{1}{2}\Big)\Big) \times \Big(0, \frac{1}{2}\Big)$$

as shown in Figure 3. As u minimizes $\mathcal{J}$ it must be affine in the remaining regions, and by the boundary values therein we obtain that

$$\frac{\partial u}{\partial x} = \pm \frac{1}{2\rho - 1},$$

and

$$f(0,1) = \frac{1}{2\rho - 1},$$

which gives β. □

Proposition 3.3 *For all $u \in \mathrm{L}^2(\Omega)$ we have $\Gamma\text{-}\liminf_{\varepsilon\to 0} J_\varepsilon(u) \geq J(u)$.*

Proof. Let $\lambda > 0$ be fixed. For $\varepsilon \leq \lambda$ we have

$$J_\varepsilon(u) \geq J_\varepsilon^\lambda(u),$$

where

$$J_\varepsilon^\lambda(u) = \begin{cases} \displaystyle\int_{\Omega\cap\varepsilon E} \Big(\Big|\frac{\partial u}{\partial x}\Big|^2 + \frac{1}{\lambda^2}\Big|\frac{\partial u}{\partial y}\Big|^2\Big)\,dx\,dy & \text{if } u \in \mathrm{H}^1(\Omega\cap\varepsilon E) \\ +\infty & \text{otherwise in } \mathrm{L}^2(\Omega). \end{cases}$$

Hence,

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} J_\varepsilon(u) \geq \Gamma\text{-}\lim_{\varepsilon\to 0} J_\varepsilon^\lambda(u).$$

By Theorem 2.2 we have

$$\Gamma\text{-}\lim_{\varepsilon\to 0} J_\varepsilon^\lambda(u) = J_\#^\lambda(u),$$

where

$$J_\#^\lambda(u) = \begin{cases} \displaystyle\int_\Omega f^\lambda(Du)\,dx\,dy & \text{if } u \in \mathrm{H}^1(\Omega) \\ +\infty & \text{otherwise in } \mathrm{L}^2(\Omega), \end{cases}$$

and $f^\lambda : \mathbb{R}^2 \to [0, +\infty)$ is the convex function satisfying the homogenization formula

$$f^\lambda(z) = \inf\Big\{\int_{(0,1)^2\cap E}\Big(\Big|\frac{\partial u}{\partial x}\Big|^2 + \frac{1}{\lambda^2}\Big|\frac{\partial u}{\partial y}\Big|^2\Big)\,dx\,dy : \ u - \langle z, x\rangle \in \mathrm{H}^1_\#((0,1)^2\cap E)\Big\} \tag{3.2}$$

for all $z \in \mathbb{R}^2$.

Let $z \in \mathbb{R}^2$ with $u - \langle z, x\rangle \in \mathrm{H}^1_\#((0,1)^2\cap E)$. By the Monotone Convergence Theorem we have

$$\sup_\lambda \int_{(0,1)^2\cap E}\Big(\Big|\frac{\partial u}{\partial x}\Big|^2 + \frac{1}{\lambda^2}\Big|\frac{\partial u}{\partial y}\Big|^2\Big)\,dx\,dy = \int_{(0,1)^2\cap E}\Big|\frac{\partial u}{\partial x}\Big|^2\,dx\,dy$$

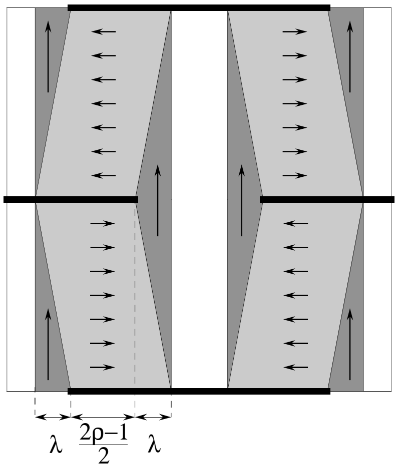

Figure 4: The test-field t for $\beta = f^\lambda(0,1)$.

if $\frac{\partial u}{\partial y} = 0$, and $+\infty$ otherwise. ¿From this we easily deduce that

$$\sup_\lambda f^\lambda(z) = f(z)$$

(see e.g. [3] Remark 7.4(iii) and Theorem 7.2) and again by the Monotone Convergence Theorem that

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} J_\varepsilon(u) \geq \sup_\lambda J^\lambda_\#(u) = J(u)$$

as desired. □

We give an alternate proof which uses an estimate for f^λ instead of f.
Alternate Proof of Proposition 3.3 Recall f^λ defined in (3.2). Clearly f^λ is a quadratic form, and by the symmetry of E we easily infer that $f^\lambda(z_1, z_2) = \alpha z_1^2 + \beta z_2^2$. It is easily verified using Jensen's inequality that $\alpha = f^\lambda(1,0) = 1$. We obtain a lower bound for $\beta = f^\lambda(0,1)$ using the following inequality which may be derived using convex duality.

$$f^\lambda(z) \geq \sup\Big\{ \int_{(0,1)^n\cap E} \Big(\langle z, t\rangle - \frac{1}{4}(t_1^2 + \lambda^2 t_2^2)\Big)\, dx\, dy,$$
$$t \in L^2_\#((0,1)^2 \cap E; \mathbb{R}^2),\ \operatorname{div} t = 0,\ t_2 = 0 \text{ on } \partial E\}\Big\}$$

where the second term in the integrand is the Fenchel transform of the integrand in the definition of f^λ. We construct a test function shown in Figure 4, first on

$(0,1/2)^2$,

$$t = \begin{cases} \dfrac{1}{\lambda(2\rho-1)}(0,1) & \left\{\frac{1-\rho}{2} - \lambda < x_1 < \frac{1-\rho}{2} - 2\lambda x_2\right\} \\ \dfrac{1}{\lambda(2\rho-1)}(0,1) & \left\{\frac{\rho}{2} < x_1 < \frac{\rho}{2} + \lambda(1-2x_2)\right\} \\ \dfrac{2}{2\rho-1}(1,0) & \left\{\frac{1-\rho}{2} - 2\lambda x_2 < x_1 < \frac{\rho}{2} + \lambda(1-2x_2)\right\} \\ 0 & \text{otherwise} \end{cases},$$

then extend it to $(0,1)^2$ using

$$\begin{gathered} t_1(x_1,x_2) = -t_1(x_1,1-x_2) = -t_1(1-x_1,x_2), \\ t_2(x_1,x_2) = t_2(x_1,1-x_2) = t_2(1-x_1,x_2), \end{gathered}$$

and finally periodically to $\mathbb{R}^2$. It is easily verified that $t \in L^2_{\#}((0,1)^2 \cap E;\mathbb{R}^2)$, $\operatorname{div} t = 0$ and $t_2 = 0$ on ∂E. Using this test function, we conclude,

$$\begin{aligned} f^\lambda(0,1) \quad &\geq \tfrac{4\lambda}{2}\left(\tfrac{1}{\lambda(2\rho-1)} - \tfrac{\lambda^2}{4}\left(\tfrac{1}{\lambda(2\rho-1)}\right)^2\right) + 4\left(\tfrac{2\rho-1+2\lambda}{4}\right)\left(-\tfrac{1}{4}\left(\tfrac{2}{2\rho-1}\right)^2\right) \\ &\geq \tfrac{1}{2\rho-1} - \lambda\left(\tfrac{5}{2}\tfrac{1}{(2\rho-1)^2}\right) \end{aligned}$$

It follows again from the Monotone Convergence Theorem that

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} J_\varepsilon(u) \geq \Gamma\text{-}\lim_{\varepsilon\to 0} J^\lambda_\varepsilon(u) = \sup_\lambda J^\lambda_{\#}(u) = J(u).$$

$\square$

3.2 Upper bound

In order to estimate the upper bound we have to construct suitable test functions. Let $A_1, A_2, B_1, B_2 \in \mathbb{R}$. We define the function $v = v^{A_2,B_2}_{A_1,B_1}$ in $\mathrm{H}^1((-1/2,1/2)^2 \setminus E)$ as that satisfying (see Figure 5):

$$v\Big(-\frac{1}{2},-\frac{1}{2}\Big) = A_1, \quad v\Big(\frac{1}{2},-\frac{1}{2}\Big) = B_1, \quad v\Big(-\frac{1}{2},\frac{1}{2}\Big) = A_2, \quad v\Big(\frac{1}{2},\frac{1}{2}\Big) = B_2, \tag{3.3}$$

$$\frac{\partial v}{\partial y} = 0 \text{ a.e. on } \Big(-\frac{1}{2},\frac{1}{2}\Big)^2 \tag{3.4}$$

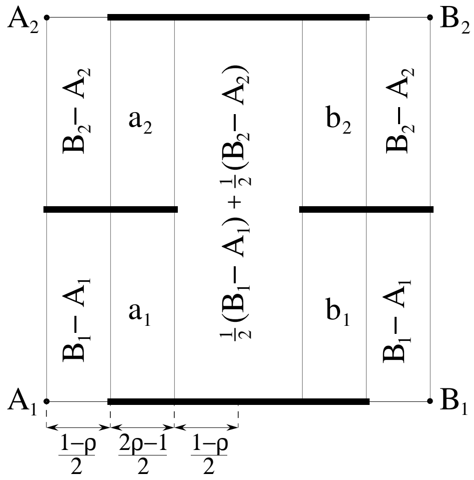

Figure 5: The test-field v

$$\frac{\partial v}{\partial x} = \begin{cases} B_1 - A_1 & \text{on } \left(-\frac{1}{2}, -\frac{\rho}{2}\right) \times \left(-\frac{1}{2}, 0\right), \\ a_1 & \text{on } \left(-\frac{\rho}{2}, \frac{\rho-1}{2}\right) \times \left(-\frac{1}{2}, 0\right), \\ b_1 & \text{on } \left(\frac{1-\rho}{2}, \frac{\rho}{2}\right) \times \left(-\frac{1}{2}, 0\right), \\ B_1 - A_1 & \text{on } \left(\frac{\rho}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{2}, 0\right), \\ \frac{1}{2}((B_2 - A_2) + (B_1 - A_1)) & \text{on } \left(-\frac{1-\rho}{2}, \frac{1-\rho}{2}\right) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \\ B_2 - A_2 & \text{on } \left(-\frac{1}{2}, -\frac{\rho}{2}\right) \times \left(0, \frac{1}{2}\right), \\ a_2 & \text{on } \left(-\frac{\rho}{2}, \frac{\rho-1}{2}\right) \times \left(0, \frac{1}{2}\right), \\ b_2 & \text{on } \left(\frac{1-\rho}{2}, \frac{\rho}{2}\right) \times \left(0, \frac{1}{2}\right), \\ B_2 - A_2 & \text{on } \left(\frac{\rho}{2}, \frac{1}{2}\right) \times \left(0, \frac{1}{2}\right), \end{cases} \tag{3.5}$$

where

$$a_1 = \tfrac{1}{2}((B_2 - A_2) + (B_1 - A_1)) + \tfrac{1}{2\rho-1}\Big(\tfrac{1-\rho}{2}(B_2 - B_1) + \tfrac{1+\rho}{2}(A_2 - A_1)\Big)$$
$$b_1 = \tfrac{1}{2}((B_2 - A_2) + (B_1 - A_1)) - \tfrac{1}{2\rho-1}\Big(\tfrac{1+\rho}{2}(B_2 - B_1) + \tfrac{1-\rho}{2}(A_2 - A_1)\Big)$$
$$a_2 = \tfrac{1}{2}((B_2 - A_2) + (B_1 - A_1)) - \tfrac{1}{2\rho-1}\Big(\tfrac{1-\rho}{2}(B_2 - B_1) + \tfrac{1+\rho}{2}(A_2 - A_1)\Big)$$

$b_2 = \frac{1}{2}((B_2 - A_2) + (B_1 - A_1)) + \frac{1}{2\rho - 1}\Big(\frac{1+\rho}{2}(B_2 - B_1) + \frac{1-\rho}{2}(A_2 - A_1)\Big)$

Proposition 3.4 *The function v defined in (3.3–3.5) satisfies:*
(a) *the traces of v on $\partial(-1/2, 1/2)^2$ are:*

$$\begin{aligned}
v(-1/2, y) = A_1, \quad v(1/2, y) = B_1 \qquad & \textit{if } y \in (-1/2, 0),\\
v(-1/2, y) = A_2, \quad v(1/2, y) = B_2 \qquad & \textit{if } y \in (0, 1/2),\\
v(x, -1/2) = A_1 + (B_1 - A_1)x \qquad & \textit{if } \frac{\rho}{2} < |x| < \frac{1}{2},\\
v(x, 1/2) = A_2 + (B_2 - A_2)x \qquad & \textit{if } \frac{\rho}{2} < |x| < \frac{1}{2},
\end{aligned}$$

(b) *we have $\dfrac{\partial v}{\partial y} = 0$ a.e., and*

$$\begin{aligned}
\int_{(-1/2,1/2)^2 \cap E} \Big|\frac{\partial v}{\partial x}\Big|^2 dx\, dy \quad = \quad & \Big(\frac{(B_2 - A_2) + (B_1 - A_1)}{2}\Big)^2\\
& + \frac{1}{2\rho - 1}\Big(\frac{(B_2 - B_1) + (A_2 - A_1)}{2}\Big)^2\\
& + C(\rho)\Big((B_2 - A_2) - (B_1 - A_1)\Big)^2,
\end{aligned}$$

where $C(\rho)$ is a constant depending only on ρ.

Proof. The proposition can be easily checked by direct computation with

$$C(\rho) = \Big(\frac{-\rho^2 + 3\rho - 1}{4(2\rho - 1)}\Big)$$

in (b). □

Proposition 3.5 *Let $u \in C^2(\overline{\Omega})$. Then there exist functions $u_\varepsilon \in \mathrm{H}^1(\Omega \cap \varepsilon E)$ such that $\lim_{\varepsilon \to 0+} J_\varepsilon(u_\varepsilon) \leq J(u)$.*

Proof. We still denote by u a fixed extension of u in $C^2(\mathbb{R}^2)$.
With fixed $\varepsilon > 0$, for all $i \in \mathbb{Z}^2$ let

$$A_1(i) = u(\varepsilon(i + (-1/2, -1/2))), \qquad B_1(i) = u(\varepsilon(i + (1/2, -1/2))),$$

$$A_2(i) = u(\varepsilon(i + (-1/2, 1/2))), \qquad B_2(i) = u(\varepsilon(i + (1/2, 1/2))),$$

and let $v_i = v$ be defined as in (3.3–3.5) with $A_1 = A_1(i)$, $B_1 = B_1(i)$, $A_2 = A_2(i)$, $B_2 = B_2(i)$. We define u_ε piecewise by setting

$$u_\varepsilon(x) = v_i\Big(\frac{x - i}{\varepsilon}\Big) \qquad \text{for } x \in \varepsilon\Big(i + \Big(-\frac{1}{2}, \frac{1}{2}\Big)^2\Big).$$

By Proposition 3.4(a) the function u_ε thus defined belongs to $\mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^2\setminus\varepsilon E)$ since the trace values of u_ε on the common boundaries of neighboring squares coincide. Moreover, if we set

$$I_\varepsilon=\{i\in\mathbb{Z}^2:\ \Omega\cap\varepsilon(i+(-1/2,1/2)^2)\neq\emptyset\},$$

by Proposition 3.4(b) we have

$$\begin{aligned}
J_\varepsilon(u_\varepsilon)\ \le\ &\sum_{i\in I_\varepsilon}\varepsilon^2\Bigg(\Big(\frac{(u(\varepsilon(i+(1/2,1/2)))-u(\varepsilon(i+(-1/2,1/2))))}{2\varepsilon}\\
&\quad+\frac{(u(\varepsilon(i+(1/2,-1/2)))-u(\varepsilon(i+(-1/2,-1/2))))}{2\varepsilon}\Big)^2\\
&+\frac{1}{2\rho-1}\Big(\frac{(u(\varepsilon(i+(1/2,1/2)))-u(\varepsilon(i+(1/2,-1/2))))}{2\varepsilon}\\
&\quad+\frac{(u(\varepsilon(i+(-1/2,1/2)))-u(\varepsilon(i+(-1/2,-1/2))))}{2\varepsilon}\Big)^2\\
&+C(\rho)\Big(\frac{(u(\varepsilon(i+(1/2,1/2)))-u(\varepsilon(i+(-1/2,1/2))))}{\varepsilon}\\
&\quad-\frac{(u(\varepsilon(i+(1/2,-1/2)))-u(\varepsilon(i+(-1/2,-1/2))))}{\varepsilon}\Big)^2\Bigg)\\
\le\ &\sum_{i\in I_\varepsilon}\varepsilon^2\Big(\Big|\frac{\partial u}{\partial x}(\varepsilon i)\Big|^2+\frac{1}{2\rho-1}\Big|\frac{\partial u}{\partial y}(\varepsilon i)\Big|^2+c\varepsilon^2\Big),
\end{aligned}$$

where we have used the Mean Value Theorem and the fact that $u\in C^2(\overline{\Omega})$ to get

$$\Big|\frac{u(\varepsilon(i+(1/2,\pm1/2)))-u(\varepsilon(i+(-1/2,\pm1/2)))}{\varepsilon}\Big|^2\le\Big|\frac{\partial u}{\partial x}(\varepsilon i)\Big|^2+c\varepsilon^2,$$
$$\Big|\frac{u(\varepsilon(i+(\pm1/2,1/2)))-u(\varepsilon(i+(\pm1/2,-1/2)))}{\varepsilon}\Big|^2\le\Big|\frac{\partial u}{\partial y}(\varepsilon i)\Big|^2+c\varepsilon^2,$$

and

$$\begin{aligned}
\Big|&\frac{(u(\varepsilon(i+(1/2,1/2)))-u(\varepsilon(i+(-1/2,1/2))))}{\varepsilon}\\
&-\frac{(u(\varepsilon(i+(1/2,-1/2)))-u(\varepsilon(i+(-1/2,-1/2))))}{\varepsilon}\Big|^2\le c\varepsilon^2.
\end{aligned}$$

Using the fact that $u\in C^2(\overline{\Omega})$ again, we obtain

$$\sum_{i\in I_\varepsilon}\varepsilon^2\Big(\Big|\frac{\partial u}{\partial x}(\varepsilon i)\Big|^2+\frac{1}{2\rho-1}\Big|\frac{\partial u}{\partial y}(\varepsilon i)\Big|^2\Big)\le\int_\Omega\Big(\Big|\frac{\partial u}{\partial x}\Big|^2+\frac{1}{2\rho-1}\Big|\frac{\partial u}{\partial y}\Big|^2\Big)\,dx\,dy+c\varepsilon^2,$$

and the proposition follows taking the limit as $\varepsilon\to0+$. $\square$

The following proposition concludes the proof of Theorem 3.1.

Proposition 3.6 *We have* $\Gamma\text{-}\limsup_{\varepsilon\to 0+} J_\varepsilon(u) \le J(u)$ *for all* $u \in \mathrm{L}^2(\Omega)$.

PROOF. Let $J''(u) = \Gamma\text{-}\limsup_{\varepsilon\to 0+} J_\varepsilon(u)$. With fixed u choose $u_j \in C^2(\overline{\Omega})$ such that $u_j \to u$ in $H^1(\Omega)$, and $\lim_j J(u_j) = J(u)$. Proposition 3.5 shows that $J''(u_j) \le J(u_j)$ for all j. Hence, by the lower semicontinuity of J'' we have

$$J''(u) \le \liminf_j J''(u_j) \le \liminf_j J(u_j) = J(u),$$

as required. □

4 Other geometries

The procedure outlined above can be repeated for other geometries of crack arrays. We include two remarks on arrays obtained by a slight variation of the example described above.

Remark 4.1 If the set E is defined as in Section 3, but with $0 \le \rho \le 1/2$ then the result of Theorem 3.1 is clearly not valid. The Γ-limit is then given by the expression

$$J(u) = \begin{cases} \displaystyle\int_\Omega \Big|\frac{\partial u}{\partial x}\Big|^2 dx\,dy & \text{if } u \in \mathrm{H}^1(\Omega) \text{ and } \dfrac{\partial u}{\partial y} = 0 \text{ a.e.} \\ +\infty & \text{otherwise,} \end{cases}$$

i.e. the effect of the cracks is not felt: the limit is 1-dimensional and its form does not depend on ρ. The upper bound is trivial, while the lower bound is obtained by comparison with the functionals with $\rho > 1/2$ considered before.

Remark 4.2 Asymmetric arrays of cracks may give rise to anisotropic limit energies. For example, we can consider a periodic array of cracks with an inclination of an angle α with respect to the x-axis.

Namely, let $-\frac{\pi}{4} < \alpha < \frac{\pi}{4}$, $\frac{1}{2} < \rho < 1$,

$$K = \Big\{ t\rho(1, \tan\alpha) : \ -\frac{1}{2} \le t \le \frac{1}{2} \Big\},$$

and define

$$E = \mathbb{R}^2 \setminus \bigcup_{i\in\mathbb{Z}^2} \Big(\Big(i + \frac{1}{2}e_1 + K\Big) \cup \Big(i + \frac{1}{2}e_2 + K\Big) \Big).$$

Note that E coincides with that defined in Section 3 if $\alpha = 0$.

Let J_ε be defined as in Section 3 with E as above. Then, following the reasoning of the proof of Theorem 3.1 it can be easily proved that the Γ-limit of the

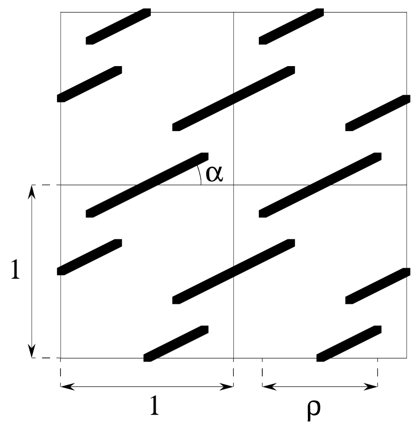

Figure 6: An array of asymmetric cracks

functionals J_ε as $\varepsilon \to 0$ is given by

$$J(u) = \begin{cases} \displaystyle\int_\Omega f(Du)\,dx\,dy & \text{if } u \in \mathrm{H}^1(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

where $f(z)$ is given by the minimum problem (3.1) in Proposition 3.2.

To compute f note that the Euler equations for the problems defining each $f(z)$ are linear. Hence, it suffices to deal with the case $z = (1,0)$ and $z = (0,1)$. In the first case, a minimizer is again $u_1(x,y) = x$, while a minimizer u_2 giving $f(0,1)$ is obtained repeating the reasoning of the proof of Proposition 3.2, and is described in Figure 7.

Finally, we easily compute

$$f(z) = \int_{(-1/2,1/2)^2} \left| z_1 \frac{\partial u_1}{\partial x} + z_2 \frac{\partial u_2}{\partial x} \right|^2 dx\,dy = z_1^2 + \frac{1}{2\rho - 1} z_2^2 + 2 z_1 z_2 \tan\alpha\,,$$

recovering the result of Theorem 3.1 when $\alpha = 0$.

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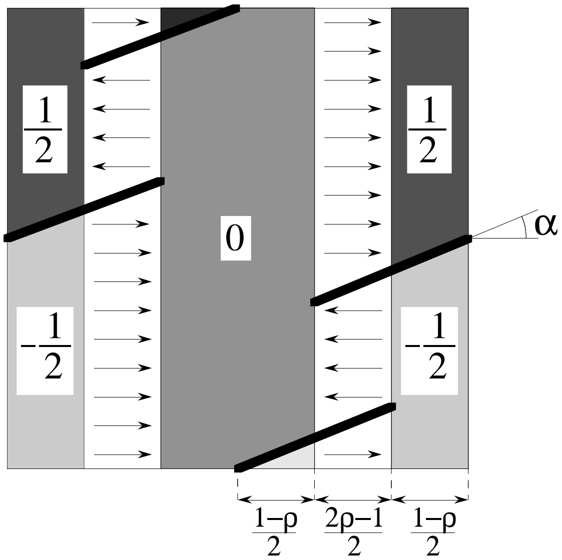

Figure 7: The minimizer for $f(0,1)$

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