THE ANISOTROPY INTRODUCED BY THE MESH IN THE FINITE ELEMENT APPROXIMATION OF THE MUMFORD-SHAH FUNCTIONAL

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ABSTRACT

We compute explicitly the anisotropy effect in the $\mathcal{H}^1$ term, generated in the approximation of the Mumford-Shah functional by finite element spaces defined on structured triangulations.

1 Introduction

As a mathematical model for image segmentation, Mumford and Shah proposed the variational problem

$$\min_{u,K} \left\{ a \int_{\Omega\setminus K} |\nabla u|^2 dx + b\mathcal{H}^1(K) + \int_\Omega |u-g|^2 dx \right\} , \tag{1}$$

where Ω is an open bounded set with Lipschitz boundary, K is a closed set in Ω, $u \in C^1(\Omega/K)$ is the regularized image, $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure, $g \in L^\infty(\Omega)$ is the original image and a, b are positive parameters. An existence result was given for of the weak formulation

$$\min_{u\in SBV(\Omega)} \left\{ a \int_\Omega |\nabla u|^2 dx + b\mathcal{H}^1(S_u) + \int_\Omega |u-g|^2 dx \right\} , \tag{2}$$

where $SBV(\Omega)$ is the space of special functions of bounded variation, S_u is the set of essential discontinuity of u and ∇u its approximate gradient. (For a detailed treatise of SBV functions see [1] or [12]). Unfortunately in general it is not possible to compute explicitly a solution of (2), so it is fundamental for the applications to have some numerical approximation of the functional.

Recently Chambolle and Dal Maso proposed in [9] a discretization of the form

$$G_\epsilon(u) = \begin{cases} \dfrac{1}{\epsilon}\displaystyle\int_\Omega f(\epsilon|\nabla u|^2)dx + \int_\Omega (u-g)^2 dx & \text{if } u \in V_{\epsilon,\theta}(\Omega) \\ +\infty & \text{if } u \in L^p(\Omega) \setminus V_{\epsilon,\theta}(\Omega) \end{cases} \tag{3}$$

where $f : [0,+\infty) \to [0,+\infty)$ is a non-decreasing continuous function such that

$$\lim_{t\to 0} \frac{f(t)}{t} = f_0 \qquad \lim_{t\to +\infty} f(t) = f_\infty \tag{4}$$

and $V_{\epsilon,\theta}(\Omega)$ is the union of all the finite element spaces defined on the triangulations of Ω such that for each element the length of the edges is between ϵ and 6ϵ and the amplitude of the angles is greater than θ. Functional (3) is to be considered an approximation of (2) because if θ is sufficently small then G_ϵ Γ-converges to the Mumford-Shah functional with $a = f_0$ and $b = f_\infty \sin(\theta)$. (For the theory of Γ-convergence see [13]).

Though the Γ-limit is exactly the Mumford-Shah functional, in practice it is not possible to solve numerically a minimization problem over the space $V_{\epsilon,\theta}(\Omega)$ which is infinite dimensional. In order to overcome this difficulty, Bourdin and Chambolle [14] could restrict the minimization on a single finite element space using an adaptive family of triangulations.

In this paper we consider the approximating functional G_ϵ no longer defined on $V_{\epsilon,\theta}(\Omega)$ but on a classical finite element space $V_\epsilon^{\,i}(\Omega)$ based on the triangulations $\mathcal{T}_\epsilon^{\,i}(\Omega)$ represented in Figure 1 and 2. In particular the elements in $\mathcal{T}_\epsilon^{\,1}(\Omega)$ are equilateral triangles and the elements of $\mathcal{T}_\epsilon^{\,2}(\Omega)$ and $\mathcal{T}_\epsilon^{\,3}(\Omega)$ are right isosceles triangles. Our aim is to prove in this framework the existence of a Γ-limit of the form

$$G(u) = f_0 \int_\Omega |\nabla u|^2 dx + f_\infty \int_{S_u} \phi_i(\nu_u)\, d\mathcal{H}^1 + \int_\Omega |u-g|^2 dx \tag{5}$$

The functional $G(u)$ is a sort of anisotropic version of (2), indeed the function $\phi_i(\nu)$, which is explicitly computed, takes into account the anistropy introduced by the geometry of the triangulation. Even if the form of the Γ-limit

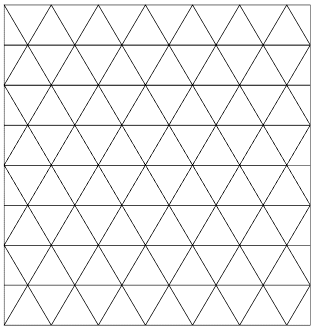

FIG 1: Triangulation $\mathcal{T}_\epsilon^{\,1}(\Omega)$

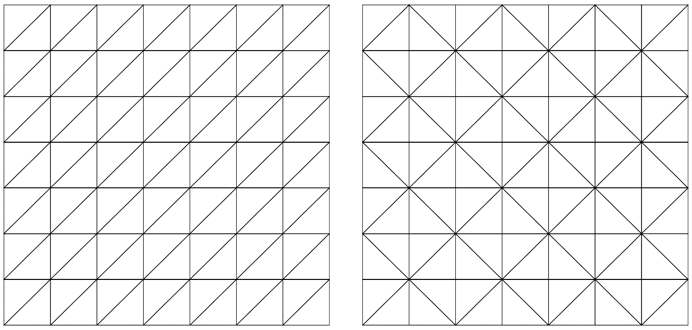

FIG 2: Triangulations $\mathcal{T}_\epsilon^{\,2}(\Omega)$ and $\mathcal{T}_\epsilon^{\,3}(\Omega)$

is not the original one, for $\mathcal{T}_\epsilon^{\,1}(\Omega)$ and $\mathcal{T}_\epsilon^{\,3}(\Omega)$ the effect of anisotropy is small, thus it is still possible to use our discretization for image segmentation.

2 Statement of the main result

First of all some notations are necessary, let $\Omega = (x_1, x_2) \times (y_1, y_2)$ be an open and bounded rectangle in $\mathbb{R}^2$, let $\mathcal{T}_\epsilon^{\,i}(\Omega)$ for $i = 1, 2, 3$ be the previous triangulations and let ϵ denote the greatest length of the edges. Note that, in order to preserve exactly the geometry of the elements, some condition should hold between ϵ, $|x_2 - x_1|$ and $|y_2 - y_1|$. It is not difficult to see that we need $2\sqrt{3}\,|y_2 - y_1|/3\epsilon \in \mathbb{N}$ and $2|x_2 - x_1|/\epsilon \in \mathbb{N}$ for $\mathcal{T}_\epsilon^{\,1}(\Omega)$, $\sqrt{2}\,|x_2 - x_1|/\epsilon \in \mathbb{N}$ and $\sqrt{2}\,|y_2 - y_1|/\epsilon \in \mathbb{N}$ for $\mathcal{T}_\epsilon^{\,2}(\Omega)$ and $\mathcal{T}_\epsilon^{\,3}(\Omega)$. Moreover let $V_\epsilon^i(\Omega)$ be the usual finite element space of piecewise affine functions on $\mathcal{T}_\epsilon^{\,i}(\Omega)$, let $\{\mathcal{T}_{\epsilon_j}^i\}$ be a sequence of triangulations with $\epsilon_j \searrow 0$ and let $f : [0, +\infty) \to [0, +\infty)$

be a non-decreasing continuous function such that

$$\lim_{t\to 0}\frac{f(t)}{t}=1 \qquad \lim_{t\to+\infty} f(t)=f_\infty<+\infty \tag{6}$$

The main result is summarized in the following

Theorem 2.1 *For each sequence of triangulations $\{\mathcal{T}^i_{\epsilon_j}\}$ there exists a convex, 1-homogeneous function $\phi_i:\mathbb{R}^2\to[0,+\infty)$ such that the functional*

$$G_{\epsilon_j}(u)=\begin{cases}\dfrac{1}{\epsilon_j}\displaystyle\int_\Omega f(\epsilon_j|\nabla u|^2)dx+\int_\Omega (u-g)^2dx & \text{if } u\in V^i_{\epsilon_j}(\Omega)\\ +\infty & \text{if } u\in L^2(\Omega)\setminus V^i_{\epsilon_j}(\Omega)\end{cases}$$

Γ-converges respect to strong L^2-topology to the "anisotropic Mumford-Shah functional"

$$G(u)=\int_\Omega|\nabla u|^2dx+f_\infty\int_{S_u}\phi_i(\nu_u)d\mathcal{H}^1+\int_\Omega(u-g)^2dx \tag{7}$$

if $u\in L^2\cap GSBV(\Omega)$ and $G(u)=+\infty$ if $u\in L^2(\Omega)\setminus GSBV(\Omega)$.

According to the standard technique of Γ-convergence, see for instance [13], the proof is given by the inequalities

$$\Gamma\text{-}\limsup_{\epsilon_j\to 0} G_{\epsilon_j}(u)\le G(u) \tag{8}$$

$$G(u)\le\Gamma\text{-}\liminf_{\epsilon_j\to 0} G_{\epsilon_j}(u) \tag{9}$$

for every $u\in L^2(\Omega)$, proved respectively in section 4 and 5.

3 Mesh anisotropies

In this section we deal with the anistropy functions $\phi_i(\nu)$ appearing in (7). For its definition we need some preliminaries. Given $\{\mathcal{T}^i_{\epsilon_j}\}$, for the computation of ϕ_i are needed a segment $S\subset\Omega$, with unit normal ν_S, and the total area of the set of triangles that covers S. To avoid problems when S intersects many elements in a single vertice, considering the set

$$\mathcal{R}=\{x\in\Omega\,:\,x=s+t\nu_S \text{ for } s\in S \text{ and } t\ge 0\} \tag{10}$$

then the set of triangles that covers S is replaced by

$$\mathcal{S}^i_{\epsilon_j}(S,\nu_S)=\{T\in\mathcal{T}^i_{\epsilon_j}(\Omega)\;:\;T\cap S\neq\emptyset \text{ and } \mathcal{L}^2(\{T\cap\mathcal{R}\})\neq 0\}\;\;. \tag{11}$$

In the following we will show that

$$\lim_{\epsilon_j\to 0}\frac{\sum_{T\in\mathcal{S}^i_{\epsilon_j}}|T|}{\epsilon_j} \tag{12}$$

exists and depends on S only throught ν_S and $\mathcal{H}^1(S)$. Moreover we will prove that a function $\phi_i(\nu)$ can be defined in such a way that the following relation

$$\phi_i(\nu)\mathcal{H}^1(S)=\lim_{\epsilon_j\to 0}\frac{\sum_{T\in\mathcal{S}^i_{\epsilon_j}}|T|}{\epsilon_j} \tag{13}$$

holds for every segment S with normal ν. In the sequel, the existence of the previous limit and the explicit form of $\phi_i(\nu)$ are treated separately for $i=1,2,3$.

Lemma 3.1 . *Let $i=1$ and let $S\subset\Omega$ be a segment with normal $\nu_S=(-\sin\alpha,\cos\alpha)$, let $\xi^1_1=(1,0)$, $\xi^1_2=(\frac{1}{2},\frac{\sqrt{3}}{2})$, $\xi^1_3=(-\frac{1}{2},\frac{\sqrt{3}}{2})$, and let the function $\phi_1(\nu):S^1\to[0,+\infty)$ be defined as*

$$\phi_1(\nu)=\begin{cases}|\langle\nu,\xi^1_1\rangle| & \text{if } 60^\circ\leq\alpha\leq 120^\circ \text{ and } -120^\circ\leq\alpha\leq -60^\circ\\[2em] |\langle\nu,\xi^1_2\rangle| & \text{if } 120^\circ\leq\alpha\leq 180^\circ \text{ and } -60^\circ\leq\alpha\leq 0^\circ\\[2em] |\langle\nu,\xi^1_3\rangle| & \text{if } 0^\circ\leq\alpha\leq 60^\circ \text{ and } -180^\circ\leq\alpha\leq -120^\circ\end{cases}$$

then the limit in (12) exists and

$$\phi_1(\nu)\mathcal{H}^1(S)=\lim_{\epsilon_j\to 0}\frac{\sum_{T\in\mathcal{S}^1_{\epsilon_j}}|T|}{\epsilon_j} \tag{14}$$

holds for any segment S with unit normal ν.

Proof. By simmetry it is sufficient to consider $0\leq\alpha\leq 60^\circ$. Moreover it will be clear from the computation that it is not rectrictive to suppose $\mathcal{H}^1(S)=1$. Finally, for every $T\in\mathcal{T}^1_{\epsilon_j}(\Omega)$ we have $|T|=\epsilon_j^2\sqrt{3}/4$ then

$$\frac{\sum_{T\in\mathcal{S}^1_{\epsilon_j}}|T|}{\epsilon_j}=\#(\mathcal{S}^1_{\epsilon_j})\,\frac{\sqrt{3}}{4}\,\epsilon_j$$

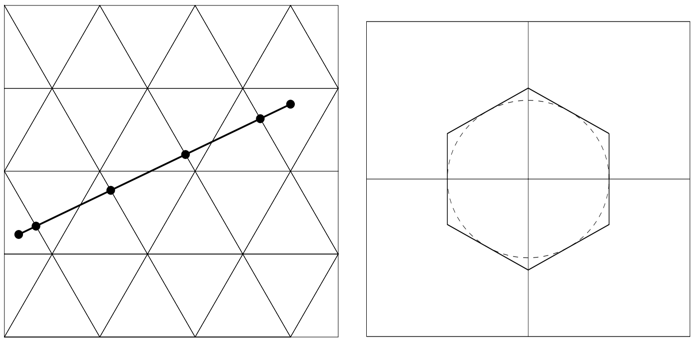

FIG 3: A segment in $\mathcal{T}_\epsilon^1(\Omega)$ and the level curve $\{\phi_1(v) = 1\}$

where $\#(\cdot)$ denotes the cardinality. We will prove that

$$\phi_1(\nu) = \lim_{\epsilon_j \to 0} \#(\mathcal{S}_{\epsilon_j}^1)\,\frac{\sqrt{3}}{4}\,\epsilon_j \tag{15}$$

Fix α and ϵ_j, take an orientation on S and let x_0 be the first point, x_{m+1} the last and $x_1, \ldots, x_m$ be the intersection points between S and the edges with slope $\tan(120°)$ (see Figure 3). It's clear that for $n = 1, \ldots, m-1$ each segment $[x_n\, x_{n+1}]$ intersect exaclty two triangles, while $[x_0\, x_1]$ and $[x_m\, x_{m+1}]$ can intersect either one or two. So we have the following estimate

$$2(m-1) + 2 \leq \#(\mathcal{S}_{\epsilon_j}^1) \leq 2(m-1) + 4 \tag{16}$$

At this point it should be clear that (15) does not depend on the particular choice of S, indeed the same estimate on $\#(\mathcal{S}_{\epsilon_j}^1)$ holds for every segment with the same normal, even when it contains a vertice of the mesh, due to the carefull definition of the set $\mathcal{S}_{\epsilon_j}^1$. To estimate m, let $h(\epsilon_j) = \mathcal{H}^1([x_1\, x_2])$ and note that for $n = 1, \ldots, m-1$ each segment $[x_n\, x_{n+1}]$ has the same lenght, then

$$\left[\frac{1}{h(\epsilon_j)}\right] - 1 \leq m \leq \left[\frac{1}{h(\epsilon_j)}\right] + 1 \tag{17}$$

where $[\cdot]$ denotes the integer part. Joining (16) and (17) we get

$$2\left[\frac{1}{h(\epsilon_j)}\right] - 2 \leq \#(\mathcal{S}_{\epsilon_j}^1) \leq 2\left[\frac{1}{h(\epsilon_j)}\right] + 4 \tag{18}$$

and we can conclude

$$\lim_{\epsilon_j \to 0} \#(\mathcal{S}_{\epsilon_j}^1)\frac{\sqrt{3}}{4}\epsilon_j = \lim_{\epsilon_j \to 0} \left[\frac{1}{h(\epsilon_j)}\right]\frac{\sqrt{3}}{2}\epsilon_j = \lim_{\epsilon_j \to 0} \frac{\sqrt{3}}{2}\frac{\epsilon_j}{h(\epsilon_j)} \tag{19}$$

By a simple trigonometric argument we have $h(\epsilon_j)\langle \xi_3^1, \nu_S \rangle = \epsilon_j \sqrt{3}/2$, then

$$\lim_{\epsilon_j \to 0} \#(\mathcal{S}^1_{\epsilon_j}) \frac{\sqrt{3}}{4} \epsilon_j = \langle \xi_3^1, \nu_S \rangle \tag{20}$$

and the proof is concluded. $\square$

Remark 3.2 *The function $\phi_1(\nu)$ can clearly be extended to any vector $v \in \mathbb{R}^2$ by 1-homogeneity, moreover it's easy to check that $\phi_1(v)$ will be convex and its level curve $\{\phi_1(v) = 1\}$ is an hexagon, represented in Figure 3. For the applications it is usefull to measure the effect of anisotropy, thus following [10] we compute the ratio $a = \frac{M}{m}$ where $M = \max_{||\nu||=1} \phi(\nu)$ and $m = \min_{||\nu||=1} \phi(\nu)$. Obviously the gratest a is the more the triangulation is anisotropic, in this case we have $a_1 = \frac{2\sqrt{3}}{3} \simeq 1.154$.*

Lemma 3.3 . *Let $i = 2$ and let $S \subset \Omega$ be a segment. with normal $\nu_S = (-\sin\alpha, \cos\alpha)$, let $\xi_1^2 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $\xi_2^2 = (0,1)$, $\xi_3^2 = (-1,0)$, and let the function $\phi_2(\nu) : S^1 \to [0, +\infty)$ be defined as*

$$\phi_2(\nu) = \begin{cases} |\langle \nu, \xi_1^2 \rangle| & \text{if } 90° \le \alpha \le 180° \text{ and } -90° \le \alpha \le 0° \\ \\ \frac{\sqrt{2}}{2} |\langle \nu, \xi_2^2 \rangle| & \text{if } 0° \le \alpha \le 45° \text{ and } -180° \le \alpha \le -135° \\ \\ \frac{\sqrt{2}}{2} |\langle \nu, \xi_3^2 \rangle| & \text{if } 45° \le \alpha \le 90° \text{ and } -135° \le \alpha \le -90° \end{cases}$$

then the limit in (12) exists and

$$\phi_2(\nu)\mathcal{H}^1(S) = \lim_{\epsilon_j \to 0} \frac{\sum_{T \in \mathcal{S}^2_{\epsilon_j}} |T|}{\epsilon_j} \tag{21}$$

holds for any segment S with unit normal ν.

Proof. The proof follows the same argument performed in the previous one. Assume that $\mathcal{H}^1(S) = 1$ and consider for α fixed a set of points $x_0, \cdots, x_{m+1}$ such that for $n = 1, \ldots, m-1$ each segment $[x_n\, x_{n+1}]$ intersect exaclty two triangles, then, using an estimate like (15) and (18) equality (21) is reduced to

$$\phi_1(\nu) = \lim_{\epsilon_j \to 0} \#(\mathcal{S}^2_{\epsilon_j}) \frac{\epsilon_j}{4} = \lim_{\epsilon_j \to 0} \frac{1}{2} \frac{\epsilon_j}{h(\epsilon_j)} \tag{22}$$

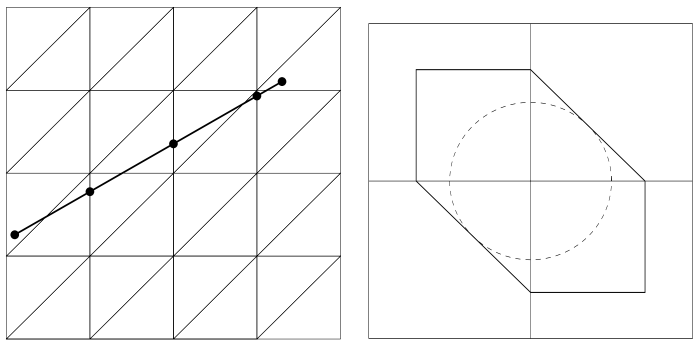

FIG 4: A segment in $\mathcal{T}_{\epsilon}^{2}(\Omega)$ and the level curve $\{\phi_2(v) = 1\}$

Take first $0 \leq \alpha \leq 45°$, the set of points $x_1, \cdots, x_m$ is defined as the intersection between S and vertical edges (see Figure 4). In this way, from $h(\epsilon_j)\langle \xi_2^2, \nu_S\rangle = \epsilon_j\sqrt{2}/2$ follows

$$\lim_{\epsilon_j \to 0} \#(\mathcal{S}^2_{\epsilon_j})\,\frac{\epsilon_j}{4} = \frac{\sqrt{2}}{2}\langle \xi_2^2, \nu_S\rangle = \phi_2(\nu_S) \tag{23}$$

For $45° \leq \alpha \leq 90°$ consider the set of intersection points between S and horizontal edges, this time $h(\epsilon_j)\langle \xi_3^2, \nu_S\rangle = \epsilon\sqrt{2}/2$ then

$$\lim_{\epsilon_j \to 0} \#(\mathcal{S}^2_{\epsilon_j})\,\frac{\epsilon_j}{4} = \frac{\sqrt{2}}{2}\langle \xi_3^2, \nu_S\rangle = \phi_2(\nu_S) \tag{24}$$

Finally, for $90° \leq \alpha \leq 180°$ define $x_1, \ldots, x_m$ as intersection between and S and the edges with slope $\tan(45°)$. In this case we have $h(\epsilon_j)\langle \xi_1^2, \nu_S\rangle = \epsilon_j 1/2$, then

$$\lim_{\epsilon_j \to 0} \#(\mathcal{S}^2_{\epsilon_j})\,\frac{\epsilon_j}{4} = \langle \xi_1^2, \nu_S\rangle = \phi_2(\nu_S) \tag{25}$$

and this concludes the proof. □

Remark 3.4 *As before $\phi_2(\nu)$ can be extended to $v \in \mathbb{R}^2$ in 1-homogeneous convex function, whose level curve $\{\phi_2(v) = 1\}$ is represented in Figure 4. This time the anisotropy factor $a = \frac{M}{m}$ is much greater, due to the particular orientation of the triangles, indeed $M = 1$ and $m = \frac{\sqrt{2}}{2}$ so $a_2 = \sqrt{2} \simeq 1.414$.*

Lemma 3.5 . *Let $i = 3$ and let $S \subset \Omega$ be a segment with normal $\nu_S = (-\sin\alpha, \cos\alpha)$, let $\zeta_k = \xi_k^2$ be the vectors defined in the previous Lemma, ζ'_k*

be the corresponding simmetric vector respect to the y-axis. Let the function $\phi_3(\nu) : S^1 \to [0, +\infty)$ *be defined as*

$$\phi_3(\nu) = \begin{cases} |\langle \nu, \zeta_1 \rangle| + \frac{\sqrt{2}}{2} |\langle \nu, \zeta_3' \rangle| & \text{if } -90^\circ \leq \alpha \leq -45^\circ \text{ and } 90^\circ \leq \alpha \leq 135^\circ \\ \\ |\langle \nu, \zeta_1 \rangle| + \frac{\sqrt{2}}{2} |\langle \nu, \zeta_3' \rangle| & \text{if } -45^\circ \leq \alpha \leq 0^\circ \text{ and } 135^\circ \leq \alpha \leq 180^\circ \\ \\ |\langle \nu, \zeta_1 \rangle| + \frac{\sqrt{2}}{2} |\langle \nu, \zeta_3' \rangle| & \text{if } 0^\circ \leq \alpha \leq 45^\circ \text{ and } -180^\circ \leq \alpha \leq -135^\circ \\ \\ |\langle \nu, \zeta_1 \rangle| + \frac{\sqrt{2}}{2} |\langle \nu, \zeta_3' \rangle| & \text{if } 45^\circ \leq \alpha \leq 90^\circ \text{ and } -135^\circ \leq \alpha \leq -90^\circ \end{cases}$$

then the limit in (12) exists and

$$\phi_3(\nu) \mathcal{H}^1(S) = \lim_{\epsilon_j \to 0} \frac{\sum_{T \in \mathcal{S}^3_{\epsilon_j}} |T|}{\epsilon_j} \tag{26}$$

holds for any segment S *with unit normal* ν.

Proof. The idea is to reduce the computation of this case to the previous one, for this reason we will consider two variants of $\mathcal{T}_\epsilon{}^3(\Omega)$ and $\mathcal{T}_\epsilon{}^2(\Omega)$, depending on the orientation of the triangles. They are represented in Figure 5 and 6 and denoted by $\mathcal{T}_\epsilon^{3a}(\Omega)$, $\mathcal{T}_\epsilon^{3b}(\Omega)$, $\mathcal{T}_\epsilon^{2a}(\Omega)$ and $\mathcal{T}_\epsilon^{2b}(\Omega)$. Consider the squares Q obtained from the vertices of the triangulation and note that the orientation of the triangles in $\mathcal{T}_\epsilon^{3a}(\Omega)$ contained in Q correspond to the orientation of the elements in $\mathcal{T}_\epsilon^{2a}(\Omega)$ or $\mathcal{T}_\epsilon^{2b}(\Omega)$ and viceversa for $\mathcal{T}_\epsilon^{3b}(\Omega)$. Thus it should be clear that for any segment S

$$\begin{aligned} \#(\mathcal{S}^{3a}_{\epsilon_j}(S, \nu_S)) + \#(\mathcal{S}^{3b}_{\epsilon_j}(S, \nu_S)) = \\ \#(\mathcal{S}^{2a}_{\epsilon_j}(S, \nu_S)) + \#(\mathcal{S}^{2b}_{\epsilon_j}(S, \nu_S)) \end{aligned} \tag{27}$$

Obviously by the simmmerty we can consider only $-90 \leq \alpha \leq -45^\circ$, in this case we show that

$$\#(\mathcal{S}^{3a}_{\epsilon_j}(S, \nu_S)) - 6 \leq \#(\mathcal{S}^{3b}_{\epsilon_j}(S, \nu_S)) \leq \#(\mathcal{S}^{3a}_{\epsilon_j}(S, \nu_S)) + 6 \tag{28}$$

Being $-90 \leq \alpha \leq -45^\circ$, taking the set of points $x_1, \ldots, x_m$ obtained by intersection with vertical edges, it is not difficult to see that for $n = 1, \cdots, m-1$ each segment $[x_n \, x_{n+1}]$ intersects the same number of triangles in $\mathcal{T}_\epsilon^{3a}(\Omega)$ and in $\mathcal{T}_\epsilon^{3b}(\Omega)$ (see Figure 7). This is enough to prove (28) because in $[x_0 \, x_1]$ and in $[x_m \, x_{m+1}]$ there are at most three elements. In conclusion

$$\lim_{\epsilon_j \to 0} \#(\mathcal{S}^{3a}_{\epsilon_j}(S, \nu_S)) \frac{1}{4} \epsilon_j = \lim_{\epsilon_j \to 0} \#(\mathcal{S}^{3b}_{\epsilon_j}(S, \nu_S)) \frac{1}{4} \epsilon_j$$

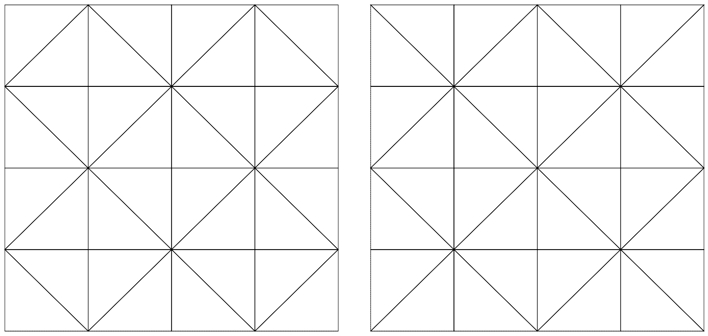

FIG 5: The triangulations $\mathcal{T}_\epsilon^{3a}(\Omega)$ and $\mathcal{T}_\epsilon^{3b}(\Omega)$

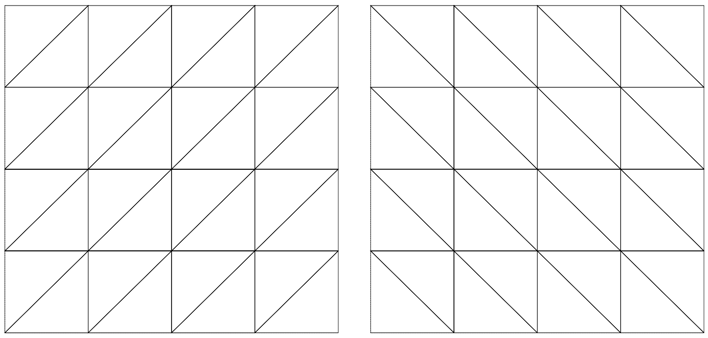

FIG 6: The triangulations $\mathcal{T}_\epsilon^{2a}(\Omega)$ and $\mathcal{T}_\epsilon^{2b}(\Omega)$

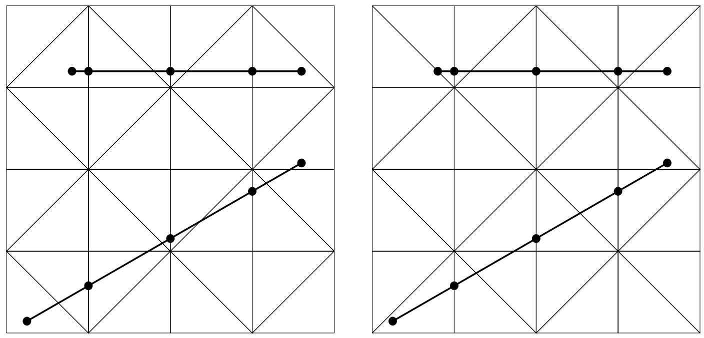

FIG 7: Two segments in $\mathcal{T}_\epsilon^{3a}(\Omega)$ and $\mathcal{T}_\epsilon^{3b}(\Omega)$

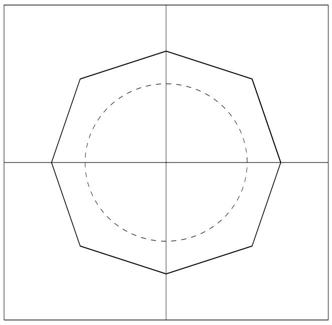

FIG 8: The level curve $\{\phi_3(v)\}$

then

$$\lim_{\epsilon_j\to 0} \#(\mathcal{S}^{3a}_{\epsilon_j}(S,\nu_S))\,\frac{1}{4}\,\epsilon_j = \frac{1}{2}\lim_{\epsilon_j\to 0}\Big(\#(\mathcal{S}^{2a}_{\epsilon_j}(S,\nu_S))\,\frac{1}{4}\epsilon_j + \#(\mathcal{S}^{2b}_{\epsilon_j}(S,\nu_S))\,\frac{1}{4}\epsilon_j\Big)$$
$$= \frac{1}{2}\Big(\phi_2(\nu_S) + \phi_2(\nu'_S)\Big) \tag{29}$$

where ν'_S is the reflexion of ν_S respect to the y-axis. ❑

Remark 3.6 *Again, the function can be extended in a 1-homogeneous convex way to any $v \in \mathbb{R}^2$. This time the anisotropy factor a_3 is much less then a_2, infact $M = \frac{\sqrt{5}}{3}$ and $m = \frac{2}{3}$ so $a_3 = \frac{\sqrt{5}}{3} \simeq 1.118$.*

Remark 3.7 *In the sequel, to prove the Γ-liminf inequality, it will be very usefull to have a representation of the functions ϕ_i in terms of the scalar products $\langle \nu, \xi^i_k\rangle$. For $i = 1$, let $c^1_k = 1\,\forall k$, then*

$$\phi_1(\nu) = \max_k\ c^1_k|\langle \nu, \xi^1_k\rangle| \tag{30}$$

For $i = 2$, let $c^2_1 = 1$ and $c^2_2 = c^2_3 = \sqrt{2}/2$, then

$$\phi_2(\nu) = \max_k\ c^2_k|\langle \nu, \xi^2_k\rangle| \tag{31}$$

For $i = 3$, let $c^3_k = c^2_k\ \ \forall k$, then we have

$$\phi_3(\nu) = \frac{1}{2}\Big(\phi_2(\nu) + \phi_2(\nu')\Big)$$

$$
\begin{aligned}
&= \frac{1}{2}\max_{k_1}\ c^3_{k_1}|\langle \nu, \zeta_{k_1}\rangle| + \frac{1}{2}\max_{k_2}\ c^3_{k_2}|\langle \nu, \zeta'_{k_2}\rangle| \\
&= \frac{1}{2}\max_{k_1,k_2}\ \Big(c^3_{k_1}|\langle \nu, \zeta_{k_1}\rangle| + c^3_{k_2}|\langle \nu, \zeta'_{k_2}\rangle|\Big) \qquad (32)
\end{aligned}
$$

where ζ'_k denotes the reflection of ζ_k respect to the y-axis. Note that by the representation of ϕ_3 in Lemma 3.5 we have the max *in (32) can be reduced over the pairs (k_1, k_2) such that $\langle \zeta_{k_1}, \zeta'_{k_2}\rangle = \sqrt{2}/2$.*

4 Γ-limsup inequality

In the sequel we will use the following notations, given $u \in L^2(\Omega)$ and a sequence of triangulations $\{\mathcal{T}^i_{\epsilon_j}\}$ for $\epsilon_j \searrow 0$, let

$$
F_{\epsilon_j}(u) = \begin{cases} \dfrac{1}{\epsilon_j}\displaystyle\int_\Omega f(\epsilon_j|\nabla u|^2)\,dx & \text{if } u \in V^i_{\epsilon_j}(\Omega) \\ +\infty & \text{if } u \in L^2(\Omega)\setminus V^i_{\epsilon_j}(\Omega) \end{cases} \qquad (33)
$$

$$
F'(u) = \Gamma\text{-}\liminf_{j\to+\infty}\ F_{\epsilon_j}(u) \qquad (34)
$$

$$
F''(u) = \Gamma\text{-}\limsup_{j\to+\infty}\ F_{\epsilon_j}(u) \qquad (35)
$$

In this section we prove that for any $u \in L^2(\Omega)\cap GSBV(\Omega)$ the Γ-lim sup inequality

$$
F''(u) \le F(u) \qquad (36)
$$

holds respect to the strong L^2-topology, where

$$
F(u) = \int_\Omega |\nabla u|^2 dx + f_\infty \int_{S_u} \phi_i(\nu_u)\,\mathrm{d}\mathcal{H}^1 \qquad (37)
$$

and ϕ_i is the anisotropy function defined in the previous section. The proof of (36) is similar to the one presented in [9].

First of all we need a set of regular functions with good density properties in $SBV(\Omega)$.

Definition 4.1 *Call $\mathcal{W}(\Omega)$ the set of $u \in SBV(\Omega)$ such that*

1. *S_u is essentially closed, i.e. $\mathcal{H}^1(\overline{S_u} \setminus S_u) = 0$.*
2. *$\overline{S_u} = \displaystyle\bigcup_{m=1}^{p} S_m$ where S_m are disjoint segments.*

3. $u \in W^{k,\infty}(\Omega \setminus \overline{S_u})$ $\forall k \in \mathbb{N}$.

The interesting result about $\mathcal{W}(\Omega)$ is described in the following lemma, presented in [5].

Lemma 4.2 . *Let $\Omega \subset \mathbb{R}^2$ be an open set with Lipschitz boundary, $u \in SBV(\Omega) \cap L^\infty(\Omega)$ such that*

$$\int_\Omega |\nabla u|^2 dx < +\infty \qquad \mathcal{H}^1(S_u) < +\infty$$

then there exist a sequence $\{w_k\} \subset \mathcal{W}(\Omega)$ such that

$$w_k \longrightarrow u \text{ strongly in } L^2(\Omega) \tag{38}$$

$$\nabla w_k \longrightarrow \nabla u \text{ strongly in } L^2(\Omega, R^n) \tag{39}$$

$$\limsup_{k\to+\infty} \|w_k\|_{L^\infty(\Omega)} \le \|u\|_{L^\infty(\Omega)} \tag{40}$$

and

$$\limsup_{k\to+\infty} \int_{S_{w_k}} \phi(\nu_{w_k}) d\mathcal{H}^1 \le \int_{S_u} \phi(\nu_u) d\mathcal{H}^1 \tag{41}$$

Before proving Proposition (4.4) we need the following result.

Lemma 4.3 . *Let T be a triangle with angles θ_i and edges L_i. Let $v : T \to \mathbb{R}$ be an affine function, then for every pair L_{i_1}, L_{i_2} we have the following estimate on the gradient*

$$|\nabla v| \le C \max\{|v'_{L_{i_1}}|, |v'_{L_{i_2}}|\} \tag{42}$$

where

$$C = \max_{i=1,2,3} \frac{\sqrt{2(1+|\cos\theta_i|)}}{|\sin\theta_i|}$$

and $v'_{L_{i_k}}$ is the slope of v restricted to L_{i_k}. In particular for C_i defined

$$C_i = \begin{cases} 2 & \text{for } i = 1 \\ \sqrt{2(2+\sqrt{2})} & \text{for } i = 2,3 \end{cases}$$

inequality (42) holds for every ϵ_j and for every $T \in \mathcal{T}^i_{\epsilon_j}(\Omega)$.

The proof is omitted, being an easy application of trigonometric formulas.

Proposition 4.4 *Let $w \in \mathcal{W}(\Omega)$, $\{\mathcal{T}^i_{\epsilon_j}\}$ a sequence of triangulations, then for every i there exist a sequence $\{v_j\}$ such that*

1. $v_j \in V^i_{\epsilon_j}(\Omega), \ \forall j$
2. $\|v_j - w\|_{L^2(\Omega)} \leq c\,\epsilon_j^2$
3. $\limsup_{j\to+\infty} F_{\epsilon_j}(v_j) \leq \int_\Omega |\nabla w|^2 dx + f_\infty \int_{S_w} \phi_i(\nu_w) d\mathcal{H}^1$

where the constant c does not depend on ϵ_j.

Proof. Fix i and let $S_u = \cup_{m=1}^p S_m$ where S_m are disjoint segments. Fix for every m a normal ν_m to S_m and let $\mathcal{S}_j = \cup_{m=1}^p \mathcal{S}^i_{\epsilon_j}(S_m, \nu_m)$ be the union of the coverings of S_m defined before. By the regularity of w we have $w \in C^\infty(\overline{\Omega \setminus \mathcal{S}_j})$ for every j. Then the sequence $\{v_j\}$, defined as the Lagrange interpolation of w in $V^i_{\epsilon_j}(\Omega)$, satisfies the required properties. First of all by the regularity of w and by standard results on finite elements, see for instance [4] Theorem 3.1.6, there exist a constant c such that the inequalities

$$\|v_j - w\|_{L^2(T)} \leq c\,\epsilon_j^2 |w|_{H^2(T)}$$

$$\|v_j - w\|_{H^1(T)} \leq c\,\epsilon_j |w|_{H^2(T)}$$

hold for every w, for every ϵ_j and for every T such that $T \not\subset \mathcal{S}_j$. Now let $\Omega_j = \Omega \setminus \mathcal{S}_j$ then we can write

$$\begin{aligned}\frac{1}{\epsilon_j}\int_\Omega f(\epsilon_j|\nabla v_j|^2)\,dx &\leq \sum_{T\in\Omega_j} |T|\frac{1}{\epsilon_j} f(\epsilon_j|\nabla v_j|^2) + f_\infty \sum_{T\in\mathcal{S}_j} |T|\frac{1}{\epsilon_j} \\ &\leq \sum_{T\in\Omega_j} |T|\frac{1}{\epsilon_j} f(\epsilon_j|\nabla v_j|^2) \\ &\quad + f_\infty \sum_{m=1}^p \Big(\sum_{T\in\mathcal{S}^i_{\epsilon_j}(S_m,\nu_m)} |T|\frac{1}{\epsilon_j}\Big)\end{aligned}$$

The proof follows from the inequalities

$$\limsup_{j\to+\infty} \sum_{T\in\Omega_j} |T|\frac{1}{\epsilon_j} f(\epsilon_j|\nabla v_j|^2) \leq \int_\Omega |\nabla u|^2\,dx \tag{43}$$

$$\limsup_{j\to+\infty} \sum_{T\in\mathcal{S}^i_{\epsilon_j}} |T|\frac{1}{\epsilon_j} = \int_{S_m} \phi_i(\nu_m)d\mathcal{H}^1 \tag{44}$$

The first can be proved as in [9] while the second follows easily from the properties of the functions $\phi_i(\nu)$. □

The previous result proves the Γ-limsup inequality for $u \in \mathcal{W}(\Omega)$, it remain to extend it to $u \in L^2(\Omega)\cap GSBV(\Omega)$. For any $u \in L^\infty(\Omega)\cap SBV(\Omega)$ there exist by Lemma (4.2) a sequence $\{w_k\} \subset \mathcal{W}(\Omega)$ such that

$$w_k \longrightarrow u \text{ strongly in } L^2(\Omega)$$

$$\nabla w_k \longrightarrow \nabla u \text{ strongly in } L^2(\Omega, R^n)$$

$$\limsup_{k\to+\infty} \int_{S_{w_k}} \phi_i(\nu_{w_k})d\mathcal{H}^1 \leq \int_{S_u} \phi_i(\nu_u)d\mathcal{H}^1$$

Then by the lower-semicontinuity of F'' respect to $L^2(\Omega)$ topology follows

$$F''(u) \leq \liminf_{k\to+\infty} F''(w_k) \leq \limsup_{k\to+\infty} F(w_k) \leq F(u)$$

and the Γ-limsup inequality holds for $u \in L^\infty(\Omega)\cap SBV(\Omega)$. The final step follows again from the lower-semicontinuity of F'', indeed denoting $u_\lambda = (u \vee \lambda) \wedge \lambda$ for $\lambda \in \mathbb{N}$ we have $u_\lambda \to u$ in $L^2(\Omega)$ and

$$F''(u) \leq \liminf_{\lambda\to+\infty} F''(u_\lambda) \leq \liminf_{\lambda\to+\infty} F(u_\lambda) = F(u)$$

where $\liminf_{\lambda\to+\infty} F(u_\lambda) = F(u)$ follows from

$$\lim_{\lambda\to+\infty} \int_\Omega |\nabla u_\lambda|dx = \int_\Omega |\nabla u|dx$$

$$\lim_{\lambda\to+\infty} \int_{S_{u_\lambda}} \phi(\nu_{u_\lambda})d\mathcal{H}^1 = \int_{S_u} \phi(\nu_u)d\mathcal{H}^1$$

The first is a direct consequence of

$$\nabla u_\lambda = \nabla u \cdot \chi_{\{|u|>\lambda\}}$$

while the second comes from the properties

$$S_{u_\lambda} \subset S_{u_{\lambda+1}} \quad S_u = \textstyle\bigcup_{\lambda=1}^{+\infty} S_{u_\lambda}$$

which follows from the definition of S_u (see [1]).

5 Γ-liminf inequality

In this section we will prove that, given a sequence $\{\mathcal{T}^i_{\epsilon_j}\}$, the Γ-liminf inequality

$$F(u) \leq F'(u)$$

holds for every $u \in L^2(\Omega)$. More precisely

Proposition 5.1 *Given $\{\mathcal{T}^i_{\epsilon_j}\}$, let $u \in L^2(\Omega)$, if $F'(u) < +\infty$ then $u \in GSBV(A)$ and*

$$\int_\Omega |\nabla u|^2 dx + f_\infty \int_{S_u} \phi_i(\nu_u)\, d\mathcal{H}^1 \leq \liminf_{j\to+\infty} \frac{1}{\epsilon_j} \int_\Omega f(\epsilon_j |\nabla u_j|^2) dx \qquad (45)$$

for every sequence $u_j \in V^i_{\epsilon_j}(\Omega)$ converging strongly to u in $L^2(\Omega)$.

The proof is based on the one presented in [9]. First of all, for every open set $A \subset \Omega$, we define the localized functionals

$$F_{\epsilon_j}(u, A) = \begin{cases} \dfrac{1}{\epsilon_j} \displaystyle\int_A f(\epsilon_j |\nabla u|^2)\, dx & \text{if } u \in V^i_{\epsilon_j}(\Omega) \\ +\infty & \text{if } u \in L^2(\Omega) \setminus V^i_{\epsilon_j}(\Omega) \end{cases}$$

$$F'(u, A) = \Gamma\text{-}\liminf_{j\to+\infty} \ F_{\epsilon_j}(u, A)$$

then we prove the following Proposition.

Proposition 5.2 *Let $i = 1, 2, 3$, given $\{\mathcal{T}^i_{\epsilon_j}\}$, $u \in L^2(\Omega)$ and A an open set in Ω, if $F'(u, A) < +\infty$ then $u \in GSBV(\Omega)$ and*

$$\mathcal{H}^1(S_u \cap A) < +\infty \qquad (46)$$

$$\int_A |\nabla u|^2 dx \leq F'(u, A) \qquad (47)$$

Moreover for $i = 1, 2$

$$f_\infty c^i_k \int_{S_u \cap A} |\langle \nu_u, \xi^i_k \rangle|\, d\mathcal{H}^1 \leq F'(u, A) \quad \forall k \qquad (48)$$

and for $i = 3$

$$\frac{1}{2} f_\infty c^3_{k_1} \int_{S_u \cap A} |\langle \nu_u, \zeta_{k_1} \rangle|\, d\mathcal{H}^1 + $$

$$+ \frac{1}{2} f_\infty c^3_{k_2} \int_{S_u \cap A} |\langle \nu_u, \zeta'_{k_2} \rangle|\, d\mathcal{H}^1 \leq F'(u, A) \qquad (49)$$

where c^i_k, ξ^i_k and ζ_k are defined in Section 3 and for every pair (k_1, k_2) such that $\langle \zeta_{k_1}, \zeta'_{k_2} \rangle = \sqrt{2}/2$.

In order to prove Proposition 5.2 we need some of technical lemma.

Lemma 5.3 *Let A be an open set in Ω, let $i = 1, 2, 3$, for every $\delta \in (0, 1)$ there exist $\alpha_\delta, \beta_\delta$ such that for every $u \in V^i_{\epsilon_j}(\Omega)$ there exist $v \in SBV(\Omega)$ satisfing*

$$(1-\delta)\int_{A_{\epsilon_j}} |\nabla v|^2 dx + \alpha_\delta \mathcal{H}^1(S_v \cap A_{\epsilon_j}) \leq F_{\epsilon_j}(u, A)$$

$$\mathcal{L}^2(\{x \in \Omega \ : \ u(x) \neq v(x)\}) \leq \beta_\delta \, \epsilon_j \, F_{\epsilon_j}(u, A)$$

where $A_\epsilon = \{x \in A \ : \ d(x, \partial A) > \epsilon\}$.

Proof. See [9], Proposition 3.3.

Lemma 5.4 *Let A be an open set in Ω and $i = 1, 2$, for every $\delta \in (0, 1)$ there exist $\alpha_\delta, \beta_\delta, \gamma_\delta$ such that for every index k and for every $u \in V^i_{\epsilon_j}(\Omega)$ there exist $v \in SBV(\Omega)$ satisfing*

$$\alpha_\delta \int_{A_{\epsilon_j}} |\nabla v|^2 dx + \beta_\delta \mathcal{H}^1(S_v \cap A_{\epsilon_j}) \leq F_{\epsilon_j}(u, A) \tag{50}$$

$$\mathcal{L}^2(\{x \in \Omega \ : \ u(x) \neq v(x)\}) \leq \gamma_\delta \, \epsilon_j \, F_{\epsilon_j}(u, A) \tag{51}$$

$$(1-\delta) f_\infty c^i_k \int_{S_v \cap A_{\epsilon_j}} |\langle \nu_v, \xi^i_k \rangle| d\mathcal{H}^1 \leq F_{\epsilon_j}(u, A) \tag{52}$$

Proof. From the properties of the function f we deduce the existence of a constant c such that

$$\min(ct, (1-\delta)f_\infty) \leq f(t) \quad \forall \, t \geq 0$$

Let $\mathcal{T}_A = \{T \in \mathcal{T}^i_{\epsilon_j}(\Omega) : T \subset A\}$ and $\sigma = \sqrt{(1-\delta)f_\infty / c}$. To build the function v we need a careful classification of the triangles in $\mathcal{T}_A$ according to the behaviour of ∇u_T: let $\mathcal{T}_{A,m}$ for $m = 0, \ldots, 3$ be the set of triangles $T \in \mathcal{T}_A$ such that the slope of u is greater than $\sigma / \sqrt{\epsilon_j}$ exactly on m edges. So we can write

$$\begin{aligned} F_{\epsilon_j}(u, A) &\geq \sum_{T \in \mathcal{T}_A} \frac{|T|}{\epsilon_j} \min(c\epsilon_j |\nabla u_T|^2, (1-\delta) f_\infty) = \\ &= c \sum_{T \in \mathcal{T}_{A,0}} |T| \min(|\nabla u_T|^2, \frac{\sigma^2}{\epsilon_j}) + (1-\delta) f_\infty \sum_{T \in \mathcal{T}_A \setminus \mathcal{T}_{A,0}} \frac{|T|}{\epsilon_j} \end{aligned}$$

If $T \in \mathcal{T}_{A,0}$ then $|\nabla u_T| \leq C_i \dfrac{\sigma}{\sqrt{\epsilon_j}}$, where C_i is the constant defined in Lemma 4.3, thus we have a lower bound on σ^2/ϵ_j in terms of $|\nabla u_T|^2$, in particular for $c' = 1/C_i^2$

$$F_{\epsilon_j}(u, A) \geq c\, c' \sum_{T \in \mathcal{T}_{A,0}} |T||\nabla u_T|^2 + (1-\delta) f_\infty \sum_{T \in \mathcal{T}_A \setminus \mathcal{T}_{A,0}} \frac{|T|}{\epsilon_j} \tag{53}$$

Now, if $T \in \mathcal{T}_{A,0}$ we take $v = u$, in this way inequality (51) holds with $\gamma_\delta = 1/(1-\delta) f_\infty$, while if $T \in \mathcal{T}_A \setminus \mathcal{T}_{A,0}$ the function v and the constants α_δ and β_δ are defined in order to satisfy the inequalities

$$\frac{|T|}{\epsilon_j} \geq c_k^i \int_{S_v \cap T} |\langle \nu_v, \xi_k^i \rangle| \mathrm{d}\mathcal{H}^1 \quad \forall k \tag{54}$$

$$(1-\delta) f_\infty \frac{|T|}{\epsilon_j} \geq \alpha_\delta \int_T |\nabla v|^2 dx + \beta_\delta \mathcal{H}^1(S_v \cap T) \tag{55}$$

which gives respectively (50) and (52). The function v can be defined as in [9], and repeating the proof of Proposition 3.4 in [9] it is not difficult to see that the same choice of v, α_δ and β_δ satisfies all the previous inequalities. $\square$

The following Remark, similar to Lemma 4.3, is very easy but it is fundamental for the proof in the case $i = 3$.

Remark 5.5 *Let T be an isosceles right triangle with edges L_i. Let $u : (L_1 \cup L_2) \to \mathbb{R}$ be a continuous function, affine on L_1 and L_2, then, by Lemma 4.3, there exists an affine function $v : T \to \mathbb{R}$ such that $v = u$ on L_1 and L_2 and*

$$|\nabla v| \leq C \max\{|u'_{L_1}|, |u'_{L_2}|\} \tag{56}$$

where $C = \sqrt{2(2+\sqrt{2})}$ and u'_{L_i} is the slope of u restricted to L_i.

Lemma 5.6 *Let A be an open set in Ω and let $i = 3$, for every $\delta \in (0,1)$ there exist $\alpha_\delta, \beta_\delta, \gamma_\delta$ such that for every pair k_1, k_2 and for every $u \in V_{\epsilon_j}^i(\Omega)$ there exist $v \in SBV(\Omega)$ satisfing (51) and*

$$\alpha_\delta \int_{A_{2\epsilon_j}} |\nabla v|^2 dx + \beta_\delta \mathcal{H}^1(S_v \cap A_{2\epsilon_j}) \leq F_{\epsilon_j}(u, A) \tag{57}$$

$$\frac{1}{2}(1-\delta)f_\infty c_{k_1}^3 \int_{S_v\cap A_{2\epsilon_j}} |\langle \nu_v, \zeta_{k_1}\rangle| d\mathcal{H}^1 +$$
$$+\frac{1}{2}(1-\delta)f_\infty c_{k_2}^3 \int_{S_v\cap A_{2\epsilon_j}} |\langle \nu_v, \zeta'_{k_2}\rangle| d\mathcal{H}^1 \leq F_{\epsilon_j}(u,A) \tag{58}$$

Proof. In this case it is not possible to define a function v on a single element in such a way that (58) is satisfied. It will be necessary to consider a group of triangles as the one represented in Figure 9a, which is the smallest periodic structure of the mesh. Let us call Q this set and let $\mathcal{Q}_A$ be the set of squares Q in $\mathcal{T}^3_{\epsilon_j}(\Omega)$ such that $Q \subset A$. It is clear that

$$A_{2\epsilon_j} \subset \Big(\cup_{Q\in\mathcal{Q}_A} Q\Big) \subset A.$$

First v is defined on ∂Q, thus for every edge $L \subset \partial Q$, let x_1, x_2 be its vertices, x_{12} its midle point and let u'_L be the slope of u on L, then

- $v \equiv u(x_1)$ on the segment $[x_1\, x_{12}]$ and $v \equiv u(x_2)$ on the segment $[x_{12}\, x_2]$ if $|u'_L| > \sigma/\sqrt{\epsilon_j}$
- $v = u$ if $|u'_L| \leq \sigma/\sqrt{\epsilon_j}$.

Note that in this way v is no longer continuous on ∂Q but its slope now is piecewise uniformly bounded by $\sigma/\sqrt{\epsilon_j}$.

Now, let T be a triangle in Q, if $|\nabla u_T| \leq \sigma/\sqrt{\epsilon_j}$ we take $v = u$ on T, hence by an estimate like (53) inequality (51) is still satisfied. Moreover this definition agrees with the one given on ∂Q, indeed on the edges of T we have $|u'_L| \leq |\nabla u_T| \leq \sigma/\sqrt{\epsilon_j}$.

At this point we have defined v on the triangles $T \subset Q$ where $|\nabla u_T| \leq \sigma/\sqrt{\epsilon_j}$ and on the boundary ∂Q. Figure 9b presents a set Q where on the lower left triangle $v = u$, then the coloured triangle and the solid lines denotes where v is already given.

Call E the set where v is still to be defined, note that on ∂E the function is assigned and its slope is piecewise controlled by $\sigma/\sqrt{\epsilon_j}$. In the following the idea is to extend the values on ∂E inside E introducing a jump set S_v, which takes into account the discontinuities on ∂Q and such that $\mathcal{H}^1(S_v \cap \partial Q) = \emptyset$, and keeping a control on the gradient by Remark 5.5. Unfortunately it is not possible to find a unique procedure for all the cases, thus we need to devide $\mathcal{Q}_A$ into five classes $\mathcal{Q}_{A,m}$ for $m = 0, \ldots, 4$ according to the number of elements $T \subset Q$ on which $|\nabla u_T| > \sigma/\sqrt{\epsilon_j}$. To avoid a repetitious proof we omit the easy part of verifing (58) and (57) and every time we describe only the construction of v. In particular the existence of the constants α_δ and β_δ will follow easily from the uniform bound on $|\nabla v|$ and $\mathcal{H}^1(S_v)$. Moreover, by Remark 3.7 we know that $\phi_3(\nu)$ can be represented as the maximum over the pair (k_1, k_2) such that $\langle \zeta_{k_1}, \zeta_{k_2} \rangle = \frac{\sqrt{2}}{2}$, then due to the simmetry of $\mathcal{T}^3_{\epsilon_j}$ and the periodicity of $\phi_3(\nu)$, it is sufficient to prove inequality (58) for the pair $\zeta_1 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $\zeta'_3 = (1, 0)$.

$Q \in \mathcal{Q}_{A,0}$. Obviously we take $v = u$ on Q, then (58) is satisfied being $S_v = \emptyset$, and the gradient is controlled by definition being $|\nabla v_T| = |\nabla u_T| \leq \sigma/\sqrt{\epsilon_j}$.

$Q \in \mathcal{Q}_{A,1}$. In this case we have four possibilities, but without loss of generality we can consider only the one represented in Figure 9c, indeed all the other cases can be reduced by simmetry to the following. Suppose that in the lower right triangle T we have $|\nabla u_T| > \sigma/\sqrt{\epsilon_j}$, then in the others (coloured) we take $v = u$. Moreover note that on $\partial T \cap \mathring{Q}$ the slope of v, being equal to the one of u, is bounded by $\sigma/\sqrt{\epsilon_j}$, otherwise there should be another triangle with gradient over the threshold and Q should not belong to $\mathcal{Q}_{A,1}$. Now consider the triangles T_1 and T_2. In both cases there are two edges, where v is already given, by Remark 5.5 it can be extended and the gradient will be controlled by $|\nabla v_{T_i}| \leq C\sigma/\sqrt{\epsilon_j}$ for $i = 1, 2$. It remain to consider T_3, also in this case we can apply the previous Remark because v now is defined on the dashed edge, but this time $|\nabla v_{T_3}| \leq C^2\sigma/\sqrt{\epsilon_j}$. By the construction it is clear that v can be discontinuous only on the bold segment and it is not difficult to see that (58) is satisfied.

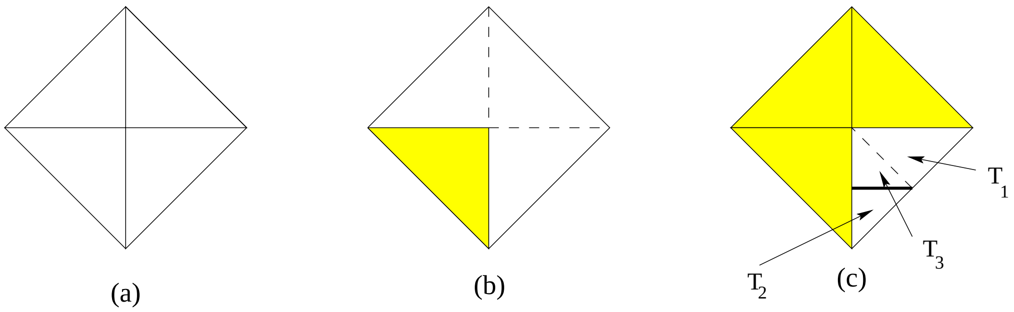

FIG 9: The set Q, an example and $Q \in \mathcal{Q}_{A,1}$.

$Q \in \mathcal{Q}_{A,2}$. Taking into account the simmetries of Q and v, we can consider only the three possibilities, represented in Figure 10. For the first it is not necessary to give datails, indeed it is sufficient to adapt the definition of v given for $Q \in \mathcal{Q}_{A,1}$.

In the second case, given v on ∂Q as described before and $v = u$ on the coloured triangles, it is defined first on T_1, T_2 and T_3, then, using the values of T_2 and T_3 on the dashed lines we complete the procedure on T_4.

The discontinuity, represented by the bold segment, satisfies (58) and the gradient controlled in the worst case by $C^2\sigma/\sqrt{\epsilon_j}$.

The last case in Figure 10c can be reduced to the previous by mean of rotation, note that the scalar products in (58) changes but the inequality is still satisfied.

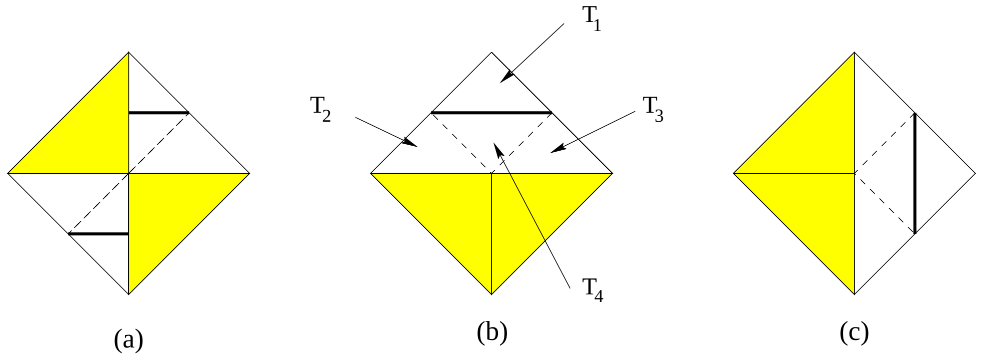

FIG 10: The three cases for $Q \in \mathcal{Q}_{A,2}$.

$Q \in \mathcal{Q}_{A,3}$. In this case we should consider two possibilities, see Figure 11, nevetheless it is easy to verify by simmetry that the same construction hold for both, thus we will give details only for the first. As before, v is given on ∂Q and on the coloured triangle, where we suppose $|\nabla u_T| \leq \sigma/\sqrt{\epsilon_j}$, then v is built as in Remark 5.5 first on the triangles T_i for $i = 1, \ldots, 4$, then on T_5 and T_6 where it is known on one edge. By construction and (56) we see that $|\nabla v_{T_i}| \leq C\sigma/\sqrt{\epsilon_j}$ if $i = 1, \ldots, 4$ and $|\nabla v_{T_j}| \leq C^2\sigma/\sqrt{\epsilon_j}$ for $j = 5, 6$.

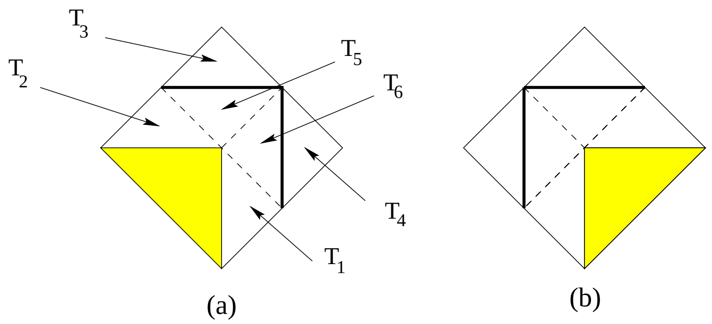

FIG 11: The two cases for $Q \in \mathcal{Q}_{A,3}$.

$Q \in \mathcal{Q}_{A,4}$. This time it is very easy to find a good function v with the required properties, indeed it is enough to extend the values on ∂Q in the sets E_i with an affine function such that $|\nabla v_{E_i}| \leq C\sigma/\sqrt{\epsilon_j}$ for every i. $\square$

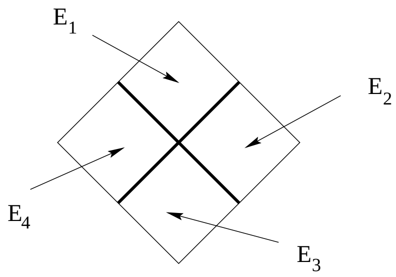

FIG 12: $Q \in \mathcal{Q}_{A,4}$.

Proof of Proposition 5.2. Consider a sequence $\{u_j\}$ converging to u in $L^2(\Omega)$ such that $u_j \in V^i_{\epsilon_j}(\Omega)$ and $\liminf_{j\to+\infty} F_{\epsilon_j}(u_j, A) < +\infty$ and, up to extracting a subsequence, that $\sup_j F_{\epsilon_j}(u_j, A) \leq c < +\infty$. Now, take $\delta \in (0,1)$, for every j let $v_j \in SBV(\Omega)$ be as in Lemma 5.3. Let $\lambda \in \mathbb{N}$ be a truncation level, we define

$$v_j^\lambda = (-\lambda \vee v_j) \wedge \lambda \quad \text{and} \quad u_j^\lambda = (-\lambda \vee u_j) \wedge \lambda$$

It's easy to see that u_j^λ converges to u^λ in $L^2(\Omega)$, then

$$\int_\Omega |v_j^\lambda - u_j^\lambda| dx \leq 2\lambda\beta_\delta\epsilon_j F_{\epsilon_j}(u_j, A) \leq (2\lambda\beta_\delta c)\epsilon_j$$

and we conclude that also v_j^λ converges to u^λ for every λ. Now, let $\eta > 0$, if ϵ_j is small enough $A_\eta \subset A_{\epsilon_j}$ and

$$(1-\delta)\int_{A_\eta} |\nabla v_j^\lambda|^2 dx + \alpha_\delta \mathcal{H}^1(S_{v_j^\lambda} \cap A_\eta) \leq F_{\epsilon_j}(u_j, A) \leq c$$

By Ambrosio's theorem we have that $u^\lambda \in SBV(A_\eta)$ for every λ and

$$(1-\delta)\int_{A_\eta} |\nabla u^\lambda|^2 dx + \alpha_\delta \mathcal{H}^1(S_{u^\lambda} \cap A_\eta) \leq \liminf_{j\to+\infty} F_{\epsilon_j}(u_j, A)$$

Since the previous inequality holds for every sequence $\{u_j\}$ and for every η we have $u^\lambda \in SBV(\Omega)$ and

$$(1-\delta)\int_A |\nabla u^\lambda|^2 dx + \alpha_\delta \mathcal{H}^1(S_{u^\lambda} \cap A) \leq F'(u, A)$$

For $\lambda \to +\infty$ and $\delta \to 0$ we get (46) and (47). It remain to consider inequality (48), it's proved exaclty in the same way using this time the function v_j given by Lemma 5.4. $\square$

Proof of proposition 5.1. From Proposition 5.4 follows (45) by a nowadays standard measure theoretic argument based on a supremum of a family of measures, see [15] or [3], and by the representations of $\phi_i(\nu)$ given in Remark 3.7. $\square$

6 Numerical examples

This section presents some numerical results, for both artificial and real life images, obtained minimizing the discrete functional

$$G_{\epsilon_j}(u) = \frac{1}{\epsilon_j}\int_\Omega \alpha\frac{2}{\pi}\arctan(\epsilon_j\frac{\beta}{\alpha}\frac{\pi}{2}|\nabla u|^2)dx + \int_\Omega (u-g)^2dx \text{ for } u \in V^i_{\epsilon_j}(\Omega) \tag{59}$$

where $\Omega = (0,1)\times(0,1)$ and the function g takes values in the interval $[0,1]$. Details about the minizing procedure and its numerical implementation are out of the pourposes here and will be presented in a forthcoming paper. By Theorem 2.1 we know that G_{ϵ_j} Γ-converges to

$$G(u) = \int_\Omega \beta|\nabla u|^2dx + \alpha\int_{S_u}\phi_i(\nu_u)d\mathcal{H}^1 + \int_\Omega (u-g)^2dx \tag{60}$$

thus, the solutions of (59) should present for ϵ_j small enough the properties of the minima of (60). In particular, according to [11], the amplitude of jump along discontinuities should be greater than a minimum contrast threshold which can be explicitly computed in terms of the coefficients α and β. For our functional, due to the presence of anisotropy, in general it is no longer possible to have a unique value of this threshold , nevertheless for rectilinear discontinuities the estimate becames

$$c = \sqrt{\frac{2\alpha\phi(\nu)}{\sqrt{\beta}}}. \tag{61}$$

Let us consider now Figure (13). Here both the original image and the parameters were choosen in order to stress the effect of anisotropy. Indeed on the white part $g \equiv 1$ and on the black one $g \equiv 0$, hence the original contrast is 1. Taking $\beta = 0.07$ and $\alpha = 0.15$, for $\mathcal{T}_\epsilon^2(\Omega)$ the anisotropy function takes

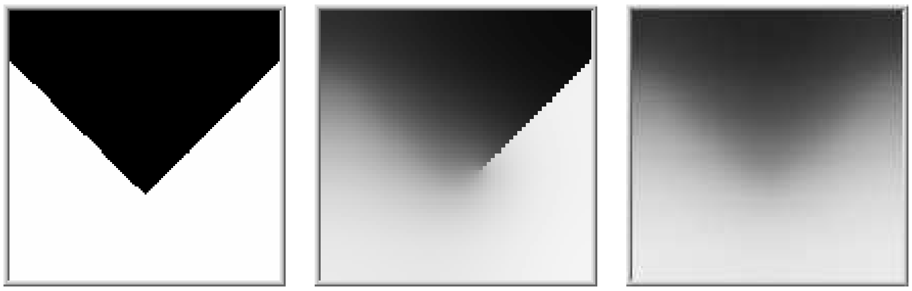

FIG 13: The original image and its segmentations using $\mathcal{T}_\epsilon^{\,2}(\Omega)$ and $\mathcal{T}_\epsilon^{\,3}(\Omega)$.

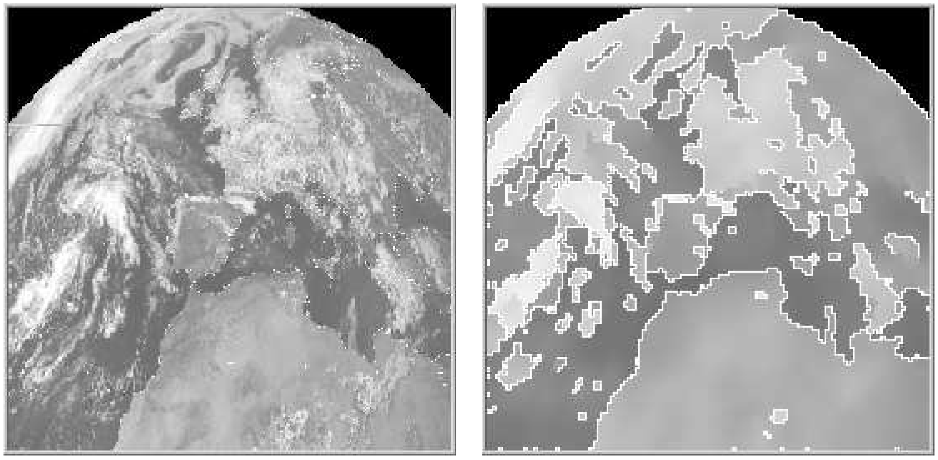

FIG 14: The original image and its segmentation.

the value 1 if $\nu = (\sqrt{2}/2, \sqrt{2}/2)$ and 0.5 if $\nu = (-\sqrt{2}/2, \sqrt{2}/2)$ then the constant c is respectively 1.06 and 0.75. For $\mathcal{T}_\epsilon^{\,3}(\Omega)$ there is a unique value, which is 0.9. According to these estimates, in the first case the solution presents a discontinuity on one side and a smooth region on the other, while in the second the original simmetry is preserved.

Finally, Figure 14 and 15 show the segmentations of real life images (white lines denotes the discontinuities).

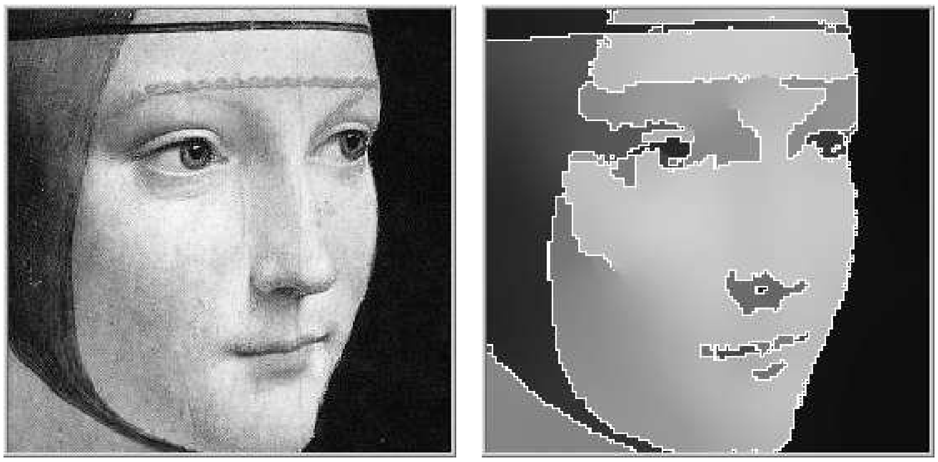

FIG 15: The original image and its segmentation.

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