Geometry of Optimal Control Problems and Hamiltonian Systems

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Preface

These notes are based on the mini-course given in June 2004 in Cetraro, Italy, in the frame of a C.I.M.E. school. Of course, they contain much more material that I could present in the 6 hours course. The idea was to explain a general variational and dynamical nature of nice and powerful concepts and results mainly known in the narrow framework of Riemannian Geometry. This concerns Jacobi fields, Morse's index formula, Levi Civita connection, Riemannian curvature and related topics.

I tried to make the presentation as light as possible: gave more details in smooth regular situations and referred to the literature in more complicated cases. There is an evidence that the results described in the notes and treated in technical papers we refer to are just parts of a united beautiful subject to be discovered on the crossroads of Differential Geometry, Dynamical Systems, and Optimal Control Theory. I will be happy if the course and the notes encourage some young ambitious researchers to take part in the discovery and exploration of this subject.

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1 Smooth optimal control problems

In these lectures we discuss some geometric constructions and results emerged from the investigation of smooth optimal control problems. We'll consider problems with integral costs and fixed endpoints. A standard formulation of such a problem is as follows: Minimize a functional

$$J_{t_0}^{t_1}(u(\cdot)) = \int_{t_0}^{t_1} \varphi(q(t), u(t))\, dt, \tag{1}$$

where

$$\dot{q}(t) = f(q(t), u(t)), \quad u(t) \in U, \quad \forall t \in [t_0, t_1], \tag{2}$$

$q(t_0) = q_0,\ q(t_1) = q_1$. Here $q(t) \in \mathbb{R}^n,\ U \subset \mathbb{R}^k$, a *control function* $u(\cdot)$ is supposed to be measurable bounded while $q(\cdot)$ is Lipschitzian; scalar function φ and vector function f are smooth. A pair $(u(\cdot), q(\cdot))$ is called an *admissible pair* if it satisfies differential equation (2) but may violate the boundary conditions.

We usually assume that Optimal Control Theory generalizes classical Calculus of Variations. Unfortunately, even the most classical geometric variational problem, the length minimization on a Riemannian manifold, cannot be presented in the just described way. First of all, even simplest manifolds, like spheres, are not domains in $\mathbb{R}^n$. This does not look as a serious difficulty: we slightly generalize original formulation of the optimal control problem assuming that $q(t)$ belongs to a smooth manifold M instead of $\mathbb{R}^n$. Then $\dot{q}(t)$ is a tangent vector to M i.e. $\dot{q}(t) \in T_{q(t)}M$ and we assume that $f(q, u) \in T_qM,\ \forall q, u$. Manifold M is called the *state space* of the optimal control problem.

Now we'll try to give a natural formulation of the length minimization problem as an optimal control problem on a Riemannian manifold M. Riemannian structure on M is (by definition) a family of Euclidean scalar products $\langle \cdot, \cdot \rangle_q$ on $T_qM,\ q \in M$, smoothly depending on q. Let $f_1(q), \ldots, f_n(q)$ be an orthonormal basis of T_qM for the Euclidean structure $\langle \cdot, \cdot \rangle_q$ selected in such a way that $f_i(q)$ are smooth with respect to q. Then any Lipschitzian curve on M satisfies a differential equation of the form:

$$\dot{q} = \sum_{i=1}^{n} u_i(t) f_i(q), \tag{3}$$

where $u_i(\cdot)$ are measurable bounded scalar functions. In other words, any Lipschitzian curve on M is an admissible trajectory of the control system (3). The Riemannian length of the tangent vector $\sum\limits_{i=1}^{n} u_i f_i(q)$ is $\left(\sum\limits_{i=1}^{n} u_i^2\right)^{1/2}$. Hence the length of a trajectory of system (3) defined on the segment $[t_0, t_1]$ is $\ell(u(\cdot)) = \int_{t_0}^{t_1} \left(\sum\limits_{i=1}^{n} u_i^2(t)\right)^{1/2} dt$. Moreover, it is easy to derive from the Cauchy–Schwarz inequality that the length minimization is equivalent to the minimization of the functional $J_{t_0}^{t_1}(u(\cdot)) = \int_{t_0}^{t_1} \sum\limits_{i=1}^{n} u_i^2(t)\, dt$. The length minimization problem is thus reduced to a specific optimal control problem on the manifold of the form (1), (2).

Unfortunately, what I've just written was wrong. It would be correct if we could select a smooth orthonormal frame $f_i(q)$, $q \in M$, $i = 1, \ldots, n$. Of course, we can always do it locally, in a coordinate neighborhood of M but, in general, we cannot do it globally. We cannot do it even on the 2-dimensional sphere: you know very well that any continuous vector field on the 2-dimensional sphere vanishes somewhere. We thus need another more flexible formulation of a smooth optimal control problem.

Recall that a *smooth locally trivial bundle* over M is a submersion $\pi : V \to M$, where all *fibers* $V_q = \pi^{-1}(q)$ are diffeomorphic to each other and, moreover, any $q \in M$ possesses a neighborhood O_q and a diffeomorphism $\Phi_q : O_q \times V_q \to \pi^{-1}(O_q)$ such that $\Phi_q(q', V_q) = V_{q'}$, $\forall q' \in O_q$. In a less formal language one can say that a smooth locally trivial bundle is a smooth family of diffeomorphic manifolds V_q (the fibers) parametrized by the points of the manifold M (the base). Typical example is the tangent bundle $TM = \bigcup\limits_{q \in M} T_q M$ with the canonical projection π sending $T_q M$ into q.

Definition. A smooth control system with the state space M is a smooth mapping $f : V \to TM$, where V is a locally trivial bundle over M and $f(V_q) \subset T_q M$ for any fiber V_q, $q \in M$. An admissible pair is a bounded[1]

[1] the term 'bounded' means that the closure of the image of the mapping is compact

measurable mapping $v(\cdot) : [t_0, t_1] \to V$ such that $t \mapsto \pi(v(t)) = q(t)$ is a Lipschitzian curve in M and $\dot{q}(t) = f(v(t))$ for almost all $t \in [t_0, t_1]$. Integral cost is a functional $J_{t_0}^{t_1}(v(\cdot)) = \int\limits_{t_0}^{t_1} \varphi(v(t))\, dt$, where φ is a smooth scalar function on V.

Remark. The above more narrow definition of an optimal control problem on M was related to the case of a *trivial bundle* $V = M \times U,\ V_q = \{q\} \times U$. For the length minimization problem we have $V = TM,\ f = \mathrm{Id},\ \varphi(v) = \langle v, v \rangle_q,\ \forall v \in T_qM,\ q \in M$.

Of course, any general smooth control system on the manifold M is locally equivalent to a standard control system on $\mathbb{R}^n$. Indeed, any point $q \in M$ possesses a coordinate neighborhood O_q diffeomorphic to $\mathbb{R}^n$ and a mapping $\Phi_q : O_q \times V_q \to \pi^{-1}(O_q)$ trivializing the restriction of the bundle V to O_q; moreover, the fiber V_q can be embedded in $\mathbb{R}^k$ and thus serve as a set of control parameters U.

Yes, working locally we do not obtain new systems with respect of those in $\mathbb{R}^n$. Nevertheless, general intrinsic definition is very useful and instructive even for a purely local geometric analysis. Indeed, we do not need to fix specific coordinates on M and a trivialization of V when we study a control system defined in the intrinsic way. A change of coordinates in M is actually a smooth transformation of the state space while a change of the trivialization results in the feedback transformation of the control system. This means that an intrinsically defined control system represents actually the whole class of systems that are equivalent with respect to smooth state and feedback transformations. All information on the system obtained in the intrinsic language is automatically invariant with respect to smooth state and feedback transformations. And this is what any geometric analysis intends to do: to study properties of the object under consideration preserved by the natural transformation group.

We denote by $L_\infty([t_0, t_1]; V)$ the space of measurable bounded mappings from $[t_0, t_1]$ to V equipped with the L_∞-topology of the uniform convergence on a full measure subset of $[t_0, t_1]$. If V would an Euclidean space, then $L_\infty([t_0, t_1]; V)$ would have a structure of a Banach space. Since V is only a smooth manifold, then $L_\infty([t_0, t_1]; V)$ possesses a natural structure of a smooth Banach manifold modeled on the Banach space $L_\infty([t_0, t_1]; \mathbb{R}^{\dim V})$.

Assume that $V \to M$ is a locally trivial bundle with the n-dimensional base and m-dimensional fibers; then V is an $(n + m)$-dimensional manifold.

Proposition I.1 *Let $f : V \to TM$ be a smooth control system; then the space $\mathcal{V}$ of admissible pairs of this system is a smooth Banach submanifold of $L_\infty([t_0, t_1]; V)$ modeled on $\mathbb{R}^n \times L_\infty([t_0, t_1]; \mathbb{R}^m)$.*

Proof. Let $v(\cdot)$ be an admissible pair and $q(t) = \pi(v(t)),\ t \in [t_0, t_1]$. There exists a Lipschitzian with respect to t family of local trivializations $R_t : O_{q(t)} \times U \to \pi^{-1}(O_{q(t)})$, where U is diffeomorphic to the fibers V_q. The construction of such a family is a boring exercise which we omit.

Consider the system

$$\dot{q} = f \circ R_t(q, u), \quad u \in U. \tag{4}$$

Let $v(t) = R_t(q(t), u(t))$; then $R_t,\ t_0 \le t \le t_1$, induces a diffeomorphism of an L_∞-neighborhood of $(q(\cdot), u(\cdot))$ in the space of admissible pairs for (4) on a neighborhood of $v(\cdot)$ in $\mathcal{V}$. Now fix $\bar{t} \in [t_0, t_1]$. For any $\hat{q}$ close enough to $q(\bar{t})$ and any $u'(\cdot)$ sufficiently close to $u(\cdot)$ in the L_∞-topology there exists a unique Lipschitzian path $q'(\cdot)$ such that $\dot{q}'(t) = f \circ R_t(q'(t), u'(t))),\ t_0 \le t \le t_1,\ q'(\bar{t}) = \hat{q}$; moreover the mapping $(\hat{q}, u'(\cdot)) \mapsto q'(\cdot)$ is smooth. In other words, the Cartesian product of a neighborhood of $q(\bar{t})$ in M and a neighborhood of $u(\cdot)$ in $L_\infty([t_0, t_1], U)$ serves as a coordinate chart for a neighborhood of $v(\cdot)$ in $\mathcal{V}$. This finishes the proof since M is an n-dimensional manifold and $L_\infty([t_0, t_1], U)$ is a Banach manifold modeled on $L_\infty([t_0, t_1], \mathbb{R}^m)$. $\square$

An important role in our study will be played by the "evaluation mappings" $F_t : v(\cdot) \mapsto q(t) = \pi(v(t))$. It is easy to show that F_t is a smooth mapping from $\mathcal{V}$ to M. Moreover, it follows from the proof of Proposition I.1 that F_t is a submersion. Indeed, $q(t) = F_t(v(\cdot))$ is, in fact a part of the coordinates of $v(\cdot)$ built in the proof (the remaining part of the coordinates is the control $u(\cdot)$.

2 Lagrange multipliers

Smooth optimal control problem is a special case of the general smooth conditional minimum problem on a Banach manifold $\mathcal{W}$. The general problem consists of the minimization of a smooth functional $J : \mathcal{W} \to \mathbb{R}$ on the level sets $\Phi^{-1}(z)$ of a smooth mapping $\Phi : \mathcal{W} \to N$, where N is a finite-dimensional manifold. In the optimal control problem we have $\mathcal{W} = \mathcal{V},\ N = M \times M,\ \Phi = (F_{t_0}, F_{t_1})$.

An efficient classical way to study the conditional minimum problem is the Lagrange multipliers rule. Let us give a coordinate free description of this rule. Consider the mapping

$$\bar{\Phi} = (J, \Phi) : \mathcal{W} \to \mathbb{R} \times N, \quad \bar{\Phi}(w) = (J(w), \Phi(w)), \ w \in \mathcal{W}.$$

It is easy to see that any point of the local conditional minimum or maximum (i.e. local minimum or maximum of J on a level set of Φ) is a critical point of $\bar{\Phi}$. I recall that w is a critical point of $\bar{\Phi}$ if the differential $D_w\bar{\Phi} : T_w\mathcal{W} \to T_{\bar{\Phi}(w)}(\mathbb{R} \times N)$ is *not* a surjective mapping. Indeed, if $D_w\bar{\Phi}$ would surjective then, according to the implicit function theorem, the image $\bar{\Phi}(O_w)$ of an arbitrary neighborhood O_w of w would contain a neighborhood of $\bar{\Phi}(w) = (J(w), \Phi(w))$; in particular, this image would contain an interval $((J(w) - \varepsilon, J(w) + \varepsilon), \Phi(w))$ that contradicts the local conditional minimality or maximality of $J(w)$.

The linear mapping $D_w\bar{\Phi}$ is not surjective if and only if there exists a nonzero linear form $\bar{\ell}$ on $T_{\bar{\Phi}(w)}(\mathbb{R} \times N)$ which annihilates the image of $D_w\bar{\Phi}$. In other words, $\bar{\ell}D_w\bar{\Phi} = 0$, where $\bar{\ell}D_w\bar{\Phi} : T_w\mathcal{W} \to \mathbb{R}$ is the composition of $D_w\bar{\Phi}$ and the linear form $\bar{\ell} : T_{\bar{\Phi}(w)}(\mathbb{R} \times N) \to \mathbb{R}$.

We have $T_{\bar{\Phi}(w)}(\mathbb{R} \times N) = \mathbb{R} \times T_{\Phi(w)}N$. Linear forms on $(\mathbb{R} \times N)$ constitute the adjoint space $(\mathbb{R} \times N)^* = \mathbb{R} \oplus T^*_{\Phi(w)}N$, where $T^*_{\Phi(w)}N$ is the adjoint space of $T_{\Phi(w)}M$ (the *cotangent space* to M at the point $\Phi(w)$). Hence $\ell = \nu \oplus \ell$, where $\nu \in \mathbb{R}$, $\ell \in T^*_{\Phi(w)}N$ and

$$\bar{\ell}D_w\bar{\Phi} = (\nu \oplus \ell)(d_wJ, D_w\Phi) = \nu d_wJ + \ell D_w\Phi.$$

We obtain the equation

$$\nu d_wJ + \ell D_w\Phi = 0. \tag{5}$$

This is the Lagrange multipliers rule: if w is a local conditional extremum, then there exists a nontrivial pair (ν, ℓ) such that equation (5) is satisfied. The pair (ν, ℓ) is never unique: indeed, if α is a nonzero real number, then the pair $(\alpha\nu, \alpha\ell)$ is also nontrivial and satisfies equation (5). So the pair is actually defined up to a scalar multiplier; it is natural to treat this pair as an element of the projective space $\mathbb{P}\left(\mathbb{R} \oplus T^*_{\Phi(w)}N\right)$ rather than an element of the linear space.

The pair (ν, ℓ) which satisfies (5) is called the *Lagrange multiplier* associated to the critical point w. The Lagrange multiplier is called *normal* if $\nu \neq 0$ and abnormal if $\nu = 0$. In these lectures we consider only normal

Lagrange multipliers, they belong to a distinguished coordinate chart of the projective space $\mathbb{P}\left(\mathbb{R} \oplus T^*_{\Phi(w)}N\right)$.

Any normal Lagrange multiplier has a unique representative of the form $(-1, \ell)$; then (5) is reduced to the equation

$$\ell D_w \Phi = d_w J. \tag{6}$$

The vector $\ell \in T^*_{\Phi(w)}N$ from equation (6) is also called a normal Lagrange multiplier (along with $(-1, \ell)$).

3 Extremals

Now we apply the Lagrange multipliers rule to the optimal control problem. We have $\Phi = (F_{t_0}, F_{t_1}) : \mathcal{V} \to M \times M$. Let an admissible pair $v \in \mathcal{V}$ be a critical point of the mapping $\left(J_{t_0}^{t_1}, \Phi\right)$, the curve $q(t) = \pi(v(t)),\ t_0 \le t \le t_1$ be the corresponding trajectory, and $\ell \in T^*_{(q(t_0),q(t_1))}(M \times M)$ be a normal Lagrange multiplier associated to $v(\cdot)$. Then

$$\ell D_v \left(F_{t_0}, F_{t_1}\right) = d_v J_{t_0}^{t_1}. \tag{7}$$

We have $T^*_{(q(t_0),q(t_1))}(M \times M) = T^*_{q(t_0)}M \times T^*_{q(t_1)}M$, hence ℓ can be presented in the form $\ell = (-\lambda_{t_0}, \lambda_{t_1})$, where $\lambda_{t_i} \in T^*_{q(t_i)}M,\ i = 0, 1$. Equation (7) takes the form

$$\lambda_{t_1} D_v F_{t_1} - \lambda_{t_0} D_v F_{t_0} = d_v J_{t_0}^{t_1}. \tag{8}$$

Note that λ_{t_1} in (8) is uniquely defined by λ_{t_0} and v. Indeed, assume that $\lambda'_{t_1} D_v F_{t_1} - \lambda_{t_0} D_v F_{t_0} = d_v J_{t_0}^{t_1}$ for some $\lambda'_{t_1} \in T^*_{q(t_1)}M$. Then $(\lambda'_{t_1} - \lambda_{t_1}) D_v F_{t_1} = 0$. Recall that F_{t_1} is a submersion, hence $D_v F_{t_1}$ is a surjective linear map and $\lambda'_{t_1} - \lambda_{t_1} = 0$.

Proposition I.2 *Equality (8) implies that for any $t \in [t_0, t_1]$ there exists a unique $\lambda_t \in T^*_{q(t)}M$ such that*

$$\lambda_t D_v F_t - \lambda_{t_0} D_v F_{t_0} = d_v J_{t_0}^{t} \tag{9}$$

and λ_t is Lipschitzian with respect to t.

Proof. The uniqueness of λ_t follows from the fact that F_t is a submersion as it was explained few lines above. Let us proof the existence. To do that we use the coordinatization of $\mathcal{V}$ introduced in the proof of Proposition I.1, in particular, the family of local trivializations $R_t : O_{q(t)} \times U \to \pi^{-1}(O_{q(t)})$. Assume that $v(t) = R_t(q(t), u(t)),\ t_0 \le t \le t_1$, where $v(\cdot)$ is the referenced admissible pair from (8).

Given $\tau \in [t_0, t_1],\ \hat{q} \in O_{q(\tau)}$ let $t \mapsto Q^t_\tau(\hat{q})$ be the solution of the differential equation $\dot{q} = R_t(q, u(t))$ which satisfies the condition $Q^\tau_\tau(\hat{q}) = \hat{q}$. In particular, $Q^t_\tau(q(\tau)) = q(t)$. Then Q^t_τ is a diffeomorphism of a neighborhood of $q(\tau)$ on a neighborhood of $q(t)$. We define a Banach submanifold $\mathcal{V}_\tau$ of the Banach manifold $\mathcal{V}$ in the following way:

$$\mathcal{V}_\tau = \{v' \in \mathcal{V} : \pi(v'(t)) = Q^t_\tau(\pi(v'(\tau))),\ \tau \le t \le t_1\}.$$

It is easy to see that $F_{t_1}\Big|_{\mathcal{V}_\tau} = Q^{t_1}_\tau \circ F_\tau\Big|_{\mathcal{V}_\tau}$ and $J^{t_1}_\tau\Big|_{\mathcal{V}_\tau} = a_\tau \circ F_\tau$, where $a(\hat{q}) = \int\limits_\tau^t \varphi\left(\Phi_t(Q^t_\tau(\hat{q}), u(t))\right) dt$. On the other hand, the set $\{v' \in \mathcal{V} : v'|_{[t_0,\tau]} \in \mathcal{V}_\tau\big|_{[t_0,\tau]}\}$ is a neighborhood of v in $\mathcal{V}$. The restriction of (8) to $\mathcal{V}_\tau$ gives:

$$\lambda_{t_1} D_v\left(Q^{t_1}_\tau \circ F_\tau\right) - \lambda_{t_0} D_v F_{t_0} = d_v J^\tau_{t_0} + d_v\left(a_\tau \circ F_\tau\right).$$

Now we apply the chain rule for the differentiation and obtain:

$$\lambda_\tau D_v F_\tau - \lambda_{t_0} D_v F_{t_0} = d_v J^\tau_{t_0},$$

where $\lambda_\tau = \lambda_{t_1} D_{q(\tau)} Q^{t_1}_\tau - d_{q(\tau)} a_\tau$. $\square$

Definition. A Lipschitzian curve $t \mapsto \lambda_t,\ t_0 \le t \le t_1$, is called a *normal extremal* of the given optimal control problem if there exists an admissible pair $v \in \mathcal{V}$ such that equality (9) holds. The projection $q(t) = \pi(\lambda_t)$ of a normal extremal is called a (normal) *extremal path* or a (normal) *extremal trajectory*.

According to Proposition I.2, normal Lagrange multipliers are just points of normal extremals. A good thing about normal extremals is that they satisfy a nice differential equation which links optimal control theory with a beautiful and powerful mathematics and, in many cases, allows to explicitly characterize all extremal paths.

4 Hamiltonian system

Here we derive equations which characterize normal extremals; we start from coordinate calculations. Given $\tau \in [t_0, t_1]$, fix a coordinate neighborhood $\mathcal{O}$

in M centered at $q(\tau)$, and focus on the piece of the extremal path $q(\cdot)$ which contains $q(\cdot)$ and is completely contained in $\mathcal{O}$. Identity (9) can be rewritten in the form

$$\lambda_t D_v F_t - \lambda_\tau D_v F_\tau = d_v J_\tau^t, \tag{10}$$

where $q(t)$ belongs to the piece of $q(\cdot)$ under consideration. Fixing coordinates and a local trivialization of V we (locally) identify our optimal control problem with a problem (1), (2) in $\mathbb{R}^n$. We have $T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n = \{(p,q) : p,q \in \mathbb{R}^n\}$, where $T_q^*\mathbb{R}^n = \mathbb{R}^n \times \{q\}$. Then $\lambda_t = \{p(t), q(t)\}$ and $\lambda_t D_v F_t \cdot = \langle p(t), D_v F_t \cdot \rangle = D_v \langle p(t), F_t \rangle$.

Admissible pairs of (2) are parametrized by $\hat{q} = F_\tau(v')$, $v' \in \mathcal{V}$, and control functions $u'(\cdot)$; the pairs have the form: $v' = (u'(\cdot), q'(\cdot; \hat{q}, u'(\cdot)))$, where $\frac{\partial}{\partial t} q'(t; \hat{q}, u'(\cdot)) = f\left(q'(t; \hat{q}, u'(\cdot)), u'(t)\right)$ for all available t and $q'(\tau; \hat{q}, u(\cdot)) = \hat{q}$. Then $F_t(v') = q'(t; \hat{q}, u'(\cdot))$.

Now we differentiate identity (10) with respect to t: $\frac{\partial}{\partial t} D_v \langle p(t), F_t \rangle = \frac{\partial}{\partial t} d_v J_\tau^t$ and change the order of the differentiation $D_v \frac{\partial}{\partial t} \langle p(t), F_t \rangle = d_v \frac{\partial}{\partial t} J_\tau^t$. We compute the derivatives with respect to t at $t = \tau$:

$$\frac{\partial}{\partial t} \langle p(t), F_t \rangle \big|_{t=\tau} = \langle \dot{p}(\tau), \hat{q} \rangle + \langle p(\tau), f(\hat{q}, u'(\tau)) \rangle, \qquad \frac{\partial}{\partial t} J_\tau^t \big|_{t=\tau} = \varphi(\hat{q}, u'(\tau)).$$

Now we have to differentiate with respect to $v'(\cdot) = (u'(\cdot), q'(\cdot))$. We however see that the quantities to differentiate depend only on the values of $u'(\cdot)$ and $q'(\cdot)$ at τ, i.e. on the finite-dimensional vector $(u'(\tau), \hat{q})$. We derive:

$$\dot{p}(\tau) + \frac{\partial}{\partial q} \langle p(\tau), f(q(\tau), u(\tau)) \rangle = \frac{\partial \varphi}{\partial q}(q(t), u(t)),$$

$$\frac{\partial}{\partial u} \langle p(\tau), f(q(\tau), u(\tau)) \rangle = \frac{\partial \varphi}{\partial u}(q(\tau), u(\tau)),$$

where $v(\cdot) = (q(\cdot), u(\cdot))$.

Of course, we can change τ and perform the differentiation at any available moment t. Finally, we obtain that (10) is equivalent to the identities

$$\dot{p}(t) + \frac{\partial}{\partial q} \left(\langle p(t), f(q(t), u(t)) \rangle - \varphi(q(t), u(t)) \right) = 0,$$

$$\frac{\partial}{\partial u} \left(\langle p(t), f(q(t), u(t)) \rangle - \varphi(q(t), u(t)) \right) = 0,$$

which can be completed by the equation $\dot{q} = f(q(t), u(t))$. We introduce a function $h(p,q,u) = \langle p, f(q,u) \rangle - \varphi(q,u)$ which is called the *Hamiltonian* of the optimal control problem (1), (2). This function permits us to present the obtained relations in a nice Hamiltonian form:

$$\begin{cases} \dot{p} = -\dfrac{\partial h}{\partial q}(p,q,u) \\ \dot{q} = \dfrac{\partial h}{\partial p}(p,q,u) \end{cases}, \quad \frac{\partial h}{\partial u}(p,q,u) = 0. \tag{11}$$

A more important fact is that system (11) has an intrinsic coordinate free interpretation. Recall that in the triple (p,q,u) neither p nor u has an intrinsic meaning; the pair (p,q) represents $\lambda \in T^*M$ while the pair (q,u) represents $v \in V$. First we consider an intermediate case $V = M \times U$ (when u is separated from q but coordinates in M are not fixed) and then turn to the completely intrinsic setting.

If $V = M \times U$, then $f : M \times U \to TM$ and $f(q,u) \in T_qM$. The Hamiltonian of the optimal control problem is a function $h : T^*M \times U \to \mathbb{R}$ defined by the formula $h(\lambda,u) = \lambda(f(q,u)) - \varphi(q,u)$, $\forall \lambda \in T_q^*M,\ q \in M,\ u \in U$. For any $u \in U$ we obtain a function $h_u \stackrel{def}{=} h(\cdot,u)$ on T^*M. The cotangent bundle T^*M possesses a canonical symplectic structure which provides a standard way to associate a *Hamiltonian vector field* to any smooth function on T^*M. We'll recall this procedure.

Let $\pi : T^*M \to M$ be the projection, $\pi(T_q^*M) = \{q\}$. The *Liouville* (or *tautological*) differential 1-form ς on T^*M is defined as follows. Let $\varsigma_\lambda : T_\lambda(T^*M) \to \mathbb{R}$ be the value of ς at $\lambda \in T^*M$, then $\varsigma_\lambda = \lambda \circ \pi_*$, the composition of $\pi_* : T_\lambda(T^*M) \to T_{\pi(\lambda)}M$ and the cotangent vector $\lambda : T_{\pi(\lambda)}M \to \mathbb{R}$. The coordinate presentation of the Liouville form is: $\varsigma_{(p,q)} = \langle p, dq\rangle = \sum\limits_{i=1}^{n} p^i dq^i$, where $p = (p^1,\ldots,p^n)$, $q = (q^1,\ldots,q^n)$. The *canonical symplectic structure* on T^*M is the differential 2-form $\sigma = d\varsigma$; its coordinate representation is: $\sigma = \sum\limits_{i=1}^{n} dp^i \wedge dq^i$. The Hamiltonian vector field associated to a smooth function $a : T^*M \to \mathbb{R}$ is a unique vector field $\vec{a}$ on T^*M which satisfies the equation $\sigma(\cdot,\vec{a}) = da$. The coordinate representation of this field is: $\vec{a} = \sum\limits_{i=1}^{n}\left(\frac{\partial a}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial a}{\partial q_i}\frac{\partial}{\partial p_i}\right)$. Equations (11) can be rewritten in the form:

$$\dot{\lambda} = \vec{h}_u(\lambda), \quad \frac{\partial h}{\partial u}(\lambda,u) = 0. \tag{12}$$

Now let V be an arbitrary locally trivial bundle over M. Consider the Cartesian product of two bundles:

$$T^*M \times_M V = \{(\lambda, v) : v \in V_q, \lambda \in T_q^*M, q \in M\}$$

that is a bundle over M whose fibers are Cartesian products of the correspondent fibers of V and T^*M. Hamiltonian of the optimal control problem takes the form $h(\lambda, v) = \lambda(f(v)) - \varphi(v)$; this is a well-defined smooth function on $T^*M \times_M U$. Let $\mathfrak{p} : T^*M \times_M V \to T^*M$ be the projection on the first factor, $\mathfrak{p} : (\lambda, v) \mapsto \lambda$. Equations (11) (or (12)) can be rewritten in the completely intrinsic form as follows: $(\mathfrak{p}^*\sigma)_v(\cdot, \dot{\lambda}) = dh$. One may check this fact in any coordinates; we leave this simple calculation to the reader.

Of course, by fixing a local trivialization of V, we turn the last relation back into a more convinient to study equation (12). A domain $\mathcal{D}$ in T^*M is called regular for the Hamiltonian h if for any $\lambda \in \mathcal{D}$ there exists a unique solution $u = \bar{u}(\lambda)$ of the equation $\frac{\partial h}{\partial u}(\lambda, u) = 0$, where $\bar{u}(\lambda)$ is smooth with respect to λ. In particular, if U is an affine space and the functions $u \mapsto h(\lambda, u)$ are strongly concave (convex) and bounded from above (below) for $\lambda \in \mathcal{D}$, then $\mathcal{D}$ is regular and $\bar{u}(\lambda)$ is defined by the relation

$$h(\lambda, \bar{u}(\lambda)) = \max_{u \in U} h(\lambda, u) \quad \left(h(\lambda, \bar{u}(\lambda)) = \min_{u \in U} h(\lambda, u) \right).$$

In the regular domain, we set $H(\lambda) = h(\lambda, \bar{u}(\lambda))$, where $\frac{\partial h}{\partial u}(\lambda, \bar{u}(\lambda)) = 0$. It is easy to see that equations (12) are equivalent to one Hamiltonian system $\dot{\lambda} = \vec{H}(\lambda)$. Indeed, the equality $d_{(\lambda, \bar{u}(\lambda))}h = d_\lambda h_{\bar{u}(\lambda)} + \frac{\partial h_{\bar{u}(\lambda)}}{\partial u} du = d_\lambda h_{\bar{u}(\lambda)}$ immediately implies that $\vec{H}(\lambda) = \vec{h}_{\bar{u}(\lambda)}(\lambda)$.

5 Second order information

We come back to the general setting of Section 2 and try to go beyond the Lagrange multipliers rule. Take a pair (ℓ, w) which satisfies equation (6). We call such pairs (normal) *Lagrangian points.* Let $\Phi(w) = z$. If w is a regular point of Φ, then $\Phi^{-1}(z) \cap O_w$ is a smooth codimension $\dim N$ submanifold of $\mathcal{W}$, for some neighborhood O_w of w. In this case w is a critical point of $J\big|_{\Phi^{-1}(z) \cap O_w}$. We are going to compute the Hessian of $J\big|_{\Phi^{-1}(z)}$ at w without resolving the constraints $\Phi(w) = z$. The formula we obtain makes sense without the regularity assumptions as well.

Let $s \mapsto \gamma(s)$ be a smooth curve in $\Phi^{-1}(z)$ such that $\gamma(0) = w$. Differentiation of the identity $\Phi(\gamma(s)) = z$ gives:

$$D_w\Phi\dot\gamma = 0, \quad D_w^2\Phi(\dot\gamma, \dot\gamma) + D_w\Phi\ddot\gamma = 0,$$

where $\dot\gamma$ and $\ddot\gamma$ are the first and the second derivatives of γ at $s = 0$. We also have:

$$\frac{d^2}{ds^2}J(\gamma(s))|_{s=0} = D_w^2 J(\dot\gamma, \dot\gamma) + D_w J\ddot\gamma \overset{\text{eq.(6)}}{=}$$
$$D_w^2 J(\dot\gamma, \dot\gamma) + \ell D_w\Phi\ddot\gamma = D_w^2 J(\dot\gamma, \dot\gamma) - \ell D_w^2\Phi(\dot\gamma, \dot\gamma).$$

Finally,

$$\mathrm{Hess}_w(J\big|_{\Phi^{-1}(z)}) = (D_w^2 J - \ell D_w^2\Phi)\big|_{\ker D_w\Phi}. \tag{13}$$

Proposition I.3 *If quadratic form (13) is positive (negative) definite, then w is a strict local minimizer (maximizer) of $J\big|_{\Phi^{-1}(z)}$.*

If w is a regular point of Φ, then the proposition is obvious but one can check that it remains valid without the regularity assumption. On the other hand, without the regularity assumption, local minimality does not imply nonnegativity of form (13). What local minimality (maximality) certainly implies is nonnegativity (nonpositivity) of form (13) on a finite codimension subspace of $\ker D_w\Phi$ (see [7, Ch. 20] and references there).

Definition. A Lagrangian point (ℓ, w) is called *sharp* if quadratic form (13) is nonnegative or nonpositive on a finite codimension subspace of $\ker D_w\Phi$.

Only sharp Lagrangian points are counted in the conditional extremal problems under consideration. Let Q be a real quadratic form defined on a linear space E. Recall that the *negative inertia index* (or the *Morse index*) $\mathrm{ind}Q$ is the maximal possible dimension of a subspace in E such that the restriction of Q to the subspace is a negative form. The *positive inertia index* of Q is the Morse index of $-Q$. Each of these indices is a nonnegative integer or $+\infty$. A Lagrangian point (ℓ, w) is sharp if the negative or positive inertia index of form (13) is finite.

In the optimal control problems, $\mathcal{W}$ is a huge infinite dimensional manifold while N usually has a modest dimension. It is much simpler to characterize Lagrange multipliers in T^*N (see the previous section) than to work directly with $J\big|_{\Phi^{-1}(z)}$. Fortunately, the information on the sign and, more generally, on the inertia indices of the infinite dimensional quadratic form (13) can also be extracted from the Lagrange multipliers or, more precisely, from the so called $\mathcal{L}$-*derivative* that can be treated as a dual to the form (13) object.

$\mathcal{L}$-derivative concerns the linearization of equation (6) at a given Lagrangian point. In order to linearize the equation we have to present its left- and right-hand sides as smooth mappings of some manifolds. No problem with the right-hand side: $w \mapsto d_w J$ is a smooth mapping from $\mathcal{W}$ to $T^*\mathcal{W}$. The variables (ℓ, w) of the left-hand side live in the manifold

$$\Phi^* T^* N = \{(\ell, w) : \ell \in T^*_{\Phi(w)},\ w \in \mathcal{W}\} \subset T^* N \times \mathcal{W}.$$

Note that $\Phi^* T^* N$ is a locally trivial bundle over $\mathcal{W}$ with the projector $\pi : (\ell, w) \mapsto w$; this is nothing else but the *induced bundle* from T^*N by the mapping Φ. We treat equation (6) as the equality of values of two mappings from $\Phi^* T^* N$ to $T^*\mathcal{W}$. Let us rewrite this equation in local coordinates.

So let $N = \mathbb{R}^m$ and $\mathcal{W}$ be a Banach space. Then $T^*N = \mathbb{R}^{m*} \times \mathbb{R}^m$ (where $T_z N = \mathbb{R}^{m*} \times \{z\}$), $T^*\mathcal{W} = \mathcal{W}^* \times \mathcal{W}$, $\Phi^* T^* N = \mathbb{R}^{m*} \times \mathbb{R}^m \times \mathcal{W}$. Surely, $\mathbb{R}^{m*} \cong \mathbb{R}^m$ but in the forthcoming calculations it is convenient to treat the first factor in the product $\mathbb{R}^{m*} \times \mathbb{R}^m$ as the space of linear forms on the second factor. We have: $\ell = (\zeta, z) \in \mathbb{R}^{m*} \times \mathbb{R}^m$ and equation (6) takes the form

$$\zeta \frac{d\Phi}{dw} = \frac{dJ}{dw}, \quad \Phi(w) = z. \tag{14}$$

Linearization of system (14) at the point (ζ, z, w) reads:

$$\zeta' \frac{d\Phi}{dw} + \zeta \frac{d^2\Phi}{dw^2}(w', \cdot) = \frac{d^2 J}{dw^2}(w', \cdot), \quad \frac{d\Phi}{dw} w' = z'. \tag{15}$$

We set

$$\mathcal{L}^0_{(\ell,w)}(\bar{\Phi}) = \{\ell' = (\zeta', z') \in T_\ell(T^*N) : \exists w' \in \mathcal{W} \text{ s.t. } (\zeta', z', w') \text{ satisfies } (15)\}.$$

Note that subspace $\mathcal{L}^0_{(\ell,w)}(\bar{\Phi}) \subset T_\ell(T^*N)$ does not depend on the choice of local coordinates. Indeed, to construct this subspace we take all $(\ell', w') \in T_{(\ell,w)}(\Phi^* T^* N)$ which satisfy the linearized equation (6) and then apply the projection $(\ell', w') \mapsto \ell'$.

Recall that $T_\ell(T^*N)$ is a symplectic space endowed with the canonical symplectic form σ_ℓ (cf. Sec. 4). A subspace $S \subset T_\ell(T^*N)$ is *isotropic* if $\sigma_\ell|_S = 0$. Isotropic subspaces of maximal possible dimension $m = \frac{1}{2} \dim T_\ell(T^*N)$ are called *Lagrangian subspaces.*

Proposition I.4 *$\mathcal{L}^0_{(\ell,w)}(\bar\Phi)$ is an isotropic subspace of $T_\ell(T^*N)$. If $\dim \mathcal{W} < \infty$, then $\mathcal{L}^0_{(\ell,w)}(\bar\Phi)$ is a Lagrangian subspace.*

Proof. First we'll prove the isotropy of $\mathcal{L}^0_{(\ell,w)}(\bar\Phi)$. Let $(\zeta',z'),(\zeta'',z'') \in T_\ell(T^*N)$. We have $\sigma_\ell((\zeta',z'),(\zeta'',z'')) = \zeta'z''-\zeta''z'$; here the symbol ζz denotes the result of the application of the linear form $\zeta \in \mathbb{R}^{m*}$ to the vector $z \in \mathbb{R}^n$ or, in the matrix terminology, the product of the row ζ and the column z. Assume that (ζ',z',w') and (ζ'',z'',w'') satisfy equations (15); then

$$\zeta'z'' = \zeta'\frac{d\Phi}{dw}w'' = \frac{d^2J}{dw^2}(w',w'') - \zeta\frac{d^2\Phi}{dw^2}(w',w''). \tag{16}$$

The right-hand side of (16) is symmetric with respect to w' and w'' due to the symmetry of second derivatives. Hence $\zeta'z''=\zeta''z'$. In other words, $\sigma_\ell((\zeta',z'),(\zeta'',z''))=0$. So $\mathcal{L}^0_{(\ell,w)}(\bar\Phi)$ is isotropic and, in particular, $\dim\left(\mathcal{L}^0_{(\ell,w)}(\bar\Phi)\right) \le m$.

Now show that the last inequality becomes the equality as soon as $\mathcal{W}$ is finite dimensional. Set $Q = \frac{d^2J}{dw^2} - \zeta\frac{d^2\Phi}{dw^2}$ and consider the diagram:

$$\zeta'\frac{d\Phi}{dw} - Q(w',\cdot) \ \overset{left}{\longleftarrow}\ (\zeta',w') \ \overset{right}{\longrightarrow}\ \left(\zeta', \frac{d\Phi}{dw}w'\right).$$

Then $\mathcal{L}^0_{(\ell,w)}(\bar\Phi) = right(\ker(left))$. Passing to a factor space if necessary we may assume that $\ker(left)\cap\ker(right)=0$; this means that:

$$\frac{d\Phi}{dw}w' \quad \& \quad Q(w',\cdot)=0 \quad \Rightarrow \quad w'=0. \tag{17}$$

Under this assumption, $\dim\mathcal{L}^0_{(\ell,w)}(\bar\Phi) = \dim\ker(left)$. On the other hand, relations (17) imply that the mapping $left : \mathbb{R}^{m*}\times\mathcal{W}\to\mathcal{W}^*$ is surjective. Indeed, if, on the contrary, the map $left$ is not surjective then there exists a nonzero vector $v \in (\mathcal{W}^*)^* = \mathcal{W}$ which annihilates the image of $left$; in other words, $\zeta'\frac{d\Phi}{dw}v - Q(w',v)=0,\ \forall \zeta',w'$. Hence $\frac{d\Phi}{dw}v=0$ & $Q(v,\cdot)=0$ that contradicts (17). It follows that $\dim\mathcal{L}^0_{(\ell,w)}(\bar\Phi) = \dim(\mathbb{R}^{m*}\times\mathcal{W}) - \dim\mathcal{W}^* = m$. $\square$

For infinite dimensional $\mathcal{W}$, the space $\mathcal{L}^0_{(\ell,w)}(\bar\Phi)$ may have dimension smaller than m due to an ill-posedness of equations (15); to guarantee dimension m one needs certain coercivity of the form $\zeta\frac{d^2\Phi}{dw^2}$. I am not going to discuss here what kind of coercivity is sufficient, it can be easily reconstructed from the proof of Proposition I.4 (see also [5]). Anyway, independently on any coercivity one can take a finite dimensional approximation of the original problem and obtain a Lagrangian subspace $\mathcal{L}^0_{(\ell,w)}(\bar\Phi)$ guaranteed by Proposition I.4. What happens with these subspaces when the approximation becomes better and better, do they have a well-defined limit (which would be unavoidably Lagrangian)? A remarkable fact is that such a limit does exist for any sharp Lagrangian point. It contains $\mathcal{L}^0_{(\ell,w)}(\bar\Phi)$ and is called the $\mathcal{L}$-derivative of $\bar\Phi$ at (ℓ,w). To formulate this result we need some basic terminology from set theoretic topology.

A partially ordered set $(\mathfrak{A},\prec)$ is a *directed set* if $\forall \alpha_1,\alpha_2\in\mathfrak{A}\ \exists\beta\in\mathfrak{A}$ such that $\alpha_1\prec\beta$ and $\alpha_2\prec\beta$. A family $\{x_\alpha\}_{\alpha\in\mathfrak{A}}$ of points of a topological space $\mathcal{X}$ indexed by the elements of $\mathfrak{A}$ is a generalized sequence in $\mathcal{X}$. A point $x\in\mathcal{X}$ is the limit of the generalized sequence $\{x_\alpha\}_{\alpha\in\mathfrak{A}}$ if for any neighborhood $\mathcal{O}_x$ of x in $\mathcal{X}$ $\exists\alpha\in\mathfrak{A}$ such that $x_\beta\in\mathcal{O}_x,\ \forall\beta\succ\alpha$; in this case we write $x=\lim\limits_{\mathfrak{A}} x_\alpha$.

Let $\mathfrak{w}$ be a finite dimensional submanifold of $\mathcal{W}$ and $w\in\mathfrak{w}$. If (ℓ,w) is a Lagrangian point for $\bar\Phi=(J,\Phi)$, then it is a Lagrangian point for $\bar\Phi|_{\mathfrak{w}}$. A straightforward calculation shows that the Lagrangian subspace $\mathcal{L}^0_{(\ell,w)}(\bar\Phi|_{\mathfrak{w}})$ depends on the tangent space $W=T_w\mathfrak{w}$ rather than on $\mathfrak{w}$, i.e. $\mathcal{L}^0_{(\ell,w)}(\bar\Phi|_{\mathfrak{w}})=\mathcal{L}^0_{(\ell,w)}(\bar\Phi|_{\mathfrak{w}'})$ as soon as $T_w\mathfrak{w}=T_w\mathfrak{w}'=W$. We denote $\Lambda_W=\mathcal{L}^0_{(\ell,w)}(\bar\Phi|_{\mathfrak{w}})$. Recall that Λ_W is an m-dimensional subspace of the $2m$-dimensional space $T_\ell(T^*N)$, i.e. Λ_W is a point of the Grassmann manifold of all m-dimensional subspaces in $T_\ell(T^*N)$.

Finally, we denote by $\mathfrak{W}$ the set of all finite dimensional subspaces of $T_w\mathcal{W}$ partially ordered by the inclusion "$\subset$". Obviously, $(\mathfrak{W},\subset)$ is a directed set and $\{\Lambda_W\}_{W\in\mathfrak{W}}$ is a generalized sequence indexed by the elements of this directed set. It is easy to check that there exists $W_0\in\mathfrak{W}$ such that $\Lambda_W\supset\mathcal{L}^0_{(\ell,w)}(\bar\Phi),\ \forall W\supset W_0$. In particular, if $\mathcal{L}^0_{(\ell,w)}(\bar\Phi)$ is m-dimensional, then $\Lambda_{W_0}=\mathcal{L}^0_{(\ell,w)}(\bar\Phi),\ \forall W\supset W_0$, the sequence Λ_W is stabilizing and $\mathcal{L}^0_{(\ell,w)}(\bar\Phi)=\lim\limits_{\mathfrak{W}}\Lambda_W$. In general, the sequence Λ_W is not stabilizing, nevertheless the following important result is valid.

Theorem I.1 *If (ℓ,w) is a sharp Lagrangian point, then there exists $\mathcal{L}_{(\ell,w)}(\bar\Phi)=\lim\limits_{\mathfrak{W}}\Lambda_W$.*

We omit the proof of the theorem, you can find this proof in paper [5] with some other results which allow to efficiently compute $\lim\limits_{\mathfrak{W}}\Lambda_W$. Lagrangian subspace $\mathcal{L}_{(\ell,w)}(\bar\Phi) = \lim\limits_{\mathfrak{W}} \Lambda_W$ is called the *$\mathcal{L}$-derivative* of $\bar\Phi = (J,\Phi)$ at the Lagrangian point (ℓ, w).

Obviously, $\mathcal{L}_{(\ell,w)}(\bar\Phi) \supset \mathcal{L}^0_{(\ell,w)}(\bar\Phi)$. One should think on $\mathcal{L}_{(\ell,w)}(\bar\Phi)$ as on a completion of $\mathcal{L}^0_{(\ell,w)}(\bar\Phi)$ by means of a kind of weak solutions to system (15) which could be missed due to the ill-posedness of the system.

Now we should explain the connection between $\mathcal{L}_{(\ell,w)}(\bar\Phi)$ and $\mathrm{Hess}_w(J\big|_{\Phi^{-1}(z)})$. We start from the following simple observation:

Lemma I.1 *Assume that* $\dim \mathcal{W} < \infty$, *$w$ is a regular point of Φ and* $\ker D_w\Phi \cap \ker(D^2_w J - \ell D^2_w \Phi) = 0$. *Then*

$$\ker \mathrm{Hess}_w(J\big|_{\Phi^{-1}(z)}) = 0 \quad \Leftrightarrow \quad \mathcal{L}_{(\ell,w)}(\bar\Phi) \cap T_\ell(T^*_z N) = 0,$$

i.e. quadratic form $\mathrm{Hess}_w(J\big|_{\Phi^{-1}(z)})$ *is nondegenerate if and only if the subspace* $\mathcal{L}_{(\ell,w)}(\bar\Phi)$ *is transversal to the fiber* $T^*_z N$.

Proof. We make computations in coordinates. First, $T_\ell(T^*_z N) = \{(\zeta', 0) : \zeta' \in \mathbb{R}^{n*}\}$; then, according to equations (15), $(\zeta', 0) \in \mathcal{L}_{(\ell,w)}(\bar\Phi)$ if and only if there exists $w \in \mathcal{W}$ such that

$$\frac{d\Phi}{dw} w' = 0, \quad \frac{d^2 J}{dw^2}(w', \cdot) - \ell \frac{d^2\Phi}{dw^2}(w', \cdot) = \zeta' \frac{d\Phi}{dw}. \tag{18}$$

Regularity of w implies that $\zeta' \frac{d\Phi}{dw} \neq 0$ and hence $w' \neq 0$ as soon as $\zeta' \neq 0$. Equalities (18) imply: $\frac{d^2 J}{dw^2}(w', v) - \ell \frac{d^2\Phi}{dw^2}(w', v) = 0, \quad \forall v \in \ker \frac{d\Phi}{dw}$, i.e. $w' \in \ker \mathrm{Hess}_w(J\big|_{\Phi^{-1}(z)})$. Moreover, our implications are invertible: we could start from a nonzero vector $w' \in \ker \mathrm{Hess}_w(J\big|_{\Phi^{-1}(z)})$ and arrive to a nonzero vector $(\zeta', 0) \in \mathcal{L}_{(\ell,w)}(\bar\Phi)$. $\square$

Remark. Condition $\ker D_w\Phi \cap \ker(D^2_w J - \ell D^2_w \Phi) = 0$ from Lemma I.1 is not heavy. Indeed, a pair (J, Φ) satisfies this condition at all its Lagrangian points if and only if 0 is a regular value of the mapping $(\zeta, w) \mapsto \zeta \frac{d\Phi}{dw} - \frac{dJ}{dw}$. Standard Transversality Theorem implies that this is true for generic pair (J, Φ).

6 Maslov index

Lemma I.1 is a starting point for a far going theory which allows to effectively compute the Morse index of the Hessians in terms of the $\mathcal{L}$-derivatives.

How to do it? Normally, extremal problems depend on some parameters. Actually, $z \in N$ is such a parameter and there could be other ones, which we do not explicitly add to the constraints. In the optimal control problems a natural parameter is the time interval $t_1 - t_0$. Anyway, assume that we have a continuous family of the problems and their sharp Lagrangian points: $\ell_\tau D_{w_\tau}\Phi_\tau = d_{w_\tau}J_\tau$, $\tau_0 \le \tau \le \tau_1$; let $\Lambda(\tau) = \mathcal{L}_{(\ell_\tau, w_\tau)}(\bar{\Phi}_\tau)$. Our goal is to compute the difference $\operatorname{ind}\operatorname{Hess}_{w_{\tau_1}}(J_{\tau_1}\big|_{\Phi_{\tau_1}^{-1}(z_{\tau_1})}) - \operatorname{ind}\operatorname{Hess}_{w_{\tau_0}}(J_{\tau_0}\big|_{\Phi_{\tau_0}^{-1}(z_{\tau_0})})$ in terms of the family of Lagrangian subspaces $\Lambda(\tau)$; that is to get a tool to follow the evolution of the Morse index under a continuous change of the parameters. This is indeed very useful since for some special values of the parameters the index could be known a'priori. It concerns, in particular, optimal control problems with the parameter $\tau = t_1 - t_0$. If $t_1 - t_0$ is very small then sharpness of the Lagrangian point almost automatically implies the positivity or negativity of the Hessian.

First we discuss the finite-dimensional case: Theorem I.1 indicates that finite-dimensional approximations may already contain all essential information. Let Q_τ be a continuous family of quadratic forms defined on a finite-dimensional vector space. If $\ker Q_\tau = 0$, $\tau_0 \le \tau \le \tau_1$, then $\operatorname{ind} Q_\tau$ is constant on the segment $[\tau_0, \tau_1]$. This is why Lemma I.1 opens the way to follow evolution of the index in terms of the $\mathcal{L}$-derivative: it locates values of the parameter where the index may change. Actually, $\mathcal{L}$-derivative allows to evaluate this change as well; the increment of $\operatorname{ind} Q_\tau$ is computed via so called Maslov index of a family of Lagrangian subspaces. In order to define this index we have to recall some elementary facts about symplectic spaces.

Let Σ, σ be a symplectic space, i.e. Σ is a $2n$-dimensional vector space and σ be a nondegenerate anti-symmetric bilinear form on Σ. The skew-orthogonal complement to the subspace $\Gamma \subset \Sigma$ is the subspace $\Gamma^\angle = \{x \in \Sigma : \sigma(x, \Gamma) = 0\}$. The nondegeneracy of σ implies that $\dim \Gamma^\angle = 2n - \dim \Gamma$. A subspace Γ is isotropic if and only if $\Gamma^\angle \supset \Gamma$; it is Lagrangian if and only if $\Gamma^\angle = \Gamma$.

Let $\Pi = span\{e_1, \ldots, e_n\}$ be a lagrangian subspace of Σ. Then there exist vectors $f_1, \ldots, f_n \in \Sigma$ such that $\sigma(e_i, f_j) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. We show this using induction with respect to n. Skew-orthogonal complement to the space $span\{e_1, \ldots, e_{n-1}\}$ contains an element f which is not skew-orthogonal to e_n; we set $f_n = \frac{1}{\sigma(e_n, f)} f$. We have

$$span\{e_n, f_n\} \cap span\{e_n, f_n\}^\angle = 0$$

and the restriction of σ to $span\{e_n, f_n\}^{\angle}$ is a nondegenerate bilinear form. Hence $span\{e_n, f_n\}^{\angle}$ is a $2(n-1)$-dimensional symplectic space with a Lagrangian subspace $span\{e_1, \ldots, e_{n-1}\}$. According to the induction assumption, there exist $f_1, \ldots, f_{n-1}$ such that $\sigma(e_i, f_j) = \delta_{ij}$ and we are done.

Vectors $e_1, \ldots, e_n, f_1, \ldots, f_n$ form a basis of Σ; in particular, $\Delta = span\{f_1, \ldots, f_n\}$ is a transversal to Π Lagrangian subspace, $\Sigma = \Pi \oplus \Delta$. If $x_i = \sum\limits_{j=1}^{n}(\zeta_i^j e_j + z_i^j f_j)$, $i = 1, 2$, and $\zeta_i = (\zeta_i^1, \ldots, \zeta_i^n)$, $z_i = (z_i^1, \ldots, z_i^n)^{\top}$, then $\sigma(x_1, x_2) = \zeta_1 z_2 - \zeta_2 z_1$. The coordinates ζ, z identify Σ with $\mathbb{R}^{n*} \times \mathbb{R}^n$; any transversal to Δ n-dimensional subspace $\Lambda \subset \Sigma$ has the following presentation in these coordinates:

$$\Lambda = \{z^{\top}, S_{\Lambda} z) : z \in \mathbb{R}^n\},$$

where S_{Λ} is an $n \times n$-matrix. The subspace Λ is Lagrangian if and only if $S_{\Lambda}^* = S_{\Lambda}$. We have:

$$\Lambda \cap \Pi = \{(z^{\top}, 0) : z \in \ker S_{\Lambda}\},$$

the subspace Λ is transversal to Π if and only if S_{Λ} is nondegenerate.

That's time to introduce some notations. Let $L(\Sigma)$ be the set of all Lagrangian subspaces, a closed subset of the Grassmannian $G_n(\Sigma)$ of n-dimensional subspaces in Σ. We set

$$\Delta^{\pitchfork} = \{\Lambda \in L(\Sigma) : \Lambda \cap \Delta = 0\},$$

an open subset of $L(\Sigma)$. The mapping $\Lambda \mapsto S_{\Lambda}$ gives a regular parametrization of $\Delta^{\pitchfork}$ by the $n(n+1)/2$-dimensional space of symmetric $n \times n$-matrices. Moreover, above calculations show that $L(\Sigma) = \bigcup\limits_{\Delta \in L(\Sigma)} \Delta^{\pitchfork}$. Hence $L(\Sigma)$ is a $n(n+1)/2$-dimensional submanifold of the Grassmannian $G_n(\Sigma)$ covered by coordinate charts $\Delta^{\pitchfork}$. The manifold $L(\Sigma)$ is called Lagrange Grassmannian associated to the symplectic space Σ. It is not hard to show that any coordinate chart $\Delta^{\pitchfork}$ is everywhere dense in $L(\Sigma)$; our calculations give also a local parametrization of its complement.

Given $\Pi \in L(\Sigma)$, the subset

$$\mathcal{M}_{\Pi} = L(\Sigma) \setminus \Pi^{\pitchfork} = \{\Lambda \in L(\Sigma) : \Lambda \cap \Pi \neq 0\}$$

is called the *train* of Π. Let $\Lambda_0 \in \mathcal{M}_{\Pi}$, $\dim(\Lambda_0 \cap \Pi) = k$. Assume that Δ is transversal to both Λ_0 and Π (i.e. $\Delta \in \Lambda_0^{\pitchfork} \cap \Pi^{\pitchfork}$). The mapping

$\Lambda \mapsto S_\Lambda$ gives a regular parametrization of the neighborhood of Λ_0 in $\mathcal{M}_\Pi$ by a neighborhood of a corank k matrix in the set of all degenerate symmetric $n \times n$-matrices. A basic perturbation theory for symmetric matrices now implies that a small enough neighborhood of Λ_0 in $\mathcal{M}_\Pi$ is diffeomorphic to the Cartesian product of a neighborhood of the origin of the cone of all degenerate symmetric $k \times k$-matrices and a $(n(n+1)-k(k+1))/2$-dimensional smooth manifold (see [1, Lemma 2.2] for details). We see that $\mathcal{M}_\Pi$ is not a smooth submanifold of $L(\Sigma)$ but a union of smooth strata, $\mathcal{M}_\Pi = \bigcup\limits_{k>0} \mathcal{M}_\Pi^{(k)}$, where $\mathcal{M}_\Pi^{(k)} = \{\Lambda \in L(\Sigma) : \dim(\Lambda \cap \Pi) = k\}$ is a smooth submanifold of $L(\Sigma)$ of codimension $k(k+1)/2$.

Let $\Lambda(\tau)$, $\tau \in [t_0, t_1]$ be a smooth family of Lagrangian subspaces (a smooth curve in $L(\Sigma)$) and $\Lambda(t_0), \Lambda(t_1) \in \Pi^{\pitchfork}$. We are going to define the intersection number of $\Lambda(\cdot)$ and $\mathcal{M}_\Pi$. It is called the Maslov index and is denoted $\mu_\Pi(\Lambda(\cdot))$. Crucial property of this index is its homotopy invariance: given a homotopy $\Lambda^s(\cdot)$, $s \in [t_0, t_1]$ such that $\Lambda^s(t_0), \Lambda^s(t_1) \in \Pi^{\pitchfork}$ $\forall s \in [0,1]$, we have $\mu_\Pi(\Lambda^0(\cdot)) = \mu_\Pi(\Lambda^1(\cdot))$.

It is actually enough to define $\mu_\Pi(\Lambda(\cdot))$ for the curves which have empty intersection with $\mathcal{M}_\Pi \setminus \mathcal{M}_\Pi^{(1)}$; the desired index would have a well-defined extension to other curves by continuity. Indeed, generic curves have empty intersection with $\mathcal{M}_\Pi \setminus \mathcal{M}_\Pi^{(1)}$ and, moreover, generic homotopy has empty intersection with $\mathcal{M}_\Pi \setminus \mathcal{M}_\Pi^{(1)}$ since any of submanifolds $\mathcal{M}_\Pi^{(k)}$, $k = 2, \dots n$ has codimension greater or equal to 3 in $L(\Sigma)$. Putting any curve in general position by a small perturbation, we obtain the curve which bypasses $\mathcal{M}_\Pi \setminus \mathcal{M}_\Pi^{(1)}$, and the invariance with respect to generic homotopies of the Maslov index defined for generic curves would imply that the value of the index does not depend on the choice of a small perturbation.

What remains is to fix a "coorientation" of the smooth hypersurface $\mathcal{M}_\Pi^{(1)}$ in $L(\Sigma)$, i. e. to indicate the "positive and negative sides" of the hypersurface. As soon as we have a coorientation, we may compute $\mu_\Pi(\Lambda(\cdot))$ for any curve $\Lambda(\cdot)$ which is transversal to $\mathcal{M}_\Pi^{(1)}$ and has empty intersection with $\mathcal{M}_\Pi \setminus \mathcal{M}_\Pi^{(1)}$. Maslov index of $\Lambda(\cdot)$ is just the number of points where $\Lambda(\cdot)$ intersects $\mathcal{M}_\Pi^{(1)}$ in the positive direction minus the number of points where this curve intersects $\mathcal{M}_\Pi^{(1)}$ in the negative direction. Maslov index of any curve with endpoints out of $\mathcal{M}_\Pi$ is defined by putting the curve in general position. Proof of the homotopy invariance is the same as for usual intersection number of a curve with a closed cooriented hypersurface (see, for instance, the nice elementary book by J. Milnor "Topology from the differential viewpoint", 1965).

The coorientation is a byproduct of the following important structure on the tangent spaces to $L(\Sigma)$. It happens that any tangent vector to $L(\Sigma)$ at the point $\Lambda \in L(\Sigma)$ can be naturally identified with a quadratic form on Λ. Her we use the fact that Λ is not just a point in the Grassmannian but an n-dimensional linear space. To associate a quadratic form on Λ to the velocity $\dot{\Lambda}(t) \in T_{\Lambda(t)}L(\Sigma)$ of a smooth curve $\Lambda(\cdot)$ we proceed as follows: given $x \in \Lambda(t)$ we take a smooth curve $\tau \mapsto x(\tau)$ in Σ in such a way that $x(\tau) \in \Lambda(\tau)$, $\forall \tau$ and $x(\tau) = x$. Then we define a quadratic form $\underline{\dot{\Lambda}}(t)(x)$, $x \in \Lambda(t)$, by the formula $\underline{\dot{\Lambda}}(t)(x) = \sigma(x, \dot{x}(t))$.

The point is that $\sigma(x, \dot{x}(t))$ does not depend on the freedom in the choice of the curve $\tau \mapsto x(\tau)$, although $\dot{x}(t)$ depends on this choice. Let us check the required property in the coordinates. We have $x = (z^\top, S_{\Lambda(t)}z)$ for some $z \in \mathbb{R}^n$ and $x(\tau) = (z(\tau)^\top, S_{\Lambda(\tau)}z(\tau))$. Then

$$\sigma(x, \dot{x}(t)) = z^\top(\dot{S}_{\Lambda(t)}z + S_{\Lambda(t)}\dot{z}) - \dot{z}^\top S_{\Lambda(t)}z = z^\top \dot{S}_{\Lambda(t)}z;$$

vector $\dot{z}$ does not show up. We have obtained a coordinate presentation of $\underline{\dot{\Lambda}}(t)$:

$$\underline{\dot{\Lambda}}(t)(z^\top, S_{\Lambda(t)}z) = z^\top \dot{S}_{\Lambda(t)}z,$$

which implies that $\dot{\Lambda} \mapsto \underline{\dot{\Lambda}}$, $\dot{\Lambda} \in T_\Lambda L(\Sigma)$ is an isomorphism of $T_\Lambda L(\Sigma)$ on the linear space of quadratic forms on Λ.

We are now ready to define the coorientation of $\mathcal{M}_\Pi^{(1)}$. Assume that $\Lambda(t) \in \mathcal{M}_\Pi^{(1)}$, i. e. $\Lambda(t) \cap \Pi = \mathbb{R}x$ for some nonzero vector $x \in \Sigma$. In coordinates, $x = (z^\top, 0)$, where $\mathbb{R}x = \ker S_{\Lambda(t)}$. It is easy to see that $\dot{\Lambda}(t)$ is transversal to $\mathcal{M}_\Pi^{(1)}$ (i. e. $\dot{S}_{\Lambda(t)}$ is transversal to the cone of degenerate symmetric matrices) if and only if $\underline{\dot{\Lambda}}(t)(x) \neq 0$ (i. e. $z^\top \dot{S}_{\Lambda(t)}z \neq 0$). Vector x is defined up to a scalar multiplier and $\underline{\dot{\Lambda}}(t)(\alpha x) = \alpha^2 \underline{\dot{\Lambda}}(t)(x)$ so that the sign of $\underline{\dot{\Lambda}}(t)(x)$ does not depend on the selection of x.

Definition. We say that $\Lambda(\cdot)$ intersects $\mathcal{M}_\Pi^{(1)}$ at the point $\Lambda(t)$ in the positive (negative) direction if $\underline{\dot{\Lambda}}(t)(x) > 0$ (< 0).

This definition completes the construction of the Maslov index. A weak point of the construction is the necessity to put the curve in general position in order to compute the intersection number. This does not look as an efficient way to thinks since putting the curve in general position is nothing else but a deliberate spoiling of a maybe nice and symmetric original object that makes even more involved the nontrivial problem of the localization of its intersection with $\mathcal{M}_\Pi$. Fortunately, just the fact that Maslov index is homotopy invariant leads to a very simple and effective way of its computation without putting things in general position and without looking for the intersection points with $\mathcal{M}_\Pi$.

Lemma I.2 *Assume that* $\Pi \cap \Delta = \Lambda(\tau) \cap \Delta = 0,\ \forall \tau \in [t_0, t_1]$. *Then* $\mu_\Pi(\Lambda(\cdot)) = \mathrm{ind} S_{\Lambda(t_0)} - \mathrm{ind} S_{\Lambda(t_1)}$, *where* $\mathrm{ind} S$ *is the Morse index of the quadratic form* $z^\top S z,\ z \in \mathbb{R}^n$.

Proof. The matrices $S_{\Lambda(t_0)}$ and $S_{\Lambda(t_0)}$ are nondegenerate since $\Lambda(t_0) \cap \Pi = \Lambda(t_1) \cap \Pi = 0$ (we define the Maslov index only for the curves whose endpoins are out of $\mathcal{M}_\Pi$). The set of nondegenerate quadratic forms with a prescribed value of the Morse index is a connected open subset of the linear space of all quadratic forms in n variables. Hence homotopy invariance of the Maslov index implies that $\mu_\Pi(\Lambda(\cdot))$ depends only on $\mathrm{ind} S_{\Lambda(t_0)}$ and $\mathrm{ind} S_{\Lambda(t_1)}$. It remains to compute μ_Π of sample curves in $\Delta^\pitchfork$, say, for segments of the curve $\Lambda(\cdot)$ such that

$$S_{\Lambda(\tau)} = \begin{pmatrix} \tau-1 & 0 & \dots & 0 \\ 0 & \tau-2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \tau-n \end{pmatrix}. \qquad \square$$

In general, given curve is not contained in the fixed coordinate neighborhood $\Delta^\pitchfork$ but any curve can be divided into segments $\Lambda(\cdot)|_{[\tau_i, \tau_{i+1}]},\ i = 0, \dots, l$, in such a way that $\Lambda(\tau) \in \Delta_i^\pitchfork\ \ \forall \tau \in [\tau_i, \tau_{i+1}]$, where $\Delta_i \in \Pi^\pitchfork,\ i = 0, \dots, l$; then $\mu_\Pi(\Lambda(\cdot)) = \sum\limits_i \mu_\Pi\left(\Lambda(\cdot)|_{[\tau_i, \tau_{i+1}]}\right)$.

Lemma I.2 implies the following useful formula which is valid for the important class of *monotone increasing curves* in the Lagrange Grassmannian, i.e. the curves $\Lambda(\cdot)$ such that $\underline{\dot{\Lambda}}(t)$ are nonnegative quadratic forms: $\underline{\dot{\Lambda}}(t) \geq 0,\ \forall t$.

Corollary I.1 *Assume that* $\underline{\dot{\Lambda}}(\tau) \geq 0,\ \forall \tau \in [t_0, t_1]$ *and* $\{\tau \in [t_0, t_1] : \Lambda(\tau) \cap \Pi \neq 0\}$ *is a finite subset of* (t_0, t_1). *Then*

$$\mu_\Pi\left(\Lambda(\cdot)\right) = \sum_{\tau \in (t_0, t_1)} \dim(\Lambda(\tau) \cap \Pi). \qquad \square$$

Corollary I.1 can be also applied to the case of monotone decreasing curves defined by the inequality $\underline{\dot{\Lambda}}(t) \leq 0,\ \forall t$; the change of parameter $t \mapsto t_0 + t_1 - t$

makes the curve monotone increasing and and change sign of the Maslov index.

Let me now recall that our interest to these symplectic playthings was motivated by the conditional minimum problems. As it was mentioned at the beginning of the section, we are going to apply this stuff to the case $\Sigma = T_{\ell_\tau}(T^*M)$, $\ell_\tau \in T^*_{z_\tau}M$, $\Pi = T_{\ell_\tau}(T^*_{z_\tau}M)$, $\Lambda(\tau) = \mathcal{L}_{(\ell_\tau,w_\tau)}(\bar{\Phi}_\tau)$, where $z_\tau = \Phi_\tau(w_\tau)$. In this case, not only Λ but also Π and even symplectic space Σ depend on τ. We thus have to define Maslov index in such situation. This is easy. We consider the bundle

$$\{(\xi,\tau) : \xi \in T_{\ell_\tau}(T^*M),\ t_0 \le \tau \le t_1\} \tag{19}$$

over the segment $[t_0, t_1]$ induced from $T(T^*M)$ by the mapping $\tau \mapsto \ell_\tau$. Bundle (19) endowed with the symplectic structure and its subbundle

$$\{(\xi,\tau) : \xi \in T_{\ell_\tau}(T^*_{z_\tau}M)\}$$

are trivial as any bundle over a segment. More precisely, let $t \in [t_0, t_1]$, $\Sigma_t = T_{\ell_t}(T^*M)$, $\Pi_t = T_{\ell_t}(T^*_{z_t}M)$; then there exists a continuous with respect to τ family of linear symplectic mappings $\Xi_\tau : T_{\ell_\tau}(T^*M) \to \Sigma_t$ such that $\Xi_\tau(T_{\ell_\tau}(T^*_{z_\tau}M)) = \Pi_t$, $t_0 \le \tau \le t_1$, $\Xi_t = \mathrm{Id}$. To any continuous family of Lagrangian subspaces $\Lambda(\tau) \subset T_{\ell_\tau}(T^*M)$, where $\Lambda(t_i) \cap \Pi_{t_i} = 0$, $i = 0, 1$, we associate a curve $\Xi_{\cdot}\Lambda(\cdot) : \tau \mapsto \Xi_\tau\Lambda(\tau)$ in the Lagrange Grassmannian $L(\Sigma_t)$ and set $\mu(\Lambda(\cdot)) \overset{def}{=} \mu_{\Pi_t}(\Xi_{\cdot}\Lambda(\cdot))$. Homotopy invariance of the Maslov index implies that $\mu_{\Pi_t}(\Xi_{\cdot}\Lambda(\cdot))$ does not depend on the choice of t and Ξ_τ.

Theorem I.2 *Assume that* $\dim \mathcal{W} < \infty$*,*

$$\bar{\Phi}_\tau = (J_\tau, \Phi_\tau) : \mathcal{W} \to \mathbb{R} \times M, \quad \tau \in [t_0, t_1]$$

is a continuous one-parametric family of smooth mappings and (ℓ_τ, w_τ) *is a continuous family of their Lagrangian points such that* $\ell_\tau \ne 0$*,* w_τ *is a regular point of* Φ_τ*, and* $\ker D_{w_\tau}\Phi_\tau \cap \ker(D^2_{w_\tau}J_\tau - \ell_\tau D^2_{w_\tau}\Phi_\tau) = 0$*,* $t_0 \le \tau \le t_1$*. Let* $z_\tau = \Phi(w_\tau)$*,* $\Lambda(\tau) = \mathcal{L}_{(\ell_\tau,w_\tau)}(\bar{\Phi}_\tau)$*. If* $\mathrm{Hess}_{w_{t_i}}(J_{t_i}\big|_{\Phi^{-1}_{t_i}(z_{t_i})})$*,* $i = 1, 2$*, are nondegenerate, then*

$$\operatorname{ind}\mathrm{Hess}_{w_{t_0}}(J_{t_0}\big|_{\Phi^{-1}_{t_0}(z_{t_0})}) - \operatorname{ind}\mathrm{Hess}_{w_{t_1}}(J_{t_1}\big|_{\Phi^{-1}_{t_1}(z_{t_1})}) = \mu(\Lambda(\cdot)).$$

Remark. If $\ell_\tau = 0$, then w_τ is a critical point of J_τ (without restriction to the level set of Φ_τ). Theorem I.2 can be extended to this situation (with the same proof) if we additionally assume that $\ker \mathrm{Hess}_{w_\tau} J_\tau = 0$ for any τ such that $\ell_\tau = 0$.

Proof. We introduce simplified notations: $A_\tau = D_{w_\tau}\Phi_\tau$, $Q_\tau = D^2_{w_\tau} J_\tau - \ell_\tau D^2_{w_\tau}\Phi_\tau$; the $\mathcal{L}$-derivative $\mathcal{L}_{(\ell_\tau, w_\tau)}(\bar{\Phi}_\tau) = \Lambda(\tau)$ is uniquely determined by the linear map A_τ and the symmetric bilinear form Q_τ. Fix local coordinates in the neighborhoods of w_τ and z_τ and set:

$$\Lambda(A, Q) = \{(\zeta, Av) : \zeta A + Q(v, \cdot) = 0\} \in L(\mathbb{R}^{n*} \times \mathbb{R}^n);$$

then $\Lambda_\tau = \Lambda(A_\tau, Q_\tau)$.

The assumption $\ker A_\tau \cap \ker Q_\tau = 0$ implies the smoothness of the mapping $(A, Q) \mapsto \Lambda(A, Q)$ for (A, Q) close enough to (A_τ, Q_τ). Indeed, as it is shown in the proof of Proposition I.4, this assumption implies that the mapping $left_\tau : (\zeta, v) \mapsto \zeta A_\tau + Q_\tau(v, \cdot)$ is surjective. Hence the kernel of the mapping

$$(\zeta, v) \mapsto \zeta A + Q(v, \cdot) \tag{20}$$

smoothly depends on (A, Q) for (A, Q) close to (A_τ, Q_τ). On the other hand, $\Lambda(A, Q)$ is the image of the mapping $(\zeta, v) \mapsto (\zeta, Av)$ restricted to the kernel of map (20).

Now we have to disclose a secret which the attentive reader already knows and is perhaps indignant with our lightness: Q_τ is not a well-defined bilinear form on $T_{w_\tau}\mathcal{W}$, it essentially depends on the choice of local coordinates in M. What are well-defined is the mapping $Q_\tau\big|_{\ker A_\tau} : \ker A_\tau \to T^*_{w_\tau}\mathcal{W}$ (check this by yourself or see [3, Subsec. 2.3]), the map $A_\tau : T_{w_\tau}\mathcal{W} \to T_{z_\tau}M$ and, of course, the Lagrangian subspace $\Lambda(\tau) = \mathcal{L}_{(\ell_\tau, w_\tau)}(\bar{\Phi}_\tau)$. By the way, the fact that $Q_\tau\big|_{\ker A_\tau}$ is well-defined guarantees that assumptions of Theorem I.2 do not depend on the coordinates choice.

Recall that any local coordinates $\{z\}$ on M induce coordinates $\{(\zeta, z) : \zeta \in \mathbb{R}^{n*}, z \in \mathbb{R}^n\}$ on T^*M and $T^*_z M = \{(\zeta, 0) : \zeta \in \mathbb{R}^{n*}\}$ in the induced coordinates.

Lemma I.3 *Given $\hat{z} \in M$, $\ell \in T^*_{\hat{z}}M \setminus \{0\}$, and a Lagrangian subspace $\Delta \in T_\ell(T^*_{\hat{z}}M)^\pitchfork \subset L(T_\ell(T^*M))$, there exist centered at $\hat{z}$ local coordinates on M such that $\Delta = \{(0, z) : z \in \mathbb{R}^n\}$ in the induced coordinates on $T_\ell(T^*M)$.*

Proof. Working in arbitrary local coordinates we have $\ell = (\zeta_0, 0)$, $\Delta = \{(Sz, z) : z \in \mathbb{R}^n\}$, where S is a symmetric matrix. In other words, Δ is the tangent space at $(\zeta_0, 0)$ to the graph of the differential of the function $a(z) = \zeta_0 z + \frac{1}{2} z^\top S z$. any smooth function with a nonzero differential can be locally made linear by a smooth change of variables. To prove the lemma it is enough to make a coordinates change which kills second derivative of the function a, for instance: $z \mapsto z + \frac{1}{2|\zeta_0|^2}(z^\top S z)\zeta_0^\top$. $\square$

We continue the proof of Theorem I.2. Lemma I.3 gives us the way to take advantage of the fact that Q_τ depends on the choice of local coordinates in M. Indeed, bilinear form Q_τ is degenerate if and only if $\Lambda_\tau \cap \{(0, z) : z \in \mathbb{R}^n\} \neq 0$; this immediately follows from the relation

$$\Lambda_\tau = \{(\zeta, A_\tau v) : \zeta A_\tau + Q_\tau(v, \cdot) = 0\}.$$

Given $t \in [t_0, t_1]$ take a transversal to $T_{\ell_t}(T^*_{z_t}M)$ and $\Lambda(t)$ Lagrangian subspace $\Delta_t \subset T_{\ell_t}(T^*M)$ and centered at z_t local coordinates in M such that $\Delta_t = \{(0, z) : z \in \mathbb{R}^n\}$ in these coordinates. Then $\Lambda(\tau)$ is transversal to $\{(0, z) : z \in \mathbb{R}^n\}$ for all τ from a neighborhood O_t of t in $[t_0, t_1]$. Selecting an appropriate finite subcovering from the covering O_t, $t \in [t_0, t_1]$ of $[t_0, t_1]$ we can construct a subdivision $t_0 = \tau_0 < \tau_1 < \ldots < \tau_k < \tau_{k+1} = t_1$ of $[t_0, t_1]$ with the following property: $\forall i \in \{0, 1, \ldots, k\}$ the segment $\{z_\tau : \tau \in [\tau_i, \tau_{i+1}]\}$ of the curve z_τ is contained in a coordinate neighborhood $\mathcal{O}^i$ of M such that $\Lambda_\tau \cap \{(0, z) : z \in \mathbb{R}^n\} = 0$ $\forall \tau \in [\tau_i, \tau_{i+1}]$ in the correspondent local coordinates.

We identify the form Q_τ with its symmetric matrix, i.e. $Q_\tau(v_1, v_2) = v_1^\top Q_\tau v_2$. Then Q_τ is a nondegenerate symmetric matrix and

$$\Lambda(\tau) = \{(\zeta, -A_\tau Q_\tau^{-1} A_\tau^\top \zeta^\top\}, \quad \tau_i \leq \tau \leq \tau_{i+1}. \tag{21}$$

Now focus on the subspace $\Lambda(\tau_i)$; it has a nontrivial intersection with $\{(\zeta, 0) : \zeta \in \mathbb{R}^{n*}\} = T_{\ell_{\tau_i}}(T^*_{z_{\tau_i}}M)$ if and only if the matrix $A_{\tau_i} Q_{\tau_i}^{-1} A_{\tau_i}^\top$ is degenerate. This is the matrix of the restriction of the nondegenerate quadratic form $v \mapsto v^\top Q_{\tau_i}^{-1} v$ to the image of the linear map $A_{\tau_i}^\top$. Hence $A_{\tau_i} Q_{\tau_i}^{-1} A_{\tau_i}^\top$ can be made nondegenerate by the arbitrary small perturbation of the map $A_{\tau_i} : T_{w_{\tau_i}}\mathcal{W} \to T_{z_{\tau_i}}M$. Such perturbations can be realized simultaneously for $i = 1, \ldots, k$ [2] by passing to a continuous family $\tau \mapsto A'_\tau$, $t_0 \leq \tau \leq t_1$,

[2] We do not need to perturb A_{t_0} and $A_{t_{k+1}}$: assumption of the theorem and Lemma I.1 guarantee the required nondegeneracy property.

arbitrary close and homotopic to the family $\tau \mapsto A_\tau$. In fact, A'_τ can be chosen equal to A_τ out of an arbitrarily small neighborhood of $\{\tau_1, \dots, \tau_k\}$. Putting now A'_τ instead of A_τ in the expression for $\Lambda(\tau)$ we obtain a family of Lagrangian subspaces $\Lambda'(\tau)$. This family is continuous (see the paragraph containing formula (20)) and homotopic to $\Lambda(\cdot)$. In particular, it has the same Maslov index as $\Lambda(\cdot)$. In other words, we can assume without lack of generality that $\Lambda(\tau_i) \cap T_{\ell_{\tau_i}}(T^*_{z_{\tau_i}}M) = 0,\ i = 0, 1, \dots, k+1$. Then $\mu(\Lambda(\cdot)) = \sum\limits_{i=0}^{k} \mu\left(\Lambda(\cdot)\big|_{[\tau_i,\tau_{i+1}]}\right)$. Moreover, it follows from (21) and Lemma I.2 that

$$\mu\left(\Lambda(\cdot)\big|_{[\tau_i,\tau_{i+1}]}\right) = \mathrm{ind}(A_{\tau_{i+1}} Q^{-1}_{\tau_{i+1}} A^\top_{\tau_{i+1}}) - \mathrm{ind}(A_{\tau_i} Q^{-1}_{\tau_i} A^\top_{\tau_i}).$$

Besides that, $\mathrm{ind} Q_{\tau_i} = \mathrm{ind} Q_{\tau_{i+1}}$ since Q_τ is nondegenerate for all $\tau \in [\tau_i, \tau_{i+1}]$ and continuously depends on τ.

Recall that $\mathrm{Hess}_{w_\tau}\left(J_\tau\big|_{\Phi^{-1}(z_\tau)}\right) = Q_\tau\big|_{\ker A_\tau}$. In order to complete proof of the theorem it remains to show that

$$\mathrm{ind} Q_\tau = \mathrm{ind}\left(Q_\tau\big|_{\ker A_\tau}\right) + \mathrm{ind}(A_\tau Q_\tau^{-1} A_\tau^\top) \qquad (22)$$

for $\tau = \tau_i, \tau_{i+1}$.

Let us rearrange the second term in the right-hand side of (22). The change of variables $v = Q_\tau^{-1} A_\tau^\top z,\ z \in \mathbb{R}^n$, implies: $\mathrm{ind}\left(A_\tau Q_\tau^{-1} A_\tau^\top\right) = \mathrm{ind}\left(Q_\tau\big|_{\{Q_\tau^{-1} A_\tau^\top z \,:\, z \in \mathbb{R}^n\}}\right)$. We have: $Q_\tau(v, \ker A_\tau) = 0$ if and only if $Q_\tau(v, \cdot) = z^\top A_\tau$ for some $z \in \mathbb{R}^n$, i.e. $v^\top Q_\tau = z^\top A_\tau,\ v = Q_\tau^{-1} A_\tau^\top z$. Hence the right-hand side of (22) takes the form

$$\mathrm{ind} Q_\tau = \mathrm{ind}\left(Q_\tau\big|_{\ker A_\tau}\right) + \mathrm{ind}\left(Q_\tau\big|_{\{v \,:\, Q_\tau(v, \ker A_\tau) = 0\}}\right)$$

and $Q_\tau\big|_{\{v: Q_\tau(v, \ker A_\tau)=0\}}$ is a nondegenerate form for $\tau = \tau_i, \tau_{i+1}$. Now equality (22) is reduced to the following elementary fact of linear algebra: If Q is a nondegenerate quadratic form on $\mathbb{R}^m$ and $E \subset \mathbb{R}^m$ is a linear subspace, then $\mathrm{ind} Q = \mathrm{ind}\left(Q|_E\right) + \mathrm{ind}\left(Q|_{E^\perp_Q}\right) + \dim(E \cap E^\perp_Q)$, where $E^\perp_Q = \{v \in \mathbb{R}^m : Q(v, E) = 0\}$ and $E \cap E^\perp_Q = \ker\left(Q|_E\right) = \ker\left(Q|_{E^\perp_Q}\right)$. $\square$

Remark. Maslov index μ_Π is somehow more than just the intersection number with $\mathcal{M}_\Pi$. It can be extended, in a rather natural way, to all continuous curves in the Lagrange Grassmannian including those whose endpoint belong to $\mathcal{M}_\Pi$. This extension allows to get rid of the annoying nondegeneracy assumption for $\mathrm{Hess}_{w_{t_i}}(J_{t_i}\big|_{\Phi_{t_i}^{-1}(z_{t_i})})$ in the statement of Theorem I.2. In general, Maslov index computes 1/2 of the difference of the signatures of the Hessians which is equal to the difference of the Morse indices in the degenerate case (see [3] for this approach).

7 Regular extremals

A combination of the finite-dimensional Theorem I.2 with the limiting procedure of Theorem I.1 and with homotopy invariance of the Maslov index allows to efficiently compute Morse indices of the Hessians for numerous infinite-dimensional problems. Here we restrict ourselves to the simplest case of a regular extremal of the optimal control problem.

We use notations and definitions of Sections 3, 4. Let $h(\lambda, u)$ be the Hamiltonian of a smooth optimal control system and $\lambda_t,\ t_0 \le t \le t_1$, be an extremal contained in the regular domain $\mathcal{D}$ of h. Then λ_t is a solution of the Hamiltonian system $\dot\lambda = \vec{H}(\lambda)$, where $H(\lambda) = h(\lambda, \bar u(\lambda)),\ \frac{\partial h}{\partial u} h(\lambda, \bar u(\lambda)) = 0$.

Let $q(t) = \pi(\lambda_t),\ t_0, \le t \le t_1$ be the extremal path. Recall that the pair $(\lambda_{t_0}, \lambda_t)$ is a Lagrangian multiplier for the conditional minimum problem defined on an open subset of the space

$$M \times L_\infty([t_0, t_1], U) = \{(q_t), u(\cdot)) : q \in M, u(\cdot) \in L_\infty([t_0, t_1], U)\},$$

where $u(\cdot)$ is control and q_t is the value at t of the solution to the differential equation $\dot q = f(q, u(\tau)),\ \tau \in [t_0, t_1]$. In particular, $F_t(q_t, u(\cdot)) = q_t$. The cost is $J_{t_0}^{t_1}(q_t, u(\cdot))$ and constraints are $F_{t_0}(q_t, u(\cdot)) = q(0),\ q_t = q(t)$.

Let us set $J_t(u) = J_{t_0}^t(q(t), u(\cdot)),\ \Phi_t(u) = F_{t_0}(q(t), u(\cdot))$. A covector $\lambda \in T^*M$ is a Lagrange multiplier for the problem (J_t, Φ_t) if and only if there exists an extremal $\hat\lambda_\tau,\ t_0 \le \tau \le t$, such that $\lambda_{t_0} = \lambda,\ \hat\lambda_t \in T^*_{q(t)}M$. In particular, λ_{t_0} is a Lagrange multiplier for the problem (J_t, Φ_t) associated to the control $u(\cdot) = \bar u(\lambda_.)$. Moreover, all sufficiently close to λ_{t_0} Lagrange multipliers for this problem are values at t_0 of the solutions $\lambda(\tau),\ t_0 \le \tau \le t$ to the Hamiltonian system $\dot\lambda = \vec{H}(\lambda)$ with the boundary condition $\lambda(t) \in T^*_{q(t)}M$.

We'll use exponential notations for one-parametric groups of diffeomorphisms generated ordinary differential equations. In particular, $e^{\tau\vec{H}} : T^*M \to$

T^*M, $\tau \in \mathbb{R}$, is a flow generated by the equation $\dot{\lambda} = \vec{H}(\lambda)$, so that $\lambda(\tau') = e^{(\tau'-\tau)\vec{H}}(\lambda(\tau)$, $\tau, \tau' \in \mathbb{R}$, and Lagrange multipliers for the problem (J_t, Φ_t) fill the n-dimensional submanifold $e^{(t_0-t)\vec{H}}\left(T^*_{q(t)}M\right)$.

We set $\bar{\Phi}_t = (J_t, \Phi_t)$; it is easy to see that the $\mathcal{L}$-derivative $\mathcal{L}_{(\lambda_{t_0},u)}(\bar{\Phi}_t)$ is the tangent space to $e^{(t_0-t)\vec{H}}\left(T^*_{q(t)}M\right)$, i.e. $\mathcal{L}_{(\lambda_{t_0},u)}(\bar{\Phi}_t) = e_*^{(t_0-t)\vec{H}}T_{\lambda_t}\left(T^*_{q(t)}M\right)$. Indeed, let us recall the construction of the $\mathcal{L}$-derivative. First we linearize the equation for Lagrange multipliers at λ_{t_0}. Solutions of the linearized equation form an isotropic subspace $\mathcal{L}^0_{(\lambda_{t_0},u)}(\bar{\Phi}_t)$ of the symplectic space $T_{\lambda_{t_0}}(T^*M)$. If $\mathcal{L}^0_{(\lambda_{t_0},u)}(\bar{\Phi}_t)$ is a Lagrangian subspace (i.e. $\dim \mathcal{L}^0_{(\lambda_{t_0},u)}(\bar{\Phi}_t) = \dim M$), then $\mathcal{L}_{(\lambda^0_{t_0},u)}(\bar{\Phi}_t) = \mathcal{L}_{(\lambda_{t_0},u)}(\bar{\Phi}_t)$, otherwise we need a limiting procedure to complete the Lagrangian subspace. In the case under consideration, $\mathcal{L}^0_{(\lambda_{t_0},u)}(\bar{\Phi}_t) = e_*^{(t_0-t)\vec{H}}T_{\lambda_t}\left(T^*_{q(t)}M\right)$ has a proper dimension and thus coincides with
$\mathcal{L}_{(\lambda_{t_0},u)}(\bar{\Phi}_t)$. We can check independently that $e_*^{(t_0-t)\vec{H}}T_{\lambda_t}\left(T^*_{q(t)}M\right)$ is Lagrangian: indeed, $T_{\lambda_t}\left(T^*_{q(t)}M\right)$ is Lagrangian and $e_*^{(t_0-t)\vec{H}} : T_{\lambda_t}(T^*M) \to T_{\lambda_{t_0}}(T^*M)$ is an isomorphism of symplectic spaces since Hamiltonian flows preserve the symplectic form.

So $t \mapsto \mathcal{L}_{(\lambda_{t_0},u)}(\bar{\Phi}_t)$ is a smooth curve in the Lagrange Grassmannian $L\left(T_{\lambda_{t_0}}(T^*M)\right)$ and we can try to compute Morse index of

$$\mathrm{Hess}_u\left(J_{t_1}\big|_{\Phi^{-1}_{t_1}(q(t_0))}\right) = \mathrm{Hess}_u\left(J^{t_1}_{t_0}\big|_{F^{-1}_{t_0}(q(t_0))\cap F^{-1}_{t_1}(q(t_1))}\right)$$

via the Maslov index of this curve. Of course, such a computation has no sense if the index is infinite.

Proposition I.5 (Legendre condition) *If quadratic form $\frac{\partial^2 h}{\partial u^2}(\lambda_t, u(t))$ is negative definite for any $t \in [t_0, t_1]$, then* $\mathrm{ind}\,\mathrm{Hess}_u\left(J_{t_1}\big|_{\Phi^{-1}_{t_1}(q(t_0))}\right) < \infty$ *and* $\mathrm{Hess}_u\left(J_t\big|_{\Phi^{-1}_{t}(q(t_0))}\right)$ *is positive definite for any t sufficiently close to (and strictly greater than) t_0. If $\frac{\partial^2 h}{\partial u^2}(\lambda_t, u(t)) \not\leq 0$ for some $t \in [t_0, t_1]$, then* $\mathrm{ind}\,\mathrm{Hess}_u\left(J_{t_1}\big|_{\Phi^{-1}_{t_1}(q(t_0))}\right) = \infty$.

We do not give here the proof of this well-known result; you can find it in many sources (see, for instance, the textbook [7]). It is based on the fact that $\frac{\partial^2 h}{\partial u^2}(\lambda_t, u(t)) = \lambda(\frac{\partial^2 f}{\partial u^2}(q(t), u(t))) - \frac{\partial^2 \varphi}{\partial u^2}(q(t), u(t))$ is the infinitesimal (for the "infinitesimally small interval" at t) version of $\lambda_{t_0} D_u^2 \Phi_{t_1} - D_u^2 J_{t_1}$ while $\mathrm{Hess}_u \left(J_{t_1}\big|_{\Phi_{t_1}^{-1}(q(t_0))} \right) = (D_u^2 J_{t_1} - \lambda_{t_0} D_w^2 \Phi_{t_1})\big|_{\ker D_u \Phi_{t_1}}$.

Next theorem shows that in the 'regular' infinite dimensional situation of this section we may compute the Morse index similarly to the finite dimensional case. The proof of the theorem requires some information about second variation of optimal control problems which is out of the scope of these notes. The required information can be found in Chapters 20, 21 of [7]. Basically, it implies that finite dimensional arguments used in the proof of Theorem I.2 are legal also in our infinite dimensional case.

We set: $\Lambda(t) = e_*^{(t_0 - t)\vec{H}} T_{\lambda_t} \left(T^*_{q(t)} M \right)$.

Theorem I.3 *Assume that* $\frac{\partial^2 h}{\partial u^2}(\lambda_t, u(t))$ *is a negative definite quadratic form and* u *is a regular point of* Φ_t, $\forall t \in (t_0, t_1]$. *Then:*

- *The form* $\mathrm{Hess}_u \left(J_{t_1}\big|_{\Phi_{t_1}^{-1}(q(t_0))} \right)$ *is degenerate if and only if* $\Lambda(t_1) \cap \Lambda(t_0) \neq 0$;
- *If* $\Lambda(t_1) \cap \Lambda(t_0) = 0$, *then there exists* $\bar{t} > t_0$ *such that*

$$\mathrm{ind}\,\mathrm{Hess}_u \left(J_{t_1}\big|_{\Phi_{t_1}^{-1}(q(t_0))} \right) = -\mu \left(\Lambda(\cdot)\big|_{[\tau, t_1]} \right), \quad \forall \tau \in (t_0, \bar{t}). \qquad \square$$

Note that Legendre condition implies monotonicity of the curve $\Lambda(\cdot)$; this property simplifies the evaluation of the Maslov index. Fix some local coordinates in M so that $T^*M \cong \{(p, q) \in \mathbb{R}^{n*} \times \mathbb{R}^n\}$.

Lemma I.4 *Quadratic form* $\underline{\dot{\Lambda}}(t)$ *is equivalent (with respect to a linear change of variables) to the form* $-\frac{\partial^2 H}{\partial p^2}(\lambda_t) = \frac{\partial \bar{u}}{\partial p}^{\top} \frac{\partial^2 h}{\partial u^2}(\lambda_t, \bar{u}(\lambda_t)) \frac{\partial \bar{u}}{\partial p}$.

Proof. Equality $\frac{\partial^2 H}{\partial p^2} = -\frac{\partial \bar{u}}{\partial p}^* \frac{\partial^2 h}{\partial u^2} \frac{\partial \bar{u}}{\partial p}$ is an easy corollary of the identities $H(p, q) = h(p, q, \bar{u}(p, q))$, $\frac{\partial h}{\partial u}\big|_{u = \bar{u}(p,q)} = 0$. Indeed, $\frac{\partial^2 H}{\partial p^2} = 2 \frac{\partial^2 h}{\partial u \partial p} \frac{\partial \bar{u}}{\partial p} + \frac{\partial \bar{u}}{\partial p}^{\top} \frac{\partial^2 h}{\partial u^2} \frac{\partial \bar{u}}{\partial p}$ and $\frac{\partial}{\partial p}\left(\frac{\partial h}{\partial u}\right) = \frac{\partial^2 h}{\partial p \partial u} + \frac{\partial^2 h}{\partial u^2} \frac{\partial \bar{u}}{\partial p} = 0$. Further, we have:

$$\frac{d}{dt}\Lambda(t) = \frac{d}{dt} e_*^{(t_0 - t)\vec{H}} T_{\lambda_t} \left(T^*_{q(t)} M \right) = e_*^{(t_0 - t)\vec{H}} \frac{d}{d\varepsilon}\Big|_{\varepsilon = 0} e_*^{-\varepsilon \vec{H}} T_{\lambda_{t+\varepsilon}} \left(T^*_{q(t+\varepsilon)} M \right).$$

Set $\Delta(\varepsilon) = e_*^{-\varepsilon\vec{H}} T_{\lambda_{t+\varepsilon}}\left(T^*_{q(t+\varepsilon)}M\right) \in L\left(T_{\lambda(t)}(T^*M)\right)$. It is enough to prove that $\underline{\dot{\Delta}(0)}$ is equivalent to $-\frac{\partial^2 H}{\partial p^2}(\lambda_t)$. Indeed, $\dot{\Lambda}(t) = e_*^{(t_0-t)\vec{H}} T_{\lambda_t}\dot{\Delta}(0)$, where

$$e_*^{(t_0-t)\vec{H}} : T_{\lambda_t}(T^*M) \to T_{\lambda_{t_0}}(T^*M)$$

is a symplectic isomorphism. The association of the quadratic form $\underline{\dot{\Lambda}}(t)$ on the subspace $\Lambda(t)$ to the tangent vector $\dot{\Lambda}(t) \in L\left(T_{\lambda_{t_0}}(T^*M)\right)$ is intrinsic, i.e. depends only on the symplectic structure on $(T_{\lambda_{t_0}}(T^*M)$. Hence $\underline{\dot{\Delta}}(0)(\xi) = \underline{\dot{\Lambda}}(t)\left(e_*^{(t_0-t)\vec{H}}\xi\right)$, $\forall \xi \in \Delta(0) = T_{\lambda_t}\left(T^*_{q(t)}M\right)$.

What remains, is to compute $\underline{\dot{\Delta}}(0)$; we do it in coordinates. We have:

$$\Delta(\varepsilon) = \left\{ (\xi(\varepsilon), \eta(\varepsilon)) : \begin{array}{rcll} \dot{\xi}(\tau) & = & \xi\frac{\partial^2 H}{\partial p \partial q}(\lambda_{t-\tau}) + \eta^\top \frac{\partial^2 H}{\partial q^2}(\lambda_{t-\tau}), & \xi(0) \in \mathbb{R}^{n*} \\ \dot{\eta}(\tau) & = & -\frac{\partial^2 H}{\partial p^2}(\lambda_{t-\tau})\xi^\top - \frac{\partial^2 H}{\partial q \partial p}(\lambda_{t-\tau})\eta, & \eta(0) = 0 \end{array} \right\},$$

$$\underline{\dot{\Delta}}(0)(\xi(0)) = \sigma\left((\xi(0), 0), (\dot{\xi}(0), \dot{\eta}(0))\right) = \xi(0)\dot{\eta}(0) = -\xi(0)\frac{\partial^2 H}{\partial p^2}(\lambda_t)\xi(0)^\top.$$

□

Now combining Lemma I.4 with Theorem I.3 and Corollary I.1 we obtain the following version of the classical "Morse formula"

Corollary I.2 *Under conditions of Theorem I.3, if* $\{\tau \in (t_0, t_1] : \Lambda(\tau) \cap \Lambda(t_0) \neq 0\}$ *is a finite subset of* (t_0, t_1)*, then*

$$\operatorname{ind} \operatorname{Hess} J_{t_1}\big|_{\Phi_{t_1}^{-1}(q(t_0))} = \sum_{\tau \in (t_0, t_1)} \dim(\Lambda(\tau) \cap \Lambda(t_0)).$$

Part II
Geometry of Jacobi curves

8 Jacobi curves

Computation of the $\mathcal{L}$-derivative for regular extremals in the last section has led us to the construction of curves in the Lagrange Grassmannians which works for all Hamiltonian systems on the cotangent bundles, independently on any optimal control problem. Set $\Delta_\lambda = T_\lambda(T^*_q M)$, where

$\lambda \in T_q^*M,\ q \in M$. The curve $\tau \mapsto e_*^{-\tau\vec{H}} \Delta_{e^{\tau\vec{H}}(\lambda)}$ in the Lagrange Grassmannian $L\left(T_\lambda(T^*M)\right)$ is the result of the action of the flow $e^{t\vec{H}}$ on the vector distribution $\{\Delta_\lambda\}_{\lambda \in T^*M}$. Now we are going to study differential geometry of these curves; their geometry will provide us with a canonical connection on T^*M associated with the Hamiltonian system and with curvature-type invariants. All that gives a far going generalization (and a dynamical interpretation) of classical objects from Riemannian geometry.

In fact, construction of the basic invariants does not need symplectic structure and the Hamiltonian nature of the flow, we may deal with more or less arbitrary pairs (*vector field*, *rank* n *distribution*) on a $2n$-dimensional manifold N. The resulting curves belong to the usual Grassmannian of all n-dimensional subspaces in the $2n$-dimensional one. We plan to work for some time in this more general situation and then come back to the symplectic framework.

In these notes we mainly deal with the case of involutive distributions (i.e. with n-foliations) just because our main motivation and applications satisfy this condition. The reader can easily recover more general definitions and construction by himself.

So we consider a $2n$-dimensional smooth manifold N endowed with a smooth foliation of rank n. Let $z \in N$, by E_z we denote the passing through z leaf of the foliation; then E_z is an n-dimensional submanifold of N. Point z has a coordinate neighborhood O_z such that the restriction of the foliation to O_z is a (trivial) fiber bundle and the fibers $E_{z'}^{loc},\ z' \in O_z$, of this fiber bundle are connected components of $E_{z'} \cap O_z$. Moreover, there exists a diffeomorphism $O_z \cong \mathbb{R}^n \times \mathbb{R}^n$, where $\mathbb{R}^n \times \{y\},\ y \in \mathbb{R}^n$, are identified with the fibers so that both the typical fiber and the base are diffeomorphic to $\mathbb{R}^n$. We denote by O_z/E^{loc} the base of this fiber bundle and by $\pi : O_z \to O_z/E^{loc}$ the canonical projection.

Let ζ be a smooth vector field on N. Then $z' \mapsto \pi_*\zeta(z'),\ z' \in E_z^{loc}$ is a smooth mapping of E_z^{loc} to $T_{\pi(z)}(O_z/E^{loc})$. We denote the last mapping by $\Pi_z(\zeta) : E_z^{loc} \to T_{\pi(z)}(O_z/E^{loc})$.

Definition. We call ζ a *lifting* field if $\Pi_z(\zeta)$ is a constant mapping $\forall z \in N$; The field ζ is called *regular* if $\Pi_z(\zeta)$ is a submersion, $z \in N$.

The flow generated by the lifting field maps leaves of the foliation in the leaves, in other words it is leaves-wise. On the contrary, the flow generated by the regular field "smears" the fibers over O_z/E^{loc}; basic examples are second order differential equations on a manifold M treated as the vector fields on the tangent bundle $TM = N$.

Let us write things in coordinates: We fix local coordinates acting in the domain $O \subset N$, which turn the foliation into the Cartesian product of vector spaces: $O \cong \{(x,y) : x, y \in \mathbb{R}^n\}$, $\pi : (x,y) \mapsto y$. Then vector field ζ takes the form $\zeta = \sum\limits_{i=1}^{n} \left(a^i \frac{\partial}{\partial x_i} + b^i \frac{\partial}{\partial y_i} \right)$, where a^i, b^i are smooth functions on $\mathbb{R}^n \times \mathbb{R}^n$. The coordinate representation of the map Π_z is: $\Pi_{(x,y)} : x \mapsto \left(b^1(x,y), \dots, b^n(x,y)\right)^\top$. Field ζ is regular if and only if $\Pi_{(x,y)}$ are submersions; in other words, if and only if $\left(\frac{\partial b^i}{\partial x_j}\right)_{i,j=1}^{n}$ is a nondegenerate matrix. Field ζ is lifting if and only if $\frac{\partial b^i}{\partial x_j} \equiv 0,\ i, j = 1, \dots, n$.

Now turn back to the coordinate free setting. The fibers E_z, $z \in N$ are integral manifolds of the involutive distribution $\mathcal{E} = \{T_z E_z : z \in N\}$. Given a vector field ζ on N, the (local) flow $e^{t\zeta}$ generated by ζ, and $z \in N$ we define the family of subspaces

$$J_z(t) = \left(e^{-t\zeta}\right)_* \mathcal{E}|_z \subset T_z N.$$

In other words, $J_z(t) = \left(e^{-t\zeta}\right)_* T_{e^{t\zeta}(z)} E_{e^{t\zeta}(z)}$, $J_z(0) = T_z E_z$.

$J_x(t)$ is an n-dimensional subspace of $T_z N$, i.e. an element of the Grassmannian $G_n(T_z N)$. We thus have (the germ of) a curve $t \mapsto J_z(t)$ in $G_n(T_z N)$ which is called a *Jacobi curve.*

Definition. We say that field ζ is *k-ample* for an interger k if $\forall z \in N$ and for any curve $t \mapsto \hat{J}_z(t)$ in $G_n(T_z N)$ with the same k-jet as $J_z(t)$ we have $\hat{J}_z(0) \cap \hat{J}_z(t) = 0$ for all t close enough but not equal to 0. The field is called *ample* if it is k-ample for some k.

It is easy to show that a field is 1-ample if and only if it is regular.

9 The cross-ratio

Let Σ be a $2n$-dimensional vector space, $v_0, v_1 \in G_n(\Sigma)$, $v_0 \cap v_1 = 0$. Than $\Sigma = v_0 + v_1$. We denote by $\pi_{v_0 v_1} : \Sigma \to v_1$ the projector of Σ onto v_1 parallel to v_0. In other words, $\pi_{v_0 v_1}$ is a linear operator on Σ such that $\pi_{v_0 v_1}\big|_{v_0} = 0$, $\pi_{v_0 v_1}\big|_{v_1} = \mathrm{id}$. Surely, there is a one-to-one correspondence between pairs of transversal n-dimensional subspaces of Σ and rank n projectors in $\mathrm{gl}(\Sigma)$.

Lemma II.1 *Let $v_0 \in G_n(\Sigma)$; we set $v_0^{\pitchfork} = \{v \in G_n(\Sigma) : v \cap v_0 = 0\}$, an open dense subset of $G_n(\Sigma)$. Then $\{\pi_{vv_0} : v \in v_0^{\pitchfork}\}$ is an affine subspace of $gl(\Sigma)$.*

Indeed, any operator of the form $\alpha\pi_{vv_0} + (1-\alpha)\pi_{wv_0}$, where $\alpha \in \mathbb{R}$, takes values in v_0 and its restriction to v_0 is the identity operator. Hence $\alpha\pi_{vv_0} + (1-\alpha)\pi_{wv_0}$ is the projector of Σ onto v_0 along some subspace.

The mapping $v \mapsto \pi_{vv_0}$ thus serves as a local coordinate chart on $G_n(\Sigma)$. These charts indexed by v_0 form a natural atlas on $G_n(\Sigma)$.

Projectors π_{vw} satisfy the following basic relations:[3]

$$\pi_{v_0v_1} + \pi_{v_1v_0} = id, \quad \pi_{v_0v_2}\pi_{v_1v_2} = \pi_{v_1v_2}, \quad \pi_{v_0v_1}\pi_{v_0v_2} = \pi_{v_0v_1}, \tag{1}$$

where $v_i \in G_n(\Sigma),\ v_i \cap v_j = 0$ for $i \neq j$. If $n = 1$, then $G_n(\Sigma)$ is just the projective line $\mathbb{RP}^1$; basic geometry of $G_n(\Sigma)$ is somehow similar to geometry of the projective line for arbitrary n as well. The group $\mathrm{GL}(\Sigma)$ acts transitively on $G_n(\Sigma)$. Let us consider its standard action on $(k+1)$-tuples of points in $G_n(\Sigma)$:

$$A(v_0, \dots, v_k) \stackrel{def}{=} (Av_0, \dots, Av_k), \quad A \in \mathrm{GL}(\Sigma),\ v_i \in G_n(\Sigma).$$

It is an easy exercise to check that the only invariants of a triple (v_0, v_1, v_2) of points of $G_n(\Sigma)$ for such an action are dimensions of the intersections: $\dim(v_i \cap v_j),\ 0 \le i \le 2$, and $\dim(v_0 \cap v_1 \cap v_2)$. Quadruples of points possess a more interesting invariant: a multidimensional version of the classical cross-ratio.

Definition. Let $v_i \in G_n(\Sigma),\ i = 0,1,2,3$, and $v_0 \cap v_1 = v_2 \cap v_3 = 0$. The cross-ratio of v_i is the operator $[v_0, v_1, v_2, v_3] \in \mathrm{gl}(v_1)$ defined by the formula:

$$[v_0, v_1, v_2, v_3] = \pi_{v_0v_1}\pi_{v_2v_3}\big|_{v_1}.$$

Remark. We do not lose information when restrict the product $\pi_{v_0v_1}\pi_{v_2v_3}$ to v_1; indeed, this product takes values in v_1 and its kernel contains v_0.

For $n = 1$, v_1 is a line and $[v_0, v_1, v_2, v_3]$ is a real number. For general n, the Jordan form of the operator provides numerical invariants of the quadruple $v_i,\ i = 0,1,2,3$.

[3] Numbering of formulas is separate in each of two parts of the paper

We will mainly use an infinitesimal version of the cross-ratio that is an invariant $[\xi_0,\xi_1]\in \mathrm{gl}(v_1)$ of a pair of tangent vectors $\xi_i\in T_{v_i}G_n(\Sigma)$, $i=0,1$, where $v_0\cap v_1=0$. Let $\gamma_i(t)$ be curves in $G_n(\Sigma)$ such that $\gamma_i(0)=v_i$, $\frac{d}{dt}\gamma_i(t)\big|_{t=0}=\xi_i$, $i=0,1$. Then the cross-ratio: $[\gamma_0(t),\gamma_1(0),\gamma_0(\tau),\gamma_1(\theta)]$ is a well defined operator on $v_1=\gamma_1(0)$ for all t,τ,θ close enough to 0. Moreover, it follows from (1) that $[\gamma_0(t),\gamma_1(0),\gamma_0(0),\gamma_1(0)] = [\gamma_0(0),\gamma_1(0),\gamma_0(t),\gamma_1(0)]=[\gamma_0(0),\gamma_1(0),\gamma_0(0),\gamma_1(t)]=id$. We set

$$[\xi_0,\xi_1]=\frac{\partial^2}{\partial t\partial\tau}[\gamma_0(t),\gamma_1(0),\gamma_0(0),\gamma_1(\tau)]\big|_{v_1}\Big|_{t=\tau=0} \tag{2}$$

It is easy to check that the right-hand side of (2) depends only on ξ_0,ξ_1 and that $(\xi_0,\xi_1)\mapsto[\xi_0,\xi_1]$ is a bilinear mapping from $T_{v_0}G_n(\Sigma)\times T_{v_1}G_n(\Sigma)$ onto $gl(v_1)$.

Lemma II.2 *Let $v_0,v_1\in G_n(\Sigma)$, $v_0\cap v_1=0$, $\xi_i\in T_{v_i}G_n(\Sigma)$, and $\xi_i=\frac{d}{dt}\gamma_i(t)\big|_{t=0}$, $i=0,1$. Then $[\xi_0,\xi_1]=\frac{\partial^2}{\partial t\partial\tau}\pi_{\gamma_1(t)\gamma_0(\tau)}\big|_{v_1}\Big|_{t=\tau=0}$ and v_1,v_0 are invariant subspaces of the operator $\frac{\partial^2}{\partial t\partial\tau}\pi_{\gamma_1(t)\gamma_0(\tau)}\big|_{v_1}\Big|_{t=\tau=0}$.*

Proof. According to the definition, $[\xi_0,\xi_1]=\frac{\partial^2}{\partial t\partial\tau}(\pi_{\gamma_0(t)\gamma_1(0)}\pi_{\gamma_0(0)\gamma_1(\tau)})\big|_{v_1}\Big|_{t=\tau=0}$. The differentiation of the identities $\pi_{\gamma_0(t)\gamma_1(0)}\pi_{\gamma_0(t)\gamma_1(\tau)}=\pi_{\gamma_0(t)\gamma_1(0)}$, $\pi_{\gamma_0(t)\gamma_1(\tau)}\pi_{\gamma_0(0)\gamma_1(\tau)}=\pi_{\gamma_0(0)\gamma_1(\tau)}$ gives the equalities:

$$\frac{\partial^2}{\partial t\partial\tau}(\pi_{\gamma_0(t)\gamma_1(0)}\pi_{\gamma_0(0)\gamma_1(\tau)})\Big|_{t=\tau=0}=-\pi_{v_0v_1}\frac{\partial^2}{\partial t\partial\tau}\pi_{\gamma_0(t)\gamma_1(\tau)}\Big|_{t=\tau=0}$$

$$=-\frac{\partial^2}{\partial t\partial\tau}\pi_{\gamma_0(t)\gamma_1(\tau)}\Big|_{t=\tau=0}\pi_{v_0v_1}.$$

It remains to mention that $\frac{\partial^2}{\partial t\partial\tau}\pi_{\gamma_1(t)\gamma_0(\tau)}=-\frac{\partial^2}{\partial t\partial\tau}\pi_{\gamma_0(\tau)\gamma_1(t)}$. $\square$

10 Coordinate setting

Given $v_i\in G_n(\Sigma)$, $i=0,1,2,3$, we coordinatize $\Sigma=\mathbb{R}^n\times\mathbb{R}^n=\{(x,y):x\in\mathbb{R}^n,y\in\mathbb{R}^n\}$ in such a way that $v_i\cap\{(0,y):y\in\mathbb{R}^n\}=0$. Then there exist $n\times n$-matrices S_i such that

$$v_i=\{(x,S_ix):x\in\mathbb{R}^n\},\quad i=0,1,2,3. \tag{3}$$

The relation $v_i \cap v_j = 0$ is equivalent to $\det(S_i - S_j) \neq 0$. If $S_0 = 0$, then the projector $\pi_{v_0v_1}$ is represented by the $2n \times 2n$-matrix $\begin{pmatrix} 0 & S_1^{-1} \\ 0 & I \end{pmatrix}$. In general, we have

$$\pi_{v_0v_1} = \begin{pmatrix} S_{01}^{-1}S_0 & -S_{01}^{-1} \\ S_1S_{01}^{-1}S_0 & -S_1S_{01}^{-1} \end{pmatrix},$$

where $S_{01} = S_0 - S_1$. Relation (3) provides coordinates $\{x\}$ on the spaces v_i. In these coordinates, the operator $[v_0, v_1, v_2, v_3]$ on v_1 is represented by the matrix:

$$[v_0, v_1, v_2, v_3] = S_{10}^{-1}S_{03}S_{32}^{-1}S_{21},$$

where $S_{ij} = S_i - S_j$.

We now compute the coordinate representation of the infinitesimal cross-ratio. Let $\gamma_0(t) = \{(x, S_tx) : x \in \mathbb{R}^n\}$, $\gamma_1(t) = \{(x, S_{1+t}x) : x \in \mathbb{R}^n\}$ so that $\xi_i = \frac{d}{dt}\gamma_i(t)\big|_{t=0}$ is represented by the matrix $\dot{S}_i = \frac{d}{dt}S_t\big|_{t=i}$, $i = 0, 1$. Then $[\xi_0, \xi_1]$ is represented by the matrix

$$\frac{\partial^2}{\partial t \partial \tau} S_{1t}^{-1}S_{t\tau}S_{\tau 0}^{-1}S_{01}\Big|_{\substack{t=0 \\ \tau=1}} = \frac{\partial}{\partial t}S_{1t}^{-1}\dot{S}_1\Big|_{t=0} = S_{01}^{-1}\dot{S}_0S_{01}^{-1}\dot{S}_1.$$

So

$$[\xi_0, \xi_1] = S_{01}^{-1}\dot{S}_0S_{01}^{-1}\dot{S}_1. \tag{4}$$

There is a canonical isomorphism $T_{v_0}G_n(\Sigma) \cong \mathrm{Hom}(v_0, \Sigma/v_0)$; it is defined as follows. Let $\xi \in T_{v_0}G_n(\Sigma)$, $\xi = \frac{d}{dt}\gamma(t)|_{t=0}$, and $z_0 \in v_0$. Take a smooth curve $z(t) \in \gamma(t)$ such that $z(0) = z_0$. Then the residue class $(\dot{z}(0) + v_0) \in \Sigma/v_0$ depends on ξ and z_0 rather than on a particular choice of $\gamma(t)$ and $z(t)$. Indeed, let $\gamma'(t)$ be another curve in $G_n(\Sigma)$ whose velocity at $t = 0$ equals ξ. Take some smooth with respect to t bases of $\gamma(t)$ and $\gamma'(t)$: $\gamma(t) = span\{e_1(t), \ldots, e_n(t)\}$, $\gamma'(t) = span\{e'_1(t), \ldots, e'_n(t)\}$, where $e_i(0) = e'_i(0)$, $i = 1, \ldots, n$; then $(\dot{e}_i(0) - \dot{e}'_i(0)) \in v_0$, $i = 1, \ldots, n$. Let $z(t) = \sum\limits_{i=1}^{n} \alpha_i(t)e_i(t)$, $z'(t) = \sum\limits_{i=1}^{n} \alpha'_i(t)e'_i(t)$, where $\alpha_i(0) = \alpha'_i(0)$. We have:

$$\dot{z}(0) - \dot{z}'(0) = \sum_{i=1}^{n} ((\dot{\alpha}_i(0) - \dot{\alpha}'_i(0))e_i(0) + \alpha'_i(0)(\dot{e}_i(0) - \dot{e}'_i(0))) \in v_0,$$

i.e. $\dot{z}(0) + v_0 = \dot{z}'(0) + v_0$.

We associate to ξ the mapping $\bar{\xi} : v_0 \to \Sigma/v_0$ defined by the formula $\bar{\xi}z_0 = \dot{z}(0) + v_0$. The fact that $\xi \to \bar{\xi}$ is an isomorphism of the linear spaces $T_{v_0}G_n(\Sigma)$ and $\mathrm{Hom}(v_0, \Sigma/v_0)$ can be easily checked in coordinates. The matrices $\dot{S}_i$ above are actually coordinate presentations of $\bar{\xi}_i,\ i = 0, 1$.

The standard action of the group $\mathrm{GL}(\Sigma)$ on $G_n(\Sigma)$ induces the action of $\mathrm{GL}(\Sigma)$ on the tangent bundle $TG_n(\Sigma)$. It is easy to see that the only invariant of a tangent vector ξ for this action is $\mathrm{rank}\bar{\xi}$ (tangent vectors are just "double points" or "pairs of infinitesimaly close points" and number $(n - \mathrm{rank}\bar{\xi})$ is the infinitesimal version of the dimension of the intersection for a pair of points in the Grassmannian). Formula (4) implies:

$$\mathrm{rank}[\xi_0, \xi_1] \leq \min\{\mathrm{rank}\bar{\xi}_0, \mathrm{rank}\bar{\xi}_1\}.$$

11 Curves in the Grassmannian

Let $t \mapsto v(t)$ be a germ at $\bar{t}$ of a smooth curve in the Grassmannian $G_n(\Sigma)$.

Definition. We say that the germ $v(\cdot)$ is *ample* if $v(t) \cap v(\bar{t}) = 0\ \forall t \neq \bar{t}$ and the operator-valued function $t \mapsto \pi_{v(t)v(\bar{t})}$ has a pole at $\bar{t}$. We say that the germ $v(\cdot)$ is *regular* if the function $t \mapsto \pi_{v(t)v(\bar{t})}$ has a simple pole at $\bar{t}$. A smooth curve in $G_n(\Sigma)$ is called ample (regular) if all its germs are ample (regular).

Assume that $\Sigma = \{(x, y) : x, y \in \mathbb{R}^n\}$ is coordinatized in such a way that $v(\bar{t}) = \{(x, 0) : x \in \mathbb{R}^n\}$. Then $v(t) = \{(x, S_t x) : x \in \mathbb{R}^n\}$, where $S(\bar{t}) = 0$ and $\pi_{v(t)v(\bar{t})} = \begin{pmatrix} I & -S_t^{-1} \\ 0 & 0 \end{pmatrix}$. The germ $v(\cdot)$ is ample if and only if the scalar function $t \mapsto \det S_t$ has a finite order root at $\bar{t}$. The germ $v(\cdot)$ is regular if and only if the matrix $\dot{S}_{\bar{t}}$ is not degenerate. More generally, the curve $\tau \mapsto \{(x, S_\tau x) : x \in \mathbb{R}^n\}$ is ample if and only if $\forall t$ the function $\tau \mapsto \det(S_\tau - S_t)$ has a finite order root at t. This curve is regular if and only if $\det \dot{S}_t \neq 0,\ \forall t$. The intrinsic version of this coordinate characterization of regularity reads: the curve $v(\cdot)$ is regular if and only if the map $\bar{\dot{v}}(t) \in \mathrm{Hom}(v(t), \Sigma/v(t))$ has rank n, $\forall t$.

Coming back to the vector fields and their Jacobi curves (see Sec. 8) one can easily check that a vector field is ample (regular) if and only if its Jacobi curves are ample (regular).

Let $v(\cdot)$ be an ample curve in $G_n(\Sigma)$. We consider the Laurent expansions at t of the operator-valued function $\tau \mapsto \pi_{v(\tau)v(t)}$,

$$\pi_{v(\tau)v(t)} = \sum_{i=-k_t}^{m} (\tau - t)^i \pi_t^i + O(\tau - t)^{m+1}.$$

Projectors of Σ on the subspace $v(t)$ form an affine subspace of $\mathrm{gl}(\Sigma)$ (cf. Lemma II.1). This fact implies that π_t^0 is a projector of Σ on $v(t)$; in other words, $\pi_t^0 = \pi_{v^\circ(t)v(t)}$ for some $v^\circ(t) \in v(t)^{\pitchfork}$. We thus obtain another curve $t \mapsto v^\circ(t)$ in $G_n(\Sigma)$, where $\Sigma = v(t) \oplus v^\circ(t)$, $\forall t$. The curve $t \mapsto v^\circ(t)$ is called the *derivative curve* of the ample curve $v(\cdot)$.

The affine space $\{\pi_{wv(t)} : w \in v(t)^{\pitchfork}\}$ is a translation of the linear space $\mathfrak{N}(v(t)) = \{\mathfrak{n} : \Sigma \to v(t) \mid \mathfrak{n}|_{v(t)} = 0\} \subset \mathrm{gl}(\Sigma)\}$ containing only nilpotent operators. It is easy to see that $\pi_t^i \in \mathfrak{N}(v(t))$ for $i \neq 0$.

The derivative curve is not necessary ample. Moreover, it may be nonsmooth and even discontinuous.

Lemma II.3 *If $v(\cdot)$ is regular then $v^\circ(\cdot)$ is smooth.*

Proof. We'll find the coordinate representation of $v^\circ(\cdot)$. Let $v(t) = \{(x, S_t x) : x \in \mathbb{R}^n\}$. Regularity of $v(\cdot)$ is equivalent to the nondegeneracy of $\dot{S}_t$. We have:

$$\pi_{v(\tau)v(t)} = \begin{pmatrix} S_{\tau t}^{-1} S_\tau & -S_{\tau t}^{-1} \\ S_t S_{\tau t}^{-1} S_\tau & -S_t S_{\tau t}^{-1} \end{pmatrix},$$

where $S_{\tau t} = S_\tau - S_t$. Then $S_{\tau t}^{-1} = (\tau - t)^{-1} \dot{S}_t^{-1} - \frac{1}{2} \dot{S}_t^{-1} \ddot{S}_t \dot{S}_t^{-1} + O(\tau - t)$ as $\tau \to t$ and

$$\pi_{v(\tau)v(t)} = (\tau - t)^{-1} \begin{pmatrix} \dot{S}_t^{-1} S_t & -\dot{S}_t^{-1} \\ S_t \dot{S}_t^{-1} S_t & -S_t \dot{S}_t^{-1} \end{pmatrix} +$$

$$\begin{pmatrix} I - \frac{1}{2} \dot{S}_t^{-1} \ddot{S}_t \dot{S}_t^{-1} S_t & \frac{1}{2} \dot{S}_t^{-1} \ddot{S}_t \dot{S}_t^{-1} \\ S_t - \frac{1}{2} S_t \dot{S}_t^{-1} \ddot{S}_t \dot{S}_t^{-1} S_t & \frac{1}{2} S_t \dot{S}_t^{-1} \ddot{S}_t \dot{S}_t^{-1} \end{pmatrix} + O(\tau - t).$$

We set $A_t = -\frac{1}{2} \dot{S}_t^{-1} \ddot{S}_t \dot{S}_t^{-1}$; then $\pi_{v^\circ(t)v(t)} = \begin{pmatrix} I + A_t S_t & -A_t \\ S_t + S_t A_t S_t & -S_t A_t \end{pmatrix}$ is smooth with respect to t. Hence $t \mapsto v^\circ(t)$ is smooth. We obtain:

$$v^\circ(t) = \{(A_t y, y + S_t A_t y) : y \in \mathbb{R}^n\}. \tag{5}$$

12 The curvature

Definition. Let v be an ample curve and v° be the derivative curve of v. Assume that v° is differentiable at t and set $R_v(t) = [\dot{v}^\circ(t), \dot{v}(t)]$. The operator $R_v(t) \in gl(v(t))$ is called the *curvature* of the curve v at t.

If v is a regular curve, then v° is smooth, the curvature is well-defined and has a simple coordinate presentation. To find this presentation, we'll use formula (4) applied to $\xi_0 = \dot{v}^\circ(t),\ \xi_1 = \dot{v}(t)$. As before, we assume that $v(t) = \{(x, S_t x) : x \in \mathbb{R}^n\}$; in particular, $v(t)$ is transversal to the subspace $\{(0, y) : y \in \mathbb{R}^n\}$. In order to apply (4) we need an extra assumption on the coordinatization of Σ: the subspace $v^\circ(t)$ has to be transversal to $\{(0, y) : y \in \mathbb{R}^n\}$ for given t. The last property is equivalent to the nondegeneracy of the matrix A_t (see (6)). It is important to note that the final expression for $R_v(t)$ as a differential operator of S must be valid without this extra assumption since the definition of $R_v(t)$ is intrinsic! Now we compute: $v^\circ(t) = \{(x, (A_t^{-1} + S_t)x) : x \in \mathbb{R}^n\}$, $\quad R_v(t) = [\dot{v}^\circ(t), \dot{v}(t)] = A_t \frac{d}{dt}(A_t^{-1} + S_t)A_t\dot{S}_t = (A_t\dot{S}_t)^2 - \dot{A}_t\dot{S}_t = \frac{1}{4}(\dot{S}_t^{-1}\ddot{S}_t)^2 - \dot{A}_t\dot{S}_t$. We also have $\dot{A}\dot{S} = -\frac{1}{2}\frac{d}{dt}(\dot{S}^{-1}\ddot{S}\dot{S}^{-1})\dot{S} = (\dot{S}^{-1})^2 - \frac{1}{2}\dot{S}^{-1}\dddot{S}$. Finally,

$$R_v(t) = \frac{1}{2}\dot{S}_t^{-1}\dddot{S}_t - \frac{3}{4}(\dot{S}_t^{-1}\ddot{S}_t)^2 = \frac{d}{dt}\left((2\dot{S}_t)^{-1}\ddot{S}_t\right) - \left((2\dot{S}_t)^{-1}\ddot{S}_t\right)^2, \qquad (6)$$

the matrix version of the Schwartzian derivative.

Curvature operator is a fundamental invariant of the curve in the Grassmannian. One more intrinsic construction of this operator, without using the derivative curve, is provided by the following

Proposition II.1 *Let v be a regular curve in $G_n(\Sigma)$. Then*

$$[\dot{v}(\tau), \dot{v}(t)] = (\tau - t)^{-2} id + \frac{1}{3}R_v(t) + O(\tau - t)$$

as $\tau \to t$.

Proof. It is enough to check the identity in some coordinates. Given t we may assume that

$$v(t) = \{(x, 0) : x \in \mathbb{R}^n\}, \quad v^\circ(t) = \{(0, y) : y \in \mathbb{R}^n\}.$$

Let $v(\tau) = \{(x, S_\tau x : x \in \mathbb{R}^n\}$, then $S_t = \ddot{S}_t = 0$ (see (5)). Moreover, we may assume that the bases of the subspaces $v(t)$ and $v^\circ(t)$ are coordinated in such a way that $\dot{S}_t = I$. Then $R_v(t) = \frac{1}{2}\dddot{S}_t$ (see (6)). On the other hand, formula (4) for the infinitesimal cross-ratio implies:

$$[\dot{v}(\tau), \dot{v}(t)] = S_\tau^{-1}\dot{S}_\tau S_\tau^{-1} = -\frac{d}{d\tau}(S_\tau^{-1}) =$$

$$-\frac{d}{d\tau}\left((\tau - t)I + \frac{(\tau - t)^3}{6}\dddot{S}_t\right)^{-1} + O(\tau - t) =$$

$$-\frac{d}{d\tau}\left((\tau - t)^{-1}I - \frac{(\tau - t)}{6}\dddot{S}_t\right) + O(\tau - t) = (\tau - t)^{-2}I + \frac{1}{6}\dddot{S}_t + O(\tau - t).$$

$\square$

Curvature operator is an invariant of the curves in $G_n(\Sigma)$ with fixed parametrizations. Asymptotic presentation obtained in Proposition II.1 implies a nice chain rule for the curvature of the reparametrized curves.

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a regular change of variables, i.e. $\dot{\varphi} \neq 0$, $\forall t$. The standard imbedding $\mathbb{R} \subset \mathbb{RP}^1 = G_1(\mathbb{R}^2)$ makes φ a regular curve in $G_1(\mathbb{R}^2)$. As we know (see (6)), the curvature of this curve is the Schwartzian of φ:

$$R_\varphi(t) = \frac{\dddot{\varphi}(t)}{2\dot{\phi}(t)} - \frac{3}{4}\left(\frac{\ddot{\varphi}(t)}{\dot{\varphi}(t)}\right)^2.$$

We set $v_\varphi(t) = v(\varphi(t))$ for any curve v in $G_n(\Sigma)$.

Proposition II.2 *Let v be a regular curve in $G_n(\Sigma)$ and $\varphi : \mathbb{R} \to \mathbb{R}$ be a regular change of variables. Then*

$$R_{v_\varphi}(t) = \dot{\varphi}^2(t)R_v(\varphi(t)) + R_\varphi(t). \tag{7}$$

Proof. We have

$$[\dot{v}_\varphi(\tau), \dot{v}_\varphi(t)] = (\tau - t)^{-2}\mathrm{id} + \frac{1}{3}R_{v_\varphi}(t) + O(\tau - t).$$

On the other hand,

$$[\dot{v}_\varphi(\tau), \dot{v}_\varphi(t)] = [\dot{\varphi}(\tau)\dot{v}(\varphi(\tau)), \dot{\varphi}(t)\dot{v}(\varphi(t))] = \dot{\varphi}(\tau)\dot{\varphi}(t)[\dot{v}(\varphi(\tau)), \dot{v}(\varphi(t))] =$$

$$\dot{\varphi}(\tau)\dot{\varphi}(t)\left((\varphi(\tau) - \varphi(t))^{-2}id + \frac{1}{3}R_v(\varphi(t)) + O(\tau - t)\right) =$$

$$\frac{\dot\varphi(\tau)\dot\varphi(t)}{((\varphi(\tau)-\varphi(t))^2}\mathrm{id}+\frac{\dot\varphi^2(t)}{3}R_v(\varphi(t))+O(\tau-t).$$

We treat φ as a curve in $\mathbb{RP}^1=G_1(\mathbb{R}^2)$. Then $[\dot\varphi(\tau),\dot\varphi(t)]=\frac{\dot\varphi(\tau)\dot\varphi(t)}{(\varphi(\tau)-\varphi(t))^2}$, see (4). The one-dimensional version of Proposition II.1 reads:

$$[\dot\varphi(\tau),\dot\varphi(t)]=(t-\tau)^{-2}+\frac{1}{3}R_\varphi(t)+O(\tau-t).$$

Finally,

$$[\dot v_\varphi(\tau),\dot v_\varphi(t)]=(t-\tau)^{-2}+\frac{1}{3}\left(R_\varphi(t)+\dot\varphi^2(t)R_v(\varphi(t))\right)+O(\tau-t).\quad\square$$

The following identity is an immediate corollary of Proposition II.2:

$$\left(R_{v_\varphi}-\frac{1}{n}(\mathrm{tr}R_{v_\varphi})\mathrm{id}\right)(t)=\dot\varphi^2(t)\left(R_v-\frac{1}{n}(\mathrm{tr}R_v)\mathrm{id}\right)(\varphi(t)).\tag{8}$$

Definition. An ample curve v is called flat if $R_v(t)\equiv 0$.

It follows from Proposition II.1 that any small enough piece of a regular curve can be made flat by a reparametrization if and only if the curvature of the curve is a scalar operator, i.e. $R_v(t)=\frac{1}{n}(\mathrm{tr}R_v(t))\mathrm{id}$. In the case of a nonscalar curvature, one can use equality (8) to define a distinguished parametrization of the curve and then derive invariants which do not depend on the parametrization.

Remark. In this paper we are mainly focused on the regular curves. See paper [6] for the version of the chain rule which is valid for any ample curve and for basic invariants of unparametrized ample curves.

13 Structural equations

Assume that v and w are two smooth curves in $G_n(\Sigma)$ such that $v(t)\cap w(t)=0,\ \forall t$.

Lemma II.4 *For any t and any $e\in v(t)$ there exists a unique $f_e\in w(t)$ with the following property: $\exists$ a smooth curve $e_\tau\in v(\tau)$, $e_t=e$, such that $\frac{d}{d\tau}e_\tau\big|_{\tau=t}=f_e$. Moreover, the mapping $\Phi_t^{vw}:e\mapsto f_t$ is linear and for any $e_0\in v(0)$ there exists a unique smooth curve $e(t)\in v(t)$ such that $e(0)=e_0$ and*

$$\dot e(t)=\Phi_t^{vw}e(t),\quad\forall t.\tag{9}$$

Proof. First we take any curve $\hat{e}_\tau \in v(\tau)$ such that $e_t = e$. Then $\hat{e}_\tau = a_\tau + b_\tau$ where $a_\tau \in v(t),\ b_\tau \in w(t)$. We take $x_\tau \in v(\tau)$ such that $x_t = \dot{a}_t$ and set $e_\tau = \hat{e}_\tau + (t-\tau)x_\tau$. Then $\dot{e}_t = \dot{b}_t$ and we put $f_e = \dot{b}_t$.

Let us prove that $\dot{b}_t$ depends only on e and not on the choice of e_τ. Computing the difference of two admissible e_τ we reduce the lemma to the following statement: if $z(\tau) \in v(\tau),\ \forall \tau$ and $z(t) = 0$, then $\dot{z}(t) \in v(t)$.

To prove the last statement we take smooth vector-functions $e^i_\tau \in v(\tau),\ i = 1, \dots, n$ such that $v(\tau) = span\{e^1_\tau, \dots, e^n_\tau\}$. Then $z(\tau) = \sum\limits_{i=1}^n \alpha_i(\tau) e^i_\tau,\ \alpha_i(t) = 0$. Hence $\dot{z}(t) = \sum\limits_{i=1}^n \dot{\alpha}_i(t) e^i_t \in v_t$.

Linearity of the map Φ^{vw}_t follows from the uniqueness of f_e. Indeed, if $f_{e^i} = \frac{d}{d\tau} e^i_\tau \big|_{\tau=t}$, then $\frac{d}{d\tau}(\alpha_1 e^1_\tau + \alpha_2 e^2_\tau)\big|_{\tau=t} = \alpha_1 f_{e^1} + \alpha_2 f_{e^2}$; hence $\alpha_1 f_{e^1} + \alpha_2 f_{e^2} = f_{\alpha_1 e^1 + \alpha_2 e^2},\ \forall e^i \in v(t),\ \alpha_i \in \mathbb{R},\ i = 1, 2$.

Now consider the smooth submanifold $V = \{(t, e) : t \in \mathbb{R},\ e \in v(t)\}$ of $\mathbb{R} \times \Sigma$. We have $(1, \Phi^{vw}_t e) \in T_{(t,e)} V$ since $(1, \Phi^{vw}_t e)$ is the velocity of a curve $\tau \mapsto (\tau, e_\tau)$ in V. So $(t, e) \mapsto (1, \Phi^{vw}_t e),\ (t, e) \in V$ is a smooth vector field on V. The curve $e(t) \in v(t)$ satisfies (9) if and only if $(t, e(t))$ is a trajectory of this vector field. Now the standard existence and uniqueness theorem for ordinary differential equations provides the existence of a unique solution to the Cauchy problem for small enough t while the linearity of the equation guarantees that the solution is defined for all t. $\square$

It follows from the proof of the lemma that $\Phi^{vw}_t e = \pi_{v(t)w(t)} \dot{e}_\tau \big|_{\tau=t}$ for any $e_\tau \in v(\tau)$ such that $v_t = e$. Let $v(t) = \{(x, S_{vt} x) : x \in \mathbb{R}^n\},\ w(t) = \{(x, S_{wt} x) : x \in \mathbb{R}^n\}$; the matrix presentation of Φ^{vw}_t in coordinates x is $(S_{wt} - S_{vt})^{-1} \dot{S}_{vt}$. Linear mappings Φ^{vw}_t and Φ^{wv}_t provide a factorization of the infinitesimal cross-ratio $[\dot{w}(t), \dot{v}(t)]$. Indeed, equality (4) implies:

$$[\dot{w}(t), \dot{v}(t)] = -\Phi^{wv}_t \Phi^{vw}_t. \tag{10}$$

Equality (9) implies one more useful presentation of the infinitesimal cross-ratio: if $e(t)$ satisfies (9), then

$$[\dot{w}(t), \dot{v}(t)] e(t) = -\Phi^{wv}_t \Phi^{vw}_t e(t) = -\Phi^{wv}_t \dot{e}(t) = -\pi_{w(t)v(t)} \ddot{e}(t). \tag{11}$$

Now let w be the derivative curve of v, $w(t) = v^\circ(t)$. It happens that $\ddot{e}(t) \in v(t)$ in this case and (11) is reduced to the *structural equation*:

$$\ddot{e}(t) = -[\dot{v}^\circ(t), \dot{v}(t)] e(t) = -R_v(t) e(t),$$

where $R_v(t)$ is the curvature operator. More precisely, we have the following

Proposition II.3 *Assume that v is a regular curve in $G_n(\Sigma)$, v° is its derivative curve, and $e(\cdot)$ is a smooth curve in Σ such that $e(t) \in v(t)$, $\forall t$. Then $\dot e(t) \in v^\circ(t)$ if and only if $\ddot e(t) \in v(t)$.*

Proof. Given t, we take coordinates in such a way that $v(t) = \{(x, 0) : x \in \mathbb{R}^n\}$, $v^\circ(t) = \{(0, y) : y \in \mathbb{R}^n\}$. Then $v(\tau) = \{(x, S_\tau x) : x \in \mathbb{R}^n\}$ for τ close enough to t, where $S_t = \ddot S_t = 0$ (see (5)).

Let $e(\tau) = \{(x(\tau), S_\tau x(\tau))\}$. The inclusion $\dot e(t) \in v^\circ(t)$ is equivalent to the equality $\dot x(t) = 0$. Further,

$$\ddot e(t) = \{\ddot x(t), \ddot S_t x(t) + 2\dot S_t \dot x(t) + S_t \ddot x(t)\} = \{\ddot x(t), 2\dot S \dot x\} \in v(t).$$

Regularity of v implies the nondegeneracy of $\dot S(t)$. Hence $\ddot e(t) \in v(t)$ if and only if $\dot x(t) = 0$. $\square$

Now equality (11) implies

Corollary II.1 *If $\dot e(t) = \Phi_t^{vv^\circ} e(t)$, then $\ddot e(t) + R_v(t) e(t) = 0$.*

Let us consider invertible linear mappings $V_t : v(0) \to v(t)$ defined by the relations $V_t e(0) = e(t)$, $\dot e(\tau) = \Phi_\tau^{vv^\circ} e(\tau)$, $0 \le \tau \le t$. It follows from the structural equation that the curve v is uniquely reconstructed from $\dot v(0)$ and the curve $t \mapsto V_t^{-1} R_V(t)$ in $\mathrm{gl}(v(0))$. Moreover, let $v_0 \in G_n(\Sigma)$ and $\xi \in T_{v_0} G_n(\Sigma)$, where the map $\bar\xi \in \mathrm{Hom}(v_0, \Sigma/v_0)$ has rank n; then for any smooth curve $t \mapsto A(t)$ in $\mathrm{gl}(v_0)$ there exists a unique regular curve v such that $\dot v(0) = \xi$ and $V_t^{-1} R_v(t) V_t = A(t)$. Indeed, let $e_i(0)$, $i = 1, \ldots, n$, be a basis of v_0 and $A(t) e_i(0) = \sum\limits_{j=1}^n a_{ij}(t) e_j(0)$. Then $v(t) = span\{e_1(t), \ldots, e_n(t)\}$, where

$$\ddot e_i(\tau) + \sum_{j=1}^n a_{ij}(\tau) e_j(\tau) = 0, \ 0 \le \tau \le t, \tag{12}$$

are uniquely defined by fixing the $\dot v(0)$.

The obtained classification of regular curves in terms of the curvature is particularly simple in the case of a scalar curvature operators $R_v(t) = \rho(t)\mathrm{id}$. Indeed, we have $A(t) = V_t^{-1} R_v(t) V_t = \rho(t)\mathrm{id}$ and system (12) is reduced to n copies of the Hill equation $\ddot e(\tau) + \rho(\tau) e(\tau) = 0$.

Recall that all $\xi \in TG_n(\Sigma)$ such that $\mathrm{rank}\bar\xi = n$ are equivalent under the action of $\mathrm{GL}(\Sigma)$ on $TG_n(\Sigma)$ induced by the standard action on the Grassmannian $G_n(\Sigma)$. We thus obtain

Corollary II.2 *For any smooth scalar function $\rho(t)$ there exists a unique, up to the action of $GL(\Sigma)$, regular curve v in $G_n(\Sigma)$ such that $R_v(t) = \rho(t)id$.*

Another important special class is that of symmetric curves.

Definition. A regular curve v is called *symmetric* if $V_t R_v(t) = R_v(t) V_t, \ \forall t$.

In other words, v is symmetric if and only the curve $A(t) = V_t^{-1} R_v(t) V_t$ in $\mathrm{gl}(v(0))$ is constant and coincides with $R_v(0)$. The structural equation implies

Corollary II.3 *For any $n \times n$-matrix A_0, there exists a unique, up to the action of $GL(\Sigma)$, symmetric curve v such that $R_v(t)$ is similar to A_0.*

The derivative curve v° of a regular curve v is not necessary regular. The formula $R_v(t) = \Phi_t^{v^\circ v} \Phi_t^{v v^\circ}$ implies that v° is regular if and only if the curvature operator $R_v(t)$ is nondegenerate for any t. Then we may compute the second derivative curve $v^{\circ\circ} = (v^\circ)^\circ$.

Proposition II.4 *A regular curve v with nondegenerate curvature operators is symmetric if and only if $v^{\circ\circ} = v$.*

Proof. Let us consider system (12). We are going to apply Proposition II.3 to the curve v° (instead of v) and the vectors $\dot{e}_i(t) \in v^\circ(t)$. According to Proposition II.3, $v^{\circ\circ} = v$ if and only if $\frac{d^2}{dt^2}\dot{e}_i(t) \in v^\circ(t)$. Differentiating (12) we obtain that $v^{\circ\circ} = v$ if and only if the functions $\alpha_{ij}(t)$ are constant. The last property is none other than a characterization of symmetric curves. $\square$

14 Canonical connection

Now we apply the developed theory of curves in the Grassmannian to the Jacobi curves $J_z(t)$ (see Sec. 8).

Proposition II.5 *All Jacobi curves $J_z(\cdot)$, $z \in N$, associated to the given vector field ζ are regular (ample) if and only if the field ζ is regular (ample).*

Proof. The definition of the regular (ample) field is actually the specification of the definition of the regular (ample) germ of the curve in the Grassmannian: general definition is applied to the germs at $t = 0$ of the curves $t \mapsto J_z(t)$. What remains is to demonstrate that other germs of these curves are regular (ample) as soon as the germs at 0 are. The latter fact follows from the identity

$$J_z(t+\tau) = e_*^{-t\zeta} J_{e^{t\zeta}(z)}(\tau) \tag{13}$$

(which, in turn, is an immediate corollary of the identity $e_*^{-(t+\tau)\zeta} = e_*^{-t\zeta} \circ e_*^{-\tau\zeta}$). Indeed, (13) implies that the germ of $J_z(\cdot)$ at t is the image of the germ of $J_{e^{t\zeta}(\tau)}(\cdot)$ at 0 under the fixed linear transformation $e_*^{-t\zeta} : T_{e^{t\zeta}(z)}N \to T_zN$. The properties of the germs to be regular or ample survive linear transformations since they are intrinsic properties. $\square$

Let ζ be an ample field. Then the derivative curves $J_z^\circ(t)$ are well-defined. Moreover, identity (13) and the fact that the construction of the derivative curve is intrinsic imply:

$$J_z^\circ(t) = e_*^{-t\zeta} J_{e^{t\zeta}(z)}^\circ(0). \tag{14}$$

The value at 0 of the derivative curve provides the splitting $T_zM = J_z(0) \oplus J_z^\circ(0)$, where the first summand is the tangent space to the fiber, $J_z(0) = T_zE_z$.

Now assume that $J_z^\circ(t)$ smoothly depends on z; this assumption is automatically fulfilled in the case of a regular ζ, where we have the explicit coordinate presentation for $J_z^\circ(t)$. Then the subspaces $J_z^\circ(0) \subset T_zN,\ z \in N$, form a smooth vector distribution, which is the direct complement to the vertical distribution $\mathcal{E} = \{T_zE_z : z \in N\}$. Direct complements to the vertical distribution are called Ehresmann connections (or just nonlinear connections, even if linear connections are their special cases). The Ehresmann connection $\mathcal{E}_\zeta = \{J_z^\circ(0) : z \in N\}$ is called the *canonical connection* associated with ζ and the correspondent splitting $TN = \mathcal{E} \oplus \mathcal{E}_\zeta$ is called the *canonical splitting.* Our nearest goal is to give a simple intrinsic characterization of $\mathcal{E}_\zeta$ which does not require the integration of the equation $\dot{z} = \zeta(z)$ and is suitable for calculations not only in local coordinates but also in moving frames.

Let $\mathcal{F} = \{F_z \subset T_zN : z \in N\}$ be an Ehresmann connection. Given a vector field ξ on E we denote $\xi_{ver}(z) = \pi_{F_zJ_z(0)}\xi,\ \xi_{hor}(z) = \pi_{J_z(0)F_z}\xi$, the "vertical" and the "horizontal" parts of $\xi(z)$. Then $\xi = \xi_{ver} + \xi_{hor}$, where ξ_{ver} is a section of the distribution $\mathcal{E}$ and ξ_{hor} is a section of the distribution $\mathcal{F}$. In general, sections of $\mathcal{E}$ are called vertical fields and sections of $\mathcal{F}$ are called horizontal fields.

Proposition II.6 *Assume that ζ is a regular field. Then $\mathcal{F} = \mathcal{E}_\zeta$ if and only if the equality*

$$[\zeta, [\zeta, \nu]]_{hor} = 2[\zeta, [\zeta, \nu]_{ver}]_{hor} \tag{15}$$

holds for any vertical vector field ν. Here $[\,,\,]$ is Lie bracket of vector fields.

Proof. The deduction of identity (15) is based on the following classical expression:

$$\frac{d}{dt}e_*^{-t\zeta}\xi = e_*^{-t\zeta}[\zeta,\xi], \tag{16}$$

for any vector field ξ.

Given $z \in N$, we take coordinates in T_zN in such a way that $T_zN = \{(x,y) : x,y \in \mathbb{R}^n\}$, where $J_z(0) = \{(x,0) : x \in \mathbb{R}^n\}$, $J_z^\circ(0) = \{(0,y) : y \in \mathbb{R}^n\}$. Let $J_z(t) = \{(x, S_t x) : x \in \mathbb{R}^n\}$, then $S_0 = \ddot{S}_0 = 0$ and $\det \dot{S}_0 \neq 0$ due to the regularity of the Jacobi curve J_z.

Let ν be a vertical vector field, $\nu(z) = (x_0, 0)$ and $\left(e_*^{-t\zeta}\nu\right)(z) = (x_t, y_t)$. Then $(x_t, 0) = \left(e_*^{-t\zeta}\nu\right)_{ver}(z)$, $(0, y_t) = \left(e_*^{-t\zeta}\nu\right)_{hor}(z)$. Moreover, $y_t = S_t x_t$ since $\left(e_*^{-t\zeta}\nu\right)(z) \in J_z(t)$. Differentiating the identity $y_t = S_t x_t$ we obtain: $\dot{y}_t = \dot{S}_t x_t + S_t \dot{x}_t$. In particular, $\dot{y}_0 = \dot{S}_0 x_0$. It follows from (16) that $(\dot{x}_0, 0) = [\zeta,\nu]_{ver}$, $(0, \dot{y}_0) = [\zeta,\nu]_{hor}$. Hence $(0, \dot{S}_0 x_0) = [\zeta,\nu]_{hor}(z)$, where, I recall, ν is any vertical field. Now we differentiate once more and evaluate the derivative at 0:

$$\ddot{y}_0 = \ddot{S}_0 x_0 + 2\dot{S}_0 \dot{x}_0 + S_0 \ddot{x}_0 = 2\dot{S}_0\dot{x}_0. \tag{17}$$

The Lie bracket presentations of the left and right hand sides of (17) are: $(0, \ddot{y}_0) = [\zeta,[\zeta,\nu]]_{hor}$, $(0, \dot{S}_0\dot{x}_0) = [\zeta,[\zeta,\nu]_{ver}]_{hor}$. Hence (17) implies identity (15).

Assume now that $\{(0,y) : y \in \mathbb{R}^n\} \neq J_z^\circ(0)$; then $\ddot{S}_0 x_0 \neq 0$ for some x_0. Hence $\ddot{y}_0 \neq 2\dot{S}_0\dot{x}_0$ and equality (15) is violated. $\square$

Inequality (15) can be equivalently written in the following form that is often more convenient for the computations:

$$\pi_*[\zeta,[\zeta,\nu]](z) = 2\pi_*[\zeta,[\zeta,\nu]_{ver}](z), \quad \forall z \in N. \tag{18}$$

Let $R_{J_z}(t) \in \mathrm{gl}(J_z(t))$ be the curvature of the Jacobi curve $J_z(t)$. Identity (13) and the fact that construction of the Jacobi curve is intrinsic imply that

$$R_{J_z}(t) = e_*^{-t\zeta} R_{J_{e^{t\zeta}(z)}}(0) e_*^{t\zeta}\big|_{J_z(t)}.$$

Recall that $J_z(0) = T_zE_z$; the operator $R_{J_z}(0) \in \mathrm{gl}(T_zE_z)$ is called *the curvature operator of the field* ζ at z. We introduce the notation: $R_\zeta(z) \stackrel{def}{=} R_{J_z}(0)$; then $R_\zeta = \{R_\zeta(z)\}_{z\in E}$ is an automorphism of the "vertical" vector bundle $\{T_zE_z\}_{z\in M}$.

Proposition II.7 *Assume that ζ is an ample vector field and $J_z^\circ(0)$ is smooth with respect to z. Let $TN = \mathcal{E} \oplus \mathcal{E}_\zeta$ be the canonical splitting. Then*

$$R_\zeta \nu = -[\zeta, [\zeta, \nu]_{hor}]_{ver} \tag{19}$$

for any vertical field ν.

Proof. Recall that $R_{J_z}(0) = [\dot{J}_z^\circ(0), \dot{J}_z(0)]$, where $[\cdot, \cdot]$ is the infinitesimal cross–ratio (not the Lie bracket!). The presentation (10) of the infinitesimal cross–ratio implies:

$$R_\zeta(z) = R_{J_z}(0) = -\Phi_0^{J_z^\circ J_z} \Phi_0^{J_z J_z^\circ},$$

where $\Phi_0^{vw} e = \pi_{v(0)w(0)} \dot{e}_0$ for any smooth curve $e_\tau \in v(\tau)$ such that $e_0 = e$. Equalities (14) and (16) imply: $\Phi_0^{J_z J_z^\circ} \nu(z) = [\zeta, \nu]_{ver}(z),\ \forall z \in M$. Similarly, $\Phi_0^{J_z^\circ J_z} \mu(z) = [\zeta, \mu]_{hor}(z)$ for any horizontal field μ and any $z \in M$. Finally,

$$R_\zeta(z)\nu(z) = -\Phi_0^{J_z^\circ J_z} \Phi_0^{J_z J_z^\circ} = -[\zeta, [\zeta, \nu]_{hor}]_{ver}(z). \qquad \square$$

15 Coordinate presentation

We fix local coordinates acting in the domain $\mathcal{O} \subset N$, which turn the foliation into the Cartesian product of vector spaces: $\mathcal{O} \cong \{(x, y) : x, y \in \mathbb{R}^n\}$, $\pi : (x, y) \mapsto y$. Then vector field ζ takes the form $\zeta = \sum_{i=1}^n \left(a^i \frac{\partial}{\partial x_i} + b^i \frac{\partial}{\partial y_i} \right)$, where a^i, b^i are smooth functions on $\mathbb{R}^n \times \mathbb{R}^n$. Below we use abridged notations: $\frac{\partial}{\partial x_i} = \partial_{x_i}$, $\frac{\partial \varphi}{\partial x_i} = \varphi_{x_i}$ etc. We also use the standard summation agreement for repeating indices.

Recall the coordinate characterization of the regularity property for the vector field ζ. Intrinsic definition of regular vector fields is done in Section 8; it is based on the mapping Π_z whose coordinate presentation is: $\Pi_{(x,y)} : x \mapsto (b^1(x, y), \ldots, b^n(x, y))^\top$. Field ζ is regular if and only if Π_y are submersions; in other words, if and only if $\left(b^i_{x_j} \right)_{i,j=1}^n$ is a non degenerate matrix.

Vector fields $\partial_{x_i},\ i = 1, \ldots, n$, provide a basis of the space of vertical fields. As soon as coordinates are fixed, any Ehresmann connection finds a unique basis of the form:

$$(\partial_{y_i})_{hor} = \partial_{y_i} + c_i^j \partial_{x_j},$$

where $c_i^j,\ i,j=1,\dots,n$, are smooth functions on $\mathbb{R}^n\times\mathbb{R}^n$. To characterize a connection in coordinates thus means to find functions c_i^j. In the case of the canonical connection of a regular vector field, the functions c_i^j can be easily recovered from identity (18) applied to $\nu=\partial_{x_i},\ i=1,\dots,n$. We'll do it explicitly for two important classes of vector fields: second order ordinary differential equations and Hamiltonian systems.

A second order ordinary differential equation

$$\dot y = x, \quad \dot x = f(x,y) \tag{20}$$

there corresponds to the vector field $\zeta=f^i\partial_{x_i}+x_i\partial_{y_i}$, where $f=(f_1,\dots,f_n)^\top$. Let $\nu=\partial_{x_i}$; then

$$[\zeta,\nu]=-\partial_{y_i}-f^j_{x_i}\partial_{x_j},\ [\zeta,\nu]_{ver}=(c_i^j-f^j_{x_i})\partial_{x_j},$$

$$\pi_*[\zeta,[\zeta,\nu]]=f^j_{x_i}\partial_{y_j},\ \pi_*[\zeta,[\zeta,\nu]_{ver}]=(f^j_{x_i}-c_i^j)\partial_{y_j}.$$

Hence, in virtue of equality (18) we obtain that $c_i^j=\frac{1}{2}f^j_{x_i}$ for the canonical connection associated with the second order differential equation (20).

Now consider a Hamiltonian vector field $\zeta=-h_{y_i}\partial_{x_i}+h_{x_i}\partial_{y_i}$, where h is a smooth function on $\mathbb{R}^n\times\mathbb{R}^n$ (a Hamiltonian). The field ζ is regular if and only if the matrix $h_{xx}=\left(h_{x_ix_j}\right)_{i,j=1}^n$ is non degenerate. We are going to characterize the canonical connection associated with ζ. Let $C=\left(c_i^j\right)_{i,j=1}^n$; the straightforward computation similar to the computation made for the second order ordinary differential equation gives the following presentation for the matrix C:

$$2\left(h_{xx}Ch_{xx}\right)_{ij}=h_{x_k}h_{x_ix_jy_k}-h_{y_k}h_{x_ix_jx_k}-h_{x_iy_k}h_{x_kx_j}-h_{x_ix_k}h_{y_kx_j}$$

or, in the matrix form:

$$2h_{xx}Ch_{xx}=\{h,h_{xx}\}-h_{xy}h_{xx}-h_{xx}h_{yx},$$

where $\{h,h_{xx}\}$ is the Poisson bracket: $\{h,h_{xx}\}_{ij}=\{h,h_{x_ix_j}\}=h_{x_k}h_{x_ix_jy_k}-h_{y_k}h_{x_ix_jx_k}$.

Note that matrix C is symmetric in the Hamiltonian case (indeed, $h_{xx}h_{yx}=(h_{xy}h_{xx})^\top$). This is not occasional and is actually guaranteed by the fact that Hamiltonian flows preserve symplectic form $dx_i\wedge dy_i$. See Section 17 for the symplectic version of the developed theory.

As soon as we found the canonical connection, formula (19) gives us the presentation of the curvature operator although the explicit coordinate expression can be bulky. Let us specify the vector field more. In the case of the Hamiltonian of a natural mechanical system, $h(x,y) = \frac{1}{2}|x|^2 + U(y)$, the canonical connection is trivial: $c_i^j = 0$; the matrix of the curvature operator is just U_{yy}.

Hamiltonian vector field associated to the Hamiltonian $h(x,y) = g^{ij}(y)x_i x_j$ with a non degenerate symmetric matrix $(g^{ij})_{i,j=1}^n$ generates a (pseudo-)Riemannian geodesic flow. Canonical connection in this case is classical Levi Civita connection and the curvature operator is Ricci operator of (pseudo-)Riemannian geometry (see [4, Sec. 5] for details). Finally, Hamiltonian $h(x,y) = g^{ij}(y)x_i x_j + U(y)$ has the same connection as Hamiltonion $h(x,y) = g^{ij}(y)x_i x_j$ while its curvature operator is sum of Ricci operator and second covariant derivative of U.

16 Affine foliations

Let $[\mathcal{E}]$ be the sheaf of germs of sections of the distribution $\mathcal{E} = \{T_z E_z : z \in N\}$ equipped with the Lie bracket operation. Then $[\mathcal{E}]_z$ is just the Lie algebra of germs at $z \in M$ of vertical vector fields. Affine structure on the foliation E is a sub-sheaf $[\mathcal{E}]^a \subset [\mathcal{E}]$ such that $[\mathcal{E}]_z^a$ is an Abelian sub-algebra of $[\mathcal{E}]_z$ and $\{\varsigma(z) : \varsigma \in [\mathcal{E}]_z^a\} = T_z E_z,\ \forall z \in N$. A foliation with a fixed affine structure is called the *affine foliation*.

The notion of the affine foliation generalizes one of the vector bundle. In the case of the vector bundle, the sheaf $[\mathcal{E}]^a$ is formed by the germs of vertical vector fields whose restrictions to the fibers are constant (i.e. translation invariant) vector fields on the fibers. In the next section we will describe an important class of affine foliations which is not reduced to the vector bundles.

Lemma II.5 *Let $\mathcal{E}$ be an affine foliation, $\varsigma \in [\mathcal{E}]_z^a$ and $\varsigma(z) = 0$. Then $\varsigma|_{E_z} = 0$.*

Proof. Let $\varsigma_1, \dots, \varsigma_n \in [\mathcal{E}]_z^a$ be such that $\varsigma_1(z), \dots, \varsigma_n(z)$ form a basis of $T_z E_z$. Then $\varsigma = b_1\varsigma_1 + \cdots + b_n\varsigma_n$, where b_i are germs of smooth functions vanishing at z. Commutativity of $[\mathcal{E}]_z^a$ implies: $0 = [\varsigma_i, \varsigma] = (\varsigma_i b_1)\varsigma_1 + \cdots + (\varsigma_i b_n)\varsigma_n$. Hence functions $b_i|_{E_z}$ are constants, i.e. $b_i|_{E_z} = 0,\ i = 1, \dots, n$. □

Lemma II.5 implies that $\varsigma \in [\mathcal{E}]_z^a$ is uniquely reconstructed from $\varsigma(z)$. This property permits to define the *vertical derivative* of any vertical vector field ν on M. Namely, $\forall v \in T_z E_z$ we set

$$D_v \nu = [\varsigma, \nu](z), \text{ where } \varsigma \in [\mathcal{E}]^a_z, \ \varsigma(z) = v.$$

Suppose ζ is a regular vector field on the manifold N endowed with the affine n-foliation. The canonical Ehresmann connection $\mathcal{E}_\zeta$ together with the vertical derivative allow to define a canonical linear connection ∇ on the vector bundle $\mathcal{E}$. Sections of the vector bundle $\mathcal{E}$ are just vertical vector fields. We set

$$\nabla_\xi \nu = [\xi, \nu]_{ver} + D_\nu(\xi_{ver}),$$

where ξ is any vector field on N and ν is a vertical vector field. It is easy to see that ∇ satisfies axioms of a linear connection. The only non evident one is: $\nabla_{b\xi}\nu = b\nabla_\xi \nu$ for any smooth function b. Let $z \in N$,$\varsigma \in [\mathcal{E}]^a_z$, and $\varsigma(z) = \nu(z)$. We have

$$\nabla_{b\xi}\nu = [b\xi, \nu]_{ver} + [\varsigma, b\xi_{ver}] =$$

$$b\left([\xi, \nu]_{ver} + [\varsigma, \xi_{ver}]\right) - (\nu b)\xi_{ver} + (\varsigma b)\xi_{ver}.$$

Hence

$$(\nabla_{b\xi}\nu)(z) = b(z)\left([\xi, \nu]_{ver}(z) + [\varsigma, \xi_{ver}](z)\right) = (b\nabla_\xi \nu)(z).$$

Linear connection ∇ gives us the way to express Pontryagin characteristic classes of the vector bundle $\mathcal{E}$ via the regular vector field ζ. Indeed, any linear connection provides an expression for Pontryagin classes. We are going to briefly recall the correspondent classical construction (see [13] for details). Let $R^\nabla(\xi, \eta) = [\nabla_\xi, \nabla_\eta] - \nabla_{[\xi,\eta]}$ be the curvature of linear connection ∇. Then $R^\nabla(\xi, \eta)\nu$ is $C^\infty(M)$-linear with respect to each of three arguments ξ, η, ν. In particular, $R^\nabla(\cdot, \cdot)\nu(z) \in \bigwedge^2(T^*_z N) \otimes T_z E_z, \ z \in N$. In other words, $R^\nabla(\cdot, \cdot) \in \mathrm{Hom}\left(\mathcal{E}, \bigwedge^2(T^*N) \otimes \mathcal{E}\right)$.

Consider the commutative exterior algebra

$$\bigwedge^{ev} N = C^\infty(N) \oplus \bigwedge^2(T^*N) \oplus \cdots \oplus \bigwedge^{2n}(T^*N)$$

of the even order differential forms on N. Then R^∇ can be treated as an endomorphism of the module $\bigwedge^{ev} N \otimes \mathcal{E}$ over algebra $\bigwedge^{ev} N$, i. e. $R^\nabla \in \mathrm{End}_{\bigwedge^{ev} N}(\bigwedge^{ev} M \otimes \mathcal{E})$. Now consider characteristic polynomial $\det(tI + \frac{1}{2\pi}R^\nabla) = t^n + \sum\limits_{i=1}^{n} \phi_i t^{n-i}$, where the coefficient ϕ_i is an order $2i$ differential form on N. All forms ϕ_i are closed; the forms ϕ_{2k-1} are exact and the forms ϕ_{2k} represent the Pontryagin characteristic classes, $k = 1, \ldots, [\frac{n}{2}]$.

17 Symplectic setting

Assume that N is a symplectic manifold endowed with a symplectic form σ. Recall that a symplectic form is just a closed non degenerate differential 2-form. Suppose E is a Lagrange foliation on the symplectic manifold (N, σ); this means that $\sigma|_{E_z} = 0, \ \forall z \in N$. Basic examples are cotangent bundles endowed with the standard symplectic structure: $N = T^*M, \ E_z = T^*_{\pi(z)}M$, where $\pi : T^*M \to M$ is the canonical projection. In this case $\sigma = d\tau$, where $\tau = \{\tau_z : z \in T^*M\}$ is the Liouville 1-form on T^*M defined by the formula: $\tau_z = z \circ \pi_*$. Completely integrable Hamiltonian systems provide another important class of Lagrange foliations. We'll briefly recall the correspondent terminology. Details can be found in any introduction to symplectic geometry (for instance, in [10]).

Smooth functions on the symplectic manifold are called Hamiltonians. To any Hamiltonian there corresponds a Hamiltonian vector field $\vec{h}$ on M defined by the equation: $dh = \sigma(\cdot, \vec{h})$. The Poisson bracket $\{h_1, h_2\}$ of the Hamiltonians h_1 and h_2 is the Hamiltonian defined by the formula: $\{h_1, h_2\} = \sigma(\vec{h}_1, \vec{h}_2) = \vec{h}_1 h_2$. Poisson bracket is obviously anti-symmetric and satisfies the Jacobi identity: $\{h_1, \{h_2, h_3\}\} + \{h_3, \{h_1, h_2\}\} + \{h_2, \{h_3, h_1\}\} = 0$. This identity is another way to say that the form σ is closed. Jacobi identity implies one more useful formula: $\overrightarrow{\{h_1, h_2\}} = [\vec{h}_1, \vec{h}_2]$.

We say that Hamiltonians $h_1, \dots, h_n$ are in involution if $\{h_i, h_j\} = 0$; then h_j is constant along trajectories of the Hamiltonian equation $\dot{z} = \vec{h}_i(z), \ i, j = 1, \dots, n$. We say that $h_1, \dots, h_n$ are independent if $d_z h_1 \wedge \dots \wedge d_z h_n \neq 0, \ z \in N$. n independent Hamiltonians in involution form a *completely integrable system*. More precisely, any of Hamiltonian equations $\dot{z} = \vec{h}_i(z)$ is completely integrable with first integrals $h_1, \dots, h_n$.

Lemma II.6 *Let Hamiltonians $h_1, \dots, h_n$ form a completely integrable system. Then the n-foliation $E_z = \{z' \in M : h_i(z') = h_i(z), \ i = 1, \dots, n\}, \quad z \in N$, is Lagrangian.*

Proof. We have $\vec{h}_i h_j = 0, \ i, j = 1, \dots, n$, hence $\vec{h}_i(z)$ are tangent to E_z. Vectors $\vec{h}_1(z), \dots, \vec{h}_n(z)$ are linearly independent, hence

$$span\{\vec{h}_1(z), \dots, \vec{h}_n(z)\} = T_z E_z.$$

Moreover, $\sigma(\vec{h}_i, \vec{h}_j) = \{h_i, h_j\} = 0$, hence $\sigma|_{E_z} = 0$. $\square$

Any Lagrange foliation possesses a canonical affine structure. Let $[\mathcal{E}]$ be the sheaf of germs of the distribution $\mathcal{E} = \{T_zE_z : z \in N\}$ as in Section 16; then $[\mathcal{E}]^a$ is the intersection of $[\mathcal{E}]$ with the sheaf of germs of Hamiltonian vector fields.

We have to check that Lie algebra $[\mathcal{E}]^a_z$ is Abelian and generates T_zE_z, $\forall z \in N$. First check the Abelian property. Let $\vec{h}_1, \vec{h}_2 \in [\mathcal{E}]^a_z$; we have $[\vec{h}_1, \vec{h}_2] = \overrightarrow{\{h_1, h_2\}}$, $\{h_1, h_2\} = \sigma(\vec{h}_1, \vec{h}_2) = 0$, since $\vec{h}_i$ are tangent to E_z and $\sigma|_{E_z} = 0$. The second property follows from the Darboux–Weinstein theorem (see [10]) which states that all Lagrange foliations are locally equivalent. More precisely, this theorem states that any $z \in M$ possesses a neighborhood O_z and local coordinates which turn the restriction of the Lagrange foliation E to O_z into the trivial bundle $\mathbb{R}^n \times \mathbb{R}^n = \{(x, y) : x, y \in \mathbb{R}^n\}$ and, simultaneously, turn $\sigma|_{O_z}$ into the form $\sum\limits_{i=1}^{n} dx_i \wedge dy_i$. In this special coordinates, the fibers become coordinate subspaces $\mathbb{R}^n \times \{y\}$, $y \in \mathbb{R}^n$, and the required property is obvious: vector fields $\frac{\partial}{\partial x_i}$ are Hamiltonian fields associated to the Hamiltonians $-y_i$, $i = 1, \ldots, n$.

Suppose ζ is a Hamiltonian field on the symplectic manifold endowed with the Lagrange foliation, $\zeta = \vec{h}$. Let $\varsigma \in [\mathcal{E}]^a_z$, $\varsigma = \vec{s}$; then $\varsigma h = \{s, h\}$. The field $\vec{h}$ is regular if and only if the quadratic form $s \mapsto \{s, \{s, h\}\}(z)$ has rank n. Indeed, in the 'Darboux–Weinstein coordinates' this quadratic form has the matrix $\{\frac{\partial^2 h}{\partial x_i \partial x_j}\}^n_{i,j=1}$.

Recall that the tangent space T_zN to the symplectic manifold N is a symplectic space endowed with the symplectic structure σ_z. An n-dimensional subspace $\upsilon \subset T_zN$ is a Lagrangian subspace if $\sigma_z|_\upsilon = 0$. The set

$$L(T_zN) = \{\upsilon \in G_n(T_zM) : \sigma_z|_\upsilon = 0\}$$

of all Lagrange subspaces of T_zM is a Lagrange Grassmannian.

Hamiltonian flow $e^{t\vec{h}}$ preserves the symplectic form, $\left(e^{t\vec{h}}\right)^* \sigma = \sigma$. Hence $\left(e^{t\vec{h}}\right)_* : T_zN \to T_{e^{t\vec{h}}(z)}N$ transforms Lagrangian subspaces in the Lagrangian ones. It follows that the Jacobi curve $J_z(t) = \left(e^{-t\vec{h}}\right)_* T_{e^{t\vec{h}}(z)}E_{e^{t\vec{h}}(z)}$ consists of Lagrangian subspaces, $J_z(t) \in L(T_zN)$.

We need few simple facts on Lagrangian Grassmannians (see Sec. 6 for the basic information and [3, Sec. 4] for a consistent description of their geometry). Let $(\Sigma, \bar{\sigma})$ be a $2n$-dimensional symplectic space and $\upsilon_0, \upsilon_1 \in$

$L(\Sigma)$ be a pair of transversal Lagrangian subspaces, $v_0 \cap v_1 = 0$. Bilinear form $\bar\sigma$ induces a non degenerate pairing of the spaces v_0 and v_1 by the rule $(e,f) \mapsto \bar\sigma(e,f),\ e \in v_0, f \in v_1$. To any basis $e_1,\dots,e_n$ of v_0 we may associate a unique dual basis $f_1,\dots,f_n$ of v_1 such that $\bar\sigma(e_i,f_j) = \delta_{ij}$. The form $\bar\sigma$ is totally normalized in the basis $e_1,\dots,e_n,f_1,\dots,f_n$ of Σ, since $\sigma(e_i,e_j) = \sigma(f_i,f_j) = 0$. It follows that symplectic group

$$\mathrm{Sp}(\Sigma) = \{A \in \mathrm{GL}(\Sigma) : \bar\sigma(Ae,Af) = \bar\sigma(e,f),\ e,f \in \Sigma\}$$

acts transitively on the pairs of transversal Lagrangian subspaces.

Next result is a 'symplectic specification' of Lemma II.1 from Section 9.

Lemma II.7 *Let $v_0 \in L(\Sigma)$; then $\{\pi_{vv_0} : v \in v_0^{\pitchfork} \cap L(\Sigma)\}$ is an affine subspace of the affine space $\{\pi_{vv_0} : v \in v_0^{\pitchfork}\}$ characterized by the relation:*

$$v \in v_0^{\pitchfork} \cap L(\Sigma) \ \Leftrightarrow\ \bar\sigma(\pi_{vv_0}\cdot,\cdot) + \bar\sigma(\cdot,\pi_{vv_0}\cdot) = \bar\sigma(\cdot,\cdot).$$

Proof. Assume that $v_1 \in v_0^{\pitchfork} \cap L(\Sigma)$. Let $e,f \in \Sigma,\ e = e_0 + e_1,\ f = f_0 + f_1$ where $e_i, f_i \in v_i,\ i = 0,1$; then

$$\bar\sigma(e,f) = \bar\sigma(e_0+e_1,f_0+f_1) = \bar\sigma(e_0,f_1) + \bar\sigma(e_1,f_0) =$$

$$\bar\sigma(e_0,f) + \bar\sigma(e,f_0) = \bar\sigma(\pi_{v_1v_0}e,f) + \bar\sigma(e,\pi_{v_1v_0}f).$$

Conversely, let $v \in v_0^{\pitchfork}$ is not a Lagrangian subspace. Then there exist $e,f \in v$ such that $\bar\sigma(e,f) \neq 0$, while $\bar\sigma(\pi_{vv_0}e,f) = \bar\sigma(e,\pi_{vv_0}f) = 0$. $\square$

Corollary II.4 *Let $v(\cdot)$ be an ample curve in $G_n(\Sigma)$ and $v^\circ(\cdot)$ be the derivative curve of $v(\cdot)$. If $v(t) \in L(\Sigma),\ \forall t$, then $v^\circ(t) \in L(\Sigma)$.*

Proof. The derivative curve v° was defined in Section 11. Recall that $\pi_{v^\circ(t)v(t)} = \pi_t^0$, where π_t^0 is the free term of the Laurent expansion

$$\pi_{v(\tau)v(t)} \approx \sum_{i=-k_t}^{\infty} (\tau - t)^i \pi_t^i.$$

The free term π_t^0 belongs to the affine hull of $\pi_{v(\tau)v(t)}$, when τ runs a neighborhood of t. Since $\pi_{v(\tau)v(t)}$ belongs to the affine space $\{\pi_{vv_0} : v \in v_0^{\pitchfork} \cap L(\Sigma)\}$, then π_t^0 belongs to this affine space as well. $\square$

We call a *Lagrange distribution* any rank n vector distribution $\{\Lambda_z \subset T_zN : z \in N\}$ on the symplectic manifold N such that $\Lambda_z \in L(T_zN),\ z \in N$.

Corollary II.5 *Canonical Ehresmann connection $\mathcal{E}_\zeta = \{J_z^\circ(0) : z \in N\}$ associated to an ample Hamiltonian field $\zeta = \vec{h}$ is a Lagrange distribution.* $\square$

It is clearly seeing in coordinates how Lagrange Grassmanian is sitting in the usual one. Let $\Sigma = \mathbb{R}^{n*} \times \mathbb{R}^n = \{(\eta, y) : \eta \in \mathbb{R}^{n*}, y \in \mathbb{R}^n\}$. Then any $v \in (\{0\} \times \mathbb{R}^n)^\pitchfork$ has a form $v = \{(y^\top, Sy) : y \in \mathbb{R}^n\}$, where S is an $n \times n$-matrix. It is easy to see that v is a Lagrangian subspace if and only if S is a symmetric matrix, $S = S^\top$.

18 Monotonicity

We continue to study curves in the Lagrange Grassmannian $L(T_zN)$, in particular, the Jacobi curves $t \mapsto \left(e^{-t\vec{H}}\right)_* T_{e^{t\vec{H}}(z)} E_{e^{t\vec{H}}(z)}$. In Section 6 we identified the velocity $\dot\Lambda(t)$ of any smooth curve $\Lambda(\cdot)$ in $L(T_zN)$ with a quadratic form $\underline{\dot\Lambda}(t)$ on the subspace $\Lambda(t) \subset T_zN$. Recall that the curve $\Lambda(\cdot)$ was called monotone increasing if $\underline{\dot\Lambda}(t) \geq 0,\ \forall t$; it is called monotone decreasing if $\underline{\dot\Lambda}(t) \leq 0$. It is called monotone in both cases.

Proposition II.8 *Set $\Lambda(t) = \left(e^{-t\vec{H}}\right)_* T_{e^{t\vec{H}}(z)} E_{e^{t\vec{H}}(z)}$; then quadratic form $\underline{\dot\Lambda}(t)$ is equivalent (up to a linear change of variables) to the form*

$$\varsigma \mapsto -(\varsigma \circ \varsigma H)(e^{t\vec{H}}(z)), \quad \varsigma \in [\mathcal{E}]^a_{e^{t\vec{H}}(z)}, \tag{21}$$

on $E_{e^{t\vec{H}}(z)}$.

Proof. Let $z_t = e^{t\vec{H}}(z)$, then

$$\frac{d}{dt}\Lambda(t) = \frac{d}{dt} e_*^{(t_0 - t)\vec{H}} T_{z_t} E_{z_t} = e_*^{(t_0-t)\vec{H}} \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} e_*^{-\varepsilon\vec{H}} T_{z_{t+\varepsilon}} E_{z_{t+\varepsilon}}.$$

Set $\Delta(\varepsilon) = e_*^{-\varepsilon\vec{H}} T_{z_{t+\varepsilon}} E_{z_{t+\varepsilon}} \in L\left(T_{z_t}N\right)$. It is enough to prove that $\underline{\dot\Delta}(0)$ is equivalent to form (21). Indeed, $\dot\Lambda(t) = e_*^{(t_0-t)\vec{H}} T_{z_t} \dot\Delta(0)$, where

$$e_*^{(t_0-t)\vec{H}} : T_{z_t}N \to T_{z_{t_0}}N$$

is a symplectic isomorphism. The association of the quadratic form $\underline{\dot\Lambda}(t)$ on the subspace $\Lambda(t)$ to the tangent vector $\dot\Lambda(t) \in L\left(T_{z_{t_0}}N\right)$ is intrinsic, i.e. depends only on the symplectic structure on $T_{z_{t_0}}N$. Hence $\dot{\underline{\Delta}}(0)(\xi) = \dot{\underline{\Delta}}(t)\left(e_*^{(t_0-t)\vec{H}}\xi\right)$, $\forall \xi \in \Delta(0) = T_{z_t}E_{z_t}$.

What remains, is to compute $\dot{\underline{\Delta}}(0)$; we do it in the Darboux–Weinstein coordinates $z = (x, y)$. We have: $\Delta(\varepsilon) =$

$$\left\{ (\xi(\varepsilon), \eta(\varepsilon)) : \begin{array}{rcl} \dot{\xi}(\tau) & = & \xi(\tau)\frac{\partial^2 H}{\partial x \partial y}(z_{t-\tau}) + \eta(\tau)^\top \frac{\partial^2 H}{\partial y^2}(z_{t-\tau}), \quad \xi(0) = \xi \in \mathbb{R}^{n*} \\ \dot{\eta}(\tau) & = & -\frac{\partial^2 H}{\partial x^2}(z_{t-\tau})\xi(\tau)^\top - \frac{\partial^2 H}{\partial y \partial x}(z_{t-\tau})\eta(\tau), \;\; \eta(0) = 0 \in \mathbb{R}^n \end{array} \right\},$$

$$\dot{\underline{\Delta}}(0)(\xi) = \sigma\left((\xi, 0), (\dot{\xi}(0), \dot{\eta}(0))\right) = \xi\dot{\eta}(0) = -\xi\frac{\partial^2 H}{\partial x^2}(z_t)\xi^\top.$$

Recall now that form (21) has matrix $\frac{\partial^2 H}{\partial x^2}(z_t)$ in the Darboux–Weinstein coordinates. $\square$

This proposition clearly demonstrates the importance of monotone curves. Indeed, monotonicity of Jacobi curves is equivalent to the convexity (or concavity) of the Hamiltonian on each leaf of the Lagrange foliation. In the case of a cotangent bundle this means the convexity or concavity of the Hamiltonian with respect to the impulses. All Hamiltonians (energy functions) of mechanical systems are like that! This is not an occasional fact but a corollary of the list action principle. Indeed, trajectories of the mechanical Hamiltonian system are extremals of the least action principle and the energy function itself is the Hamiltonian of the correspondent regular optimal control problem as it was considered in Section 7. Moreover, it was stated in Section 7 that convexity of the Hamiltonian with respect to the impulses is necessary for the extremals to have finite Morse index. It turns out that the relation between finiteness of the Morse index and monotonicity of the Jacobi curve has a fundamental nature. A similar property is valid for any, not necessary regular, extremal of a finite Morse index. Of course, to formulate this property we have first to explain what are Jacobi curve for non regular extremals. To do that, we come back to the very beginning; indeed, Jacobi curves appeared first as the result of calculation of the $\mathcal{L}$-derivative at the regular extremal (see Sections 7, 8). On the other hand, $\mathcal{L}$-derivative is well-defined for any extremal of the finite Morse index as it follows from Theorem I.1. We thus come to the following construction in which we use notations and definitions of Sections 3, 4.

Let $h(\lambda, u)$ be the Hamiltonian of a smooth optimal control system, λ_t, $t_0 \leq t \leq t_1$, an extremal, and $q(t) = \pi(\lambda_t)$, $t_0, \leq t \leq t_1$ the extremal path.

Recall that the pair $(\lambda_{t_0}, \lambda_t)$ is a Lagrangian multiplier for the conditional minimum problem defined on an open subset of the space

$$M \times L_\infty([t_0, t_1], U) = \{(q_t, u(\cdot)) : q \in M, u(\cdot) \in L_\infty([t_0, t_1], U)\},$$

where $u(\cdot)$ is control and q_t is the value at t of the solution to the differential equation $\dot{q} = f(q, u(\tau)),\ \tau \in [t_0, t_1]$. In particular, $F_t(q_t, u(\cdot)) = q_t$. The cost is $J_{t_0}^{t_1}(q_t, u(\cdot))$ and constraints are $F_{t_0}(q_t, u(\cdot)) = q(0),\ q_t = q(t)$.

Let us set $J_t(u) = J_{t_0}^{t}(q(t), u(\cdot)),\ \Phi_t(u) = F_{t_0}(q(t), u(\cdot))$. A covector $\lambda \in T^*M$ is a Lagrange multiplier for the problem (J_t, Φ_t) if and only if there exists an extremal $\hat{\lambda}_\tau,\ t_0 \le \tau \le t$, such that $\lambda_{t_0} = \lambda,\ \hat{\lambda}_t \in T^*_{q(t)}M$. In particular, λ_{t_0} is a Lagrange multiplier for the problem (J_t, Φ_t) associated to the control $u(\cdot) = \bar{u}(\lambda.)$.

Assume that $\operatorname{ind}\operatorname{Hess}_u \left(J_{t_1}\big|_{\Phi_{t_1}^{-1}(q(t_0))} \right) \le \infty,\ t_0 \le t \le t_1$ and set $\bar{\Phi}_t = (J_t, \Phi_t)$. The curve

$$t \mapsto \mathcal{L}_{(\lambda_{t_0}, u)}(\bar{\Phi}_t), \quad t_0 \le t \le t_1$$

in the Lagrange Grassmannian $L\left(T_{\lambda_{t_0}}(T^*M)\right)$ is called the Jacobi curve associated to the extremal $\lambda_t,\ t_0 \le t \le t_1$.

In general, the Jacobi curve $t \mapsto \mathcal{L}_{(\lambda_{t_0}, u)}(\bar{\Phi}_t)$ is not smooth, it may even be discontinues, but it is monotone decreasing in a sense we are going to briefly describe now. You can find more details in [2] (just keep in mind that similar quantities may have opposite signs in different papers; sign agreements vary from paper to paper that is usual for symplectic geometry). Monotone curves in the Lagrange Grassmannian have analytic properties similar to scalar monotone functions: no more than a countable set of discontinuity points, right and left limits at every point, and differentiability almost everywhere with semi-definite derivatives (nonnegative for monotone increasing curves and nonpositive for decreasing ones). True reason for such a monotonicity is a natural monotonicity of the family $\bar{\Phi}_t$. Indeed, let $\tau < t$, then $\bar{\Phi}_\tau$ is, in fact, the restriction of $\bar{\Phi}_t$ to certain subspace: $\bar{\Phi}_\tau = \bar{\Phi}_t \circ \mathfrak{p}_\tau$, where $\mathfrak{p}_\tau(u)(s) = \begin{cases} u(s) & , \quad s < \tau \\ \tilde{u}(s) & , \quad s > \tau \end{cases}$. One can define the Maslov index of a (maybe discontinues) monotone curve in the Lagrange Grassmannian and the relation between the Morse and Maslov index indices from Theorem I.3 remains true.

In fact, Maslov index is a key tool in the whole construction. The starting point is the notion of a *simple curve.* A smooth curve $\Lambda(\tau),\ \tau_0 \le \tau \le \tau_1$, in the Lagrange Grassmannian $L(\Sigma)$ is called simple if there exists $\Delta \in L(\Sigma)$ such that $\Delta \cap \Lambda(\tau) = 0,\ \forall \tau \in [\tau_0, \tau_1]$; in other words, the entire curve is contained in one coordinate chart. It is not hard to show that any two points of $L(\Sigma)$ can be connected by a simple monotone increasing (as well as monotone decreasing) curve. An important fact is that the Maslov index $\mu(\Lambda_\Pi(\cdot))$ of a simple monotone increasing curve $\Lambda(\tau),\ \tau_0 \le \tau \le \tau_1$ is uniquely determined by the triple $(\Pi, \Lambda(\tau_0), \Lambda(\tau_1))$; i.e. it has the same value for all simple monotone increasing curves connecting $\Lambda(\tau_0)$ with $\Lambda(\tau_1)$. A simple way to see this is to find an intrinsic algebraic expression for the Maslov index preliminary computed for some simple monotone curve in some coordinates. We can use Lemma I.2 for this computation since the curve is simple. The monotonic increase of the curve implies that $S_{\Lambda(t_1)} > S_{\Lambda(t_0)}$.

Exercise. Let S_0, S_1 be nondegenerate symmetric matrices and $S_1 \ge S_0$. Then $\mathrm{ind} S_0 - \mathrm{ind} S_1 = \mathrm{ind}\left(S_0^{-1} - S_1^{-1}\right)$.

Let $x \in (\Lambda(\tau_0) + \Lambda(\tau_1) \cap \Pi$ so that $x = x_0 + x_1$, where $x_i \in \Lambda(\tau_i),\ i = 0, 1$. We set $\mathfrak{q}(x) = \sigma(x_1, x_0)$. If $\Lambda(\tau_0) \cap \Lambda(\tau_1) = 0$, then $\Lambda(\tau_0) + \Lambda(\tau_1) = \Sigma$, x is any element of Π and x_0, x_1 are uniquely determined by x. This is not true if $\Lambda(\tau_0) \cap \Lambda(\tau_1) \ne 0$ but $\mathfrak{q}(x)$ is well-defined anyway: $\sigma(x_1, x_2)$ depends only on $x_0 + x_1$ since σ vanishes on $\Lambda(\tau_i),\ i = 0, 1$.

Now we compute $\mathfrak{q}$ in coordinates. Recall that

$$\Lambda(\tau_i) = \{(y^\top, S_{\Lambda(\tau_i)} y) : y\mathbb{R}^n\},\ i = 0, 1,\ \Pi = \{y^\top, 0) : y \in \mathbb{R}^n\}.$$

We have

$$\mathfrak{q}(x) = y_1^\top S_{\Lambda(\tau_0)} y_0 - y_0^\top S_{\Lambda(\tau_1)} y_1,$$

where $x = (y_0^\top + y_1^\top, 0)$, $S_{\Lambda(\tau_0)} y_0 + S_{\Lambda(\tau_1)} y_1 = 0$. Hence $y_1 = -S^{-1}_{\Lambda(tau_1)} S_{\Lambda(\tau_0)} y_0$ and

$$\mathfrak{q}(x) = -y_0^\top S_{\Lambda(\tau_0)} y_0 - \left(S_{\Lambda(\tau_0)} y_0\right)^\top S^{-1}_{\Lambda(\tau_1)} S_{\Lambda(\tau_0)} y_0 = y^\top \left(S^{-1}_{\Lambda(\tau_0)} - S^{-1}_{\Lambda(\tau_1)}\right) y,$$

where $y = S_{\Lambda(\tau_0)} y_0$. We see that the form $\mathfrak{q}$ is equivalent, up to a linear change of coordinates, to the quadratic form defined by the matrix $S^{-1}_{\Lambda(\tau_0)} - S^{-1}_{\Lambda(\tau_1)}$. Now we set

$$\mathrm{ind}_\Pi(\Lambda(\tau_0), \Lambda(\tau_1)) \stackrel{def}{=} \mathrm{ind}\, \mathfrak{q}.$$

The above exercise and Lemma I.2 imply the following:

Lemma II.8 *If $\Lambda(\tau),\ \tau_0 \le \tau \le \tau_1$, is a simple monotone increasing curve, then*

$$\mu(\Lambda(\cdot)) = \mathrm{ind}_\Pi(\Lambda(\tau_0), \Lambda(\tau_1)).$$

Note that definition of the form $\mathfrak{q}$ does not require transversality of $\Lambda(\tau_i)$ to Π. It is convenient to extend definition of $\mathrm{ind}_\Pi(\Lambda(\tau_0),\Lambda(\tau_1))$ to this case. General definition is as follows:

$$\mathrm{ind}_\Pi(\Lambda_0,\Lambda_1)=\mathrm{ind}\,\mathfrak{q}+\frac{1}{2}(\dim(\Pi\cap\Lambda_0)+\dim(\Pi\cap\Lambda_1))-\dim(\Pi\cap\Lambda_0\cap\Lambda_1).$$

The Maslov index also has appropriate extension (see [3, Sec.4]) and Lemma II.8 remains true.

Index $\mathrm{ind}_\Pi(\Lambda_0,\Lambda_1)$ satisfies the triangle inequality:

$$\mathrm{ind}_\Pi(\Lambda_0,\Lambda_2)\le\mathrm{ind}_\Pi(\Lambda_0,\Lambda_1)+\mathrm{ind}_\Pi(\Lambda_1,\Lambda_2).$$

Indeed, the right-hand side of the inequality is equal to the Maslov index of a monotone increasing curve connecting Λ_0 with Λ_2, i.e. of the concatenation of two simple monotone increasing curves. Obviously, the Maslov index of a simple monotone increasing curve is not greater than the Maslov index of any other monotone increasing curve connecting the same endpoints.

The constructed index gives a nice presentation of the Maslov index of any (not necessary simple) monotone increasing curve $\Lambda(t),\ t_0\le t\le t_1$:

$$\mu_\Pi(\Lambda(\cdot))=\sum_{i=0}^{l}\mathrm{ind}_\Pi(\Lambda(\tau_i),\Lambda(\tau_{i+1})),\tag{22}$$

where $t_0=\tau_0<\tau_1<\cdots<\tau_l<\tau_{l+1}=t_1$ and $\Lambda\big|_{[\tau_i,\tau_{i+1}]}$ are simple pieces of the curve $\Lambda(\cdot)$. If the pieces are not simple, then the right-hand side of (22) gives a low bound for the Maslov index (due to the triangle inequality).

Let now $\Lambda(t),\ t_0\le t\le t_1$, be a smooth curve which is *not* monotone increasing. Take any subdivision $t_0=\tau_0<\tau_1<\cdots<\tau_l<\tau_{l+1}=t_1$ and compute the sum $\sum\limits_{i=0}^{l}\mathrm{ind}_\Pi(\Lambda(\tau_i),\Lambda(\tau_{i+1}))$. This sum inevitably goes to infinity when the subdivision becomes finer and finer. The reason is as follows: $\mathrm{ind}_\Pi(\Lambda(\tau_i),\Lambda(\tau_{i+1}))>0$ for any simple piece $\Lambda\big|_{[\tau_i,\tau_{i+1}]}$ such that $\underline{\dot\Lambda}(\tau)\not\geq 0,\ \forall\tau\in[\tau_i,\tau_{i+1}]$ and $\mu_\Pi(\Lambda\big|_{[\tau_i,\tau_{i+1}]}=0$. I advise reader to play with the one-dimensional case of the curve in $L(\mathbb{R}^2)=S^1$ to see better what's going on.

This should now be clear how to manage in the general nonsmooth case. Take a curve $\Lambda(\cdot)$ (an arbitrary mapping from $[t_0,t_1]$ into $L(\Sigma)$). For any finite subset $\mathcal{T}+\{\tau_1,\ldots,\tau_k\}\subset[t_0,t_1]$, where $t_0=\tau_0<\tau_1<\cdots<\tau_l<$

$\tau_{l+1} = t_1$, we compute the sum $I_\Pi^{\mathcal{T}} = \sum\limits_{i=0}^{l} \mathrm{ind}_\Pi(\Lambda(\tau_i), \Lambda(\tau_{i+1}))$ and then find supremum of these sums for all finite subsets: $I_\Pi(\Lambda(\cdot)) = \sup\limits_{\mathcal{T}} I_\Pi^{\mathcal{T}}$. The curve $\Lambda(\cdot)$ is called monotone increasing if $I_\Pi^{\mathcal{T}} < \infty$; it is not hard to show that the last property does not depend on Π and that monotone increased curves enjoy listed above analytic properties. A curve $\Lambda(\cdot)$ is called monotone decreasing if inversion of the parameter $t \mapsto t_0 + t_1 - t$ makes it monotone increasing.

We set $\mu(\Lambda(\cdot)) = I_\Pi(\Lambda(\cdot))$ for any monotone increasing curve and $\mu(\Lambda(\cdot)) = -I_\Pi(\hat{\Lambda}(\cdot))$ for a monotone decreasing one, where $\hat{\Lambda}(t) = \Lambda(t_0 + t_1 - t)$. The defined in this way Maslov index of a discontinues monotone curve equals the Maslov index of the continues curve obtained by gluing all discontinuities with simple monotone curves of the same direction of monotonicity.

If $\Lambda(t) = \mathcal{L}_{(\lambda_{t_0},u)}(\bar{\Phi}_t)$ is the Jacobi curve associated to the extremal with a finite Morse index, then $\Lambda(\cdot)$ is monotone decreasing and its Maslov index computes $\mathrm{ind}\,\mathrm{Hess}_u\left(J_{t_1}\big|_{\Phi_{t_1}^{-1}(q(t_0))}\right)$ in the way similar to Theorem I.3. Of course, these nice things have some value only if we can effectively find Jacobi curves for singular extremals: their definition was too abstract. Fortunately, this is not so hard; see [5] for the explicit expression of Jacobi curves for a wide class of singular extremals and, in particular, for singular curves of rank 2 vector distributions (these last Jacobi curves have found important applications in the geometry of distributions, see [11, 14]).

One more important property of monotonic curves is as follows.

Lemma II.9 *Assume that $\Lambda(\cdot)$ is monotone and right-continues at t_0, i.e. $\Lambda(t_0) = \lim\limits_{t \searrow t_0} \Lambda(t)$. Then $\Lambda(t_0) \cap \Lambda(t) = \bigcap\limits_{t_0 \le \tau \le t} \Lambda(t)$ for any t sufficiently close to (and greater than) t_0.*

Proof. We may assume that $\Lambda(\cdot)$ is monotone increasing. Take centered at $\Lambda(t_0)$ local coordinates in the Lagrange Grassmannian; the coordinate presentation of $\Lambda(t)$ is a symmetric matrix $S_{\Lambda(t)}$, where $S_{\Lambda(t_0)} = 0$ and $t \mapsto y^\top S_{\Lambda(t)} y$ is a monotone increasing scalar function $\forall y \in \mathbb{R}^n$. In particular, $\ker S_{\Lambda(t)} = \Lambda(t) \cap \Lambda(t_0)$ is a monotone decreasing family of subspaces. $\square$

We set $\Gamma_t = \bigcap\limits_{t_0 \le \tau \le t} \Lambda(\tau)$, a monotone decreasing family of isotropic subspaces. Let $\Gamma = \max\limits_{t > t_0} \Gamma_t$, then $\Gamma_t == \Gamma$ for all $t > t_0$ sufficiently close to t_0. We have: $\Lambda(t) = \Lambda(t)^\angle$ and $\Lambda(t) \supset \Gamma$ for all $t > t_0$ close enough to t_0; hence

$\Gamma_t^{\angle} \supset \Lambda(t)$. In particular, $\Lambda(t)$ can be treated as a Lagragian subspace of the symplectic space $\Gamma^{\angle}/\Gamma$. Moreover, Lemma II.9 implies that $\Lambda(t)\cap\Lambda(t_0) = \Gamma$. In other words, $\Lambda(t)$ is transversal to $\Lambda(t_0)$ in $\Gamma^{\angle}/\Gamma$. In the case of a real-analytic monotone curve $\Lambda(\cdot)$ this automatically implies that $\Lambda(\cdot)$ is an ample curve in $\Gamma^{\angle}/\Gamma$. Hence any nonconstant monotone analytic curve is reduced to an ample monotone curve. It becoms ample after the factorization by a fixed (motionless) subspace.

19 Comparizon theorem

We come back to smooth regular curves after the deviation devoted to a more general perspective.

Lemma II.10 *Let $\Lambda(t)$, $t \in [t_0, t_1]$ be a regular monotone increasing curve in the Lagrange Grassmannian $L(\Sigma)$. Then $\{t \in [t_0, t_1] : \Lambda(t) \cap \Pi \neq 0\}$ is a finite subset of $[t_0, t_1]$ $\forall \Pi \in L(\Sigma)$. If t_0 and t_1 are out of this subset, then*

$$\mu_\Pi(\Lambda(\cdot)) = \sum_{t\in(t_0,t_1)} \dim(\Lambda(t)\cap\Pi).$$

Proof. We have to proof that $\Lambda(t)$ may have a nontrivial intersection with Π only for isolated values of t; the rest is Lemma I.1. Assume that $\Lambda(t)\cap\Pi \neq 0$ and take a centered at Π coordinate neighborhood in $L(\Sigma)$ which contains $\Lambda(t)$. In these coordinates, $\Lambda(\tau)$ is presented by a symmetric matrix $S_\Lambda(\tau)$ for any τ sufficiently close to t and $\Lambda(\tau)\cap\Pi = \ker S_{\Lambda(\tau)}$. Monotonicity and regularity properties are equivalent to the inequality $\dot S_{\Lambda(\tau)} > 0$. In particular, $y^\top \dot S_{\Lambda(t)} y > 0$ $\forall y \in \ker S_{\Lambda(t)} \setminus \{0\}$. The last inequality implies that $S_{\Lambda(\tau)}$ is a nondegenerate for all τ sufficiently close and not equal to t.

Definition. Parameter values τ_0, τ_1 are called conjugate for the continues curve $\Lambda(\cdot)$ in the Lagrange Grassmannian if $\Lambda(\tau_0)\cap\Lambda(\tau_1) \neq 0$; the dimension of $\Lambda(\tau_0)\cap\Lambda(\tau_1)$ is the *multiplicity* of the conjugate parameters.

If $\Lambda(\cdot)$ is a regular monotone increasing curve, then, according to Lemma II.9, conjugate points are isolated and the Maslov index $\mu_{\Lambda(t_0)}\left(\Lambda\big|_{[t,t_1]}\right)$ equals the sum of multiplicities of the conjugate to t_0 parameter values located in (t, t_1). If $\Lambda(\cdot)$ is the Jacobi curve of an extremal of an optimal control problem, then this Maslov index equals the Morse index of the extremal; this is why conjugate points are so important.

Given a regular monotone curve $\Lambda(\cdot)$, the quadratic form $\underline{\dot{\Lambda}}(t)$ defines an Euclidean structure $\langle\cdot,\cdot\rangle_{\dot{\Lambda}(t)}$ on $\Lambda(t)$ so that $\underline{\dot{\Lambda}}(t)(x) = \langle x, x\rangle_{\dot{\Lambda}(t)}$. Let $R_\Lambda(t) \in \mathrm{gl}(\Lambda(t))$ be the curvature operator of the curve $\Lambda(\cdot)$; we define the *curvature quadratic form* $r_\lambda(t)$ on $\Lambda(t)$ by the formula:

$$r_\Lambda(t)(x) = \langle R_\Lambda(t)x, x\rangle_{\dot{\Lambda}(t)}, \quad x \in \Lambda(t).$$

Proposition II.9 *The curvature operator $R_\Lambda(t)$ is a self-adjoint operator for the Euclidean structure $\langle\cdot,\cdot\rangle_{\dot{\Lambda}(t)}$. The form $r_\Lambda(t)$ is equivalent (up to linear changes of variables) to the form $\underline{\dot{\Lambda}}^\circ(t)$, where $\Lambda^\circ(\cdot)$ is the derivative curve.*

Proof. The statement is intrinsic and we may check it in any coordinates. Fix t and take Darboux coordinates $\{(\eta, y) : \eta \in \mathbb{R}^{n*}, y \in \mathbb{R}^n\}$ in Σ in such a way that $\Lambda(t) = \{(y^\top, 0) : y \in \mathbb{R}^n\}$, $\Lambda^\circ(t) = \{(0, y) : y \in \mathbb{R}^n\}$, $\underline{\dot{\Lambda}}(t)(y) = y^\top y$. Let $\Lambda(\tau) = \{(y^\top, S_\tau y) : y \in \mathbb{R}^n\}$, then $S_t = 0$. Moreover, $\dot{S}(t)$ is the matrix of the form $\underline{\dot{\Lambda}}(t)$ in given coordinates, hence $\dot{S}_t = I$. Recall that $\Lambda^\circ(\tau) = \{(y^\top A_\tau, y + S_\tau A_\tau y) : y \in \mathbb{R}^n\}$, where $A_\tau = -\frac{1}{2}\dot{S}_\tau^{-1}\ddot{S}_\tau\dot{S}_\tau^{-1}$ (see (5)). Hence $\ddot{S}_t = 0$. We have: $R_\Lambda(t) = \frac{1}{2}\,\dddot{S}_t$, $r_\Lambda(t)(y) = \frac{1}{2}y^\top\,\dddot{S}_t\,y$,

$$\underline{\dot{\Lambda}}^\circ(t)(y) = \sigma\Big((0, y), (y^\top \dot{A}_t, 0)\Big) = -y^\top \dot{A}_t y = \frac{1}{2}y^\top\,\dddot{S}_t\,y.$$

So $r_\Lambda(t)$ and $\underline{\dot{\Lambda}}^\circ(t)$ have equal matrices for our choice of coordinates in $\Lambda(t)$ and $\Lambda^\circ(t)$. The curvature operator is self-adjoint since it is presented by a symmetric matrix in coordinates where form $\underline{\dot{\Lambda}}(t)$ is the standard inner product. $\square$

Proposition II.9 implies that the curvature operators of regular monotone curves in the Lagrange Grassmannian are diagonalizable and have only real eigenvalues.

Theorem II.1 *Let $\Lambda(\cdot)$ be a regular monotone curve in the Lagrange Grassmannian $L(\Sigma)$, where* $\dim\Sigma = 2n$.

- *If all eigenvalues of $R_\Lambda(t)$ do not exceed a constant $c \geq 0$ for any t from the domain of $\Lambda(\cdot)$, then $|\tau_1 - \tau_0| \geq \frac{\pi}{\sqrt{c}}$ for any pair of conjugate parameter values τ_0, τ_1. In particular, If all eigenvalues of $R_\Lambda(t)$ are nonpositive $\forall t$, then $\Lambda(\cdot)$ does not possess conjugate parameter values.*

- *If* $\mathrm{tr}R_\Lambda(t) \geq nc$ *for some constant* $c > 0$ *and* $\forall t$, *then, for arbitrary* $\tau_0 \leq t$, *the segment* $[t, t + \frac{\pi}{\sqrt{c}}]$ *contains a conjugate to* τ_0 *parameter value as soon as this segment is contained in the domain of* $\Lambda(\cdot)$.

Both estimates are sharp.

Proof. We may assume without lack of generality that $\Lambda(\cdot)$ is ample monotone increasing. We start with the case of nonpositive eigenvalues of $R_\Lambda(t)$. The absence of conjugate points follows from Proposition II.9 and the following

Lemma II.11 *Assume that* $\Lambda(\cdot)$ *is an ample monotone increasing (decreasing) curve and* $\Lambda^\circ(\cdot)$ *is a continues monotone decreasing (increasing) curve. Then* $\Lambda(\cdot)$ *does not possess conjugate parameter values and there exists a* $\lim\limits_{t\to+\infty} \Lambda(t) = \Lambda_\infty$.

Proof. Take some value of the parameter τ_0; then $\Lambda(\tau_0)$ and $\Lambda^\circ(\tau_0)$ is a pair of transversal Lagrangian subspaces. We may choose coordinates in the Lagrange Grassmannian in such a way that $S_{\Lambda(\tau))} = 0$ and $S_{\Lambda^\circ(\tau_0)} = I$, i.e. $\Lambda(\tau_0)$ is represented by zero $n \times n$-matrix and $\Lambda^\circ(\tau_0)$ by the unit matrix. Monotonicity assumption implies that $t \mapsto S_{\Lambda(t)}$ is a monotone increasing curve in the space of symmetric matrices and $t \mapsto S_{\Lambda^\circ(t)}$ is a monotone decreasing curve. Moreover, transversality of $\Lambda(t)$ and $\Lambda^\circ(t)$ implies that $S_{\Lambda^\circ(t)} - S_{\Lambda(t)}$ is a nondegenerate matrix. Hence $0 < S_{\Lambda(t)} < S_{\Lambda^\circ(t)} \leq I$ for any $t > \tau_0$. In particular, $\Lambda(t)$ never leaves the coordinate neighborhood under consideration for $T > \tau_0$, the subspace $\Lambda(t)$ is always transversal to $\Lambda(\tau_0)$ and has a limit Λ_∞, where $S_{\Lambda_\infty} = \sup\limits_{t\geq\tau_0} S_{\Lambda(t)}$. $\square$

Now assume that the eigenvalues of $R_\Lambda(t)$ do not exceed a constant $c > 0$. We are going to reparametrize the the curve $\Lambda(\cdot)$ and to use the chain rule (7). Take some $\bar{t}$ in the domain of $\Lambda(\cdot)$ and set

$$\varphi(t) = \frac{1}{\sqrt{c}}\left(\arctan(\sqrt{c}t) + \frac{\pi}{2}\right) + \bar{t}, \quad \Lambda_\varphi(t) = \Lambda(\varphi(t)).$$

We have: $\varphi(\mathbb{R}) = \left(\bar{t}, \bar{t} + \frac{\pi}{\sqrt{c}}\right)$, $\dot{\varphi}(t) = \frac{1}{ct^2+1}$, $R_\varphi(t) = -\frac{c}{(ct^2+1)^2}$. Hence, according to the chain rule (7), the operator

$$R_{\Lambda_\varphi}(t) = \frac{1}{(ct^2+1)^2}\left(R_\Lambda(\varphi(t)) - cI\right)$$

has only nonpositive eigenvalues. Already proved part of the theorem implies that Λ_φ does not possess conjugate values of the parameter. In other words, any length $\frac{\pi}{\sqrt{c}}$ interval in the domain of $\Lambda(\cdot)$ is free of conjugate pairs of the parameter values.

Assume now that $\mathrm{tr}R_\Lambda(t) \geq nc$. We will prove that the existence of $\Delta \in L(\Sigma)$ such that $\Delta \cap \Lambda(t) = 0$ for all $t \in [\bar{t}, \tau]$ implies that $\tau - \bar{t} < \frac{\pi}{\sqrt{c}}$. We'll prove it by contradiction. If there exists such a Δ, then $\Lambda\big|_{[\bar{t},\tau]}$ is completely contained in a fixed coordinate neighborhood of $L(\Sigma)$, therefore the curvature operator $R_\Lambda(t)$ is defined by the formula (6). Put $B(t) = (2\dot{S}_t)^{-1}\ddot{S}_t$, $b(t) = \mathrm{tr}B(t)$, $t \in [\bar{t}, \tau]$. Then

$$\dot{B}(t) = B^2(t) + R_\Lambda(t), \quad \dot{b}(t) = \mathrm{tr}B^2(t) + \mathrm{tr}R_\Lambda(t).$$

Since for an arbitrary symmetric $n \times n$-matrix A we have $\mathrm{tr}A^2 \geq \frac{1}{n}(\mathrm{tr}A)^2$, the inequality $\dot{b} \geq \frac{b^2}{n} + nc$ holds. Hence $b(t) \geq \beta(t)$, $\bar{t} \leq t \leq \tau$, where $\beta(\cdot)$ is a solution of the equation $\dot{\beta} = \frac{\beta^2}{n} + nc$, i.e. $\beta(t) = n\sqrt{c}\tan(\sqrt{c}(t - t_0))$. The function $b(\cdot)$ together with $\beta(\cdot)$ are bounded on the segment $[\bar{t}, \tau]$. Hence $\tau - t \leq \frac{\pi}{\sqrt{c}}$.

To verify that the estimates are sharp, it is enough to consider regular monotone curves of constant curvature. $\square$

20 Reduction

We consider a Hamiltonian system on a symplectic manifold N endowed with a fixed Lagrange foliation E. Assume that $g : N \to \mathbb{R}$ is a first integral of our Hamiltonian system, i.e. $\{h, g\} = 0$.

Lemma II.12 *Let $z \in N$, $g(z) = c$. The leaf E_z is transversal to $g^{-1}(c)$ at z if and only if $\vec{g}(z) \notin T_zE_z$.*

Proof. Hypersurface $g^{-1}(c)$ is not transversal to $g^{-1}(c)$ at z if and only if

$$d_zg(T_zE_z) = 0 \;\Leftrightarrow\; \sigma(\vec{g}(z), T_zE_z) = 0 \;\Leftrightarrow\; \vec{g}(z) \in (T_zE_z)^\angle = T_zE_z. \quad \square$$

If all points of some level $g^{-1}(c)$ satisfy conditions of Lemma II.12, then $g^{-1}(c)$ is a (2n-1)-dimensional manifold foliated by $(n-1)$-dimensional submanifolds $E_z \cap g^{-1}(c)$. Note that $\mathbb{R}\vec{g}(z) = \ker \sigma\big|_{T_zg^{-1}(c)}$, hence

$\Sigma^g_z \stackrel{def}{=} T_z g^{-1}(c)/\mathbb{R}\vec{g}(z)$ is a $2(n-1)$-dimensional symplectic space and $\Delta^g_z \stackrel{def}{=} T_z\left(E_z \cap g^{-1}(c)\right)$ is a Lagrangian subspace in L^g_z, i.e. $\Delta^g_z \in L(\Sigma^g_z)$.

The submanifold $g^{-1}(c)$ is invariant for the flow $e^{t\vec{h}}$. Moreover, $e^{t\vec{h}}_*\vec{g} = \vec{g}$. Hence $e^{t\vec{h}}_*$ induces a symplectic transformation $e^{t\vec{h}}_* : \Sigma^g_z \to \Sigma^g_{e^{t\vec{h}}(z)}$. Set $J^g_z(t) = e^{-t\vec{h}}_*\Delta^g_{e^{t\vec{h}}(z)}$. The curve $t \mapsto J^g_z(t)$ in the Lagrange Grassmannian $L(\Sigma^g_z)$ is called a *reduced Jacobi curve* for the Hamiltonian field $\vec{h}$ at $z \in N$.

The reduced Jacobi curve can be easily reconstructed from the Jacobi curve $J_z(t) = e^{-t\vec{h}}_*\left(T_{e^{t\vec{h}}(z)}E_{e^{t\vec{h}}(z)}\right) \in L(T_zN)$ and vector $\vec{g}(z)$. An elementary calculation shows that

$$J^g_z(t) = J_z(t) \cap \vec{g}(z)^{\angle} + \mathbb{R}\vec{g}(z).$$

Now we can temporary forget the symplectic manifold and Hamiltonians and formulate everything in terms of the curves in the Lagrange Grassmannian. So let $\Lambda(\cdot)$ be a smooth curve in the Lagrange Grassmannian $L(\Sigma)$ and γ a one-dimensional subspace in Σ. We set $\Lambda^\gamma(t) = \Lambda(t) \cap \gamma^{\angle} + \gamma$, a Lagrange subspace in the symplectic space $\gamma^{\angle}/\gamma$. If $\gamma \not\subset \Lambda(t)$, then $\Lambda^\gamma(\cdot)$ is smooth and $\underline{\dot{\Lambda}}^\gamma(t) = \underline{\dot{\Lambda}}(t)\big|_{\Lambda(t)\cap\gamma^{\angle}}$ as it easily follows from the definitions. In particular, monotonicity of $\Lambda(\cdot)$ implies monotonicity of $\Lambda^\gamma(\cdot)$; if $\Lambda(\cdot)$ is regular and monotone, then $\Lambda^\gamma(\cdot)$ is also regular and monotone. The curvatures and the Maslov indices of $\Lambda(\cdot)$ and $\Lambda^\gamma(\cdot)$ are related in a more complicated way. The following result is proved in [9].

Theorem II.2 *Let $\Lambda(t),\ t \in [t_0, t_1]$ be a smooth monotone increasing curve in $L(\Sigma)$ and γ a one-dimensional subspace of Σ such that $\gamma \not\subset \Lambda(t),\ \forall t \in [t_0, t_1]$. Let $\Pi \in L(\Sigma),\ \gamma \not\subset \Pi,\ \Lambda(t_0) \cap \Pi = \Lambda(t_1) \cap \Pi = 0$. Then*

- $\mu_\Pi(\Lambda(\cdot)) \le \mu_{\Pi^\gamma}(\Lambda^\gamma(\cdot)) \le \mu_\Pi(\Lambda(\cdot)) + 1$.
- *If $\Lambda(\cdot)$ is regular, then $r_{\Lambda^\gamma}(t) \ge r_\Lambda(t)\big|_{\Lambda(t)\cap\gamma^{\angle}}$ and*
  $\mathrm{rank}\left(r_{\Lambda^\gamma}(t) - r_\Lambda(t)\big|_{\Lambda(t)\cap\gamma^{\angle}}\right) \le 1$.

The inequality $r_{\Lambda^\gamma}(t) \ge r_\Lambda(t)\big|_{\Lambda(t)\cap\gamma^{\angle}}$ turns into the equality if $\gamma \subset \Lambda^\circ(t),\ \forall t$. Then $\gamma \subset \ker \underline{\dot{\Lambda}}^\circ(t)$. According to Proposition II.9, to γ there corresponds a one-dimensional subspace in the kernel of $r_\Lambda(t)$; in particular, $r_\Lambda(t)$ is degenerate.

Return to the Jacobi curves $J_z(t)$ of a Hamiltonian field $\vec{h}$. There always exists at least one first integral: the Hamiltonian h itself. In general, $\vec{h}(z) \notin J_z^\circ(0)$ and the reduction procedure has a nontrivial influence on the curvature (see [8, 9] for explicit expressions). Still, there is an important class of Hamiltonians and Lagrange foliations for which the relation $\vec{h}(z) \in J_z^\circ(0)$ holds $\forall z$. These are homogeneous on fibers Hamiltonians on cotangent bundles. In this case the generating homotheties of the fibers Euler vector field belongs to the kernel of the curvature form.

21 Hyperbolicity

Definition. We say that a Hamiltonian function h on the symplectic manifold N is regular with respect to the Lagrange foliation E if the functions $h\big|_{E_z}$ have nondegenerate second derivatives at z, $\forall z \in N$ (second derivative is well-defined due to the canonical affine structure on E_z). We say that h is monotone with respect to E if $h\big|_{E_z}$ is a convex or concave function $\forall z \in N$.

Typical examples of regular monotone Hamiltonians on the cotangent bundles are energy functions of natural mechanical systems. Such a function is the sum of the kinetic energy whose Hamiltonian system generates the Riemannian geodesic flow and a "potential" that is a constant on the fibers function. Proposition II.8 implies that Jacobi curves associated to the regular monotone Hamiltonians are also regular and monotone. We'll show that negativity of the curvature operators of such a Hamiltonian implies the hyperbolic behavior of the Hamiltonian flow. This is a natural extension of the classical result about Riemannian geodesic flows.

Main tool is the structural equation derived in Section 13. First we'll show that this equation is well coordinated with the symplectic structure. Let $\Lambda(t)$, $t \in \mathbb{R}$, be a regular curve in $L(\Sigma)$ and $\Sigma = \Lambda(t) \oplus \Lambda^\circ(t)$ the correspondent canonical splitting. Consider the structural equation

$$\ddot{e}(t) + R_\Lambda(t)e(t) = 0, \quad \text{where } e(t) \in \Lambda(t),\ \dot{e}(t) \in \Lambda^\circ(t), \tag{23}$$

(see Corollary II.1).

Lemma II.13 *The mapping $e(0) \oplus \dot{e}(0) \mapsto e(t) \oplus \dot{e}(t)$, where $e(\cdot)$ and $\dot{e}(\cdot)$ satisfies (23), is a symplectic transformation of Σ.*

Proof. We have to check that $\sigma(e_1(t), e_2(t)),\ \sigma(\dot e_1(t), \dot e_2(t)),\ \sigma(e_1(t), \dot e_2(t))$ do not depend on t as soon as $e_i(t), \dot e_i(t),\ i = 1, 2$, satisfy (23). First two quantities vanish since $\Lambda(t)$ and $\Lambda^\circ(t)$ are Lagrangian subspaces. The derivative of the third quantity vanishes as well since $\ddot e_i(t) \in \Lambda(t)$. $\square$

Let h be a regular monotone Hamiltonian on the symplectic manifold N equipped with a Lagrange foliation E. As before, we denote by $J_z(t)$ the Jacobi curves of $\vec h$ and by $J_z^h(t)$ the reduced to the level of h Jacobi curves (see previous Section). Let $R(z) = R_{J_z}(0)$ and $R^h(z) = R_{J_z^h}(0)$ be the curvature operators of $J_z(\cdot)$ and $J_z^h(\cdot)$ correspondently. We say that the Hamiltonian field $\vec h$ has a negative curvature at z with respect to E if all eigenvalues of $R(z)$ are negative. We say that $\vec h$ has a negative reduced curvature at z if all eigenvalues of R_z^h are negative.

Proposition II.10 *Let $z_0 \in N,\ z_t = e^{t\vec h}(z)$. Assume that that $\overline{\{z_t : t \in \mathbb{R}\}}$ is a compact subset of N and that N is endowed with a Riemannian structure. If $\vec h$ has a negative curvature at any $z \in \overline{\{z_t : t \in \mathbb{R}\}}$, then there exists a constant $\alpha > 0$ and a splitting $T_{z_t}N = \Delta_{z_t}^+ \oplus \Delta_{z_t}^-$, where $\Delta_{z_t}^\pm$ are Lagrangian subspaces of $T_{z_t}N$ such that $e_*^{\tau\vec h}(\Delta_{z_t}^\pm) = \Delta_{z_{t+\tau}}^\pm\ \forall\, t, \tau \in \mathbb{R}$ and*

$$\|e_*^{\pm\tau\vec h}\zeta_\pm\| \geq e^{\alpha\tau}\|\zeta_\pm\| \quad \forall\, \zeta \in \Delta_{z_t}^\pm,\ \tau \geq 0,\ t \in \mathbb{R}. \tag{24}$$

Similarly, if $\vec h$ has a negative reduced curvature at any $z \in \overline{\{z_t : t \in \mathbb{R}\}}$, then there exists a splitting $T_{z_t}(h^{-1}(c)/\mathbb{R}h(z_t)) = \hat\Delta_{z_t}^+ \oplus \hat\Delta_{z_t}^-$, where $c = h(z_0)$ and $\hat\Delta_{z_t}^\pm$ are Lagrangian subspaces of $T_{z_t}(h^{-1}(c)/\mathbb{R}h(z_t))$ such that $e_*^{\tau\vec h}(\hat\Delta_{z_t}^\pm) = \hat\Delta_{z_{t+\tau}}^\pm\ \forall\, t, \tau \in \mathbb{R}$ and $\|e_*^{\pm\tau\vec h}\zeta_\pm\| \geq e^{\alpha\tau}\|\zeta_\pm\| \quad \forall\, \zeta \in \hat\Delta_{z_t}^\pm,\ \tau \geq 0,\ t \in \mathbb{R}$.

Proof. Obviously, the desired properties of $\Delta_{z_t}^\pm$ and $\hat\Delta_{z_t}^\pm$ do not depend on the choice of the Riemannian structure on N. We'll introduce a special Riemannian structure determined by h. The Riemannian structure is a smooth family of inner products $\langle\cdot,\cdot\rangle_z$ on T_zN, $z \in N$. We have $T_zN = J_z(0) \oplus J_z^\circ(0)$, where $J_z(0) = T_zE_z$. Replacing h with $-h$ if necessary we may assume that $h\big|_{E_z}$ is a strongly convex function. First we define $\langle\cdot,\cdot\rangle_z\big|_{J_z(0)}$ to be equal to the second derivative of $h\big|_{E_z}$. Symplectic form σ induces a nondegenerate pairing of $J_z(0)$ and $J_z^\circ(0)$. In particular, for any $\zeta \in J_z(0)$ there exists a unique $\zeta^\circ \in J_z^\circ(0)$ such that $\sigma(\zeta^\circ,\cdot)\big|_{J_z(0)} = \langle\zeta,\cdot\rangle_z\big|_{J_z(0)}$. There exists a unique extension of the inner product $\langle\cdot,\cdot\rangle_z$ from $J_z(0)$ to the whole T_zN with the following properties:

- $J_z^\circ(0)$ is orthogonal to $J_z(0)$ with respect to $\langle\cdot,\cdot\rangle_z$;
- $\langle\zeta_1,\zeta_2\rangle_z = \langle\zeta_1^\circ,\zeta_2^\circ\rangle_z\ \forall\,\zeta_1,\zeta_2\in J_z(0)$.

We'll need the following classical fact from Hyperbolic Dynamics (see, for instance, [12, Sec. 17.6]).

Lemma II.14 *Let $A(t)$, $t\in\mathbb{R}$, be a bounded family of symmetric $n\times n$-matrices whose eigenvalues are all negative and uniformly separated from 0. Let $\Gamma(t,\tau)$ be the fundamental matrix of the $2n$-dimensional linear system $\dot{x}=-y$, $\dot{y}=A(t)x$, where $x,y\in\mathbb{R}^n$, i.e.*

$$\frac{\partial}{\partial t}\Gamma(t,\tau)=\left(\begin{smallmatrix}0 & -I\\ A & 0\end{smallmatrix}\right)\Gamma(t,\tau),\quad \Gamma(\tau,\tau)=\left(\begin{smallmatrix}I & 0\\ 0 & I\end{smallmatrix}\right). \tag{25}$$

Then there exist closed conic neighborhoods C_Γ^+, C_Γ^-, where $C_\Gamma^+\cap C_\Gamma^-=0$, of some n-dimensional subspaces of $\mathbb{R}^{2n}$ and a constant $\alpha>0$ such that

$$\Gamma(t,\tau)C_\Gamma^+\subset C_\Gamma^+,\quad |\Gamma(t,\tau)\xi_+|\geq e^{\alpha(\tau-t)}|\xi_+|,\ \forall\,\xi_+\in C_\Gamma^+,\,t\leq\tau,$$

and

$$\Gamma(t,\tau)C_\Gamma^-\subset C_\Gamma^-,\quad |\Gamma(t,\tau)\xi_-|\geq e^{\alpha(t-\tau)}|\xi_-|,\ \forall\,\xi_-\in C_\Gamma^-,\,t\geq\tau.$$

The constant α depends only on upper and lower bounds of the eigenvalues of $A(t)$. $\square$

Corollary II.6 *Let $C_\Gamma^\pm$ be the cones described in Lemma II.14; then $\Gamma(0,\pm t)C_\Gamma^\pm\subset\Gamma(0;\pm\tau)C_\Gamma^\pm$ for any $t\geq\tau\geq 0$ and the subsets $K_\Gamma^\pm=\bigcap\limits_{t\geq 0}\Gamma(0,t)C_\Gamma^\pm$ are Lagrangian subspaces of $\mathbb{R}^n\times\mathbb{R}^n$ equipped with the standard symplectic structure.*

Proof. The relations $\Gamma(\tau,t)C_\Gamma^+\subset C_\Gamma^+$ and $\Gamma(\tau,t)C_\Gamma^-\subset C_\Gamma^-$ imply:

$$\Gamma(0,\pm t)C_\Gamma^\pm=\Gamma(0,\pm\tau)\Gamma(\pm\tau,\pm t)C_\Gamma^\pm\subset\Gamma(0,\pm\tau)C_\Gamma^\pm.$$

In what follows we'll study K_Γ^+; the same arguments work for K_Γ^-. Take vectors $\zeta,\zeta'\in K_\Gamma^+$; then $\zeta=\Gamma(0,t)\zeta_t$ and $\zeta'=\Gamma(0,t)\zeta_t'$ for any $t\geq 0$ and some $\zeta_t,\zeta_t'\in C_\Gamma^+$. Then, according to Lemma II.14, $|\zeta_t|\leq e^{-\alpha t}|\zeta|$, $|\zeta_t'|\leq e^{-\alpha t}|\zeta'|$, i.e. ζ_t and ζ_t' tend to 0 as $t\to+\infty$. On the other hand,

$$\sigma(\zeta,\zeta')=\sigma(\Gamma(0,t)\zeta_t,\Gamma(0,t)\zeta_t')=\sigma(\zeta_t,\zeta_t')\quad\forall t\geq 0$$

since $\Gamma(0,t)$ is a symplectic matrix. Hence $\sigma(\zeta,\zeta')=0$.

We have shown that K_Γ^+ is an isotropic subset of $\mathbb{R}^n\times\mathbb{R}^n$. On the other hand, K_Γ^+ contains an n-dimensional subspace since C_Γ^+ contains one and $\Gamma(0,t)$ are invertible linear transformations. Isotropic n-dimensional subspace is equal to its skew-orthogonal complement, therefore K_Γ^+ is a Lagrangian subspace. $\square$

Take now a regular monotone curve $\Lambda(t)$, $t\in\mathbb{R}$ in the Lagrange Grassmannian $L(\Sigma)$. We may assume that $\Lambda(\cdot)$ is monotone increasing, i.e. $\dot{\Lambda}(t)>0$. Recall that $\underline{\dot{\Lambda}}(t)(e(t))=\sigma(e(t),\dot{e}(t))$, where $e(\cdot)$ is an arbitrary smooth curve in Σ such that $e(\tau)\in\Lambda(\tau)$, $\forall\tau$. Differentiation of the identity $\sigma(e_1(\tau),e_2(\tau))=0$ implies: $\sigma(e_1(t),\dot{e}_2(t))=-\sigma(\dot{e}_1(t),e_2(t))=\sigma(e_2(t),\dot{e}_1(t))$ if $e_i(\tau)\in\Lambda(\tau)$, $\forall\tau$, $i=1,2$. Hence the Euclidean structure $\langle\cdot,\cdot\rangle_{\dot{\Lambda}(t)}$ defined by the quadratic form $\underline{\dot{\Lambda}}(t)$ reads: $\langle e_1(t),e_2(t)\rangle_{\dot{\Lambda}(t)}=\sigma(e_1(t),\dot{e}_2(t))$.

Take a basis $e_1(0),\ldots,e_n(0)$ of $\Lambda(0)$ such that the form $\underline{\dot{\Lambda}}(t)$ has the unit matrix in this basis, i.e. $\sigma(e_i(0),\dot{e}_j(0))=\delta_{ij}$. In fact, vectors $\dot{e}_j(0)$ are defined modulo $\Lambda(0)$; we can normalize them assuming that $\dot{e}_i(0)\in\Lambda^\circ(0)$, $i=1,\ldots,n$. Then $e_1(0),\ldots,e_n(0),\dot{e}_1(0),\ldots,\dot{e}_n(0)$ is a Darboux basis of Σ. Fix coordinates in Σ using this basis: $\Sigma=\mathbb{R}^n\times\mathbb{R}^n$, where $\left(\begin{smallmatrix}x\\ y\end{smallmatrix}\right)\in\mathbb{R}^n\times\mathbb{R}^n$ is identified with $\sum\limits_{j=1}^n\left(x^je_j(0)+y^j\dot{e}_j(0)\right)\in\Sigma$, $x=(x^1,\ldots,x^n)^\top$, $y=(y^1,\ldots,y^n)^\top$.

We claim that there exists a smooth family $A(t)$, $t\in\mathbb{R}$, of symmetric $n\times n$ matrices such that $A(t)$ has the same eigenvalues as $R_\Lambda(t)$ and

$$\Lambda(t)=\Gamma(0,t)\left(\begin{smallmatrix}\mathbb{R}^n\\ 0\end{smallmatrix}\right),\quad \Lambda^\circ(t)=\Gamma(0,t)\left(\begin{smallmatrix}0\\ \mathbb{R}^n\end{smallmatrix}\right),\quad \forall t\in\mathbb{R}$$

in the fixed coordinates, where $\Gamma(t,\tau)$ satisfies (25). Indeed, let $e_i(t)$, $i=1,\ldots,n$, be solutions to the structural equations (23). Then

$$\Lambda(t)=span\{e_1(t),\ldots,e_n(t)\},\quad \Lambda^\circ(t)=span\{\dot{e}_1(t),\ldots,\dot{e}_n(t)\}.$$

Moreover, $\ddot{e}_i(t)=-\sum\limits_{i=1}^n a_{ij}(t)e_j(t)$, where $A(t)=\{a_{ij}(t)\}_{i,j=1}^n$ is the matrix of the operator $R_\Lambda(t)$ in the 'moving' basis $e_1(t),\ldots,e_n(t)$. Lemma I.13 implies that $\langle e_i(t),e_j(t)\rangle_{\dot{\Lambda}(t)}=\sigma(e_i(t),\dot{e}_j(t))=\delta_{ij}$. In other words, the Euclidean structure $\langle\cdot,\cdot\rangle_{\dot{\Lambda}(t)}$ has unit matrix in the basis $e_1(t),\ldots,e_n(t)$. Operator $R_\Lambda(t)$ is self-adjoint for the Euclidean structure $\langle\cdot,\cdot\rangle_{\dot{\Lambda}(t)}$ (see Propositon II.9). Hence matrix $A(t)$ is symmetric.

Let $e_i(t) = \left(\begin{smallmatrix} x_i(t) \\ y_i(t) \end{smallmatrix}\right) \in \mathbb{R}^n \times \mathbb{R}^n$ in the fixed coordinates. Make up $n \times n$-matrices $X(t) = (x_1(t), \ldots, x_n(t))$, $Y(t) = (y_1(t), \ldots, y_n(t))$ and a $2n \times 2n$-matrix $\left(\begin{smallmatrix} X(t) & \dot{X}(t) \\ Y(t) & \dot{Y}(t) \end{smallmatrix}\right)$. We have

$$\frac{d}{dt}\begin{pmatrix} X & \dot{X} \\ Y & \dot{Y} \end{pmatrix}(t) = \begin{pmatrix} X & \dot{X} \\ Y & \dot{Y} \end{pmatrix}(t)\begin{pmatrix} 0 & -A(t) \\ I & 0 \end{pmatrix}, \quad \begin{pmatrix} X & \dot{X} \\ Y & \dot{Y} \end{pmatrix}(0) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Hence $\left(\begin{smallmatrix} X & \dot{X} \\ Y & \dot{Y} \end{smallmatrix}\right)(t) = \Gamma(t,0)^{-1} = \Gamma(0,t)$.

Let now $\Lambda(\cdot)$ be the Jacobi curve, $\Lambda(t) = J_{z_0}(t)$. Set $\xi_i(z_t) = e_*^{t\vec{h}} e_i(t)$, $\eta_i(z_t) = e_*^{t\vec{h}} \dot{e}_i(t)$; then

$$\xi_1(z_t), \ldots, \xi_n(z_t), \eta_1(z_t), \ldots, \eta_n(z_t) \tag{26}$$

is a Darboux basis of $T_{z_t}N$, where $J_{z_t}(0) = span\{\xi_1(z_t), \ldots, \xi_n(z_t)\}$, $J^\circ_{z_t}(0) = span\{\eta_1(z_t), \ldots, \eta_n(z_t)\}$. Moreover, the basis (26) is orthonormal for the inner product $\langle \cdot, \cdot \rangle_{z_t}$ on $T_{z_t}N$.

The intrinsic nature of the structural equation implies the translation invariance of the construction of the frame (26): if we would start from z_s instead of z_0 and put $\Lambda(t) = J_{z_s}(t)$, $e_i(0) = \xi_i(z_s)$, $\dot{e}_i(0) = \eta_i(z_s)$ for some $s \in \mathbb{R}$, then we would obtain $e_*^{t\vec{h}} e_i(t) = \xi_i(z_{s+t})$, $e_*^{t\vec{h}} \dot{e}_i(t) = \eta_i(z_{s+t})$.

The frame (26) gives us fixed orthonormal Darboux coordinates in $T_{z_s}N$ for $\forall\, s \in \mathbb{R}$ and the correspondent symplectic $2n \times 2n$-matrices $\Gamma_{z_s}(\tau, t)$. We have: $\Gamma_{z_s}(\tau, t) == \Gamma_{z_0}(s+\tau, s+t)$; indeed, $\Gamma_{z_s}(\tau, t)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ is the coordinate presentation of the vector

$$e_*^{(\tau - t)\vec{h}}\left(\sum_i x^i \xi^i(z_{s+t}) + y^i \eta_i(z_{s+t})\right)$$

in the basis $\xi_i(z_{s+\tau})$, $\eta_i(z_{s+\tau})$. In particular,

$$|\Gamma_{z_s}(0,t)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)| = \left\| e_*^{-t\vec{h}}\left(\sum_i x^i \xi^i(z_{s+t}) + y^i \eta_i(z_{s+t})\right)\right\|_{z_s}. \tag{27}$$

Recall that $\xi_1(z_\tau), \ldots, \xi_n(z_\tau), \eta_1(z_\tau), \ldots, \eta_n(z_\tau)$ is an orthonormal frame for the scalar product $\langle \cdot, \cdot \rangle_{z_\tau}$ and $\|\zeta\|_{z_\tau} = \sqrt{\langle \zeta, \zeta \rangle_{z_\tau}}$.

We introduce the notation :

$$\lfloor W \rfloor_{z_s} = \left\{ \sum_i x^i \xi^i(z_s) + y^i \eta_i(z_s) : \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in W \right\},$$

for any $W \subset \mathbb{R}^n \times \mathbb{R}^n$. Let $C^{\pm}_{\Gamma_{z_0}}$ be the cones from Lemma II.14. Then

$$e_*^{-\tau\vec{h}} \lfloor \Gamma_{z_s}(0,t) C^{\pm}_{\Gamma_{z_0}} \rfloor_{z_{s-\tau}} = \lfloor \Gamma_{z_{s-\tau}}(0,t+\tau) C^{\pm}_{\Gamma_{z_0}} \rfloor_{z_{s-\tau}}, \quad \forall\, t, \tau, s. \tag{28}$$

Now set $K^+_{\Gamma_{z_s}} = \bigcap\limits_{t\geq 0} C^+_{\Gamma_{z_0}}$, $K^-_{\Gamma_{z_s}} = \bigcap\limits_{t\leq 0} C^-_{\Gamma_{z_0}}$ and $\Delta^{\pm}_{z_s} = \lfloor K^{\mp}_{\Gamma_{z_s}} \rfloor_{z_s}$. Corollary II.6 implies that $\Delta^{\pm}_{z_s}$ are Lagrangian subspaces of $T_{z_s}N$. Moreover, it follows from (28) that $e_*^{t\vec{h}} \Delta^{\pm}_{z_s} = \Delta^{\pm}_{z_{s+t}}$, while (28) and (27) imply inequalities (24).

This finishes the proof of the part of Proposition II.10 which concerns Jacobi curves $J_z(t)$. We leave to the reader a simple adaptation of this proof to the case of reduced Jacobi curves $J^h_z(t)$. $\square$

Remark. Constant α depends, of course, on the Riemannian structure on N. In the case of the special Riemannian structure defined at the beginning of the proof of Proposition II.10 this constant depends only on the upper and lower bounds for the eigenvalues of the curvature operators and reduced curvature operators correspondently (see Lemma II.14 and further arguments).

Let e^{tX} $, t \in \mathbb{R}$ be the flow generated by the the vector field X on a manifold M. Recall that a compact invariant subset $W \subset M$ of the flow e^{tX} is called a hyperbolic set if there exists a Riemannian structure in a neighborhood of W, a positive constant α, and a splitting $T_zM = E^+_z \oplus E^-_z \oplus \mathbb{R}X(z)$, $z \in W$, such that $X(z) \neq 0$, $e_*^{tX} E^{\pm}_z = E^{\pm}_{e^{tX}(z)}$, and $\|e_*^{\pm tX}\zeta^{\pm}\| \geq e^{\alpha t}\|\zeta^{\pm}\|$, $\forall t \geq 0$, $\zeta^{\pm} \in E^{\pm}_z$. Just the fact some invariant set is hyperbolic implies a rather detailed information about asymptotic behavior of the flow in a neighborhood of this set (see [12] for the introduction to Hyperbolic Dynamics). The flow e^{tX} is called an Anosov flow if the entire manifold M is a hyperbolic set.

The following result is an immediate corollary of Proposition II.10 and the above remark.

Theorem II.3 *Let h be a regular monotone Hamiltonian on N, $c \in \mathbb{R}$, $W \subset h^{-1}(c)$ a compact invariant set of the flow $e^{t\vec{h}}$, $t \in \mathbb{R}$, and $d_zh \neq 0$, $\forall z \in W$. If $\vec{h}$ has a negative reduced curvature at every point of W, then W is a hyperbolic set of the flow $e^{t\vec{h}}\big|_{h^{-1}(c)}$.* $\square$

This theorem generalizes a classical result about geodesic flows on compact Riemannian manifolds with negative sectional curvatures. Indeed, if N is the cotangent bundle of a Riemannian a Riemannian manifold and $e^{t\vec{h}}$

is the geodesic flow, then negativity of the reduced curvature of $\vec{h}$ means simply negativity of the sectional Riemannian curvature. In this case, the Hamiltonian h is homogeneous on the fibers of the cotangent bundle and the restrictions $e^{t\vec{h}}\big|_{h^{-1}(c)}$ are equivalent for all $c > 0$.

The situation changes if h is the energy function of a general natural mechanical system on the Riemannian manifold. In this case, the flow and the reduced curvature depend on the energy level. Still, negativity of the sectional curvature implies negativity of the reduced curvature at $h^{-1}(c)$ for all sufficiently big c. In particular, $e^{t\vec{h}}\big|_{h^{-1}(c)}$ is an Anosov flow for any sufficiently big c; see [8, 9] for the explicit expression of the reduced curvature in this case.

Theorem II.3 concerns only the reduced curvature while the next result deals with the (not reduced) curvature of $\vec{h}$.

Theorem II.4 *Let h be a regular monotone Hamiltonian and W a compact invariant set of the flow $e^{t\vec{h}}$. If $\vec{h}$ has a negative curvature at any point of W, then W is a finite set and each point of W is a hyperbolic equilibrium of the field $\vec{h}$.*

Proof. Let $z \in W$; the trajectory $z_t = e^{t\vec{h}}(z),\ t \in \mathbb{R}$, satisfies conditions of Proposition II.10. Take the correspondent splitting $T_{z_t}N = \Delta^+_{z_t} \oplus \Delta^-_{z_t}$. In particular, $\vec{h}(z_t) = \vec{h}^+(z_t) + \vec{h}^-(z_t)$, where $\vec{h}^\pm(z_t) \in \Delta^\pm_{z_t}$.

We have $e^{\tau\vec{h}}_*\vec{h}(z_t) = \vec{h}(z_{t+\tau})$. Hence

$$\|\vec{h}(z_{t+\tau})\| = \|e^{\tau\vec{h}}_*\vec{h}(z_t)\| \geq \|e^{\tau\vec{h}}_*\vec{h}^+(z_t)\| - \|e^{\tau\vec{h}}_*\vec{h}^-(z_t)\|$$

$$\geq e^{\alpha\tau}\|\vec{h}^+(z_t)\| - e^{-\alpha\tau}\|\vec{h}^-(z_t)\|, \quad \forall \tau \geq 0.$$

Compactness of $\overline{\{z_t : t \in \mathbb{R}\}}$ implies that $\vec{h}^+(z_t)$ is uniformly bounded; hence $\vec{h}^+(z_t) = 0$. Similarly, $\|\vec{h}(z_{t-\tau}\| \geq e^{\alpha\tau}\|\vec{h}^-(z_t)\| - e^{-\alpha\tau}\|\vec{h}^+(z_t)\|$ that implies the equality $\vec{h}^-(z_t) = 0$. Finally, $\vec{h}(z_t) = 0$. In other words, $z_t \equiv z$ is an equilibrium of $\vec{h}$ and $T_zN = \Delta^+_z \oplus \Delta^-_z$ is the splitting of T_zN into the repelling and attracting invariant subspaces for the linearization of the flow $e^{t\vec{h}}$ at z. Hence z is a hyperbolic equilibrium; in particular, z is an isolated equilibrium of $\vec{h}$. $\square$

We say that a subset of a finite dimensional manifold is bounded if it has a compact closure.

Corollary II.7 *Assume that h is a regular monotone Hamiltonian and $\vec{h}$ has everywhere negative curvature. Then any bounded semi-trajectory of the system $\dot{z} = \vec{h}(z)$ converges to an equilibrium with the exponential rate while another semi-trajectory of the same trajectory must be unbounded.* $\square$

Typical Hamiltonians which satisfy conditions of Corollary II.7 are energy functions of natural mechanical systems in $\mathbb{R}^n$ with a strongly concave potential energy. Indeed, in this case, the second derivative of the potential energy is equal to the matrix of the curvature operator in the standard Cartesian coordinates (see Sec. 15).

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