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Lecture 2. Singularities of plane curves

Many thanks to Professor Andrei Agrachev for inviting me to SISSA and very useful for me math discussions, and many thanks to all listeners of my lectures for all questions and comments, and for the patience (the lectures were twice longer than scheduled). The notes of lecture 1 and lecture 3 (lecture 1: Singularity theory: ideology and main notions; lecture 3: How to compute mini-versal unfoldings? Versality theorem) will be published a bit later. Thanks a lot to Alesha Remizov who took all the job on scanning and publishing my hand-written notes.

Trieste, October 6, 2010

Lecture 2. Singularities of plane curves

(1)

1. If I ask you if the curve

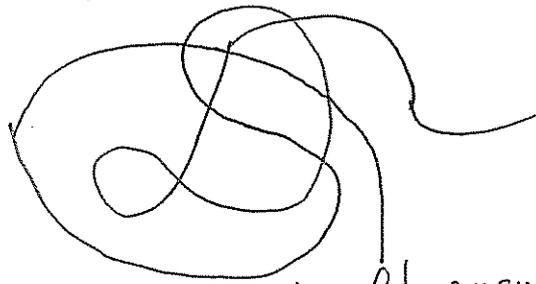
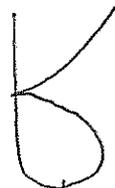
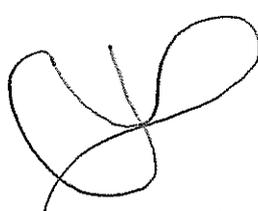
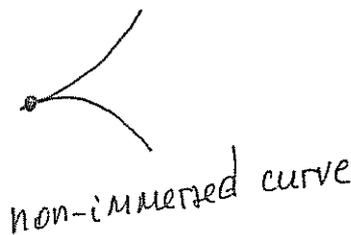
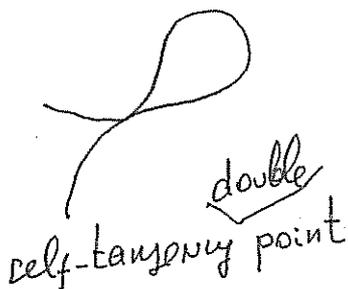
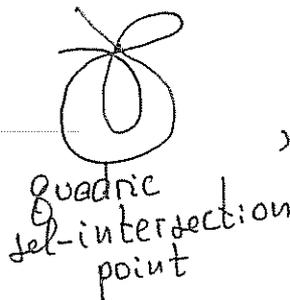
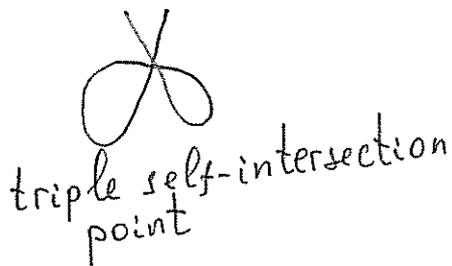


fig 1

is generic or not, you should answer that this question is wrong. A right question is as follows (since there are no generic objects, there are generic properties): find, in the space of all curves, an open and dense set (as small as possible) such that the given curve belongs to this set.

Answer: This open and dense set consists of immersed curves without self-intersection points and of immersed curves with double transversal self-intersection points; all other curves (see fig 2) are excluded



more degenerate cases

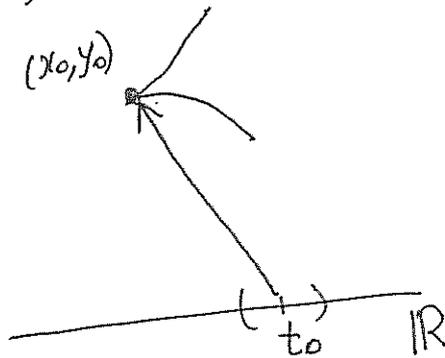
fig. 2

2. What is a ^{plane} curve? Our analysis is local, so the question is as follows: how to describe a curve near (in a small neighbourhood) of a fixed point

(2)

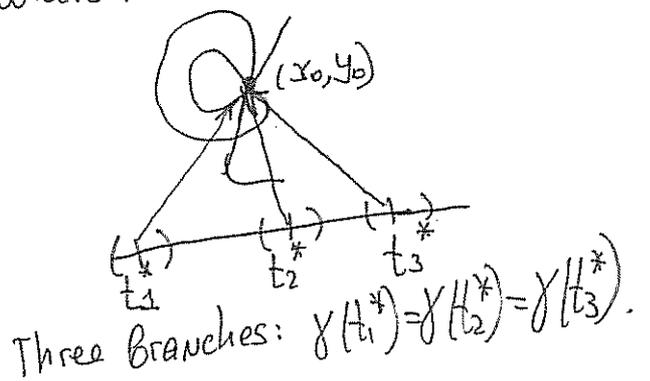


There are two classical (natural) ways. The first way is to describe a curve as the zero level of a function $\{ (x, y) : f(x, y) = 0 \}$, defined near a fixed point (x_0, y_0) .⁽¹⁾ The second way is to describe a curve as the image of a map $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$. If the curve has one branch then we need to know (for local analysis of the curve near a fixed point (x_0, y_0)) the map γ only in a neighbourhood of a point $t_0^* \in \mathbb{R}$ such that $\gamma(t_0^*) = (x_0, y_0)$.⁽²⁾

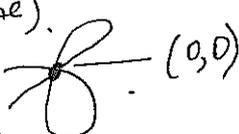


But if the curve has several branches intersecting at (x_0, y_0) then a single germ is not enough, we need a multi-germ, i.e. we need to know γ in a neighbourhood of N points $t_1^*, \dots, t_N^* \in \mathbb{R}$, where N is the number of branches:

- (1) i.e. the germ of f at (x_0, y_0)
- (2) i.e. the germ of γ at t_0^*



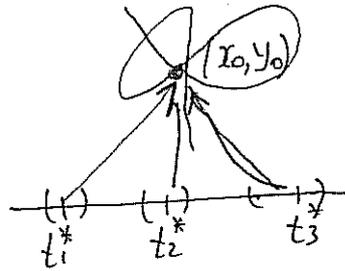
3. These two ways of describing a curve near a fixed point (as the zero level of a function and as a multi-germ) are, in analytic category, in 1-1 correspondence. Nevertheless, the theory of singularities depends on which of the two ways is chosen. The thing is that studying singularities requires distinguishing generic case, case of codim 1, 2, ..., versal unfoldings (see lecture 1), and all this stuff requires topology in the space of objects we deal with. If we describe a curve as the zero level of a function $f(x,y)$ then our objects are maps $\mathbb{R}^2 \rightarrow \mathbb{R}$, defined up to multiplication by a non-vanishing function. If we describe a curve by a multigerms then our objects are tuples of map germs $\gamma_1: \mathbb{R} \rightarrow \mathbb{R}^2, \dots, \gamma_N: \mathbb{R} \rightarrow \mathbb{R}^2$. The correspondence between the two ways of describing a plane curve does NOT respect natural topologies, and therefore the results depend on the chosen way. (Some of results, concerning cases of codim 0 ~~and 1~~, and some cases of higher codimension, are the same).

To see it, consider the case of triple point  (0,0). It is clear that if we deal with multigerms this case has codimension 1. On the other hand, if we describe the singularity by a function $f(x,y)$ then this function must have the form $f(x,y) = f_1(x,y) \cdot f_2(x,y) \cdot f_3(x,y)$ where $f_i(0,0) = 0$. The class of such functions has a very big codimension in the space of all functions because the Taylor series of $f(x,y)$ starts with terms of order ≥ 3 .

4. I will deal in this lecture with multi-germs, i.e. with parameterized curves.

5. What is the natural equivalence (local) for multigerms?

(4)



$$\begin{aligned} \gamma_1 &: \text{nbhd of } t_1^* \rightarrow \text{nbhd } (x_0, y_0) \\ \gamma_2 &: \text{nbhd of } t_2^* \rightarrow \text{nbhd } (x_0, y_0) \\ \gamma_3 &: \text{nbhd of } t_3^* \rightarrow \text{nbhd } (x_0, y_0) \end{aligned}$$

It is a local change of coordinates t_1, t_2, t_3 (defined near t_1^*, t_2^*, t_3^*) - reparameterization of the branches, and a local change of coordinates x, y near (x_0, y_0) . The reparameterization of the branches does not change the curve (more precisely, it does not change the image of a parameterized curve). The change of coordinates x, y near (x_0, y_0) (= local diffeomorphism of the (x, y) -plane) does not change the main qualitative properties of the curve. Therefore it is worth to deal with equivalence defined by these transformations:

a multi-germ $\rightarrow \gamma_1(t_1), \dots, \gamma_N(t_N)$

$$\begin{array}{ccc} x = x(t_1) & x = x(t_2) & \dots & x = x(t_N) \\ y = y(t_1) & y = y(t_2) & \dots & y = y(t_N) \end{array}$$

is equivalent to a multi-germ

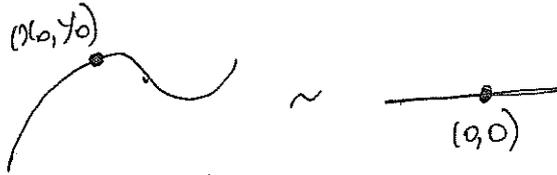
$$\begin{array}{ccc} x = \tilde{x}(t_1) & x = \tilde{x}(t_2) & \dots & x = \tilde{x}(t_N) \\ y = \tilde{y}(t_1) & y = \tilde{y}(t_2) & \dots & y = \tilde{y}(t_N) \end{array}$$

if $\Phi_1(x(\phi(t_i))) = \tilde{x}(t_i)$

$$\begin{aligned} \Psi_1(x(\phi(t_i)), y(\phi(t_i))) &= \tilde{x}(t_i) \\ \Psi_2(x(\phi(t_i)), y(\phi(t_i))) &= \tilde{y}(t_i), \quad i=1, \dots, N \end{aligned}$$

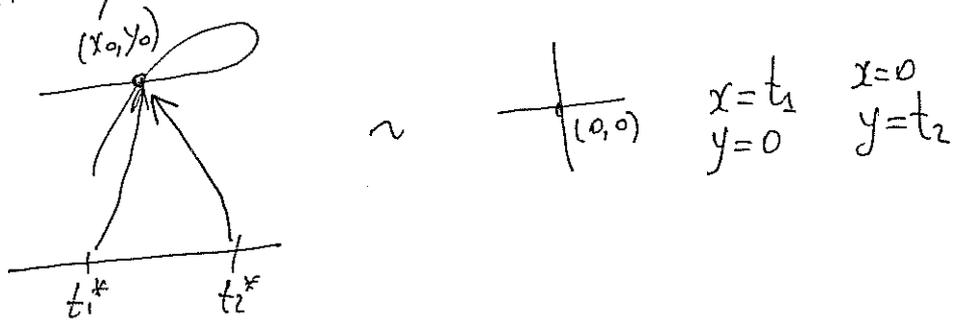
for some local diffeomorphism $(x, y) \rightarrow (\Psi_1(x, y), \Psi_2(x, y))$ and some local diffeomorphisms $t_i \rightarrow \phi(t_i)$.

6. We must start with the case of $\text{codim} = 0$.
It is clear that the germ of a smooth curve at an immersed non-self-intersection point



is equivalent to $x = t_1$
 $y = 0$.

The other part of our generic case is an immersed transversal self-intersection point



It is described by multigerms

$$\begin{aligned} x &= f_1(t_1) & x &= g_1(t_2) \\ y &= f_2(t_1) & y &= g_2(t_2) \end{aligned}$$

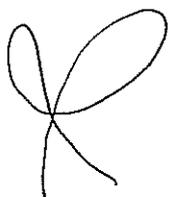
We may assume ⁽¹⁾ $t_1^* = t_2^* = 0$ (but t_1 and t_2 are different independent local coordinates), $(x_0, y_0) = (0, 0)$.
Since the branches are immersed, one has $(f_1'(0), f_2'(0)) \neq (0, 0)$, $(g_1'(0), g_2'(0)) \neq (0, 0)$ and we may assume ⁽¹⁾ $f_1'(0) \neq 0$. Then the first branch reduces to (= is equivalent to)

$$\begin{aligned} x &= t_1 & x &= \tilde{g}_1(t_2) \\ y &= 0 & y &= \tilde{g}_2(t_2) \end{aligned}$$

The transversality of the branches means $g_2'(0) \neq 0$, and it is very easy to get the normal form $x = t_1$; $x = 0$, $y = 0$; $y = t_2$.

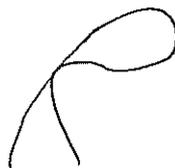
⁽¹⁾ may assume = if this is not so, there is an equivalent multigerms for which this is so.

7. Now we have a right to consider case of codimension 1. (6)
 It consists of three subcases:



triple transversal immersed self-intersection point

A



self-tangency point, immersed, with simple (order 1) tangency

B



non-immersed point satisfying certain genericity conditions (to be understood)

C

8. Case A The singularity is described by multigerms

$$\begin{array}{lll} x = f_1(t_1) & x = f_2(t_2) & x = f_3(t_3) \\ y = g_1(t_1) & y = g_2(t_2) & y = g_3(t_3) \end{array}$$

and again we assume that t_i are local coordinates near $0 \in \mathbb{R}$ and (x, y) are local coordinates near $(0, 0)$.

Using the normal form for the generic case we can reduce this multigerms to

$$\begin{array}{lll} x = t_1 & x = 0 & x = f(t_3) \\ y = 0 & y = t_2 & y = g(t_3) \end{array}$$

The fact that the third branch is immersed and transversal to the first two branches means $f'(0) \neq 0, g'(0) = 0$. Reparameterizing t_3 we get

a (preliminary) normal form

$$\begin{array}{lll} x = t_1 & x = 0 & x = t_3 \\ y = 0 & y = t_2 & y = h(t_3), \end{array} \quad h(t_3) = \lambda t_3 + o(t_3)$$

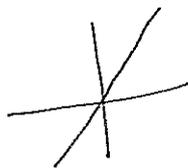
Simple exercise

Simple exercise 1

- 1) Show that λ can be reduced to 1 by scaling (some of) the coordinates x, y, t_1, t_2, t_3 .
- 2) Prove that $0(t_3)$ can be "rilled" by a change of (some of) the coordinates x, y, t_1, t_2, t_3 .

Thus the normal form for case A is

$$\begin{array}{l} x = t_1 \\ y = 0 \end{array} \quad \begin{array}{l} x = 0 \\ y = t_2 \end{array} \quad \begin{array}{l} x = t_3 \\ y = t_3 \end{array} \quad ;$$



Since case A is a codim 1 case, we need, except normal form, a versal deformation (see lect. 1) which contains 1 parameter (since codim = 1).

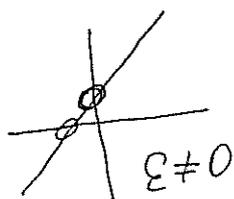
Exercise 2 Prove (in the same way as exercise 1)

that

$$\begin{array}{l} x = t_1 \\ y = 0 \end{array} \quad \begin{array}{l} x = 0 \\ y = t_2 \end{array} \quad \begin{array}{l} x = t_3 \\ y = t_3 + \epsilon \end{array}$$

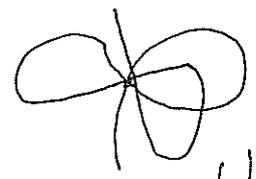
is a versal deformation.

For $\epsilon \neq 0$ the singularity "decomposes" into two generic singularities, two transversal self-intersection immersed points:



9. Before considering codim 1 case B,
 consider codim 2 case of quadric
self-intersection transversal immersed point.

(*)



Do we have a right to consider this case
 before considering cases B and C of codim 1?
 Yes, because our results will concern open set
 of curves, though not dense. If each of our
 results, along with previously obtained results,
 concerns open set of objects - we are OK (density
 is in general not necessary).

10. So, what happens for case (*)? It seems we can
 put a "good" normal form arriving as in Exercise 1,
 but this is not exactly so.

Exercise 3

1). Obtain a preliminary normal form
 $x = t_1$ $x = 0$ $x = t_3$ $x = t_4$
 $y = 0$ $y = t_2$ $y = t_3$ $y = f(t_4)$,

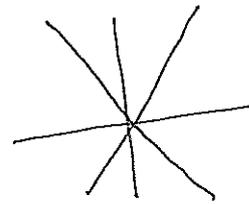
$f(t_4) = \lambda t_4 + o(t_4), \quad \lambda \neq 0, \lambda \neq 1.$

2). Check that λ cannot be changed by
 scaling the coordinates x, y, t_i .

3). If you do not know what is the cross-ratio invariant (in the classification of 4 lines through 0 in $\mathbb{R}^2 = 4$ points in $\mathbb{R}P^1$) - learn it, and prove that in the classification of curves under consideration λ is an invariant. (9)

4). Prove that λ is a complete invariant, i.e. $o(t_4)$ can be killed. Therefore one has exact normal form

$$\begin{array}{cccc} x=t_1 & x=0 & x=t_3 & x=t_4 \\ y=0 & y=t_2 & y=t_3 & y=\lambda t_4 \end{array}$$



5). Prove that ^{one of} the versal deformation of this normal form is

$$\begin{array}{cccc} x=t_1 & x=0 & x=t_3 & x=t_4 \\ y=0 & y=t_2 & y=t_3 & y=(\lambda+\epsilon_1)t_4 + \epsilon_2. \end{array}$$

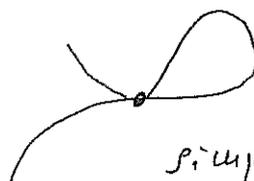
Problem 4 Try to obtain classification and versal deformations for the case of transversal immersed self-intersection point with 5 (or more) branches.

11. Now let us consider ^{immersed} double self-tangency points

(10)



tangency of order 3
(like x^4):
topologically the same as simple tangency, but: not smoothly versal deformations are different!



simple tangencies (order 1)



tangency of order 2 (like graph of x^3 is tangent to the x-axis).

It is described by multiform
 $x = t_1 \quad x = t_2 \quad f(t_2) = t_2^N + o(t_2^N)$
 $y = 0 \quad y = f(t_2), \quad N \geq 2$
 (obvious preliminary normal form). Here the order of tangency is $N-1$.

One of the basic results of singularity theory, used in almost all problems of singularity theory, is the normal form of function $f(x) = x^N + o(x^N)$ of one variable (which is easy) and the versal deformation of this normal form (which is hard; one needs "general" difficult theorem, discussed below).

This result is as follows.

(11)

Theorem. Any function germ $f(x)$ at the point $x_0=0$ of the form $f(x) = x^N + o(x^N)$ can be reduced by a change of the coordinate $x \in \mathbb{R}^1$ to x^N (i.e. $o(x^N)$ can be "killed") and the versal deformation of this normal form (with respect to change of coordinates) is

$$x^N + \epsilon_1 + \epsilon_2 x + \dots + \epsilon_{N-1} x^{N-2}$$

Note that the number of parameters of the versal deformation, $N-1$, is correct since the set of ~~functions $g(x) = x^N + o(x^N)$, $g_N \neq 0$~~ singularity $f(x) = x^N + o(x^N)$ represents the case

$$\exists x_0 : f'(x_0) = f''(x_0) = \dots = f^{(N-1)}(x_0) = 0$$

$$f(x_0) = 0$$

which is N equations for one unknown x_0 , \Rightarrow this case has codimension $N-1$.

Corollary. The singularity of a parameterized curve at an immersed transversal double order N tangency point of its image is described by the normal form

$$\begin{array}{l} x = t_1 \\ y = 0 \end{array} \quad \begin{array}{l} x = t_2 \\ y = t_2^{N+1} \end{array}$$

and its versal deformation is

$$\begin{array}{l} x = t_1 \\ y = 0 \end{array} \quad \begin{array}{l} x = t_2 \\ y = \epsilon_1 + \epsilon_2 t_2 + \epsilon_3 t_2^2 + \dots + \epsilon_N t_2^{N-1} \end{array}$$

I do not have time to discuss bifurcation diagrams

(how the points of local max and local min meet "paste" and "kill one the other" or "go through" when we move by a certain curve in the \mathbb{E} -space, and decomposing the \mathbb{E} -space on domains corresponding to various numbers of local max and min). I will discuss bifurcation diagram for other singularities: non-immersed singularities.

12. The next (and the last) case in this lecture is non-immersed singularity with one branch

$$\begin{array}{l} \nearrow \\ (0,0) \\ \searrow \end{array} \quad \begin{array}{l} x = f(t) \\ y = g(t) \end{array} \quad f'(0) = g'(0) = 0.$$

The normal form and versal deformation depends on terms of degree ≥ 2 in the Taylor series of $f(t)$ and $g(t)$. Let us assume

(a) $(f''(0), g''(0)) \neq (0, 0).$

Then (using the theorem at page 11) it is easy to ~~get~~ obtain the preliminary normal form

$$\begin{array}{l} x = t^2 \\ y = h(t), \quad h(0) = h'(0) = h''(0) = 0. \end{array}$$

Let us make one more assumption, in terms of this preliminary normal form

(b) $h'''(0) \neq 0.$

Then, using the fact that

(*) any smooth (C^∞) function $h(t)$ can be decomposed into even and odd part: $h(t) = h_1(t^2) + t h_2(t^2)$, h_i are functions of one variable,

and making a change of coordinates

(12)

$x \rightarrow x, y \rightarrow y - \varphi(x)$
with a suitable function $\varphi(x)$, it is easy to
get the normal form

$$\begin{aligned}x &= t^2 \\ y &= t^3 f(t^2), f(0) \neq 0.\end{aligned}$$

Now, introduce a new coordinate

$$\tilde{y} = \frac{y}{f(x)}$$

(well-defined since $f(0) \neq 0$) we obtain the
final normal form

$$x = t^2, y = t^3$$

(called cusp or semi-cubic parabola).

Our result requires describing the genericity
assumptions (a) and (b) in "canonical" way
(for that we need the space of jets, jet-extension of
a map which is also the main ingredient of
Thom transversality theorem) or, at least,
we have to prove that (a) and (b) do not
depend on the choice of coordinates (i.e. if
they hold for some curve then they hold for
any equivalent curve). The latter task is easy
(the first task is not).

A difficult part is not the normal form $(t^2 | t^3)$
but its versal deformation. We are in case of
codimension 1 (a) and (b) ~~are~~ are "open"
assumptions), therefore any

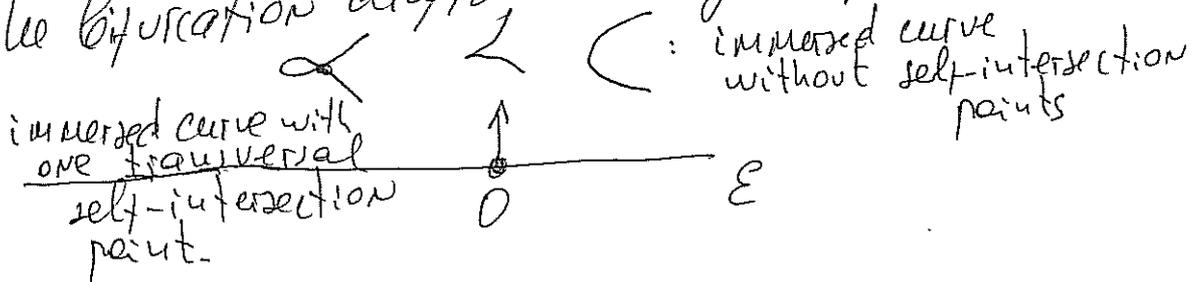
versal deformation has one parameter.

(14)

Theorem ^{One of} the versal deformation of the cusp (t^2, t^3)

is $x = t^2$
 $y = t^3 + \epsilon t$.

The bifurcation diagram is very simple:



13. Now let us refuse from genericity assumption (b), but let us preserve genericity assumption (a), page 12, so that

$$x(t) = t^2$$

$$y(t) = a_4 t^4 + a_5 t^5 + \dots$$

Obviously a_4 can be reduced to 0:

$$x(t) = t^2, \quad y(t) = a_5 t^5 + o(t^5)$$

let us make genericity assumption in terms of this (preliminary) normal form:

(c) $a_5 \neq 0$.

Exercise 5 Prove that in this case one has the normal form $x = t^2, y = t^5$ (degenerate cusp)

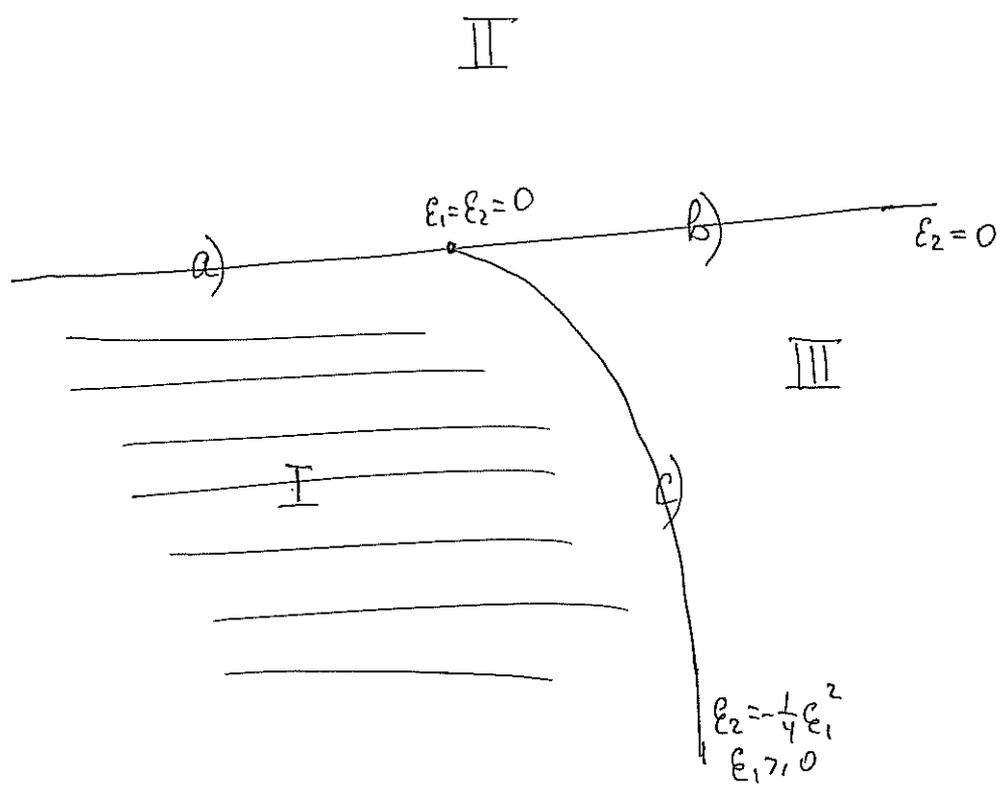
It is codim 2 case \Rightarrow any versal deformation has 2 parameters.

Theorem One of the versal deformations of (t^2, t^5)

is $x = t^2, y = t^5 - \epsilon_1 t^3 - \epsilon_2 t$

("_" at ϵ_1, ϵ_2 for convenience only).

The singularities can be easily analyzed and the following bifurcation diagram holds (which includes the versal deformation of a non-degenerate cusp: a) \rightarrow II \searrow I and the versal deformation of a curve with immersed simple self-tangency point: c) \rightarrow III \searrow I.



Open (generic) cases in the plane (ϵ_1, ϵ_2) :

- I: curve without singularities
- II: one singular point: transversal self-intersection point
- III: two singular points: transversal immersed self-intersection pts:

Boundary cases:

- a) between I and II ($\epsilon_1 < 0, \epsilon_2 = 0$): non-degenerate cusp
- b) between II and III ($\epsilon_1 > 0, \epsilon_2 = 0$): non-degenerate cusp and an immersed transversal self-intersection point:
- c) between II and III: ($\epsilon_2 = -\frac{1}{4}\epsilon_1^2, \epsilon_1 > 0$): self-tangency point:
(immersed: simple tangency)