A Short Introduction to Young Measures

Andrea Braides

Lecture Notes of part of a course held at SISSA in 1999/2000.

1. Weak* convergence of measures

Before introducing the basic notions about weak* convergence of measures we recall a few simple facts about lower semicontinuous functions.

Lower semicontinuous functions

In this section X is an arbitrary metric space.

Definition 1.1. We recall that a function $f : X \to \overline{\mathbb{R}}$ is *lower semicontinuous* (l.s.c. for short) if $f(x) = \liminf_{y\to x} f(y)$ for all $x \in X$. If f is any function, its *lower semicontinuous envelope* $\overline{f}$ is the greatest l.s.c. function not greater than f. We recall that $\overline{f}(x) = \liminf_{y\to x} f(y)$.

Definition 1.2. If $f : X \to \overline{\mathbb{R}}$, then for all $\lambda \geq 0$ and $p > 0$ we define the *Yosida transform* $T^p_\lambda f : X \to \overline{\mathbb{R}}$ by setting

$$T^p_\lambda f(x) = \inf\{f(y) + \lambda(d(x, y))^p : \ y \in X\}.$$

If $p = 1$, then we shall write $T_\lambda f$ in place of $T^p_\lambda f$; i.e.,

$$T_\lambda f(x) = \inf\{f(y) + \lambda d(x, y) : \ y \in X\}.$$

Remark 1.3. (a) Whatever be p and f, if $0 \leq \lambda \leq \eta$ then

$$T^p_0 f = \inf_X f \leq T^p_\lambda f \leq T^p_\eta f \leq f\,, \qquad \text{and} \qquad T^p_\lambda f \leq T^p_\lambda g \text{ if } f \leq g.$$

(b) Let f be bounded from below and proper; then

(i) for all x_1, $x_2 \in X$

$$|T_\lambda f(x_1) - T_\lambda f(x_2)| \leq \lambda\, d(x_1, x_2).$$

Moreover, if $d(x, y) < +\infty$ for all $x, y \in X$ and $p \geq 1$ then we have the following local Lipschitz condition: for all $x_1, x_2 \in X$

$$|T^p_\lambda f(x_1) - T^p_\lambda f(x_2)| \leq (1 + \lambda)c(1 + (d(x_1, x_0))^{p-1} + (d(x_2, x_0))^{p-1})d(x_1, x_2),$$

where c is a suitable constant and x_0 is any point in X fixed; in particular, if $X = \mathbb{R}^n$, and we take $x_0 = 0$, then we have

$$|T^p_\lambda f(x_1) - T^p_\lambda f(x_2)| \leq (1 + \lambda)c(1 + |x_1|^{p-1} + |x_2|^{p-1})|x_1 - x_2|\,;$$

(ii) if $0 < p \leq 1$ then $T^p_\lambda f$ is the greatest function not greater than f satisfying the Hölder condition

$$|T^p_\lambda f(x_1) - T^p_\lambda f(x_2)| \leq \lambda(d(x_1, x_2))^p\,;$$

in particular, $T_\lambda f$ is the greatest Lipschitz function with Lipschitz constant λ not greater than f.

Proposition 1.4. *Let $f : X \to \overline{\mathbb{R}}$ be bounded from below; then we have*

$$\overline{f}(x) = \sup_{\lambda>0} T^p_\lambda f(x)$$

for all $x \in X$.

In particular, Proposition 1.4 shows that any l.s.c. function f bounded from below can be seen as the pointwise supremum of a family of continuous functions (take $p = 1$ and recall Remark 1.6). The condition of boundedness from below can be replaced by $f(x) \geq c - \lambda_0(d(x, x_0))^p$ for some λ_0, x_0 for all x. Moreover, if f is convex then by Remark 1.3(c) and by a convolution argument it can be seen as the pointwise supremum of a family of Lipschitz and smooth convex functions.

Weak convergence of measures*

Let X be a metric, locally compact, σ-compact and separable space, $C_0(X) := \{ u : X \to \mathbb{R}\,\text{continuous} : \forall\varepsilon > 0\ \exists K \subseteq X \text{ compact} : |u(x)| < \varepsilon \text{ for } x \in X\setminus K\}$. By the assumptions on X, we have $(C_0(X))^* \simeq \mathcal{M}_b(X)$ in the sense that for all $L \in (C_0(X))^*$ there exists $\mu \in \mathcal{M}_b(X)$ such that

$$L(u) = \int_X u\, d\mu$$

for all $u \in C_0(X)$.

We consider $\mathcal{M}_b(X)$ endowed with the weak* topology $\sigma(\mathcal{M}_b(X), C_0(X))$, therefore we write $\nu_h \stackrel{*}{\rightharpoonup} \nu$ to indicate that ν_h *converges weakly** to ν in the sense of the following definition.

Definition 1.5. Let (ν_j) be a sequence of Borel measures. We say that the sequence (ν_j) *converges weakly** to ν if

$$\int_X f\, d\nu = \lim_j \int_X f\, d\nu_j$$

for all $f \in C_0(X)$.

From the Banach-Steinhaus and the Banach-Alaoglu-Bourbaki theorems it is easy to deduce the following.

Theorem 1.6. *The sequence (ν_j) converges weakly* to ν if and only if $\sup_j \nu_j(X) < +\infty$ and there exists a dense subset $D \subset C_0(X)$ such that*

$$\lim_j \int_X u\, d\nu_j = \int_X u\, d\nu \qquad \forall u \in D.$$

Moreover, every sequence (ν_j) such that $(\nu_j(X))$ is bounded admits a weakly converging subsequence, and the map $\nu \mapsto \nu(X)$ is l.s.c. with respect to weak* convergence.* The last statement in Theorem 1.6 follows immediately from the equality

$$\nu(X) = \sup\Big\{ \int_X u\, d\nu : u \in C_0(X),\ |u| \leq 1 \Big\},$$

and the fact that the supremum of a family of (semi)continuous functions is a semicontinuous function.

Proposition 1.7. *Let ν_j be positive Borel measures. If (ν_j) converges weakly* to ν then*

$$\int_X f\,d\nu \le \liminf{}_j \int_X f\,d\nu_j$$

for all l.s.c. function $f : X \to [0,+\infty]$, and

$$\int_X f\,d\nu \ge \limsup{}_j \int_X f\,d\nu_j$$

for all upper semicontinuous function $f : X \to [0,+\infty)$ with compact support.

Proof. Let $f : X \to [0,+\infty]$ be lower semicontinuous, and let $f_\lambda = T_\lambda f$ be as in Definition 1.2. Then we have $f = \sup_\lambda f_\lambda$ by Proposition 1.4. Let $(\phi_k) \subset C_0^\infty(X)$ be an increasing sequence converging to 1 in X. Then

$$\int_X \phi_k f_\lambda\,d\nu = \lim{}_j \int_X \phi_k f_\lambda\,d\nu_j \le \liminf{}_j \int_X u\,d\nu_j.$$

Letting $k \to +\infty$ and $\lambda \to +\infty$ Beppo-Levi's theorem yields the desired inequality.

If $f : X \to [0,+\infty)$ is upper semicontinuous with compact support in X then $-f = \sup_\lambda T_\lambda(-f)$, so that we can write $f = \inf_\lambda f_\lambda$, where $f_\lambda = -T_\lambda(-f)$. Moreover the support of f_λ is compact in X for λ large. Then

$$\int_X f_\lambda\,d\nu \ge \limsup{}_j \int_X f_\lambda\,d\nu_j \ge \limsup{}_j \int_X u\,d\nu_j.$$

Letting $\lambda \to +\infty$ we complete the proof. □

Remark 1.8. As a particular case of proposition 1.7 we have

$$\nu(U) \le \liminf{}_j \nu_j(U) \tag{1.1}$$

for all open subset $U \subset X$, and

$$\nu(K) \ge \limsup{}_j \nu_j(K) \tag{1.2}$$

for all compact $K \subset X$. Moreover, from (1.1)–(1.2) we obtain that

$$\nu(U) = \lim{}_j \nu_j(U),$$

for all open set U with compact closure in X such that $\nu(\partial U) = 0$.

2. Families of measures

In all that follows Ω and S are metric, locally compact, σ-compact and separable spaces, $\mathcal{F}$ denotes a σ-algebra on Ω containing $\mathcal{B}(\Omega)$, the Borel sets of Ω, and $\mu : \mathcal{F} \to [0,+\infty)$ is a complete non-atomic, measure, strictly positive on non-empty open sets, belonging to the space $\mathcal{M}_b(\Omega)$ of bounded Borel regular measures on Ω.

We set:

$$\mathcal{P}(X) := \{\text{probability measures on } X\} = \{\lambda \in \mathcal{M}_b^+(X) \,:\, \lambda(X) = 1\}$$
$$\mathcal{SP}(X) := \{\text{subprobability measures on } X\} = \{\lambda \in \mathcal{M}_b^+(X) \,:\, \lambda(X) \le 1\}$$

where $\mathcal{M}_b^+(X) := \{\lambda \in \mathcal{M}_b(X) \,:\, \lambda \ge 0\}$.

Definition 2.1. A family $\{\nu_x\}_{x\in\Omega}$ in $\mathcal{M}_b(S)$ is said to be *measurable* if for every $A \in \mathcal{B}(S)$ $\Omega \ni x \mapsto \nu_x(A) \in \mathrm{I\!R}$ is $\mathcal{F}$-measurable.

Definition 2.2. *A measurable family of probability* (resp. subprobability) is a measurable family such that $\nu_x \in \mathcal{P}(S)$ (resp. $\mathcal{SP}(S)$), for every $x \in \Omega$. $\mathcal{P}(\Omega,\mu,S)$ (resp. $\mathcal{SP}(\Omega,\mu,S)$) denotes the set of such families.

Finally we recall the notion of image of a measure.

Definition 2.3. Given a measurable space $(X,\mathcal{F})$ and a topological space (Y,τ), let $\theta : X \to Y$ be a measurable function and let λ be a positive measure on $(X,\mathcal{F})$. *The image of* λ by θ is the measure ν on $(Y,\mathcal{B}(Y))$ defined by

$$\nu(B) := \lambda \circ \theta^{-1}(B) \ , \qquad B \in \mathcal{B}(Y).$$

Remark 2.4. If $\psi : Y \to \mathrm{I\!R}$ is ν-integrable or measurable and positive, we have

$$\int_X \psi(\theta(x))\, d\lambda = \int_Y \psi(y)\, d\nu \ .$$

3. Young Measures

Definition 3.1. $\lambda \in \mathcal{M}_b^+(\Omega \times S)$ is a *Young measure* with respect to μ if

$$\lambda(A \times S) = \mu(A) \ , \qquad \forall A \in \mathcal{F}.$$

$\mathcal{Y}(\Omega,\mu,S)$ will denote the set of such measures.

Definition 3.2. $\lambda \in \mathcal{M}_b^+(\Omega \times S)$ is a *Young submeasure* with respect to μ if

$$\lambda(A \times S) \le \mu(A) \ , \qquad \forall A \in \mathcal{F}.$$

$\mathcal{SY}(\Omega,\mu,S)$ will denote the set of such measures.

Remark 3.3. If $\nu \in \mathcal{Y}(\Omega,\mu,S)$, then $\mu = \nu \circ \pi_\Omega^{-1}$, where $\pi_\Omega : \Omega \times S \to \Omega, (x,\xi) \mapsto x$, is the projection on Ω. Thus, if $\psi : \Omega \to \mathrm{I\!R}$ is measurable and positive or μ-integrable, then

$$\int_\Omega \psi\, d\mu = \int_{\Omega\times S} \psi\, d\nu.$$

Definition 3.4. Let $u : \Omega \to S$ be measurable. The *Young measure associated to* u is the measure ν in $\mathcal{Y}(\Omega,\mu,S)$ defined by

$$\int_{\Omega\times S} \psi(x,\xi)\, d\nu := \int_\Omega \psi(x,u(x))\, d\mu \ , \qquad \psi \in C_0(\Omega \times S).$$

Remark 3.5. Let ν and u be as in the previous definition. Then $\nu = \mu \circ \theta^{-1}$, where $\theta : \Omega \to \Omega \times S$ is defined by

$$\theta(x) := (x, u(x))$$

for all $x \in \Omega$.

Remark 3.6. Let u_h , $u_\infty : \Omega \to S$ be measurable functions with $u_h \to u_\infty \quad \mu$-a.e. Then $\nu^h \overset{*}{\rightharpoonup} \nu^\infty$, where ν^h , ν^∞ are the Young measures associated to u_h and u_∞, respectively. Indeed, by Lebesgue's Theorem, if $\psi \in C_0(\Omega \times S)$ we have

$$\begin{aligned} \lim_h \int_{\Omega\times S} \psi \, d\nu^h &= \lim_h \int_\Omega \psi(x, u_h(x))\, d\mu \\ &= \int_\Omega \psi(x, u_\infty(x))\, d\mu = \int_{\Omega \times S} \psi \, d\nu^\infty . \end{aligned}$$

Remark 3.7. It can easily be checked that

$$\begin{aligned} \mathcal{SY}(\Omega, \mu, S) = \Big\{ \nu \in \mathcal{M}_b^+(\Omega \times S) \ : \ & \nu(\Omega \times S) \le \mu(\Omega) \text{ and} \\ & \forall \alpha \in C_0(\Omega),\ \alpha \ge 0,\ \forall \theta \in C_0(\Omega \times S) \ : \\ & \theta(x, \xi) \le \alpha(x),\ \int_{\Omega\times S} \theta \, d\nu \le \int_\Omega \alpha \, d\mu \Big\} \end{aligned} \tag{3.1}$$

and, if S is compact,

$$\mathcal{Y}(\Omega, \mu, S) = \Big\{ \nu \in \mathcal{SY}(\Omega, \mu, S) \ : \ \forall \alpha \in C_0(\Omega),\ \alpha \ge 0,\ \int_\Omega \alpha \, d\mu = \int_{\Omega \times S} \alpha \, d\nu \Big\}. \tag{3.2}$$

If we consider $\mathcal{Y}(\Omega, \mu, S)$ and $\mathcal{SY}(\Omega, \mu, S)$ endowed with the weak* topology of $\mathcal{M}_b(\Omega \times S)$, we have the following compactness results.

Theorem 3.8. *$\mathcal{SY}(\Omega, \mu, S)$ is compact and metrizable.*

Proof. Note that $\mathcal{SY}(\Omega, \mu, S) \subseteq U := \{\lambda \in \mathcal{M}_b(\Omega \times S) \ : \ |\lambda|(\Omega \times S) \le \mu(\Omega)\}$. Since $C_0(\Omega \times S)$ is separable, U is metrizable and, by the Banach-Alaoglu Theorem, it is compact. Then it suffices to prove that $\mathcal{SY}(\Omega, \mu, S)$ is sequentially closed in U. Let $(\nu^h) \subseteq \mathcal{SY}(\Omega, \mu, S)$ be such that $\nu^h \overset{*}{\rightharpoonup} \nu$.

If $\theta \in C_0(\Omega \times S)$, $\alpha \in C_0(\Omega)$, $\alpha \ge 0$ and $\theta \le \alpha$, we have, by (3.1),

$$\int_{\Omega \times S} \theta \, d\nu^h \le \int_\Omega \alpha \, d\mu \ ,$$

thus, letting $h \to +\infty$,

$$\int_{\Omega \times S} \theta \, d\nu \le \int_\Omega \alpha \, d\mu.$$

Hence $\nu \in \mathcal{SY}(\Omega, \mu, S)$. $\square$

Theorem 3.9. *$\mathcal{Y}(\Omega,\mu,S)$ is compact $\Longleftrightarrow$ S is compact.*

Proof. If S is compact, then, given $\alpha \in C_0(\Omega)$, the function $\beta(x,\xi) := \alpha(x)$ belongs to $C_0(\Omega \times S)$. Let $(\nu^h) \subseteq \mathcal{Y}(\Omega,\mu,S)$ be such that $\nu^h \overset{*}{\rightharpoonup} \nu$. Then, by (3.2),

$$\int_{\Omega\times S} \beta\, d\nu^h = \int_\Omega \alpha\, d\mu$$

and, letting $h \to +\infty$,

$$\int_{\Omega\times S} \beta\, d\nu = \int_\Omega \alpha\, d\mu,$$

hence $\nu \in \mathcal{Y}(\Omega,\mu,S)$.

Conversely, suppose S is not compact and consider a sequence $(s_h) \subseteq S$ which does not admit convergent subsequences. Then, if ν^h are the Young measures associated to the constants s_h, for each $\psi \in C_0(\Omega,\mu,S)$ we have

$$\lim_h \int_{\Omega\times S} \psi(x,\xi)\, d\nu^h = \lim_h \int_\Omega \psi(x,s_h)\, d\mu = 0,$$

hence $\nu^h \overset{*}{\rightharpoonup} 0 \notin \mathcal{Y}(\Omega.\mu,S)$. $\square$

Now we give some examples illustrating the behaviour of Young measures associated to sequences of functions.

Examples 3.10. In the following examples we suppose that $\Omega = [0,1]$, μ is the Lebesgue measure and $S = \mathbb{R}$.

1) Let

$$u_h(x) = \begin{cases} 1 & \text{if } x \in \left[\dfrac{k}{2^h}, \dfrac{k+1}{2^h}\right],\ k \text{ even} \\ -1 & \text{otherwise} \end{cases}$$

We have that $u_h \overset{*}{\rightharpoonup} 0$ in $\mathrm{L}^\infty(0,1)$, while $\nu^h \overset{*}{\rightharpoonup} \frac{1}{2}(dx \otimes \delta_1) + \frac{1}{2}(dx \otimes \delta_{-1})$. Hence, the weak* convergence in $\mathrm{L}^\infty(\Omega)$ of the functions u_h to a function u does not imply the weak* convergence in $\mathcal{M}_b(\Omega \times S)$ of the associated Young measures. Moreover, note that there is no a.e. convergent subsequence $(u_{h_k})_{k\in\mathbb{N}}$ (see Remark 3.6), since $\|u_k - u_h\|_{\mathrm{L}^1(0,1)} = 1$ for $h \neq k$.

2) Let u_h be as in example 1). Define

$$v_h(x) = \begin{cases} u_h(x) & \text{if } h \text{ is even} \\ \dfrac{1}{2} u_h(x) & \text{if } h \text{ is odd.} \end{cases}$$

Then $v_h \overset{*}{\rightharpoonup} 0$ in $\mathrm{L}^\infty(0,1)$, but the sequence (v_h) of the associated Young measures is not convergent. Indeed

$$\nu^{2h} \overset{*}{\rightharpoonup} \frac{1}{2}(dx \otimes \delta_1) + \frac{1}{2}(dx \otimes \delta_{-1})$$
$$\nu^{2h+1} \overset{*}{\rightharpoonup} \frac{1}{2}(dx \otimes \delta_{\frac{1}{2}}) + \frac{1}{2}(dx \otimes \delta_{-\frac{1}{2}}).$$

3) Let $u_h(x) = \sin hx$.

It can be proved that (ν^h) converges to the measure carried by [0,1]×[-1,1] whose density with respect to the 2-dimensional Lebesgue measure is

$$f(\xi) = \frac{1}{\pi}\frac{1}{\sqrt{1-\xi^2}}.$$

4) Let $u_h(x) = \xi_h \chi_{[0,\frac{1}{h}]}(x)$, with $\xi_h \in \mathbb{R}$.

Whatever be ξ_h, $\nu^h \stackrel{*}{\rightharpoonup} \nu_0$, where $\nu_0 = dx \otimes \delta_0$ is the Young measure associated to the function identically 0.

5) Let $u_h(x) \equiv h$.

ν^h converges to the null measure.

Note that in examples (1)–(3), the values of the functions u_h belong to the compact set [-1,1], so that (ν^h) is relatively compact in $\mathcal{Y}(\Omega,\mu,S)$ by Theorem 3.9, since we may assume $S = [-1,1]$. In examples (4) and (5),we may not assume S compact, but in (4), the sequence (u_h) is tight (see Definition 5.1 below), a sufficient condition to the precompactness of (ν^h), as we will show later on.

Disintegration

We introduce now the Disintegration Theorem, which allow us to view Young measures as measurable families of probability.

Theorem 3.11. (Disintegration Theorem) *The map* $\Phi : \mathcal{P}(\Omega,\mu,S) \to \mathcal{Y}(\Omega,\mu,S)$ *which associates to* $\lambda = \{\lambda_x\}_{x\in\Omega}$ *the measure* ν *defined by*

$$\nu(A) := \int_\Omega \left(\int_S \chi_A(x,\xi)\, d\lambda_x(\xi) \right) d\mu(x) \qquad A \in \mathcal{F} \otimes \mathcal{B}(S) \tag{3.3}$$

induces a bijection between $\mathcal{P}(\Omega,\mu,S)/\mathcal{R}$ *and* $\mathcal{Y}(\Omega,\mu,S)$ *where*

$$\lambda^1 \mathcal{R} \lambda^2 \iff \lambda^1_x = \lambda^2_x \quad \textit{for } \mu\textit{-a.e. } x \in \Omega\,.$$

Proof. It is easy to verify that Φ is compatible with $\mathcal{R}$.

Note that, by (3.3), we have

$$\int_{\Omega\times S} f\, d\nu = \int_\Omega \left(\int_S f(x,\xi)\, d\lambda_x(\xi) \right) d\mu(x)$$

for any $f : \Omega \times S \to \mathbb{R}$ measurable and positive or ν-integrable.

Φ is one-to-one. Indeed let λ^1, λ^2 be such that $\Phi(\lambda^1) = \Phi(\lambda^2)$, then, set $\psi(x,\xi) := \chi_E(x)\,\phi(\xi)$ with $E \in \mathcal{F}$ and $\phi \in C_0(S)$, we have

$$\int_E \left(\int_S \phi(\xi)\, d\lambda^1_x(\xi) \right) d\mu(x) = \int_E \left(\int_S \phi(\xi)\, d\lambda^2_x(\xi) \right) d\mu(x).$$

Thus, since E is arbitrary,

$$\int_S \phi(\xi)\, d\lambda^1_x(\xi) = \int_S \phi(\xi)\, d\lambda^2_x(\xi)$$

for μ-a.e. x, for all $\phi \in C_0(S)$; *i.e.* $\lambda_x^1 = \lambda_x^2$ μ-a.e.

We now show that Φ is surjective. If $\nu \in \mathcal{Y}(\Omega, \mu, S)$, then, for any $\varepsilon > 0$, we can determine a family $(S_i)_{i\in\mathbb{N}}$ of pairwise disjoint measurable sets in S with $\operatorname{diam}(S_i) < \varepsilon$ and $\nu(\Omega \times (S \setminus \bigcup_{i\in\mathbb{N}} S_i)) = 0$ and a family $(B_j)_{j\in\mathbb{N}}$ of pairwise disjoint open sets in Ω such that $\mu(\Omega \setminus \bigcup_{j\in\mathbb{N}} B_j) = 0$, $\operatorname{diam}(B_j) < \varepsilon$ and $\mu(\partial B_j) = 0$.

Choose $\xi_i \in S_i$ for any $i \in \mathbb{N}$ and set

$$\lambda_x^\varepsilon := \sum_{i\in\mathbb{N}} \frac{\nu(B_j \times S_i)}{\mu(B_j)} \delta_{\xi_i} \qquad \text{if } x \in B_j.$$

$\lambda^\varepsilon = \{\lambda_x^\varepsilon\}_{x\in\Omega} \in \mathcal{P}(\Omega, \mu, S)$, then $\nu^\varepsilon := \Phi(\lambda^\varepsilon)$ is well defined. Note that

$$\nu\Big(\bigcup_{(j,i): B_j \times S_i \subseteq E} B_j \times S_i \Big) \le \nu^\varepsilon(E) \le \nu\Big(\bigcup_{(j,i): B_j \times S_i \supseteq E} B_j \times S_i \Big)$$

for all $E \in \mathcal{F} \otimes \mathcal{B}(S)$, thus, for any open set A with $\nu(\partial A) = 0$,

$$\lim_{\varepsilon \to 0} \nu^\varepsilon(A) = \nu(A).$$

Then, by Remark 1.8 and since ν is Borel regular, we can deduce that $\nu^\varepsilon \overset{*}{\rightharpoonup} \nu$. On the other hand, since $\{\lambda^\varepsilon\}_{\varepsilon\in\Omega}$ is bounded in $\mathrm{L}^\infty(\Omega, \mu, \mathcal{M}_b(S))$, we may suppose that λ^ε converges to $\lambda \in \mathcal{SP}(\Omega, \mu, S)$ in the weak* topology of the dual pair $(\mathrm{L}^\infty(\Omega, \mu, \mathcal{M}_b(S)), \mathrm{L}^1(\Omega, \mu, C_0(S)))$.

Let $\psi \in C_0(\Omega, \times S)$. In particular $\psi \in \mathrm{L}^1(\Omega, \mu, C_0(S))$, thus

$$\begin{aligned} &\int_{\Omega\times S} \psi \, d\nu = \lim_{\varepsilon\to 0} \int_{\Omega\times S} \psi \, d\nu^\varepsilon \\ &= \lim_{\varepsilon\to 0} \int_\Omega \left(\int_S \psi(x, \xi) \, d\lambda_x^\varepsilon(\xi) \right) d\mu = \int_\Omega \left(\int_S \psi(x, \xi) \, d\lambda_x(\xi) \right) d\mu. \end{aligned}$$

Then, by the construction of λ, we deduce that $\lambda \in \mathcal{P}(\Omega, \mu, S)$ and $\nu = \Phi(\lambda)$. $\square$

Remark 3.12. By the argument of the previous theorem, it can be proved that $\mathcal{SP}(\Omega, \mu, S)/\mathcal{R}$ is in bijection with $\mathcal{SY}(\Omega, \mu, S)$.

Remark 3.13. If ν is the Young measure associated to a measurable function u, then $\nu \simeq \{\delta_{u(x)}\}_{x\in\Omega}$.

Remark 3.14. By Theorem 3.11, it can be deduced that the weak* topology on $\mathcal{Y}(\Omega, \mu, S)$ of the dual pair $(\mathcal{M}_b(\Omega \times S), C_0(\Omega \times S))$ is equivalent to the weak* topology of the dual pair $(\mathrm{L}^\infty(\Omega, \mu, \mathcal{M}_b(S)), \mathrm{L}^1(\Omega, \mu, C_0(S)))$.

Proposition 3.15. *The Young measures associated to measurable functions are dense in* $\mathcal{Y}(\Omega, \mu, S)$.

Proof. Taking into account the proof of the Disintegration Theorem, it suffices to approximate measures $\lambda = \{\lambda_x\}_{x\in\Omega}$ with $\lambda_x = \lambda$ independent of x given by

$$\lambda := \sum_{i\in\mathbb{N}} c_i \delta_{\xi_i}\ , \quad c_i \in (0,1),\ \sum_{i\in\mathbb{N}} c_i = 1,\ \xi_i \in S.$$

Let one of such measures be fixed. For $h \in \mathbb{N}$ we can determine, by a covering argument, a countable family of pairwise disjoint measurable sets $(D^h_j)_{j\in\mathbb{N}}$ with $\mathrm{diam}(D^h_j) < \frac{1}{h}$ for all $j \in \mathbb{N}$ and $\mu(\Omega \setminus \bigcup D^h_j) = 0$. For $j \in \mathbb{N}$ fixed, we can find a partition D^{hi}_j of D^h_j such that $\mu(D^{hi}_j) = c_i \mu(D^h_j)$. Set $E^{hi} := \bigcup_{j\in\mathbb{N}} D^{hi}_j$ and define

$$u_h(x) = \xi_i \qquad x \in E^{hi}.$$

Since, letting $h \to +\infty$,

$$\chi_{E^{hi}} \overset{*}{\rightharpoonup} c_i \text{ in } \mathrm{L}^\infty(\Omega),$$

it is easy to verify that $\{\delta_{u_h(x)}\}_{x\in\Omega} \overset{*}{\rightharpoonup} \{\lambda_x\}_{x\in\Omega} = \lambda$ in $\mathcal{Y}(\Omega, \mu, S)$. $\square$

Integrands

Let us introduce, now, the following classes of integrands, that are measurable functions from $\Omega \times S$ to $\mathbb{R}$:

- *normal integrands*

$$\mathcal{G}(\Omega, \mathcal{F}, S) := \{\psi : \Omega \times S \to \overline{\mathbb{R}} \text{ measurable } : \ \psi(x, \cdot) \text{ is l.s.c. } \forall x\}$$

- *positive normal integrands*

$$\mathcal{G}_+(\Omega, \mathcal{F}, S) := \{\psi \in \mathcal{G}(\Omega, \mathcal{F}, S) : \psi \geq 0\}$$

- *Caratheodory integrands*

$$\mathcal{G}_C(\Omega, \mathcal{F}, S) := \{\psi \in \mathcal{G}(\Omega, \mathcal{F}, S) :\ \psi(x, \cdot) \in C_b(S)\, \forall x$$

$$\text{and } x \mapsto ||\psi(x, \cdot)||_{\mathrm{L}^\infty(S)} \text{ is } \mu\text{-integrable}\}$$

$$\mathcal{G}_{C_0}(\Omega, \mathcal{F}, S) := \{\psi \in \mathcal{G}(\Omega, \mathcal{F}, S) :\ \psi(x, \cdot) \in C_0(S)\, \forall x\}$$

Lemma 3.16. If $\psi \in \mathcal{G}_+(\Omega, \mathcal{F}, S)$, then there exists a sequence $(\theta_h) \subseteq \mathcal{G}_{C_0}(\Omega, \mathcal{F}, S)$ such that

$$\psi(x, \xi) = \sup_{h \in \mathbb{N}} \theta_h(x, \xi) \qquad \forall (x, \xi) \in \Omega \times S.$$

Proof. Suppose $\psi \not\equiv +\infty$ and set

$$\psi_h(x, \xi) := (T_h \psi(x, \cdot))(\xi) := \inf_{z \in S} \{\psi(x, z) + h d(z, \xi)\},$$

where d is the metric on S.

By Remark 1.3 and Proposition 1.4, $\psi_h(x, \cdot) \in C(S)$ for any fixed x in Ω and $\sup_{h \in \mathbb{N}} \psi_h(x, \xi) = \psi(x, \xi)$. Moreover for any $\xi \in S$ and $\alpha \in \mathbb{R}$, the set

$$\begin{aligned} \{x : \psi_h(x, \xi) < \alpha\} &= \{x : \exists z \in S : \psi(x, z) + h d(z, \xi) < \alpha\} \\ &= \pi_\Omega(\{(x, z) : \psi(x, z) + h d(z, \xi) < \alpha\}) \end{aligned}$$

belongs to $\mathcal{F}$, by the Measurable Projection Theorem. Then $\psi(\cdot, \xi)$, is $\mathcal{F}$-measurable. Since $\psi_h(x, \cdot)$ is continuous, ψ_h is $\mathcal{F} \otimes \mathcal{B}(S)$-measurable.

Let (φ_h) be an increasing and positive sequence in $C_0(S)$ which converges to 1 pointwise. Then the thesis of the lemma follows by taking

$$\theta_h(x, \xi) := \min\{h \varphi_h(\xi),\ \psi_h(x, \xi)\}.$$

If $\psi \equiv +\infty$, it suffices to take $\theta_h(x, \xi) := h \varphi_h(\xi)$. □

Proposition 3.17. *If $\psi \in \mathcal{G}_+(\Omega, \mathcal{F}, S)$, then $\nu \mapsto \int_{\Omega \times S} \psi \, d\nu$ is weakly* l.s.c. on $\mathcal{SY}(\Omega, \mu, S)$.*

Proof. Given (θ_h) as in the previous lemma, we have

$$\int_{\Omega \times S} \psi \, d\nu = \sup_{h \in \mathbb{N}} \int_{\Omega \times S} \theta_h \, d\nu.$$

Then the thesis follows from the fact that for any $h \in \mathbb{N}$ $\nu \mapsto \int_{\Omega \times S} \theta_h \, d\nu$ is weakly* continuous on $\mathcal{SY}(\Omega, \mu, S)$, since $\theta_h \in \mathrm{L}^1(\Omega, \mu, C_0(S))$ (see Remark 3.14). □

Proposition 3.18. *Let $u_h : \Omega \to S$ be measurable functions and let ν^h be the associated Young measures. If $\nu^h \overset{*}{\rightharpoonup} \nu$ in $\mathcal{SY}(\Omega, \mu, S)$, then for μ-a.e. x, ν^x is carried by*

$$K\text{-}\limsup_{h \to +\infty} (\{u_h(x)\}) := \bigcap_{k \in \mathbb{N}} \overline{\{u_h(x) : h \geq k\}}.$$

Proof. For each $k \in \mathbb{N}$ set

$$\Gamma_k(x) := \overline{\{u_h(x) : h \geq k\}} \qquad x \in \Omega$$

$$\psi_k(x, \xi) := \begin{cases} 0 & \xi \in \Gamma_k(x) \\ +\infty & \text{otherwise.} \end{cases}$$

Since $\psi_k \in \mathcal{G}_+(\Omega, \mu, S)$, then, by Proposition 3.17,

$$\begin{aligned} 0 \leq \int_{\Omega \times S} \psi_k \, d\nu &\leq \liminf_h \int_{\Omega \times S} \psi_k \, d\nu^h \\ &= \liminf_{h \in \mathbb{N}} \int_\Omega \psi_k(x, u_h(x)) \, d\mu = 0\,, \end{aligned}$$

hence for μ-a.e. x in Ω $\int_\Omega \psi_k(x, \xi) \, d\nu_x(\xi) = 0$ *i.e.* ν_x is carried by $\Gamma_k(x)$. By the arbitrariness of $k \in \mathbb{N}$, the thesis follows. □

4 Narrow Topology and Characterization of Precompact Sets in $\mathcal{Y}(\Omega, \mu, S)$

In this section we will endow $\mathcal{Y}(\Omega, \mu, S)$ with a new topology, finer than the weak* topology, and we will give a characterization of precompact sets.

Definition 4.1. The *narrow topology* on $\mathcal{Y}(\Omega, \mu, S)$ is the weakest topology such that

$$\nu \mapsto \int_{\Omega \times S} \psi \, d\nu \tag{4.1}$$

is continuous for any $\psi \in \mathcal{G}_C(\Omega, \mathcal{F}, S)$. We will say that a sequence (ν^h) converges to ν narrowly if

$$\lim_{h \to \infty} \int_{\Omega \times S} \psi \, d\nu^h = \int_{\Omega \times S} \psi \, d\nu$$

for all $\psi \in \mathcal{G}_C(\Omega, \mathcal{F}, S)$.

Remark 4.2. If S is compact, it is clear that the narrow topology on $\mathcal{Y}(\Omega, \mu, S)$ coincides with the weak* topology.

Proposition 4.3. *Let $u_h, u_\infty : \Omega \to S$ be measurable and let ν^h, ν^∞ be the Young measures associated. Then*

$$u_h \to u_\infty \quad \textit{in measure} \iff \nu^h \to \nu^\infty \quad \textit{narrowly.}$$

Proof. ($\Longrightarrow$) If $u_h \to u_\infty$ in measure, then for any subsequence $(u_{h_k})_{k \in \mathbb{N}}$ there exists $(u_{n_{k_j}})_{j \in \mathbb{N}}$ which converges to u_∞ a.e.

By Lebesgue's Theorem,

$$\lim_j \int_\Omega \psi(x, u_{h_{k_j}}(x)) \, d\mu = \int_\Omega \psi(x, u_\infty(x)) \, d\mu$$

for all $\psi \in \mathcal{G}_C(\Omega, \mathcal{F}, S)$. By Urysohn's property, $\nu^h \to \nu^\infty$ narrowly.
($\Longleftarrow$) Suppose $\nu^h \to \nu^\infty$ narrowly and set

$$\psi(x, \xi) := \min\{1, d(\xi, u_\infty(x))\}.$$

$\psi \in \mathcal{G}_C(\Omega, \mathcal{F}, S)$, then

$$\lim_h \int_\Omega \psi(x, u_h(x)) \, d\mu = \int_\Omega \psi(x, u_\infty(x)) \, d\mu = 0.$$

For any $\varepsilon > 0$ fixed, let $M_\varepsilon^h := \{x : d(u_h(x), u_\infty(x)) \geq \varepsilon\}$. We have

$$\varepsilon \mu(M_\varepsilon^h) \leq \int_{M_\varepsilon^h} \psi(x, u_h(x)) \, d\mu \leq \int_\Omega \psi(x, u_h(x)) \, d\mu,$$

hence $\lim_h \mu(M_\varepsilon^h) = 0$ $\square$

In the following $\widehat{S}$ will be a compact metric space, in which S is imbedded.

Proposition 4.4. *Let $\psi \in \mathcal{G}_+(\Omega, \mathcal{F}, S)$. Then*

(i) there exists $(\psi_h) \subseteq \mathcal{G}_C(\Omega, \mathcal{F}, \widehat{S})$ such that $\psi(x,\xi) = \sup_{h\in\mathbb{N}} \psi_h(x,\xi)$;

(ii) $\nu \to \int_{\Omega\times S} \psi\, d\nu$ is narrowly l.s.c.

Proof. Arguing as in Lemma 3.16, we can define

$$\psi_h(x,\xi) := (T_h\psi(x,\cdot))(\xi) = \inf_{z\in S}\{\psi(x,z) + h\hat{d}(z,\xi)\}$$

for $(x,z) \in \Omega \times \widehat{S}$, where $\hat{d}$ is the metric on $\widehat{S}$, which extends d. Then (i) is proved by this choice of ψ_h.

(ii) follows immediately from Proposition 3.17, taking into account that the narrow topology is finer than the weak* topology. $\square$

Proposition 4.5. *The narrow topology on $\mathcal{Y}(\Omega, \mu, S)$ is the weakest topology τ such that*

$$\nu \mapsto \int_{\Omega\times S} \psi\, d\nu$$

is continuous for any $\psi \in \mathcal{G}_C(\Omega, \mathcal{F}, \widehat{S})$.

Proof. Since $\mathcal{G}_C(\Omega, \mathcal{F}, \widehat{S}) \subseteq \mathcal{G}_C(\Omega, \mathcal{F}, S)$ the narrow topology is, a priori, finer than τ. Than it suffices to prove that for any $\psi \in \mathcal{G}_C(\Omega, \mathcal{F}, S)$ fixed, the map defined by (4.1) is continuous with respect to the topology τ. Set

$$\alpha := \|\psi(x,\cdot)\|_{L^\infty(S)}$$

$$g(x,\xi) := \psi(x,\xi) + \alpha(x).$$

g is in $\mathcal{G}_+(\Omega, \mathcal{F}, S)$, hence, by Proposition 4.4, the map

$$\nu \mapsto \int_{\Omega\times S} g\, d\nu$$

is l.s.c. with respect to τ. Since

$$\int_{\Omega\times S} \psi\, d\nu = \int_{\Omega\times S} g\, d\nu - \int_\Omega \alpha\, d\mu,$$

$\nu \mapsto \int_{\Omega\times S} \psi\, d\nu$ is τ-l.s.c.

By the same arguments it is easy to prove that $\nu \mapsto -\int_{\Omega\times S} \psi\, d\nu$ is τ-l.s.c., then the thesis follows. $\square$

To investigate precompact sets in $\mathcal{Y}(\Omega, \mu, S)$ we first need to introduce a result of characterization of precompact sets in $\mathcal{M}_b(S)$, endowed with the weakest topology such that

$$\lambda \mapsto \int_S \varphi\, d\lambda$$

is continuous for any $\varphi \in C_b(S)$. Such topology will be still denoted by narrow topology. If S is compact it coincides with the weak* topology of $\mathcal{M}_b(S)$.

Theorem 4.6. (Prokhorov Theorem) *Let $\mathcal{K}$ be bounded in $\mathcal{M}_b^+(S)$. Then $\mathcal{K}$ is narrowly precompact in $\mathcal{M}_b(S)$ if and only if*

$$\forall\varepsilon>0\ \exists K_\varepsilon\subset S\,\text{compact}\,:\,\lambda(S\setminus K_\varepsilon)\le\varepsilon\quad\forall\lambda\in\mathcal{K}. \tag{4.2}$$

Any set in $\mathcal{M}_b^+(S)$ satisfying (4.2) will be said "tight in the sense of Prokhorov".

Proof. Let $\mathcal{K}$ be tight in the sense of Prokhorov and let (λ^h) be a sequence in $\mathcal{K}$. For each $n\in\mathbb{N}$, there exists a compact set K_n in S such that

$$\lambda^h(S\setminus K_n)\le\frac{1}{n}\quad\forall h\in\mathbb{N}.$$

It is not restrictive to suppose $K_n\subseteq K_{n+1}\quad\forall n\in\mathbb{N}$.

Set

$$\mu_h^n:=\lambda^h\llcorner K_n$$

(i.e. $\mu_h^n(E):=\lambda^h(E\cap K_n)$ for all $E\in\mathcal{B}(S)$). For all $n\in\mathbb{N}$ (μ_h^n) is bounded in $\mathcal{M}_b(S)$, hence precompact with respect to the weak* topology, which coincides in this case with the narrow topology. By a diagonal process we determine an increasing sequence in $\mathcal{M}_b^+(S)$ such that, up to a subsequence,

$$\mu_h^n\to\mu^n\text{ narrowly for }h\to+\infty.$$

Set $\lambda:=\sup_{n\in\mathbb{N}}\mu^n$. If $f\in C_b(S)$, we have

$$\begin{aligned}\Big|\int_S f\,d\lambda^h-\int_S f\,d\lambda\Big|&\le\Big|\int_{S\setminus K_n}f\,d\lambda^h\Big|+\Big|\int_S f\,d\mu_h^n-\int_S f\,d\lambda\Big|\\&\le\|f\|_{\mathrm{L}^\infty(S)}\frac{1}{n}+\Big|\int_S f\,d\mu_h^n-\int_S f\,d\lambda\Big|,\end{aligned}$$

thus

$$\limsup_h\Big|\int_S f\,d\lambda^h-\int_S f\,d\lambda\Big|\le\frac{c}{n}+\Big|\int_S f\,d\mu^n-\int_S f\,d\lambda\Big|\ \forall n\in\mathbb{N},$$

hence

$$\limsup_h\Big|\int_S f\,d\lambda^h-\int_S f\,d\lambda\Big|=0.$$

By the arbitrariness of f in $C_b(S)$, $\lambda^h\to\lambda$ narrowly.

Conversely, suppose $\mathcal{K}$ is not tight, i.e.

$$\exists\varepsilon>0\,:\,\forall K\text{compact in }S\ \exists\lambda\in\mathcal{K}\,:\,\lambda(S\setminus K)>\varepsilon. \tag{4.3}$$

Let (K_n) be an increasing sequence of compact sets in S such that $\bigcup_{n\in\mathbb{N}}\overset{\circ}{K}_n=S$. By (4.3), there exists $(\lambda^n)\subseteq\mathcal{K}$ such that

$$\lambda^n(S\setminus K_n)>\varepsilon\ \forall n\in\mathbb{N}.$$

In particular $\lambda^n(\overset{\circ}{K}_n)\le\lambda^n(S)-\varepsilon$. Suppose, by contradiction, that there exists a subsequence (λ^{n_k}) such that $\lambda^{n_k}\to\lambda$ narrowly. In particular $\lim_k\lambda^{n_k}(S)=\lambda(S)$. Fixed $n\in\mathbb{N}$, for $n_k\ge n$ we have

$$\lambda^{n_k}(\overset{\circ}{K}_n)\le\lambda^{n_k}(\overset{\circ}{K}_{n_k})\le\lambda^{n_k}(S)-\varepsilon,$$

then

$$\lambda(\overset{\circ}{K}_n)\le\liminf_k\lambda^{n_k}(\overset{\circ}{K}_n)\le\liminf_k(\lambda^{n_k}(S)-\varepsilon)=\lambda(S)-\varepsilon.$$

Letting $n\to\infty$, we obtain the contradiction $\lambda(S)\le\lambda(S)-\varepsilon$. □

Furthermore, the following characterization of tight sets in $\mathcal{M}_b(S)$ holds.

Proposition 4.7. *$\mathcal{K} \subseteq \mathcal{M}_b^+(S)$ is tight if and only if there exists $\varphi : S \to [0, +\infty]$ such that*

(i) $\forall t \in [0, +\infty)$ $\{\varphi \le t\}$ is compact (φ is inf-compact) ;

(ii) $\sup_{\lambda\in\mathcal{K}} \int_S \varphi \, d\lambda < +\infty$.

Proof. Let $\varphi : S \to [0, \infty]$ satisfy (i), (ii) and let $\varepsilon > 0$ be fixed. We set

$$K_\varepsilon := \left\{\varphi \le \frac{M}{\varepsilon}\right\}$$

with $M := \sup_{\lambda\in\mathcal{K}} \int_S \varphi \, d\lambda$. Then, for any $\lambda \in \mathcal{K}$

$$\lambda(S \setminus K_\varepsilon) \le \frac{\varepsilon}{M} \int_S \varphi \, d\lambda \le \varepsilon,$$

i.e. $\mathcal{K}$ is tight.

Conversely, let $\mathcal{K}$ be tight and let (K_n) be an increasing sequence of compact sets of S such that

$$\lambda(S \setminus K_n) \le 2^{-n} \quad \forall\lambda \in \mathcal{K}, \forall n \in \mathbb{N}.$$

If $\varphi := \sum_{n=1}^{\infty} \chi_{S\setminus K_n}$, then

$$\int_S \varphi \, d\lambda = \sum_{n=1}^{\infty} \int_S \chi_{S\setminus K_n} \, d\lambda \le 1 \quad \forall\lambda \in \mathcal{K}.$$

Since φ is $\mathbb{N} \cup \{+\infty\}$-valued, we have

$$\{\varphi \le t\} = \{\varphi \le [t]\} = K_{[t]+1},$$

where [t] denotes the integral part of t. $\square$

Definition 4.8. $\mathcal{H} \subseteq \mathcal{Y}(\Omega, \mu, S)$ is said to be *tight* if

$$\forall\varepsilon > 0 \ \exists K_\varepsilon \subseteq S \text{ compact} : \sup_{\nu\in\mathcal{H}} \nu(\Omega \times (S \setminus K_\varepsilon)) \le \varepsilon,$$

i.e. $\mathcal{K} := \{\nu \circ \pi_S^{-1} : \nu \in \mathcal{H}\}$ is tight in the sense of Prokhorov.

Theorem 4.9. *$\mathcal{H} \subseteq \mathcal{Y}(\Omega, \mu, S)$ is narrowly precompact if and only if $\mathcal{H}$ is tight.*

Proof. ($\Longrightarrow$) Let $\phi : \mathcal{Y}(\Omega, \mu, S) \to \mathcal{M}_b^+(S)$ be defined by

$$\phi(\nu) := \nu \circ \pi_S^{-1}.$$

If $\varphi \in C_b(S)$ then $\varphi \circ \pi_S \in \mathcal{G}_C(\Omega, \mathcal{F}, S)$ and, by Remark 2.4,

$$\int_S \varphi \, d(\nu \circ \pi_S^{-1}) = \int_{\Omega\times S} \varphi \circ \pi_S \, d\nu,$$

so ϕ is narrowly continuous. Hence $\mathcal{K} := \phi(\mathcal{H})$ is narrowly precompact in $\mathcal{M}_b^+(S)$. By Theorem 4.6, $\mathcal{K}$ is tight in the sense of Prokhorov, *i.e.* $\mathcal{H}$ is tight.

($\Longleftarrow$) Let $(\nu^h) \subseteq \mathcal{H}$ and $\hat{\nu}^h \subseteq \mathcal{Y}(\Omega, \mu, \widehat{S})$ be defined by

$$\hat{\nu}^h(E) := \nu^h(E \cap (\Omega \times S)) \quad E \in \mathcal{F} \otimes \mathcal{B}(\widehat{S}).$$

Since $\widehat{S}$ is compact, there exists $(\hat{\nu}^{h_k})$ such that $\hat{\nu}^{h_k} \to \hat{\nu}$ narrowly in $\mathcal{Y}(\Omega, \mu, \widehat{S})$.

For each $j \in \mathbb{N}$, there exists a compact set $K_j \subseteq S$ such that

$$\nu^h(\Omega \times (S \setminus K_j)) \leq \frac{1}{j} \quad \forall h \in \mathbb{N},$$

i.e.

$$\hat{\nu}^h(\Omega \times (\widehat{S} \setminus K_j)) \leq \frac{1}{j} \quad \forall h \in \mathbb{N},$$

so that

$$\hat{\nu}(\Omega \times (\widehat{S} \setminus K_j)) \leq \liminf_h \hat{\nu}^h(\Omega \times (\widehat{S} \setminus K_j)) \leq \frac{1}{j};$$

hence, $\hat{\nu}$ is carried by $\Omega \times (\cup_{j \in \mathbb{N}} K_j)$. □

5. Sequences of Young Measures associated to measurable functions

In what follows (u_h) is a sequence of measurable functions from Ω to S and (ν^h) isthe sequence of associated Young measures.

Definition 5.1. (u_h) is said to be *tight* if (ν^h) is tight *i.e.* $\forall \varepsilon > 0 \quad \exists K_\varepsilon \subseteq S$ compact : $\mu(\{x : u_h(x) \notin K_\varepsilon\}) \leq \varepsilon \ \ \forall h \in \mathbb{N}$.

Note that we can apply Proposition 4.7 with e.g. $\varphi(z) = |z|$, which implies that a bounded sequence in L^1 is tight.

Theorem 5.2. *If $\nu^h \to \nu$ narrowly and $\psi \in \mathcal{G}(\Omega, \mathcal{F}, S)$ is such that $(\psi^-(\cdot, u_h(\cdot)))$ is equiintegrable, then*

(i) $\liminf_h \int_\Omega \psi(x, u_h(x))\, d\mu < \infty \Rightarrow \int_{\Omega \times S} \psi^+ \, d\nu < \infty$;

(ii) $\int_{\Omega \times S} \psi \, d\nu \leq \liminf_h \int_\Omega \psi(x, u_h(x))\, d\mu$.

Proof. Without loss of generality we can suppose that there exists $\lim_h \int_\Omega \psi(x, u_h(x))\, d\mu$. Let $r \geq 0$ be fixed and $\psi_r := \max\{-r, \psi\}$. As $(\psi_r + r) \in \mathcal{G}_+(\Omega, \mathcal{F}, S)$, by Proposition 4.4, we have

$$\int_{\Omega \times S} (\psi_r + r)\, d\nu \leq \liminf_h \int_\Omega (\psi_r(x, u_h(x)) + r)\, d\mu$$

and then

$$\int_{\Omega \times S} \psi_r \, d\nu \leq \liminf_h \int_\Omega \psi_r(x, u_h(x))\, d\mu \tag{5.1}$$

(i) follows from the equiintegrability hypothesis and from (5.1) by choosing $r = 0$.

Let $A_{h,r} := \{x : \psi(x, u_h(x)) < -r\} = \{\psi < \psi_r\}$ and let $\varepsilon > 0$ be fixed. Then, by equiintegrability hypothesis, for r large enough, we have

$$\int_{A_{h,r}} \psi(x, u_h(x))\, d\mu \geq -\varepsilon,$$

hence

$$\begin{aligned}\int_\Omega \psi(x, u_h(x))\, d\mu &= \int_{A_{h,r}} \psi(x, u_h(x))\, d\mu + \int_{\Omega\setminus A_{h,r}} \psi_r(x, u_h(x))\, d\mu \\ &\geq -\varepsilon + \int_\Omega \psi_r(x, u_h(x))\, d\mu.\end{aligned}$$

By (5.1), as $\psi_r \geq \psi$,

$$\begin{aligned}\int_{\Omega\times S} \psi\, d\nu &\leq \liminf_h \int_\Omega \psi_r(x, u_h(x))\, d\mu \\ &\leq \liminf_h \int_\Omega \psi(x, u_h(x))\, d\mu + \varepsilon.\end{aligned}$$

Then (ii) follows. □

Corollary 5.3. *Let $\nu^h \to \nu$ narrowly and let $\beta \in \mathcal{G}(\Omega, \mu, S)$ be such that $\beta(x, \cdot) \in C(S)$ $\forall x \in \Omega$ and $(\beta(\cdot, u_h(\cdot)))$ is equiintegrable. Then β is ν-integrable and*

$$\int_{\Omega\times S} \beta\, d\nu = \lim_h \int_\Omega \beta(x, u_h(x))\, d\mu.$$

Proof. It follows from Theorem 5.2, with $\psi = \beta$ or $-\beta$. □

An important consequence of the previous result is the following.

Theorem 5.4. *If $\nu^h \to \nu$ narrowly and $\varphi \in C(S)$ is such that $(\varphi \circ u_h)$ is equiintegrable, then*

(i) for μ-*a.e.* x φ is ν_x-integrable and

$$\int_\Omega \left(\int_S |\varphi(\xi)|\, d\nu_x(\xi)\right) d\mu(x) < \infty\ ;$$

(ii) $\varphi \circ u_h \rightharpoonup \overline{\varphi}$ in $\mathrm{L}^1(\Omega)$, with $\bar{\varphi}(x) := \int_S \varphi(\xi)\, d\nu_x(\xi)$.

Proof. (i) follows immediately from Corollary 5.3.

Let $g \in \mathrm{L}^\infty(\Omega)$, then $\beta(x, \xi) := g(x)\varphi(\xi)$ satisfies the hypothesis of Corollary 5.3, as we have

$$\int_E \beta(x, u_h(x)\, d\mu \leq \|g\|_{\mathrm{L}^\infty(\Omega)} \int_E |\varphi \circ u_h(x)|\, d\mu\,, \quad E \in \mathcal{F}.$$

This implies

$$\int_{\Omega\times S} \beta\, d\nu = \lim_h \int_\Omega \beta(x, u_h(x))\, d\mu = \lim_h \int_\Omega g(x)\varphi(u_h(x))\, d\mu.$$

On the other hand,

$$\int_{\Omega\times S} \beta\, d\nu = \int_\Omega g(x) \left(\int_S \varphi(\xi)\, d\nu_x(\xi)\right) d\mu = \int_\Omega g(x)\bar{\varphi}(x)\, d\mu\,,$$

then (ii) follows. □

Remark 5.5. If S is compact and $\nu^h \overset{*}{\rightharpoonup} \nu$, then, by the arguments of the proof of the previous theorem, we can verify that $\varphi \circ u_h \overset{*}{\rightharpoonup} \bar{\varphi} \quad$ in $\mathrm{L}^\infty(\Omega)$.

Young measures on $\Omega \times \mathbb{R}^N$

In what follows, we will suppose $(u_h) \subseteq \mathrm{L}^p(\Omega, \mu, \mathbb{R}^N)$, with $p \in [1, +\infty]$, and, by using the previous results, we will state some criteria of strong convergence.

Proposition 5.6. *Let* $(u_h) \in \mathrm{L}^\infty(\Omega)$ *be such that* $u_h \overset{*}{\rightharpoonup} u$ *in* $\mathrm{L}^\infty(\Omega)$ *and let* $\nu^{h_k} \overset{*}{\rightharpoonup} \nu$ *(note that, as* (u_h) *is bounded in* $\mathrm{L}^\infty(\Omega)$*, the sequence is tight and, by Remark 4.11, weak* convergence is equivalent to narrow convergence). Then*

$$u(x) = \int_{\mathbb{R}} \xi \, d\nu_x(\xi).$$

Proof. It follows from Remark 5.5, with $\varphi(\xi) := \xi$. $\square$

Proposition 5.7. (criterion of strong convergence in L^p) *Let* $u_h \overset{*}{\rightharpoonup} u$ *in* $\mathrm{L}^\infty(\Omega, \mu, \mathbb{R}^N)$ *and* $\nu^h \overset{*}{\rightharpoonup} \nu$*. Then*

$$u_h \to u \quad \textit{strongly in } \mathrm{L}^p(\Omega, \mu, \mathbb{R}^N), \ \forall p \in [1, +\infty) \Longleftrightarrow \nu_x = \delta_{u(x)} \ \mu\textit{-a.e.}$$

Proof. ($\Longrightarrow$) If $u_h \to u$ in $\mathrm{L}^p(\Omega, \mu, \mathbb{R}^N)$, then for every $\varphi \in C(\mathbb{R}^N)$ $\varphi \circ u_h \to \varphi \circ u$ in $\mathrm{L}^p(\Omega)$. On the other hand, by Remark 5.5, $\varphi \circ u_h \rightharpoonup \bar{\varphi}$ in $\mathrm{L}^p(\Omega)$, hence

$$\varphi(u(x)) = \bar{\varphi}(x) = \int_{\mathbb{R}^N} \varphi(\xi) \, d\nu_x(\xi) \quad \mu\textit{-a.e.},$$

i.e. $\nu_x = \delta_{u(x)}$ μ-a.e.

($\Longleftarrow$) Clearly it suffices to prove it for $p > 1$. If $\nu_x = \delta_{u(x)}$ μ-a.e., by Theorem 5.4 with $\varphi(\xi) := |\xi|^p$, we have

$$|u_h|^p \rightharpoonup |u|^p \quad \text{in } \mathrm{L}^1(\Omega)$$

and this implies

$$\lim_h \|u_h\|_{\mathrm{L}^p(\Omega)} = \|u\|_{\mathrm{L}^p(\Omega)} \tag{5.2}$$

As $u_h \rightharpoonup u$ in $\mathrm{L}^p(\Omega, \mu, \mathbb{R}^N)$, it follows from (5.2) that $u_h \to u$ in $\mathrm{L}^p(\Omega, \mu, \mathbb{R}^N)$. $\square$

Remark 5.8. If (u_h) is bounded in $\mathrm{L}^1(\Omega, \mu, \mathbb{R}^N)$, then it is tight: it suffices to apply Proposition 4.7 with $\varphi(\xi) := |\xi|$.

Theorem 5.9. *Let* (u_h) *be equiintegrable in* $\mathrm{L}^1(\Omega, \mu, \mathbb{R}^N)$*. Then there exists a subsequence* (ν^{h_k}) *such that:*

(1) $\nu^{h_k} \to \nu$ narrowly;

(2) $\int_{\mathbb{R}^N} \xi \, d\nu_x(\xi) < \infty \quad \mu$-a.e.;

(3) $u_{h_k} \rightharpoonup u(x) := \int_{\mathbb{R}^n} \xi \, d\nu_x(\xi)$ in $\mathrm{L}^1(\Omega, \mu, \mathbb{R}^N)$.

Proof. As (u_h) is bounded in $\mathrm{L}^1(\Omega, \mu, \mathbb{R}^N)$, then, by Remark 5.8, (u_h) is tight, hence we have (1).

(2) follows from Theorem 5.4(i) with $\varphi(\xi) := |\xi|$, while (3) follows from Theorem 5.4(ii), with $\varphi(\xi) := \xi_j$, $j = 1, \ldots, N$. $\square$

Theorem 5.10. (criterion of strong convergence in L^1) *Let $u_h \rightharpoonup u$ in $\mathrm{L}^1(\Omega,\mu,\mathrm{I\!R}^N)$. If for every (ν^{h_k}) such that $\nu^{h_k} \to \nu$ narrowly, ν is the Young measure associated to a measurable function $v : \Omega \to \mathrm{I\!R}^N$, then $v = u$ and $u_h \to u$ in $\mathrm{L}^1(\Omega,\mu,\mathrm{I\!R}^N)$.*

Proof. $v \in \mathrm{L}^1(\Omega,\mu,\mathrm{I\!R}^N)$. Indeed

$$\int_\Omega |v|\,d\mu = \int_{\Omega\times S} |\xi|\,d\nu \le \liminf_h \int_\Omega |u_h(x)|\,d\mu < \infty.$$

If $\nu^{h_k} \to \{\delta_{v(x)}\}_{x\in\Omega}$ narrowly, then, by the previous theorem, $u_h \rightharpoonup v$ in $\mathrm{L}^1(\Omega,\mu,\mathrm{I\!R}^N)$, hence $v = u$.

By Proposition 4.3. $u_{h_k} \to u$ in measure and then, up to a subsequence,

$$u_{h_k} \to u \quad \mu\text{-a.e.} \tag{5.3}$$

The assertion of the theorem follows from (5.3) and the equiintegrability hypothesis. □

Theorem 5.11. (Weak lower semicontinuity in L^1) *Let $u_h \rightharpoonup u$ in $\mathrm{L}^1(\Omega,\mu,\mathrm{I\!R}^N)$ and let $v_h, v : \Omega \to S$ be measurable functions, with $v_h \to v$ in measure. If $\psi \in \mathcal{G}(\Omega,\mathcal{F},S\times\mathrm{I\!R}^N)$ is such that*

(i) *$\psi(x,v(x),\cdot)$ is convex μ-a.e. x;*

(ii) *$(\psi^-(\cdot,v_h(\cdot),u_h(\cdot)))$ is equiintegrable;*

then

(a) $\liminf_h \int_\Omega \psi(x,v_h(x),u_h(x))\,d\mu < +\infty$

$\Rightarrow \int_\Omega \psi^+(x,v(x),u(x))\,d\mu < +\infty;$

(b) $\int_\Omega \psi(x,v(x),u(x))\,d\mu \le \liminf_h \int_\Omega \psi(x,v_h(x),u_h(x))\,d\mu.$

Proof. We can suppose, up to subsequences, that

- $\exists \lim_h \int_\Omega \psi(x,v_h(x),u_h(x))\,d\mu$
- $\{\delta_{v_h(x)}\}_{x\in\Omega} \to \{\delta_{v(x)}\}_{x\in\Omega}$ narrowly in $\mathcal{Y}(\Omega,\mu,S)$
- $\{\delta_{u_h(x)}\}_{x\in\Omega} \to \{\nu_x\}_{x\in\Omega}$ narrowly in $\mathcal{Y}(\Omega,\mu,\mathrm{I\!R}^N)$.

Then, it can be checked that

$$\{\delta_{v_h(x)}\otimes\delta_{u_h(x)}\}_{x\in\Omega} \to \{\delta_{v(x)}\otimes\nu_x\}_{x\in\Omega} \ \text{ narrowly in } \mathcal{Y}(\Omega,\mu,S\times\mathrm{I\!R}^N). \tag{5.4}$$

By Theorem 5.2, we have

$$\liminf_h \int_\Omega \psi(x,v_h(x),u_h(x))\,d\mu < +\infty \Rightarrow$$

$$\int_\Omega \left(\int_{\mathrm{I\!R}^N} \psi^+(x,v(x),\xi)\,d\nu_x(\xi)\right)\,d\mu < +\infty \tag{5.5}$$

$$\int_\Omega \left(\int_{\mathrm{I\!R}^N} \psi(x,v(x),\xi)\,d\nu_x(\xi)\right)\,d\mu \le \liminf_h \int_\Omega \psi(x,v_h(x),u_h(x))\,d\mu. \tag{5.6}$$

On the other hand, by Jensen's inequality,

$$
\begin{aligned}
&\int_\Omega \left(\int_{\mathbb{R}^N} \psi(x, v(x), \xi)\, d\nu_x(\xi) \right) d\mu \\
(5.7) \qquad &\geq \int_\Omega \psi \left(x, v(x), \int_{\mathbb{R}^N} \xi\, d\nu_x(\xi) \right) d\mu \\
&= \int_\Omega \psi(x, v(x), u(x))\, d\mu,
\end{aligned}
$$

hence, (b) follows from (5.6) and (5.7), while (a) follows from (5.5) and (5.7), with ψ replaced by ψ^+. $\square$

Finally, we give another result concerning strong convergence in L^1.

Theorem 5.12. (Visintin) *If $u_h \rightharpoonup u$ in $\mathrm{L}^1(\Omega, \mu, \mathbb{R}^N)$, then*

$$u(x) \in \overline{co}\,(K\text{-}\limsup_h (u_h(x))) := C(x) \quad \mu\text{-a.e.}\, x.$$

If $u(x)$ is an extreme point of $C(x)$ for μ-a.e. x, then $u_h \to u$ strongly in $\mathrm{L}^1(\Omega, \mu, \mathbb{R}^N)$.

Proof. By Theorem 5.9, we have

$$u(x) = \int_{\mathbb{R}^N} \xi\, d\nu_x(\xi), \tag{5.8}$$

with ν_x carried by K-$\limsup_h \{u_h(x)\}$ (see Proposition 3.18), hence $u(x) \in C(x)$.

If $u(x)$ is an extreme point of $C(x)$, then there exists a sequence (D_h) of closed convex sets in $C(x)$, such that $\bigcup_{h \in \mathbb{N}} D_h = C(x) \setminus \{u(x)\}$. Let $h \in \mathbb{N}$ be fixed. To obtain a contradiction, suppose $\nu_x(D_h) > 0$. If $\nu_x(D_h)$ were equal to 1, then $u(x)$ would belong to D_h, by (5.8). So $0 < \nu_x(D_h) < 1$ and we can define

$$\lambda_1 := \frac{\nu_x \llcorner D_h}{\nu_x(D_h)}, \qquad \lambda_2 := \frac{\nu_x \llcorner (\mathbb{R}^N \setminus D_h)}{1 - \nu_x(D_h)}.$$

Thus we have

$$u(x) = \int_{\mathbb{R}^N} \xi\, d\nu_x(\xi) = \nu_x(D_h) \int_{\mathbb{R}^N} \xi\, d\lambda_1 + (1 - \nu_x(D_h)) \int_{\mathbb{R}^N} \xi\, d\lambda_2,$$

with $u(x) \neq \int_{\mathbb{R}^N} \xi\, d\lambda_1$, as $\int_{\mathbb{R}^N} \xi\, d\lambda_1 \in D_h$. So we get a contradiction since $u(x)$ is extremal. Then $\nu_x(D_h) = 0$ for any $h \in \mathbb{N}$, hence $\nu_x(\{u(x)\}) = 1$, *i.e.* $\nu_x = \delta_{u(x)}$. The second part of the assertion follows from Theorem 5.10. $\square$

6. The Biting Lemma.

If (u_h) is a bounded sequence in $\mathrm{L}^1(\Omega, \mu, \mathbb{R}^n)$, then it is tight, so there exists a subsequence (u_{h_k}) such that $\delta_{u_{h_k}} \to \nu$ narrowly. In this chapter we will investigate the connection between (u_h) and the function $u(x) := \int_{\mathbb{R}^n} \xi\, d\nu_x(\xi)$.

Theorem 6.1. (Biting Lemma) *Let (u_h) be bounded in $\mathrm{L}^1(\Omega, \mu, \mathbb{R}^n)$. Then there exists a subsequence (u_{h_k}) and a decreasing sequence of sets (A_k) with $\lim_k \mu(A_k) = 0$ such that $(\chi_{\Omega \setminus A_k} u_{h_k})$ is equiintegrable.*

Proof. For $j \in \mathbb{N}$ let

$$g_j(t) := \sup_{h \geq j} \int_{\{|u_h| > t\}} |u_h| \, d\mu.$$

$g_j(\cdot)$ is bounded and decreasing, then there exists

$$\alpha := \lim_{t \to +\infty} g_0(t) = \lim_{t \to +\infty} g_j(t) < +\infty.$$

Let (t_k) be a strictly increasing sequence with $t_k \to +\infty$ such that $\alpha \leq g_0(t_k) \leq \alpha + \frac{1}{k}$, for each $k \in \mathbb{N}$. Since $g_j(t) \geq \alpha$ for every $t \in \mathbb{R}$, $j \in \mathbb{N}$, there exists an increasing sequence of integers (m_k) such that $m_k \geq j$ and

$$\int_{\{|u_{m_k}| > t_k\}} |u_{m_k}| \, d\mu \geq \alpha - \frac{1}{k}.$$

Let $B_k := \{|u_{m_k}| > t_k\}$, then

$$t_k \mu(B_k) \leq \int_{B_k} |u_{m_k}| \, d\mu \leq \sup_{h \in \mathbb{N}} \int_{\Omega} |u_h| \, d\mu \leq C,$$

hence $\lim_k \mu(B_k) = 0$. Let us show that $(\chi_{\Omega \setminus B_k} u_{m_k})$ is equiintegrable. Set

$$g(t) := \sup_{k \in \mathbb{N}} \int_{(\Omega \setminus B_k) \cap \{|u_{m_k}| \geq t\}} |u_{m_k}| \, d\mu.$$

Since $g(\cdot)$ is decreasing, it suffices to verify that $\lim_j g(t_j) = 0$. We have

$$\begin{aligned} g(t_j) &= \sup_{k \geq j} \int_{\{t_j \leq |u_{m_k}| \leq t_k\}} |u_{m_k}| \, d\mu \\ &= \sup_{k \geq j} \Big(\int_{\{t_j \leq |u_{m_k}|\}} |u_{m_k}| \, d\mu - \int_{\{t_k < |u_{m_k}|\}} |u_{m_k}| \, d\mu \Big) \\ &\leq \sup_{k \geq j} \Big(\alpha + \frac{1}{j} - (\alpha - \frac{1}{k})\Big) = \frac{2}{j} \to 0. \end{aligned}$$

Choose (t_k) such that $\mu(B_k) \leq 2^{-k}$ and set $A_k := \cup_{j \geq k} B_j$. Then (A_k) is decreasing, $\lim_k \mu(A_k) = 0$ and $(\chi_{\Omega \setminus A_k} u_{m_k})$ is still equiintegrable. $\square$

Remark 6.2. The previous Lemma can be reformulated as follows: if (u_h) is bounded in $\mathrm{L}^1(\Omega, \mu, \mathbb{R}^n)$, then there exists (u_{h_k}) such that

$$\forall \delta > 0 \ \exists A : \mu(A) < \delta \text{ and } u_{h_k} \rightharpoonup u \text{ in } \mathrm{L}^1(\Omega \setminus A, \mu, \mathbb{R}^n).$$

Corollary 6.3. *Let (u_h) be bounded in $\mathrm{L}^1(\Omega,\mu,\mathbb{R}^n)$. Then there exists a subsequence (u_{h_k}) such that $\nu_{h_k} := \{\delta_{u_{h_k}(x)}\}_{x\in\Omega} \to \nu$ narrowly and $v_k := \chi_{\Omega\setminus A_k} u_{h_k} \rightharpoonup u$ in $\mathrm{L}^1(\Omega,\mu,\mathbb{R}^n)$ with $u(x) = \int_{\mathbb{R}^n} \xi \, d\nu_x(\xi)$ a.e..*

Proof. By the Biting Lemma and the tightness of (u_h), it suffices to show that $\tau_k := \{\delta_{v_k(x)}\}_{x\in\Omega}$ converges narrowly to ν. For any $\psi \in \mathcal{G}_C(\Omega,\mathcal{F},\mathbb{R}^n)$, we have

$$\begin{aligned} \Big|\int_{\Omega\times\mathbb{R}} \psi \, d\tau_k - \int_{\Omega\times\mathbb{R}} \psi \, d\nu_{h_k}\Big| &= \Big|\int_{A_k} (\psi(x,0) - \psi(x,u_{h_k}))\, d\mu\Big| \\ &\le \mu(A_k) 2 \|\psi\|_{\mathrm{L}^\infty(\Omega\times\mathbb{R}^n)} \to 0, \end{aligned}$$

then $\tau_k \to \nu$ narrowly. □

Definition 6.4. Let $u_h, u \in \mathrm{L}^1(\Omega,\mu,\mathbb{R}^n)$, $h \in \mathbb{N}$. We say that (u_h) *biting converges* to u (and we write $u_h \xrightarrow{b} u$), if there exists a decreasing subsequence of sets (A_k), with $\lim_k(A_k) = 0$ such that $u_h \rightharpoonup u$ in $\mathrm{L}^1(\Omega\setminus A_k,\mu,\mathbb{R}^n)$, $\forall k \in \mathbb{N}$.

We give another characterization of the limit function u. Let us consider now a sequence of continuous functions $g_k : \mathbb{R}^n \to \mathbb{R}^n$ with the following properties:

(1) $g_k(\xi) \to \xi$ as $k \to +\infty$, $\forall \xi \in \mathbb{R}^n$;

(2) $|g_k(\xi)| \le c(1+|\xi|)$ $\forall k \in \mathbb{N}, \forall \xi \in \mathbb{R}^n$;

(3) $\lim_{|\xi|\to+\infty} \frac{|g_k(\xi)|}{|\xi|} = 0$ $\forall k \in \mathbb{N}$.

(For example one can think of truncations:

$$g_k(\xi) = \begin{cases} \min\{|\xi|, k\} \dfrac{\xi}{|\xi|} & \xi \ne 0 \\ 0 & \xi = 0 \end{cases}).$$

Proposition 6.5. *Let (u_h) be bounded in $\mathrm{L}^1(\Omega,\mu,\mathbb{R}^n)$ with $u_h \xrightarrow{b} u$ and $\nu_h := \{\delta_{u_h(x)}\}_{x\in\Omega} \to \nu$ narrowly. Then there exists (v_k) in $\mathrm{L}^1(\Omega,\mu,\mathbb{R}^n)$ such that*

(i) $g_k(u_h) \rightharpoonup v_k$ for $h \to +\infty$ weakly in $\mathrm{L}^1(\Omega,\mu,\mathbb{R}^n)$;

(ii) $v_k \to u$ for $k \to +\infty$ strongly in $\mathrm{L}^1(\Omega,\mu,\mathbb{R}^n)$.

Proof. Let $k \in \mathbb{N}$ be fixed. We have

$$\int_{\{|u_h|>t\}} |g_k(u_h)| \, d\mu \le \sup_{|\xi|>t} \frac{|g_k(\xi)|}{|\xi|} \sup_{h\in\mathbb{N}} \|u_h\|_{\mathrm{L}^1(\Omega)},$$

thus

$$\lim_{t\to+\infty} \sup_{h\in\mathbb{N}} \int_{\{|u_h|>t\}} |g_k(u_h)| \, d\mu = 0.$$

Since for t large enough $|g_k(t)| < |\xi|$ for all $\xi > t$, we have

$$\{|u_h| > t\} \supseteq \{|g_k(u_h)| > t\},$$

$(g_k(u_h))$ is equiintegrable. Then, by Theorem 5.4,

$$g_k(u_h) \rightharpoonup v_k \text{ for } h \to +\infty \text{ weakly in } \mathrm{L}^1(\Omega,\mu,\mathbb{R}^n),$$

where $v_k(x) := \int_{\mathbb{R}^n} g_k(\xi)\, d\nu_x(\xi)$. Since

$$
\begin{aligned}
&|g_k(\xi)| \le c(1+|\xi|),\\
&g_k(\xi) \to \xi,
\end{aligned}
$$

by Lebesgue's Theorem, $\lim_k v_k(x) = \int_{\mathbb{R}^n} \xi\, d\nu_x(\xi) = u(x)$. Let $F(x) := \int_{\mathbb{R}^n} |\xi|\, d\nu_x(\xi)$. By Theorem 5.4, $F \in \mathrm{L}^1(\Omega)$. We have

$$
\begin{aligned}
&v_k \to u(x),\\
&|v_k(x)| \le c(1+F(x)),
\end{aligned}
$$

then, by Lebesgue's Theorem, $v_k \to u$ in $\mathrm{L}^1(\Omega,\mu,\mathbb{R}^n)$. □

Examples 6.6.

1) We point out that, for a bounded sequence (u_h) in $\mathrm{L}^1(\Omega,\mu,\mathbb{R}^n)$, what we stated in Remark 2.5.2 does not imply that for any $\delta > 0$ we can find $A \subseteq \Omega$ with $\mu(A) < \delta$ such that (u_h) is weakly converging in $\mathrm{L}^1(\Omega \setminus A,\mu,\mathbb{R}^n)$.
Take for example $u_{jk} : (0,1) \to \mathbb{R}$ defined by

$$
u_{jk} = \begin{cases} \dfrac{1}{q_k} & x \in (q_j - q_k, q_j + q_k) \cap (0,1) \\ 0 & \text{otherwise,} \end{cases}
$$

where $(q_j) = \mathbf{Q} \cap (0,+\infty)$. For each $j,k \in \mathbb{N}$ we have $\int_{(0,1)} |u_{jk}|\, dx \le 2$. What we are going to show is that, given $A \subseteq (0,1)$ with $|A| > 0$, for any $t > 0$ there exist infinitely many (j,k) such that

$$
\int_{\{|u_{jk}|>t\}\cap A} |u_{jk}|\, dx \ge 1.
$$

Indeed, let $\overline{x} \in A$ be a Lebesgue point for χ_A, *i.e.*

$$
\lim_{\rho\to 0} \frac{|A \cap (\overline{x}-\rho, \overline{x}+\rho)|}{2\rho} = 1,
$$

then there exists $\rho \in (0, \frac{1}{t})$ such that

$$
|A \cap (\overline{x}-\rho, \overline{x}+\rho)| \ge 2\rho\frac{3}{4}.
$$

Let (q_{j_h}), (q_{k_h}) with $j_h \ne k_h$ $\forall h \in \mathbb{N}$, be such that

$$
\begin{aligned}
&q_{j_h} \to \overline{x};\\
&q_{k_h} \to \rho;\\
&q_{k_h} < \frac{1}{t},
\end{aligned}
$$

then

$$
\begin{aligned}
\int_{\{|u_{j_h k_h}|>t\}\cap A} |u_{j_h k_h}|\, dx &= \frac{1}{q_{k_h}} |(q_{j_h} - q_{k_h}, q_{j_h} + q_{k_h}) \cap A| \\
&\to \frac{1}{\rho} |(\overline{x}-\rho, \overline{x}+\rho) \cap A| \ge \frac{3}{2}.
\end{aligned}
$$

2) Let (u_h) be bounded in $L^1(\Omega)$. By Theorem 2.1.3 and by the Biting Lemma, up to a subsequence, $\mu_h := u_h dx \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}_b(\Omega)$ and $u_h \overset{b}{\to} u$. But this does not imply that $\mu = udx$ as the following example shows. Let $u_j : (0,1) \to \mathbb{R}$ be defined by

$$u_j(x) = \begin{cases} \frac{j^2}{2} & x \in [\frac{k}{j+1} - \frac{1}{j^3}, \frac{k}{j+1} + \frac{1}{j^3}],\ k = 1, ..., j \\ 0 & \text{otherwise.} \end{cases}$$

Set $A_k := \{x : |u_k(x)| \neq 0\}$, $\lim_k |A_k| = 0$, hence $u_j \overset{b}{\to} u \equiv 0$. On the other hand $u_j dx \overset{*}{\rightharpoonup} dx$.

Essential References

S. Müller, Variational models for microstructure and phase transitions, to appear.

P. Pedregal, *Parametrized Measures and Variational Principles.* Progress in Nonlinear Differential Equations and their Applications. **30**. Basel, Birkhäuser, 1997.

M. Valadier, Young measures, in *Methods of Nonconvex Analysis, Lect. 1st Sess. CIME, Varenna/Italy 1989, Lect. Notes Math.* **1446**, 152-188.

L.C. Young, Generalized curves and the existence of an attained absolute minimum in the calculus of variations, *Comptes Rendus Soc. Sci. Lettres Varsovie* **30** (1937), 212-234