FRACTURE MODELS AS Γ-LIMITS OF DAMAGE MODELS

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Abstract. We analyze the asymptotic behavior of a variational model for damaged elastic materials. This model depends on two small parameters, which govern the width of the damaged regions and the minimum elasticity constant attained in the damaged regions. When these parameters tend to zero, we find that the corresponding functionals Γ-converge to a functional related to fracture mechanics. The corresponding problem is brittle or cohesive, depending on the asymptotic ratio of the two parameters.

1. Introduction. In damage models for linearly elastic materials the elasticity tensor depends continuously and monotonically on an internal variable $v$. This is a function defined on the reference configuration $\Omega \subset \mathbb{R}^n$ with values in the interval $[0,1]$. We assume that $v = 1$ corresponds to the original elastic material, while $v = 0$ represents the totally damaged material, with vanishing elasticity tensor. In the regions where $v$ is very small, the displacement gradient $\nabla u$ of the solution $u$ of a stationary elastic problem can be very large, so we expect that $u$ develops a jump discontinuity when $v$ tends to zero.

To study this problem in a definite setting, we consider a damage model depending on a small parameter $\epsilon$, and investigate the behavior of the displacement $u_\epsilon$ and of the internal variable $v_\epsilon$ as $\epsilon \to 0$. We assume that the model forces damage concentration, so that $v_\epsilon \to 1$ in $L^1(\Omega)$. We assume also that $\min v_\epsilon \to 0$. In other words, there are regions with smaller and smaller volume where the elasticity tensor tends to zero. It is possible that $\nabla u_\epsilon$ becomes larger and larger in these regions and that $u_\epsilon$ converges in $L^1(\Omega)$ to some function $u$ that exhibits jump discontinuities along sets of codimension one. We expect that $u$ can be considered as the displacement obtained in some fracture model.

In this paper we consider this problem in the simplest situation: the antiplane case for a homogeneous and isotropic material. Then the displacement $u$ is scalar, and the elasticity tensor reduces to a single constant. We also assume that the material remains isotropic during the damage process, so that the internal variable $v$ can be chosen equal to the elasticity coefficient of the damaged material, up to a multiplicative constant.

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This paper is dedicated to Igor V. Skrypnik.
The stored elastic energy corresponding to the displacement $u$ and to the internal variable $v$ is then given by
\[ \int_{\Omega} v |\nabla u|^2 dx. \]
The energy dissipated by the damage process has the form
\[ \int_{\Omega} \phi_{\epsilon}(v) dx, \]
where $\phi_{\epsilon} : [0, 1] \to \mathbb{R}$ is strictly decreasing and satisfies $\phi_{\epsilon}(1) = 0$. The total energy for the damage model is then
\[ \int_{\Omega} v |\nabla u|^2 dx + \int_{\Omega} \phi_{\epsilon}(v) dx. \]

To force damage concentration as $\epsilon \to 0$ we assume that $\phi_{\epsilon} = \frac{1}{\epsilon} \psi$ for some strictly decreasing function $\psi : [0, 1] \to \mathbb{R}$ satisfying $\psi(1) = 0$. In the limit as $\epsilon \to 0$ this leads to the condition $v = 1$ $\mathcal{L}^n$-a.e. in $\Omega$. Under this assumption the total energy is
\[ \int_{\Omega} v |\nabla u|^2 dx + \frac{1}{\epsilon} \int_{\Omega} \psi(v) dx. \]

It is reasonable to require that neighboring points are similarly damaged. For this reason we assume that the internal variable $v$ satisfies an inequality like
\[ |\nabla v| \leq a_\epsilon \text{ a.e. on } \Omega, \]
for a certain constant $a_\epsilon$. In this paper we develop the case $a_\epsilon = 1/\epsilon$. The arguments for our proofs show that, in this case, the damaged regions $\{v_\epsilon < \frac{1}{2}\}$ tend to concentrate around sets of codimension one, on strips whose width is proportional to $\epsilon$.

Finally, we suppose that the material is never totally damaged. This leads to the condition
\[ \eta_\epsilon \leq v \leq 1, \]
where $\eta_\epsilon$ is a positive constant. We assume that $\eta_\epsilon / \epsilon \to \alpha$, with $\alpha \in [0, \infty]$.

In this model a solution of the stationary damage problem is a minimizer of $1$ under the constraints $2$ and $3$. To study their asymptotic behavior as $\epsilon \to 0$ we fix two sequences $\epsilon_k > 0$ and $\eta_k > 0$, with $\epsilon_k \to 0$ and $\eta_k \to 0$, and we determine the $\Gamma$-limit in $L^1(\Omega) \times L^1(\Omega)$ of the sequence of functionals defined by
\[ F_k(u, v) := \int_{\Omega} v |\nabla u|^2 dx + \frac{1}{\epsilon_k} \int_{\Omega} \psi(v) dx \]
if $u \in H^1(\Omega)$ and $v$ satisfies the constraints
\[ \eta_k \leq v \leq 1 \quad \text{and} \quad |\nabla v| \leq \frac{1}{\epsilon_k} \mathcal{L}^n \text{-a.e. on } \Omega, \]
and by $F_k(u, v) := +\infty$ if these conditions are not satisfied. We assume that $\eta_k / \epsilon_k \to \alpha$, with $\alpha \in [0, \infty]$. Moreover we assume that $\Omega$ is a bounded open set with Lipschitz boundary and that $\psi$ satisfies a very mild technical condition, which is fulfilled in the standard examples $\psi(z) = 1 - z^\beta$, with $\beta > 0$.

The $\Gamma$-limit depends on $\alpha$. Its domain is contained in the spaces $GSBV^2(\Omega)$, $SBV^2(\Omega)$, or $H^1(\Omega)$, depending on the value of $\alpha$. For the definition of the first two spaces we refer to 7 in Section 2, which contains also a short account on the notation.
used for free discontinuity problems. To describe the $\Gamma$-limit when $0 < \alpha < \infty$ we introduce the functional $\Phi_\alpha : L^1(\Omega) \to [0, +\infty)$ defined as follows:

$$\Phi_\alpha(u) := \begin{cases} \int_\Omega |\nabla u|^2 \, dx + a \mathcal{H}^{n-1}(J_u) + b_\alpha \int_{J_u} |u| \, d\mathcal{H}^{n-1} & \text{if } u \in SBV^2(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$a := 2 \int_0^1 \psi(s) \, ds \quad \text{and} \quad b_\alpha := 2(\alpha \psi(0))^{1/2}.$$  

In the limiting cases $\alpha = 0$ and $\alpha = \infty$ we define

$$\Phi_0(u) := \begin{cases} \int_\Omega |\nabla u|^2 \, dx + a \mathcal{H}^{n-1}(J_u) & \text{if } u \in GSBV^2(\Omega) \cap L^1(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$\Phi_\infty(u) := \begin{cases} \int_\Omega |\nabla u|^2 \, dx & \text{if } u \in H^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

First we prove the following theorem (see Theorem 3.1).

**Theorem 1.1.** The $\Gamma$-limit of $(F_k)$ in $L^1(\Omega) \times L^1(\Omega)$ is given by

$$F_\alpha(u, v) := \begin{cases} \Phi_\alpha(u) & \text{if } v = 1 \mathcal{L}^n\text{-a.e. in } \Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

For $\alpha = 0$ the limit functional is used to determine stationary solutions in some brittle fracture models (see [7]). For $0 < \alpha < \infty$ the limit functional is related to fracture models with a cohesive zone. It is also used in the study of plastic slips (see [4]). For $\alpha = \infty$ the limit functional corresponds to an elasticity problem without cracks.

In the case $\alpha = 0$ our result is similar to the approximation of $\Phi_0$ obtained in [6] (see also [5]) by means of the functionals

$$G_\epsilon(u, v) = \int_\Omega (v^2 + \eta_\epsilon) |\nabla u|^2 \, dx + \epsilon \int_\Omega |\nabla v|^2 \, dx + \frac{1}{4 \epsilon} \int_\Omega (v - 1)^2 \, dx. \quad (5)$$

In our result the integral term $\epsilon \int_\Omega |\nabla v|^2 \, dx$ is replaced by the constraint $|\nabla v| \leq 1/\epsilon$.

To our knowledge, no result has been proved for 5 in the case $0 < \alpha < \infty$. In [4] the functional $\Phi_\alpha$, with $0 < \alpha < \infty$, has been obtained as $\Gamma$-limit of the sequence

$$G_\epsilon(u, v) = \int_\Omega (v^2 + \eta_\epsilon) |\nabla u|^2 \, dx + \epsilon \int_\Omega |\nabla v|^2 \, dx + \frac{a}{4 \epsilon} \int_\Omega (v - 1)^2 \, dx + b_\alpha \int_\Omega (v - 1)^2 |\nabla u| \, dx.$$

Theorem 1.1 enables us to prove the following result about the convergence of minimizers of some variational problems involving the functionals $F_k$ and $F_\alpha$ (see Theorem 7.1).

**Theorem 1.2.** Let $q > 1$. For every $k$, let $(u_k, v_k)$ be a minimizer of the functional

$$\int_\Omega v |\nabla u|^2 \, dx + \frac{1}{\epsilon_k} \int_\Omega \psi(v) \, dx + \int_\Omega |u - g|^q \, dx \quad (6)$$

with the constraints 4. Then $v_k \rightharpoonup 1$ in $L^1(\Omega)$ and a subsequence of $(u_k)$ converges in $L^q(\Omega)$ to a minimizer $u$ of the following limit problem:

$$\min_{u \in SBV^2(\Omega)} \left( \int_\Omega |\nabla u|^2 \, dx + a \mathcal{H}^{n-1}(J_u) + b_\alpha \int_{J_u} |u| \, d\mathcal{H}^{n-1} + \int_\Omega |u - g|^q \, dx \right).$$
if $0 < \alpha < \infty$; whereas in the extreme cases $\alpha = 0$ and $\alpha = \infty$ the limit problems are

$$
\min_{u \in GSBV^2(\Omega)} \left( \int_\Omega |\nabla u|^2 dx + a\mathcal{H}^{n-1}(J_u) + \int_\Omega |u - g|^q dx \right),
$$

$$
\min_{u \in H^1(\Omega)} \left( \int_\Omega |\nabla u|^2 dx + \int_\Omega |u - g|^q dx \right),
$$

respectively. Moreover for $0 \leq \alpha \leq \infty$ the minimum values of (6) with the constraints $\frac{q}{\alpha}$ tend to the minimum value of the limit problem.

The paper is composed of seven sections. After a brief introduction, in Section 2 we fix the notation for functions with bounded variation. In Section 3 we state the $\Gamma$-convergence result, which is proved in Sections 4, 5, and 6. In particular in Section 4 we face the one-dimensional problem in the cases $\alpha = 0, 0 < \alpha < \infty$, and $\alpha = \infty$; in Section 5 we prove the $\Gamma$-lim inf inequality in the $n$-dimensional case by a slicing argument, whereas in Section 6 we prove the $\Gamma$-lim sup inequality through the construction of a recovery sequence. Finally we deal with the convergence of minimisers in Section 7.

2. Notation and Preliminaries. Let $n \geq 1$ be a fixed integer. The Lebesgue measure and the $k$-dimensional Hausdorff measure in $\mathbb{R}^n$ are denoted by $\mathcal{L}^n$ and $\mathcal{H}^k$. For the general properties of the Hausdorff measure we refer to [13] and [12].

The open ball of $\mathbb{R}^n$ with centre $x$ and radius $r$ is indicated by $B(x, r)$ or $B_r(x)$; if $x = 0$, we write also $B_r$ in place of $B_r(0)$. The Lebesgue measure of the unit ball of $\mathbb{R}^n$ and the $(n-1)$-dimensional Hausdorff measure of its boundary are denoted by $\omega_n$ and $S^{n-1}$. Moreover let $d(x, E)$ be the Euclidean distance of the point $x$ from the set $E \subset \mathbb{R}^n$, let $diam(E)$ be the diameter of $E$, and let $E \triangle F$ be the symmetric difference of $E$ and $F$. The symbols $\lor$ and $\land$ denote the maximum and the minimum operators respectively.

For the general theory of $BV$-functions we refer to [3], [13], [14] and [12]; here we just recall the notation and some results we use in the sequel.

Let $\Omega$ be an open subset of $\mathbb{R}^n$. For every $u \in BV(\Omega)$ the distributional gradient $Du$, the one-sided approximate limits $u^+$ and $u^-$, the approximate differential $\nabla u$ and the jump set $J_u$ are defined in [3, Sections 3.1, 3.6]. Moreover the jump function $u^+ - u^-$ is denoted by $|u|$.

The jump set $J_u$ is countably $\mathcal{H}^{n-1}$-rectifiable according to [3, Definition 2.57]. Moreover there exists a Borel function $\nu : J_u \to S^{n-1}$ such that the vector $\nu$ is normal to $J_u$ in the sense that, if $M$ is a $C^1$-manifold of dimension $n-1$, then $\nu(x)$ is normal to $M$ for $\mathcal{H}^{n-1}$-a.e. $x \in M \cap J_u$. In particular the triplet $(u^+(x), u^-(x), \nu(x))$ is uniquely determined up to a change of sign of $\nu(x)$ and a simultaneous interchange between $u^+(x)$ and $u^-(x)$.

If $u \in BV(\Omega)$ then

$$
Du = D^a u + D^j u + D^c u,
$$

where $D^a u$ is absolutely continuous and $D^j u + D^c u$ is singular with respect to the Lebesgue measure; in particular $D^j u$ denotes the jump derivative of $u$ and

$$
D^j u = (u^+ - u^-)\nu|\mathcal{H}^{n-1}[J_u],
$$

whereas $D^c u$ is the Cantor part of the derivative of $u$ (see [3, Section 3.9]). In particular the approximate differential $\nabla u$ coincides with the density of $D^a u$. 
The spaces $SBV(\Omega)$, $GBV(\Omega)$, $GSBV(\Omega)$ are defined as in [3]. We recall that a $GBV$-function is weakly approximately differentiable $L^n$-a.e. in $\Omega$ (see [3, Definition 4.31, Theorem 4.34]). Since an approximately differentiable function $u$ is also weakly approximately differentiable and the approximate differential coincides with the weak approximate differential $L^n$-a.e. in $\Omega$, we also denote the weak approximate differential by $\nabla u$.

Let $p \in ]1, +\infty[$; let us define

$$SBV^p(\Omega) := \{ u \in SBV(\Omega) : \nabla u \in L^p(\Omega, \mathbb{R}^n) \text{ and } \mathcal{H}^{n-1}(J_u) < +\infty \},$$

$$GSBV^p(\Omega) := \{ u \in GSV(\Omega) : \nabla u \in L^p(\Omega, \mathbb{R}^n) \text{ and } \mathcal{H}^{n-1}(J_u) < +\infty \}.$$  \hspace{1cm} (7)

In the case $\Omega \subset \mathbb{R}$, if $u \in SBV^2(\Omega)$, then $u \in H^1(\Omega \setminus J_u)$. Conversely, if $\Omega \subset \mathbb{R}$ and there exists a finite set $F$ such that $u \in H^1(\Omega \setminus F)$, then $u \in SBV^2(\Omega)$ and $J_u \subset F$.

Finally for the $\Gamma$-convergence theory we refer to [11].

3. \indent {\bf The $\Gamma$-Convergence Result.} Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ and let $\epsilon_k > 0$, $\epsilon_k \to 0$. We shall study the $\Gamma$-limit in $L^1(\Omega) \times L^1(\Omega)$ of the sequence of functionals $F_k : L^1(\Omega) \times L^1(\Omega) \to [0, +\infty]$ defined by

$$F_k(u, v) := \begin{cases} \int_\Omega |\nabla u|^2 \, dx + \frac{1}{\epsilon_k} \int_\Omega \psi(v) \, dx & \text{if } (u, v) \in H^1(\Omega) \times V_k, \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\psi \in C^1([0, 1]) \text{ is strictly decreasing with } \psi(1) = 0,$$

$$V_k := \left\{ v \in W^{1,\infty}(\Omega) : \eta_k \leq v \leq 1, \ |\nabla v| \leq \frac{1}{\epsilon_k} \mathcal{L}^n \text{-a.e. in } \Omega \right\},$$

with $\eta_k \geq 0$, $\eta_k \to 0$. We assume that for every $c \geq 0$

the equation $s^2 \psi'(s) = -c$ has a finite number of solutions.  \hspace{1cm} (11)

We assume that the limit

$$\alpha := \lim_{k \to \infty} \frac{\eta_k}{\epsilon_k}$$

exists. For $0 < \alpha < \infty$ let $\Phi_{\alpha} : L^1(\Omega) \to [0, +\infty]$ is defined by

$$\Phi_{\alpha}(u) := \begin{cases} \int_\Omega |\nabla u|^2 \, dx + aH^{n-1}(J_u) + b_\alpha \int_{J_u} |u|dH^{n-1} & \text{if } u \in SBV^2(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$a := 2 \int_0^1 \psi(s) \, ds \quad \text{and} \quad b_\alpha := 2(\alpha \psi(0))^2.$$  \hspace{1cm} (14)

In the limiting cases $\alpha = 0$ and $\alpha = \infty$ we define

$$\Phi_0(u) := \begin{cases} \int_\Omega |\nabla u|^2 \, dx + aH^{n-1}(J_u) & \text{if } u \in GSBV^2(\Omega) \cap L^1(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

$$\Phi_\infty(u) := \begin{cases} \int_\Omega |\nabla u|^2 \, dx & \text{if } u \in H^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

We are now in a position to state the main result of the section.
Theorem 3.1. Assume (8)-(12) and assume that \( \Omega \) has Lipschitz boundary. The \( \Gamma \)-limit of \( (F_k) \) in \( L^1(\Omega) \times L^1(\Omega) \) exists and is given by
\[
F_\alpha(u, v) := \begin{cases} 
\Phi_\alpha(u) & \text{if } v = 1 \text{ \( \mathcal{L}^n \)-a.e. in } \Omega, \\
+\infty & \text{otherwise}.
\end{cases}
\] (16)

Theorem 3.1 is an immediate consequence of the estimates for the functionals
\[
F'_\alpha := \Gamma- \liminf_{k \to \infty} F_k \quad \text{and} \quad F''_\alpha := \Gamma- \limsup_{k \to \infty} F_k
\] (17)
contained in the following results.

Theorem 3.2. Assume (8)-(12). Let \( (u, v) \in L^1(\Omega) \times L^1(\Omega) \) such that \( F'_\alpha(u, v) < +\infty \). Then \( v = 1 \) \( \mathcal{L}^n \)-a.e. in \( \Omega \), \( u \in GSBV^2(\Omega) \cap L^1(\Omega) \), and
\[
\Phi_\alpha(u) \leq F'_\alpha(u, 1).
\] (18)

Theorem 3.3. Assume (8)-(12) and assume that \( \Omega \) has Lipschitz boundary. Let \( u \in GSBV^2(\Omega) \cap L^1(\Omega) \). Then the following estimate holds
\[
F''_\alpha(u, 1) \leq \Phi_\alpha(u).
\] (19)

Theorem 3.3 will be proved in Sections 4 and 6. Theorem 3.2 can be obtained as a consequence of the following proposition that will be proved in Sections 4 and 5.

Proposition 1. Assume (8)-(12). Let \( (u_k, v_k) \) be a sequence in \( L^1(\Omega) \times L^1(\Omega) \) such that
\[
(u_k, v_k) \to (u, v) \text{ in } L^1(\Omega) \times L^1(\Omega),
\]
\[
(F_k(u_k, v_k)) \text{ is bounded.}
\] (20)
Then \( u \in GSBV^2(\Omega) \cap L^1(\Omega) \), \( v = 1 \) \( \mathcal{L}^n \)-a.e. in \( \Omega \), and
\[
\int_{\Omega} |v_k| \|u_k\|^2 dx \leq \liminf_{k \to \infty} \int_{\Omega} v_k \|\nabla u_k\|^2 dx,
\] (22)
\[
\alpha \mathcal{H}^{n-1}(J_u) \leq \liminf_{k \to \infty} \frac{1}{\epsilon_k} \int_{\Omega} \psi(v_k) dx,
\] (23)
\[
\Phi_\alpha(u) \leq \liminf_{k \to \infty} \int_{\Omega} \left[ v_k \|\nabla u_k\|^2 + \frac{1}{\epsilon_k} \psi(v_k) \right] dx.
\] (24)

Remark 1. Estimates (22) and (23) cannot be deduced from (24), so that they require a direct proof.

Let us show that Proposition 1 implies Theorem 3.2.

Proof of Theorem 3.2. If \( F'_\alpha(u, v) < +\infty \), there exists a sequence \( (u_k, v_k) \) such that \( (u_k, v_k) \to (u, v) \) in \( L^1(\Omega) \times L^1(\Omega) \) and
\[
\liminf_{k \to \infty} F_k(u_k, v_k) = F'_\alpha(u, v).
\] (21)
Passing to a subsequence, not relabeled, we can assume that \( F_k(u_k, v_k) \to F'_\alpha(u, v) \), so that (21) holds. By Proposition 1 we have that \( u \in GSBV^2(\Omega) \cap L^1(\Omega) \), \( v = 1 \) \( \mathcal{L}^n \)-a.e. in \( \Omega \), and (18) follows from (24).

Remark 2. The hypothesis of Lipschitz boundary for \( \Omega \) is used only to state the estimate from above in the case \( n > 1 \). Indeed, in the proof of that estimate we shall use a local reflection argument and the approximation Theorem 6.1 which is proved under this hypothesis.
In the case $n = 1$ the functional $\Phi_\alpha$ used to express the $\Gamma$-limit $F_\alpha$ is finite only on the space $SBV^2(\Omega)$. This is stated in the following proposition which will be proved at the end of Section 4. Note that the last statements of Section 2 imply $SBV^2(\Omega) \subset L^\infty(\Omega)$ in the one-dimensional case.

**Proposition 2.** Let $\Omega \subset \mathbb{R}$ be a bounded open set. Then $\Phi_0(u) < +\infty$ if and only if $u \in SBV^2(\Omega)$.

**Remark 3.** Let us note that, if $n > 1$, then the inequality $F_\alpha(u, 1) < +\infty$ does not imply $u \in BV(\Omega)$ nor $u \in L^2(\Omega)$. Indeed, let $\Omega$ be a bounded open set in $\mathbb{R}^n$ and consider a sequence of pairwise disjoint balls $B_{r_i}(x_i)$, contained in $\Omega$, with centres $x_i$ and radii $r_i := 2^{-i}$. Moreover assume that also the balls $B_{3r_i}(x_i)$ are contained in $\Omega$ and pairwise disjoint. Let $u \in L^1(\Omega)$ be defined by

$$u(x) := \begin{cases} a_i & \text{if } x \in B_{r_i}(x_i), \\ 0 & \text{otherwise}, \end{cases}$$

where $a_i := 2^{(n-1)i}$. Clearly $u \in L^1(\Omega) \setminus L^2(\Omega)$. Moreover $u$ belongs to $GSBV(\Omega)$ but does not to $BV(\Omega)$ since

$$|D^j u| (\Omega) = \sum_{i=1}^{+\infty} a_i r_i^{-n-1} = +\infty.$$

Let $\sigma \geq 2$, $\epsilon_k := 2^{-nk}$, and $\eta_k := \epsilon_k^\sigma$; this implies $\alpha = 0$. Let us show that $F'_\alpha(u, 1) < +\infty$. To this aim let us consider $\delta_k := 2^{nk(1-\sigma)}$ and let us define $u_k$ as $a_i$ in $B_{r_i-\delta_k}(x_i)$, 0 out of $B_{r_i+\delta_k}(x_i)$, and with constant slope in $B_{r_i+\delta_k}(x_i) \setminus B_{r_i-\delta_k}(x_i)$, for $i \leq k$; we set $u_k := 0$ otherwise. Let $v_k$ be defined as $\eta_k$ in $B_{r_i+\delta_k}(x_i) \setminus B_{r_i-\delta_k}(x_i)$, with constant slope in $(B_{r_i+\delta_k+\epsilon_k(1-\eta_k)}(x_i) \setminus B_{r_i+\delta_k}(x_i)) \cup (B_{r_i-\delta_k}(x_i) \setminus B_{r_i-\delta_k-\epsilon_k(1-\eta_k)}(x_i))$, for $i \leq k$, and as 1 otherwise. Note that $(u_k, v_k) \in H^1(\Omega) \times V_k$ and $(u_k, v_k) \to (u, 1)$ in $L^1(\Omega) \times L^1(\Omega)$. A direct computation shows that

$$\liminf_{k \to \infty} F_k(u_k, v_k) < +\infty,$$

so that $F_\alpha(u, 1) < +\infty$.

**4. Proof of the $\Gamma$-Convergence Result in the Case $n = 1$.**

**Proof of Proposition 1.** It is sufficient to prove the statement when $\Omega$ is an interval, since the left-hand sides of (22), (23) and (24) are $\sigma$-additive with respect to $\Omega$, whereas the right-hand sides are $\sigma$-superadditive. Therefore we can assume $\Omega = [0, 1]$.

Let $(u_k, v_k)$ be a sequence satisfying (20) and (21) with bounding constant $c$. Note that $\psi(v_k) \to 0$ in $L^1(\Omega)$ by (8) and (21); as $\psi(v_k) \to \psi(v)$ in $L^1(\Omega)$ we deduce $v = 1$ $L^1$-a.e. on $\Omega$.

**Proof of (22).** It is not restrictive to assume that the lower limit in the right-hand side of (22) is actually a limit. Let us divide the proof into two steps.

(a) Since $v_k$ is a Lipschitz function, the set

$$B_k = \{ x \in \overline{\Omega} : v_k(x) > 1/2 \}$$

is relatively open in $\overline{\Omega}$. By Chebyshev inequality we get

$$\psi(1/2)\mathcal{L}^1(B_k) \leq \int_0^1 \psi(v_k) dx,$$
so that (8) and (21) imply
\[ \mathcal{L}^1(B_k^c) \to 0. \]
(26)
We write
\[ B_k = \bigcup_{1 \leq j \leq N_k} I^k_j \cup \bigcup_{j > N_k} J^k_j, \]
(27)
where \( I^k_1, \ldots, I^k_{N_k} \) are the connected components of \( B_k \) such that \( \mathcal{L}^1(I^k_j) \geq \epsilon_k/4 \), whereas \( J^k_j \) are the connected components satisfying the opposite inequality. Let \( a^k_j \) and \( b^k_j \) be the end points of the interval \( I^k_j \). By changing the numeration, we may assume that \( 0 \leq a^k_1 \leq b^k_1 \leq b^k_2 < b^k_2 < \cdots < a^k_{N_k} \leq b^k_{N_k} \leq 1 \). Moreover we set \( b^k_0 := 0 \) and \( a^k_{N_k+1} := 1 \).

By definition \( v_k \leq 1/2 \) on \( B_k^c \); moreover \( v_k \leq 3/4 \) on each \( J^k_j \), since at least one end point belongs to \( B_k^c \), the length of \( J^k_j \) is less than \( \epsilon_k/4 \), and \( |\nabla v_k| \leq 1/\epsilon_k \mathcal{L}^1 \)-a.e. in \( \Omega \) by (8), (9), and (21). Then \( v_k \leq 3/4 \) in \([a^k_j, a^k_{j+1}]\) for \( j = 0, \ldots, N_k \). From this estimate and from (21) it follows that
\[ \sum_{j > N_k} \mathcal{L}^1(J^k_j) \leq \frac{\epsilon_k c}{C_1}, \]
(28)
where \( C_1 := \psi(3/4) \).

Let us show that \((N_k)\) is bounded. To this aim we choose a point \( r_j \) in each interval \([b^k_{j-1}, a^k_j]\). We have \( v_k \leq 7/8 \) in \([r_j - \frac{\epsilon_k}{4k}, r_j + \frac{\epsilon_k}{4k}]\), since \( v_k(r_j) \leq 3/4 \) and \(|\nabla v_k| \leq 1/\epsilon_k \mathcal{L}^1 \)-a.e. in \( \Omega \). Then
\[ \frac{1}{\epsilon_k} \int_{r_j - \frac{\epsilon_k}{4k}}^{r_j + \frac{\epsilon_k}{4k}} \psi(v_k) dx \geq C_2, \]
where \( C_2 := 1/4 \psi(7/8) \). We note that the intervals \([r_j - \frac{\epsilon_k}{4k}, r_j + \frac{\epsilon_k}{4k}]\) are pairwise disjoint, since \( \mathcal{L}^1(I^k_j) \geq \epsilon_k/4 \). By summing on the index \( j \) we find
\[ C_2(N_k + 1) \leq c. \]
This shows that \((N_k)\) is a bounded sequence of integers. Up to subsequences, we can assume \( N_k = N \) for a certain \( N \); by compactness we can also assume the existence of the limits
\[ \lim_{k \to \infty} b^k_j =: b_j \quad \text{and} \quad \lim_{k \to \infty} a^k_j =: a_j, \]
(29)
with \( 0 = b_0 \leq a_1 \leq b_1 \leq \cdots \leq a_N \leq b_N \leq a_{N+1} = 1 \). Now, by (26) and (28) we have
\[ \sum_{j=0}^N (a^k_{j+1} - b^k_j) = \mathcal{L}^1(B_k^c) + \sum_{j > N} \mathcal{L}^1(J^k_j) \to 0; \]
(30)
it follows that \( b_j = a_j+1 \), for \( j = 0, \ldots, N \). Let \( 0 = x_0 < x_1 < \cdots < x_m = 1 \) be an increasing enumeration of the set \( \{b_0, \ldots, b_N\} \).

Let \( \sigma > 0 \) be such that \( x_{i-1} + \sigma < x_i - \sigma \) for \( i = 1, \ldots, m \). For large values of \( k \) we have \( a^k_j, b^k_j \not\in [x_{i-1} + \sigma, x_i - \sigma] \). Using (30) and (29), we can deduce that for every \( k \) and every \( i \) there exists \( j \) such that
\[ [x_{i-1} + \sigma, x_i - \sigma] \subset [a^k_j, b^k_j]; \]
therefore \( v_k > 1/2 \) in \([x_{i-1} + \sigma, x_i - \sigma] \), for \( i = 1, \ldots, m \). By (8) and (21) we find
\[ \int_{x_{i-1}^+}^{x_i^-} |\nabla u_k|^2 dx \leq 2c, \]
(31)
i.e., \((\nabla u_k)\) is bounded in \(L^2(x_{i-1} + \sigma, x_i - \sigma)\), for \(i = 1, \ldots, m\).

(b) Using the Poincaré-Wirtinger Inequality, we deduce from (20) and (31) that 
\((u_k)\) is bounded in \(H^1([x_{i-1} + \sigma, x_i - \sigma])\). This ensures that \(u \in H^1(x_{i-1} + \sigma, x_i - \sigma)\) and that \(u_k \rightharpoonup u\) weakly in \(H^1(x_{i-1} + \sigma, x_i - \sigma)\).

By the Severini-Egorov Theorem for every \(\mu > 0\) there exists a measurable set \(A_\mu \subset [x_{i-1} + \sigma, x_i - \sigma]\), with \(\mathcal{L}^1(A_\mu) < \mu\), such that, up to a subsequence, \(v_k \to 1\) uniformly in \([x_{i-1} + \sigma, x_i - \sigma] \setminus A_\mu\). Then, fixed \(\delta > 0\), we have \(v_k > 1 - \delta\) in \([x_{i-1} + \sigma, x_i - \sigma] \setminus A_\mu\) for large \(k\). By the weak lower semicontinuity of the \(L^2\)-norm, we have

\[
(1 - \delta) \int_{[x_{i-1} + \sigma, x_i - \sigma] \setminus A_\mu} |\nabla u|^2 \, dx \leq \liminf_{k \to \infty} \int_{[x_{i-1} + \sigma, x_i - \sigma]} u_k |\nabla u_k|^2 \, dx.
\]

We pass to the limit first as \(\delta \to 0\) and then as \(\mu \to 0\); adding on the index \(i\) we find

\[
\sum_{i=1}^m \int_{[x_{i-1} + \sigma, x_i - \sigma]} |\nabla u|^2 \, dx \leq \liminf_{k \to \infty} \sum_{i=1}^m \int_{[x_{i-1} + \sigma, x_i - \sigma]} u_k |\nabla u_k|^2 \, dx.
\]

(32)

As \(\sigma \to 0\), from (21) we obtain \(u \in H^1(x_{i-1}, x_i)\) for \(i = 1, \ldots, m\). Inequality (22) follows.

Proof of (23). If \(u\) is continuous in a certain \(x_i\), then \(u \in H^1(x_{i-1}, x_{i+1})\) and we can remove \(x_i\) from the list. Therefore it is not restrictive to assume that every \(x_i\) is a jump point for \(u\), for \(i = 1, \ldots, m - 1\), so that \(\mathcal{H}^0(J_u) = m - 1\). Fix \(\sigma > 0\) such that \(2\sigma < x_i - x_{i-1}\) for every \(i\) and let \(\delta_k^i = \min\{v_k(x) : x \in [x_i - \frac{\sigma}{2}, x_i + \frac{\sigma}{2}]\}\).

Let us prove that \(\delta_k^i \to 0\) as \(k \to \infty\); by contradiction, we suppose that there exists a subsequence of \((\delta_k^i)\), not relabeled, and a constant \(K > 0\) such that \(\delta_k^i > K\) for every \(k\), i.e., \(v_k > K > 0\) in \([x_i - \frac{\sigma}{2}, x_i + \frac{\sigma}{2}]\). By repeating the argument used in steps (a) and (b) we find that \(u \in H^1(x_i - \frac{\sigma}{2}, x_i + \frac{\sigma}{2})\) and this contradicts the assumption that \(x_i\) is a jump point.

Now let \(t_k^i\) be a minimum point for \(v_k\) in \([x_i - \frac{\sigma}{2}, x_i + \frac{\sigma}{2}]\). For large value of \(k\) we have \([t_k^i - \epsilon_k(1 - \delta_k^i), t_k^i + \epsilon_k(1 - \delta_k^i)] \subset [x_i - \sigma, x_i + \sigma]\). Since \(v_k(t_k^i) = \delta_k^i\) and \(|\nabla v_k| \leq 1/\epsilon_k\) \(L^1\)-a.e. in \(\Omega\), it follows that \(v_k \leq \frac{1}{\epsilon_k} |x - t_k^i| + \delta_k^i\). Since \(\psi\) is decreasing we deduce

\[
2 \int_{\delta_k^i}^1 \psi(s) \, ds = \frac{1}{\epsilon_k} \int_{t_k^i - \epsilon_k(1 - \delta_k^i)}^{t_k^i + \epsilon_k(1 - \delta_k^i)} \psi\left(\frac{|x - t_k^i|}{\epsilon_k} + \delta_k^i\right) \, dx \leq \frac{1}{\epsilon_k} \int_{x_i - \sigma}^{x_i + \sigma} \psi(v_k) \, dx;
\]

adding with respect to \(i\) and passing to the lower limit we obtain (23).

Proof of (24). In the case \(\alpha = 0\) inequality (24) is obtained by adding (22) and (23).

Let \(\alpha > 0\). Up to subsequences, we have \(u_k \rightharpoonup u\) \(L^1\)-a.e. on \(\Omega\); we write \(J_u = \{x_1 \ldots x_{m-1}\}\), where \(0 = x_0 < x_1 < \cdots < x_{m-1} < x_m = 1\), and we choose \(\sigma > 0\), with \(2\sigma < x_i - x_{i-1}\), such that

\[
u_k(x_i - \sigma) \to u(x_i - \sigma) \quad \text{and} \quad u_k(x_{i-1} + \sigma) \to u(x_{i-1} + \sigma) \quad \text{for } i = 1, \ldots, m. \quad (33)
\]

We want to estimate from below the integrals

\[
I_k^i := \int_{x_{i-1} - \sigma}^{x_i + \sigma} v_k(\nabla u_k)^2 \, dx + \frac{1}{\epsilon_k} \int_{x_{i-1} - \sigma}^{x_i + \sigma} \psi(v_k) \, dx. \quad (34)
\]
To this aim fix $1 \leq i \leq m - 1$ and for $k$ large we define

\[ W_k := \{ w \in W^1,\infty(x_i - \sigma, x_i + \sigma), \ w(x_i - \sigma) = u_k(x_i - \sigma), \ w(x_i + \sigma) = u_k(x_i + \sigma) \}, \]

\[ Z_k := \{ z \in W^1,\infty(x_i - \sigma, x_i + \sigma), \ \eta_k \leq z \leq 1, \ |\nabla z| \leq 1/\epsilon_k \ L^1\text{-a.e. on } |x_i - \sigma, x_i + \sigma| \}, \]

\[ H_k(w, z) := \int_{x_i - \sigma}^{x_i + \sigma} z |\nabla w|^2 \, dx + \frac{1}{\epsilon_k} \int_{x_i - \sigma}^{x_i + \sigma} \psi(z) \, dx, \quad \text{for } (w, z) \in W_k \times Z_k, \]

\[ h_k(z) := \min_{w \in W_k} H_k(w, z). \]

By elementary computation we find that this minimum is achieved and that

\[ h_k(z) = \frac{\beta_k^2}{\int_{x_i - \sigma}^{x_i + \sigma} z \, dx} + \frac{1}{\epsilon_k} \int_{x_i - \sigma}^{x_i + \sigma} \psi(z) \, dx, \quad (35) \]

where

\[ \beta_k := |u_k(x_i + \sigma) - u_k(x_i - \sigma)|. \quad (36) \]

Let $z_k$ be a minimum point for $h_k$ in $Z_k$. It follows from the definition of $h_k$ and from (34) that

\[ h_k(z_k) \leq I_k^i. \quad (37) \]

We note that $h_k$ is invariant with respect to symmetric rearrangements of $z$ (see [2], [9], [13], [15], [16]), therefore we can assume that $z_k$ is symmetric with respect to $x_i$ and nondecreasing on $[x_i, x_i + \sigma]$. Now we want to prove that $z_k$ is piecewise affine.

First of all, by monotonicity and continuity, the sets

\[ A_k := \{ z_k = \eta_k \} \cap [x_i, x_i + \sigma] \quad \text{and} \quad B_k := \{ z_k = 1 \} \cap [x_i, x_i + \sigma] \]

are closed intervals of $[x_i, x_i + \sigma]$. Let us define

\[ C_k := \{ \eta_k < z_k < 1, \ |\nabla z_k| < 1/\epsilon_k \} \cap [x_i, x_i + \sigma], \]

\[ U_{j,k} := \{ \eta_k + 1/2 < z_k < 1 - 1/2 \} \cap [x_i, x_i + \sigma], \quad E_{j,k} = \{ |\nabla z_k| < 1/\epsilon_k - 1/2 \} \cap U_{j,k}, \]

so that $C_k$ is the union of the sets $E_{j,k}$ for $j \in \mathbb{N}$. For every $j$, $U_{j,k}$ is open in $[x_i, x_i + \sigma]$ and $E_{j,k}$ is measurable. Suppose $L^1(C_k) > 0$ and fix $j$ such that $L^1(E_{j,k}) > 0$; let $\varphi$ be a Lipschitz function such that

\[ \{ \varphi \neq 0 \} \subset U_{j,k} \quad \text{and} \quad |\nabla \varphi| \leq 1 \leq E_{j,k} \ L^1\text{-a.e. on } \mathbb{R}; \quad (38) \]

then $z_k + t \varphi \in Z_k$ for $t$ small enough. So $0$ is a a minimizer for the function $t \mapsto h_k(z_k + t \varphi)$ and, imposing that $0$ is a critical point, we find

\[ \int_{U_{j,k}} \left[ \frac{\lambda_k}{z_k} + \frac{\psi'(z_k)}{\epsilon_k} \right] \varphi \, dx = 0, \quad (39) \]

where $\lambda_k := \beta_k^2 (2 \int_{x_i + \sigma}^{x_i} \frac{1}{z_k} \, dx)^{-2}$.

Let us prove that

\[ \frac{\lambda_k}{z_k} + \frac{\psi'(z_k)}{\epsilon_k} = 0 \ \ L^1\text{-a.e. on } E_{j,k}, \quad (40) \]

arguing by contradiction. Let

\[ E_{j,k}^+ := E_{j,k} \cap \left\{ \frac{\lambda_k}{z_k} + \frac{\psi'(z_k)}{\epsilon_k} > 0 \right\} \]
and suppose \( \mathcal{L}^1(E_{j,k}^+) > 0 \). By the continuity of \( z_k \) and \( \psi' \) and by the Lebesgue Differentiation Theorem there exist \( x_0 \in E_{j,k}^+ \) and \( \delta > 0 \) such that

\[
[x_0 - \delta, x_0 + \delta] \subset U_{j,k} \cap \left\{ \frac{\lambda_k}{\varepsilon_k^2} + \frac{\psi'(z_k)}{\varepsilon_k} > 0 \right\} \quad \text{and} \quad \mathcal{L}^1(E_{j,k} \cap [x_0 - \delta, x_0 + \delta]) > 0.
\]

Now let \( y \) be such that

\[
\mathcal{L}^1(E_{j,k} \cap [x_0 - \delta, y]) = \mathcal{L}^1(E_{j,k} \cap [y, x_0 + \delta]),
\]

and let

\[
\theta(x) := \mathcal{L}^1(E_{j,k} \cap [x_0 - \delta, y] \cap [x_0 - \delta, x]) - \mathcal{L}^1(E_{j,k} \cap [x_0 - \delta, x] \cap [y, x_0 + \delta]),
\]

for \( x \in [x_i, x_i + \sigma] \). In particular \( \theta \) is a Lipschitz function satisfying (38), so that (39) implies

\[
\int_{x_0 - \delta}^{x_0 + \delta} \left[ \frac{\lambda_k}{\varepsilon_k^2} + \frac{\psi'(z_k)}{\varepsilon_k} \right] \theta dx = 0;
\]

since \( \theta \geq 0 \), \( \theta(y) > 0 \), and \( \frac{\lambda_k}{\varepsilon_k^2} + \frac{\psi'(z_k)}{\varepsilon_k} > 0 \) in \( [x_0 - \delta, x_0 + \delta] \) the integral in (41) is positive and we get a contradiction. This concludes the proof of (40).

From (40) it follows that \( z_k \) maps \( C_k \) into the set of solutions of the equation \( s^2 \psi'(s) = -\lambda_k \varepsilon_k \), where \( \lambda_k \varepsilon_k \) is infinitesimal since \( (\lambda_k) \) is bounded. Then, assumption (11) implies that \( z_k \) takes only a finite number of different values on \( C_k \) and, by monotonicity and continuity, \( C_k \) is a finite union of intervals. It follows that \([x_i, x_i + \sigma] \) can be written as union of a finite number of intervals, where either \( z_k \) is constant or \( \nabla z_k = 1/\varepsilon_k \).

We now estimate from below \( h_k(z_k) \). In order to simplify the computation, we suppose that \( z_k \) assumes a unique value \( \xi_k \) in \( C_k \), \( \eta_k < \xi_k < 1 \), so that \( C_k \) is an interval. Let \( \alpha_k := \mathcal{L}^1(A_k) \) and \( \gamma_k := \mathcal{L}^1(C_k) \); since \( \nabla z_k = 1/\varepsilon_k \) in \([x_i, x_i + \sigma] \setminus (A_k \cup B_k \cup C_k) \), the measure of \([x_i, x_i + \sigma] \setminus (A_k \cup B_k \cup C_k) \) is \( -\varepsilon_k \eta_k + \varepsilon_k \) so that \( \mathcal{L}^1(B_k) = \sigma - \gamma_k - \alpha_k + \varepsilon_k \eta_k - \varepsilon_k \).

By (35) we get

\[
h_k(z_k) = \beta^2_k \frac{2}{\alpha_k \eta_k + 2 \gamma_k \varepsilon_k} + \frac{\alpha_k \psi(\eta_k)}{\varepsilon_k} + \frac{2 \gamma_k \psi(\xi_k)}{\varepsilon_k} + \kappa_k \\
\geq \frac{2}{\alpha_k \eta_k + 2 \gamma_k \varepsilon_k} \left( \alpha_k + \gamma_k \right) \frac{\psi(\xi_k)}{\varepsilon_k} + \kappa_k,
\]

where \( \zeta_k = 2\sigma + 2 \varepsilon_k \eta_k - 2 \varepsilon_k - 2 \varepsilon_k \log \eta_k \) and \( \kappa_k = 2 \int_{\eta_k}^1 \psi(s)ds \).

The map

\[
t \mapsto \frac{\beta^2_k}{t + \zeta_k} + \frac{\eta_k \psi(\xi_k)}{\varepsilon_k} t + \kappa_k
\]

can be estimated differently in the cases \( \alpha = \infty \) and \( 0 < \alpha < \infty \).

If \( \alpha = \infty \), by (21), (34), and (37) we find

\[
\beta^2_k \zeta_k \leq h_k(z_k) \leq T_k \leq c.
\]

By (33), this implies, as \( k \to \infty \),

\[
\frac{(u(x_i + \sigma) - u(x_i - \sigma))^2}{2\sigma} \leq c.
\]
As $\sigma \to 0$, we obtain $||u(x_i)|| = 0$; this contradicts our assumption that $x_i$ is a jump point and proves that $H^0(J_u) = 0$, so that $u \in H^1(\Omega)$ and (24) follows from (22).

If $0 < \alpha < \infty$ we have

$$2\beta_k \left( \frac{\psi(\xi_k)}{1 - \eta_k \varepsilon_k} \right)^{\frac{1}{2}} - \frac{\psi(\xi_k)}{1 - \eta_k \varepsilon_k} \varepsilon_k + \kappa = h_k(z_k) \leq I_k,$$

then taking $k \to \infty$ and summing on the index $i$ we get

$$\sum_{i=1}^{m-1} \left[ (\alpha \psi(0))^{\frac{1}{2}} |u(x_i + \sigma) - u(x_i - \sigma)| - \alpha \psi(0) \sigma + \int_0^1 \psi(s) ds \right] \leq \sum_{i=1}^{m-1} \liminf_{k \to \infty} \int_{x_i - \sigma}^{x_i + \sigma} \left[ \psi(v_k) \frac{\nabla u_k}{\varepsilon_k} \right] dx.$$

By adding (32) and (42), as $\sigma \to 0$, we obtain (24).

**Proof of Theorem 3.3.** By Proposition 2 it is sufficient to take $u \in SBV^2(\Omega)$. We are going to construct a recovery sequence converging to $(u, 1)$ in $L^1(\Omega) \times L^1(\Omega)$.

The case $\alpha = \infty$ is trivial since the right-hand side of (19) is finite if and only if $u \in H^1(\Omega)$ and in this case it is sufficient to choose the recovery sequence identically equal to $(u, 1)$.

Now we suppose $\alpha < \infty$. In order to simplify the discussion we assume $u$ has only one jump point $x$. Let $(\delta_k^a)$ be an infinitesimal sequence and let

$$A_k := [x - \delta_k^a, x + \delta_k^a] \quad \text{and} \quad B_k := [x - \delta_k^a - \varepsilon_k(1 - \eta_k), x + \delta_k^a + \varepsilon_k(1 - \eta_k)];$$

moreover let us define $v_k$ by $\eta_k$ in $A_k$, by 1 out of $B_k$, and connecting linearly in $B_k \setminus A_k$; finally let us define $u_k$ by $u$ out of $A_k$ and linking linearly in $A_k$.

Then $(u_k, v_k) \in H^1(\Omega) \times V_k$ and $(u_k, v_k) \to (u, 1)$ in $L^1(\Omega) \times L^1(\Omega)$. We have

$$\lim_k \int_{\Omega \setminus A_k} \left( v_k |\nabla u_k|^2 + \frac{1}{\varepsilon_k} \psi(v_k) \right) dx = \int_{\Omega} |\nabla u|^2 dx + 2 \int_0^1 \psi(s) ds,$$

$$\int_{A_k} \left( v_k |\nabla u_k|^2 + \frac{1}{\varepsilon_k} \psi(v_k) \right) dx = \frac{\eta_k}{2\delta_k} (u(x + \delta_k) - u(x - \delta_k))^2 + 2\psi(\eta_k) \frac{\delta_k}{\varepsilon_k}. \tag{43}$$

If $\alpha = 0$ we take $\delta_k^a$ such that $\eta_k / \delta_k^a \to 0$ and $\delta_k^a / \varepsilon_k \to 0$; by this choice the integral in (43) converges to 0. Whereas if $0 < \alpha < \infty$ we define $\delta_k^a := \frac{1}{2} (\frac{\alpha}{\psi(0)})^{\frac{1}{2}} ||u(x)|| \varepsilon_k$ and the integral in (43) tends to $b_\alpha ||u(x)||$.

We conclude this section by proving Proposition 2 on the effective domain of the $\Gamma$-limit in dimension one.

**Proof of Proposition 2.** Let $u \in GSBV^2(\Omega) \cap L^1(\Omega)$. Since $J_u$ is a finite set it is sufficient to prove that $u \in H^1(\Omega \setminus J_u)$. Let $M > 0$ and let $u^M$ be the truncated function $u^M := (-M \land u) \land M$. From $u^M \in GSBV^2(\Omega) \cap L^\infty(\Omega)$ we deduce $u^M \in SBV^2(\Omega)$. This fact implies $u^M \in H^1(\Omega \setminus J_u)$ since $J_u \subset J_u$. The sequence $(u^M)$ is bounded in $H^1(\Omega \setminus J_u)$ and $u^M \to u$ in $L^1(\Omega)$, therefore we conclude $u \in H^1(\Omega \setminus J_u)$. \qed
5. **Proof of the Estimate from Below in the Case** $n > 1$. In this section we use the slicing argument to prove the estimate from below (18) when $n > 1$. For every $\xi \in S^{n-1}$ we define

$$\Pi^\xi := \{ y \in \mathbb{R}^n : y \cdot \xi = 0 \} \quad \text{and} \quad \Omega^\xi_y := \{ t \in \mathbb{R} : y + t\xi \in \Omega \},$$

for every $y \in \Pi^\xi$. For every $u : \Omega \to \mathbb{R}$ we define $u^\xi_y : \Omega^\xi_y \to \mathbb{R}$ by

$$u^\xi_y(t) := u(y + t\xi).$$

For the main properties of slicing we refer to [3, Section 3.11]. We collect here only few properties used in this section.

If $(u_k)$ is a sequence in $L^1(\Omega)$ such that $u_k \to u$ in $L^1(\Omega)$ then for every $\xi \in S^{n-1}$ there exists a subsequence $(u_{k_j})$ such that, for $H^{n-1}$-a.e. $y \in \Pi^\xi$,

$$(u_{k_j})^\xi_y \to u^\xi_y \text{ in } L^1(\Omega).$$

If $u \in BV(\Omega)$, then for every $\xi \in S^{n-1}$ the following properties hold:

$$\int_{J_u} |\nu_u \cdot \xi| dH^{n-1}(y) = \int_{\Pi^\xi} \mathcal{H}^0((J_u)^\xi_y) dH^{n-1}(y), \quad (44)$$

$$\int_{J_u} |\nu_u \cdot \xi||u| dH^{n-1}(y) = \int_{\Pi^\xi} \left( \int_{(J_u)^\xi_y} |u|^\xi y dH^0(t) \right) dH^{n-1}(y), \quad (45)$$

for $H^{n-1}$-a.e. $y \in \Pi^\xi$ we have $|\nabla(u^\xi_y)| = |(\nabla u)^\xi_y| \leq |(\nabla u)^\xi_y| \mathcal{L}^1$-a.e. on $\Omega^\xi_y$.\quad (46)

Moreover for every $\xi \in S^{n-1}$ and for $H^{n-1}$-a.e. $y \in \Pi^\xi$ we have

$$(J_u)^\xi_y = J_{u^\xi_y} \quad \text{and} \quad |u|^\xi_y = |u^\xi_y| \quad \text{on } \Omega^\xi_y. \quad (47)$$

We also make use of the fine properties of $GBV$-functions collected in [3, Theorem 4.34].

As we have already seen in Section 4, in order to obtain the $\Gamma$-lim inf inequality it is sufficient to prove Proposition 1.

**Proof of Proposition 1.** Let $(u_k, v_k)$ be a sequence satisfying (20) and (21) with bounding constant $c$; as in the one-dimensional case we can deduce that $v \equiv 1 \mathcal{L}^n$-a.e. in $\Omega$. In the first part of the proof we assume that $(u_k)$ is bounded in $L^\infty(\Omega)$ and we want to prove that $u \in SBV^2(\Omega)$ in this case.

**Proof of (22) in the bounded case.** Given $\xi \in S^{n-1}$, we extract a subsequence $(u_r, v_r)$ of $(u_k, v_k)$ such that

$$((u_r)^\xi_y, (v_r)^\xi_y) \to (u^\xi_y, 1) \text{ in } L^1(\Omega^\xi_y) \times L^1(\Omega^\xi_y) \text{ for } H^{n-1}$-a.e. } y \in \Pi^\xi, \quad (48)$$

and

$$\lim_{r \to \infty} \int_\Omega v_r |\nabla u_r \cdot \xi|^2 \, dx = \liminf_{k \to \infty} \int_\Omega v_k |\nabla u_k \cdot \xi|^2 \, dx. \quad (49)$$

Let $0 < \kappa < 1$; by the Fubini Theorem and (46) we can write

$$\int_{\Pi^\xi} \left[ \int_{\Omega^\xi_y} \left( (v_r)^\xi_y |\nabla((u_r)^\xi_y)|^2 + \frac{\kappa}{\epsilon_r} \psi(v_r)^\xi_y \right) dt \right] dH^{n-1}(y) \leq \int_{\Omega} \left( v_r |\nabla u_r|^2 + \frac{\kappa}{\epsilon_r} \psi(v_r) \right) \, dx \leq c,$$

where the last inequality follows from (21). From the Fatou Lemma it follows that

$$\int_{\Pi^\xi} \liminf_{r \to \infty} \left[ \int_{\Omega^\xi_y} \left( (v_r)^\xi_y |\nabla((u_r)^\xi_y)|^2 + \frac{\kappa}{\epsilon_r} \psi(v_r)^\xi_y \right) dt \right] dH^{n-1}(y) \leq c;$$
then for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$

$$\liminf_{r \to \infty} \int_{\Omega^\xi_y} \left( (v_r)^\xi_y \right| |\nabla((u_r)^\xi_y)|^2 + \frac{\kappa}{\epsilon_r} \psi((v_r)^\xi_y) \right) dt < +\infty. \quad (50)$$

Let $F_{y,r}$ be the one-dimensional functional on the set $\Omega^\xi_y$, defined by

$$F_{y,r}(w, z) := \begin{cases} \int_{\Omega^\xi_y} z |\nabla w|^2 dt + \frac{1}{\epsilon_r} \int_{\Omega^\xi_y} \varphi(z) dt & \text{if } (w, z) \in H^1(\Omega^\xi_y) \times V_{y,r}, \\ +\infty & \text{otherwise}, \end{cases}$$

where $\varphi := \kappa \psi$

and $V_{y,r} := \left\{ z \in H^1(\Omega^\xi_y) \times W^{1,\infty}(\Omega^\xi_y) : \eta_r \leq z \leq 1, |\nabla z| \leq \frac{1}{\epsilon_r} \mathcal{L}^1\text{-a.e. in } \Omega^\xi_y \right\}$. \quad (52)

The corresponding $\Gamma$-lim inf will be denoted by $F'_{y,\alpha}$.

For $0 < \alpha < \infty$ let $\Phi_{y,\alpha} : L^1(\Omega^\xi_y) \to [0, +\infty]$ be defined by

$$\Phi_{y,\alpha}(w) := \begin{cases} \int_{\Omega^\xi_y \setminus J_w} |\nabla w|^2 dx + a \mathcal{H}^0(J_w) + \beta_\alpha \int_{J_w} |w| d\mathcal{H}^0 & \text{if } w \in SBV^2(\Omega^\xi_y) \\ +\infty & \text{otherwise}, \end{cases}$$

where $a$ is as in (14) and $\beta_\alpha := 2(\alpha \varphi(0))^{1/2}$. In the limiting cases $\alpha = 0$ and $\alpha = \infty$ we define

$$\Phi_{y,0}(w) := \begin{cases} \int_{\Omega^\xi_y \setminus J_w} |\nabla w|^2 dx + a \mathcal{H}^0(J_w) & \text{if } w \in SBV^2(\Omega^\xi_y) \cap L^1(\Omega^\xi_y) \\ +\infty & \text{otherwise}, \end{cases}$$

$$\Phi_{y,\infty}(w) := \begin{cases} \int_{\Omega^\xi_y \setminus J_w} |\nabla w|^2 dx & \text{if } w \in H^1(\Omega^\xi_y) \\ +\infty & \text{otherwise}. \end{cases}$$

Thanks to (46) we have $|\nabla ((v_r)^\xi_y)| \leq 1/\epsilon_r \mathcal{L}^1\text{-a.e. in } \Omega^\xi_y$ and then

$$F_{y,r}((u_r)^\xi_y, (v_r)^\xi_y) = \int_{\Omega^\xi_y} (v_r)^\xi_y |\nabla((u_r)^\xi_y)|^2 dt + \frac{\kappa}{\epsilon_r} \int_{\Omega^\xi_y} \psi((v_r)^\xi_y) dt.$$ 

By (50), for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ we can find a subsequence $(u_m, v_m)$ of $(u_r, v_r)$ such that

$$\lim_{m \to \infty} \int_{\Omega^\xi_y} \left( (v_m)^\xi_y \right| |\nabla((u_m)^\xi_y)|^2 + \frac{\kappa}{\epsilon_m} \psi((v_m)^\xi_y) \right) dt =$$

$$= \liminf_{r \to \infty} \int_{\Omega^\xi_y} \left( (v_r)^\xi_y \right| |\nabla((u_r)^\xi_y)|^2 + \frac{\kappa}{\epsilon_r} \psi((v_r)^\xi_y) \right) dt < +\infty, \quad (53)$$

so that (48) and (53) in particular imply

$$F'_{y,\alpha}(u^\xi_y, 1) \leq \lim_{m \to \infty} F_{y,m}((u_m)^\xi_y, (v_m)^\xi_y) < +\infty,$$

for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$. Applying Theorem 3.2 in the case $n = 1$ (and Proposition 2) we obtain that $u^\xi_y \in SBV^2(\Omega^\xi_y)$,

$$\Phi_{y,\alpha}(u^\xi_y) \leq F'_{y,\alpha}(u^\xi_y, 1), \quad (54)$$

and that (22) is true for $((u_m)^\xi_y, (v_m)^\xi_y)$. 

Now let us prove that $u \in SBV(\Omega)$. Let $M < +\infty$ be such that $\|u_m\|_{L^\infty(\Omega)} \leq M$ for every $m$. Then decomposing the derivative of $u_\xi$ (see [3, Section 3.9]) we get

$$|D(u_\xi^m)(\Omega_\xi^m)| = \int_{\Omega_\xi^m \setminus J_{u_\xi}^m} |\nabla(u_\xi^m)| dt + \sum_{J_{u_\xi}^m} |[u_\xi^m]| \leq$$

$\leq L^1(\Omega_\xi^m) + \int_{\Omega_\xi^m \setminus J_{u_\xi}^m} |\nabla(u_\xi^m)|^2 dt + 2M \mathcal{H}^0(J_{u_\xi}^m) \leq A[1 + F'_{y,\alpha}(u_\xi^m, 1)],$

where in the last inequality $A := \text{diam}(\Omega) + 1 + \frac{2M}{\alpha}$ and we have used (54). Since $(u_r)$ does not depend on $y$, we can integrate on the projection $\Pi^\xi(\Omega)$ of $\Omega$ on $\Pi^\xi$ and we obtain

$$\int_{\Pi^\xi(\Omega)} |D(u_\xi^m)(\Omega_\xi^m)| d\mathcal{H}^{n-1}(y) \leq$$

$$\leq A \mathcal{H}^{n-1}(\Pi^\xi(\Omega)) + A \int_{\Pi^\xi} \liminf_{r \to \infty} F_{y,r}((u_r)_y^\xi, (v_r)_y^\xi) d\mathcal{H}^{n-1}(y) \leq$$

$$\leq A \mathcal{H}^{n-1}(\Pi^\xi(\Omega)) + Ac < +\infty.$$

By taking $\xi = e_1, \ldots, e_n$, the elements of the canonical basis of $\mathbb{R}^n$, we get $u \in BV(\Omega)$ by [3, Remark 3.104]; since $u_\xi^m \in SBV^2(\Omega_\xi^m)$, we obtain also $u \in SBV(\Omega)$ by [3, Theorem 3.108].

From (22) applied to $((u_m)_y^\xi, (v_m)_y^\xi)$ and from (53) it follows that

$$\int_{\Omega_\xi^m \setminus J_{u_\xi}^m} |\nabla(u_\xi^m)|^2 dt \leq \liminf_{r \to \infty} \left[ \int_{\Omega_\xi^m} \left( (v_r)_y^\xi |\nabla((u_r)_y^\xi)|^2 + \frac{\kappa}{\epsilon_r} (\psi(v_r)_y^\xi) \right) dt \right] d\mathcal{H}^{n-1}(y),$$

for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$. Integrating on $\Pi^\xi$ and applying the Fatou Lemma we get

$$\int_{\Pi^\xi} \left[ \int_{\Omega_\xi^m \setminus J_{u_\xi}^m} |\nabla(u_\xi^m)|^2 dt \right] d\mathcal{H}^{n-1}(y) \leq$$

$$\leq \liminf_{r \to \infty} \int_{\Pi^\xi} \left[ \int_{\Omega_\xi^m} \left( (v_r)_y^\xi |\nabla((u_r)_y^\xi)|^2 + \frac{\kappa}{\epsilon_r} (\psi(v_r)_y^\xi) \right) dt \right] d\mathcal{H}^{n-1}(y) \leq$$

$$\leq \liminf_{k \to \infty} \int_{\Pi^\xi} \left[ \int_{\Omega_\xi^m} (v_k)_y^\xi |\nabla((u_k)_y^\xi)|^2 dt \right] d\mathcal{H}^{n-1}(y) + \kappa c,$n

where the last inequality follows from (21) and (49). We observe that $(u_k, v_k)$ does not depend on $\kappa$; as $\kappa \to 0$ in the previous inequality we find

$$\int_{\Omega} |\nabla u \cdot \xi|^2 dx \leq \liminf_{k \to \infty} \int_{\Omega} v_k |\nabla u_k \cdot \xi|^2 dx,$$

using (46) and the Fubini Theorem. By taking $\xi = e_1, \ldots, e_n$ and summimg the results we obtain (22).

Proof of (23) in the bounded case. Given $\xi \in S^{n-1}$, the first subsequence $(u_r, v_r)$ of $(u_k, v_k)$ is now chosen so that (48) holds and (49) is replaced by

$$\lim_{r \to \infty} \int_{\Pi^\xi} \left[ \int_{\Omega_\xi^m} \frac{1}{\epsilon_r} (\psi(v_r)_y^\xi) dt \right] d\mathcal{H}^{n-1}(y) =$$

$$= \liminf_{k \to \infty} \int_{\Pi^\xi} \left[ \int_{\Omega_\xi^m} \frac{1}{\epsilon_k} ((v_k)_y^\xi) dt \right] d\mathcal{H}^{n-1}(y).$$

(56)
Let $0 < \kappa < 1$; by the Fubini Theorem and the Fatou Lemma we find
\[
\int_{\Omega} \liminf_{r \to \infty} \left[ \int_{\Omega_r^y} \left( \kappa (u_r)_y^\xi \left| \nabla ((u_r)_y^\xi) \right|^2 + \frac{1}{\epsilon_r} (\psi(v_r)_y^\xi) \right) dt \right] d\mathcal{H}^{n-1}(y) \leq c
\]
and this implies, for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega^\xi$,
\[
\liminf_{r \to \infty} \int_{\Omega_r^y} \left( \kappa (u_r)_y^\xi \left| \nabla ((u_r)_y^\xi) \right|^2 + \frac{1}{\epsilon_r} (\psi(v_r)_y^\xi) \right) dt < +\infty.
\]
It follows that for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega^\xi$ there exists a subsequence $(u_m, v_m)$ of $(u_r, v_r)$ such that
\[
\lim_{m \to \infty} \int_{\Omega_r^y} \left( \kappa (u_m)_y^\xi \left| \nabla ((u_m)_y^\xi) \right|^2 + \frac{1}{\epsilon_m} (\psi(v_m)_y^\xi) \right) dt = \liminf_{r \to \infty} \int_{\Omega_r^y} \left( \kappa (v_r)_y^\xi \left| \nabla ((u_r)_y^\xi) \right|^2 + \frac{1}{\epsilon_r} (\psi(v_r)_y^\xi) \right) dt < +\infty. \quad (57)
\]

Let us consider the one-dimensional functional $F_{y,r}$ defined in (51), where $\varphi$ is now set equal to $\psi$ and $V_{y,r}$ is as in (52).

By (48) and (57) the sequence $F_{y,m}((\kappa^{1/2}u_m)_y^\xi, (v_m)_y^\xi)$ is bounded, so that Theorem 3.2 in the case $n = 1$ implies that inequality (23) holds for $((\kappa^{1/2}u_m)_y^\xi, (v_m)_y^\xi)$; using formula (57) we get
\[
a \mathcal{H}^0(J_{u_y^\xi}) \leq \liminf_{r \to \infty} \int_{\Omega_r^y} \left( \kappa (v_r)_y^\xi \left| \nabla ((u_r)_y^\xi) \right|^2 + \frac{1}{\epsilon_r} (\psi(v_r)_y^\xi) \right) dt.
\]
Let us observe that $(u_r)$ does not depend on $y$. Then we can integrate on $\Omega^\xi$ both sides of the previous inequality and apply the Fatou Lemma
\[
a \int_{\Omega^\xi} \mathcal{H}^0(J_{u_y^\xi}) d\mathcal{H}^{n-1}(y) \leq \liminf_{r \to \infty} \int_{\Omega^\xi} \left[ \int_{\Omega_r^y} \left( \kappa (v_r)_y^\xi \left| \nabla ((u_r)_y^\xi) \right|^2 + \frac{1}{\epsilon_r} (\psi(v_r)_y^\xi) \right) dt \right] d\mathcal{H}^{n-1}(y) \leq \liminf_{k \to \infty} \int_{\Omega^\xi} \left[ \int_{\Omega_r^y} \frac{1}{\epsilon_k} (\psi(v_k)_y^\xi) dt \right] d\mathcal{H}^{n-1}(y) + \kappa c,
\]
by (21) and (56). As $\kappa \to 0$, using (44) and (47) we find
\[
a \int_{J_u} |\nu_u \cdot \xi| d\mathcal{H}^{n-1} \leq \liminf_{k \to \infty} \frac{1}{\epsilon_k} \int_{\Omega} \psi(v_k) dx \leq c. \quad (58)
\]
Applying (58) with $\xi = e_1, \ldots, e_n$ we get $\mathcal{H}^{n-1}(J_u) < +\infty$. Since we have already proved that $u \in SBV^2(\Omega)$, we deduce from (21) and (22) that $u \in SBV^2(\Omega)$.

In order to obtain (23) we use a particular case of the localization method developed in [8, Theorem 2.3.1]. First we note that (58) holds also for an open set $A \subset \Omega$, hence
\[
a \int_{J_u \cap A} |\nu_u \cdot \xi| d\mathcal{H}^{n-1} \leq \liminf_{k \to \infty} \frac{1}{\epsilon_k} \int_{A} \psi(v_k) dx. \quad (59)
\]
Since $\nu_u$ is a Borel function with values in $S^{n-1}$, there exists a sequence $(\omega_j)$ of simple functions with values in $S^{n-1}$ converging to $\nu_u$ pointwise $\mathcal{H}^{n-1}$-a.e. in $J_u$. 

}\end{document}
We can write \( \omega_j = \xi^i_j 1_{B^i_j} + \cdots + \xi^{m_j}_j 1_{B^{m_j}_j} \), where \( \xi^i_j \) are unit vectors and \( B^1_j, \ldots, B^{m_j}_j \) is a Borel partition of \( J_u \). By the Dominated Convergence Theorem we have
\[
\lim_{j \to \infty} \sum_{i=1}^{m_j} \int_{B^i_j} |\nu_u \cdot \xi^i_j| d\mathcal{H}^{n-1} = \mathcal{H}^{n-1}(J_u). \quad (60)
\]

For every \( j \) we can find \( A^1_j, \ldots, A^{m_j}_j \) a family of pairwise disjoint open subsets of \( \Omega \) such that \( \mathcal{H}^{n-1}(A^i_j \cap J_u \setminus B^i_j) \leq 1/(jm_j) \). Then (60) holds with \( B^i_j \) replaced by \( J_u \cap A^i_j \). By (59)
\[
a \sum_{i=1}^{m_j} \int_{J_u \cap A^i_j} |\nu_u \cdot \xi^i_j| d\mathcal{H}^{n-1} \leq \liminf_{k \to \infty} \frac{1}{\epsilon_k} \int_{\Omega} \psi(v_k)dx,
\]
we obtain (23) as \( j \to \infty \).

**Proof of (24) in the bounded case.** If \( \alpha = 0 \) inequality (24) can be obtained by adding (22) and (23).

When \( \alpha = \infty \) by Theorem 3.2 in the case \( n = 1 \) we get \( u^\xi \in H^1(\Omega^\xi) \) for every \( \xi \) and for \( \mathcal{H}^{n-1} \)-a.e. \( y \in \Pi^\xi \). Since \( u \in SBV^2(\Omega) \), this implies \( u \in H^1(\Omega) \) by [3, Theorem 3.108], therefore (24) follows from (22).

Let now \( 0 < \alpha < \infty \). Given \( \xi \in S^{n-1} \), we choose a subsequence \((u_r, v_r)\) of \((u_k, v_k)\) such that (48) holds and
\[
\lim_{r \to \infty} \int_{\Omega} \left[ \int_{\Omega^\xi} \left( (v_r)^\xi |\nabla(u_r)^\xi| + \frac{1}{\epsilon_r} (\psi(v_r)^\xi) \right) dt \right] d\mathcal{H}^{n-1}(y) =
\]
\[
\leq \liminf_{k \to \infty} \int_{\Omega} \left[ \int_{\Omega^\xi} \left( (v_k)^\xi |\nabla(u_k)^\xi| + \frac{1}{\epsilon_k} (\psi(v_k)^\xi) \right) dt \right] d\mathcal{H}^{n-1}(y).
\]

By (46), using the Fubini Theorem and the Fatou Lemma we get
\[
\int_{\Omega} \liminf_{r \to \infty} \int_{\Omega^\xi} \left( (v_r)^\xi |\nabla((u_r)^\xi)| + \frac{1}{\epsilon_r} (\psi(v_r)^\xi) \right) dt d\mathcal{H}^{n-1}(y) \leq c
\]
and then for \( \mathcal{H}^{n-1} \)-a.e. \( y \in \Pi^\xi \) we have
\[
\liminf_{r \to \infty} \int_{\Omega^\xi} \left( (v_r)^\xi |\nabla((u_r)^\xi)| + \frac{1}{\epsilon_r} (\psi(v_r)^\xi) \right) dt < +\infty.
\]

Let \( F_{y,r} \) be the one-dimensional functional defined in (51), where \( \varphi := \psi \) and \( V_{y,r} \) is as in (52). For \( \mathcal{H}^{n-1} \)-a.e. \( y \in \Pi^\xi \) we can find a subsequence \((u_m, v_m)\) of \((u_r, v_r)\) such that
\[
\lim_{m \to \infty} \int_{\Omega^\xi} \left( (v_m)^\xi |\nabla((u_m)^\xi)| + \frac{1}{\epsilon_m} (\psi(v_m)^\xi) \right) dt =
\]
\[
= \liminf_{r \to \infty} \int_{\Omega^\xi} \left( (v_r)^\xi |\nabla((u_r)^\xi)| + \frac{1}{\epsilon_r} (\psi(v_r)^\xi) \right) dt < +\infty; \quad (61)
\]
then Theorem 3.2 in the case \( n = 1 \) implies
\[
\Phi_{y,\alpha}(u^\xi_y) \leq \liminf_{r \to \infty} F_{y,r}((u_r)^\xi_y, (v_r)^\xi_y).
\]
Let us observe that \((u_r)\) does not depend on \(y\); integrating on \(\Pi^\xi\) both sides of the previous inequality and applying the Fatou Lemma we get

\[
\int_{\Pi^\xi} \Phi_{y,a}(u^\xi_y) d\mathcal{H}^{n-1}(y) \leq \liminf_{k \to \infty} \int_{\Omega^\xi} \left( \int_0^t \left( (v_k)_y^\xi \nabla ((u_k)_y^\xi)^2 + \frac{1}{\varepsilon_k} (\psi(v_k))_y^\xi \right) dt \right) d\mathcal{H}^{n-1}(y). \tag{62}
\]

We now apply the localization method to the measure \(\mu = \mathcal{L}^n|\Omega + \mathcal{H}^{n-1}|J_u\) instead of \(\mathcal{H}^{n-1}|J_u\). Since (62) holds with an open set \(A \subset \Omega\) in place of \(\Omega\), by (44)-(47) and by the Fubini Theorem we get

\[
\int_A \left| \nabla u \cdot \xi_j^2 \right| 1_{\Omega \setminus J_u} + |\nu_u : \xi_j|(a + b_n|u|)1_{J_u} \, d\mu \leq \liminf_{k \to \infty} \int_A \left( v_k |\nabla u_k \cdot \xi_j^2 + \frac{1}{\varepsilon_k} \psi(v_k) \right) \, dx. \tag{63}
\]

Let us define \(\omega := \nu_u\) on \(J_u\), \(\omega := \nabla u / |\nabla u|\) on \(\{ \nabla u \neq 0 \} \setminus J_u\), and \(\omega := e_1\) elsewhere. Since \(\omega\) is a \(\mu\)-measurable function with values in \(S^{n-1}\), there exists a sequence \((\omega_j)\) of simple functions with values in \(S^{n-1}\), converging to \(\omega\) \(\mu\)-a.e. in \(\Omega\). We can write \(\omega_j = (\xi_1^j B_1^j + \cdots + \xi_m^j B_m^j)\), where \(\xi_j^i\) are unit vectors and \(B_1^j, \ldots, B_m^j\) is a Borel partition of \(\Omega\). By the Dominated Convergence Theorem we have

\[
\lim_{j \to \infty} \sum_{i=1}^{m_j} \int_{B_i^j} \left| \nabla u \cdot \xi_j^i \right| 1_{\Omega \setminus J_u} + |\nu_u : \xi_j^i|(a + b_n|u|)1_{J_u} \, d\mu = \Phi_\alpha(u). \tag{64}
\]

For every \(j\) we can find a family \(A_j^1, \ldots, A_j^{m_j}\) of pairwise disjoint open subsets of \(\Omega\) such that \(\mu(A_j^i \Delta B_i^j) \leq 1/(jm_j)\). Then (64) holds with \(B_i^j\) replaced by \(A_i^j\). By (63) we find

\[
\sum_{i=1}^{m_j} \int_{A_i^j} \left| \nabla u \cdot \xi_j^i \right| 1_{\Omega \setminus J_u} + |\nu_u : \xi_j^i|(a + b_n|u|)1_{J_u} \, d\mu \leq \liminf_{k \to \infty} \int_\Omega \left( v_k |\nabla u_k \cdot \xi_j^i + \frac{1}{\varepsilon_k} \psi(v_k) \right) \, dx
\]

and we obtain (24) as \(j \to \infty\).

\textit{The general case.} We now remove the assumption that \((u_k)\) is bounded in \(L^\infty(\Omega)\). Let us fix \(M > 0\) and let us consider the sequence of truncated functions \(u_k^M = (-M \vee u) \wedge M\). We have that \(u_k^M \to u^M\) in \(L^1(\Omega)\), \(v_k \to 1\) in \(L^1(\Omega)\), and by (21)

\[
F_k(u_k^M, v_k) \leq F_k(u_k, v_k) \leq c.
\]

From the proof in the bounded case it follows that \(u^M \in SBV^2(\Omega)\) and that

\[
\int \left| \nabla u^M \right|^2 \, dx \leq \liminf_{k \to \infty} \int \left| \nabla u_k^M \right|^2 \, dx, \tag{65}
\]

\[
a\mathcal{H}^{n-1}(J_{u^M}) \leq \liminf_{k \to \infty} \frac{1}{\varepsilon_k} \int_\Omega \psi(v_k) \, dx. \tag{66}
\]

This implies \(u \in GSBV(\Omega)\). As \(\left| \nabla u^M \right| = |\nabla u| 1_{\{|u| \leq M\}}\) by Theorem [3, Theorem 4.34], using the Monotone Convergence Theorem we obtain

\[
\int \left| \nabla u \right|^2 \, dx = \lim_{M \to \infty} \int \left| \nabla u^M \right|^2 \, dx,
\]

\[
\int \left| \nabla u \right|^2 \, dx \leq \liminf_{k \to \infty} \int \left| \nabla u_k \right|^2 \, dx,
\]

which together with (65) proves (22). Moreover, taking \( M \to \infty \) in (66) we find (23). Therefore \( u \in GSBV^2(\Omega) \).

Let us prove now (24). When \( \alpha = 0 \), this inequality can be obtained by adding (22) and (23).

Assume \( \alpha = \infty \). In this case we prove that \( u \in H^1(\Omega) \). The proof in the bounded case, applied to \((u_k^M, v_k)\), gives \( u^M \in H^1(\Omega) \) and by (65) we have that the sequence \((\nabla u^M)\) is bounded in \( L^2(\Omega, \mathbb{R}^n) \); using the Poincaré-Wirtinger Inequality and the fact that \( u^M \rightharpoonup u \) in \( L^1(\Omega) \) we obtain \( u \in H^1(\Omega) \). Inequality (24) follows from (22).

Let \( 0 < \alpha < \infty \). The proof in the bounded case, applied to \((u_k^M, v_k)\), gives

\[
\Phi_\alpha(u^M) \leq \liminf_{k \to \infty} \int_\Omega [v_k|\nabla u_k|^2 + \frac{1}{\epsilon_k}\psi(v_k)] dx \leq c. \tag{67}
\]

Since \( u^M \in SBV^2(\Omega) \), inequality (67) gives \(|Du^M| |\Omega| \leq L^\alpha(\Omega) + \epsilon \max(1, 1/b_\alpha)\) for every \( M > 0 \). From \( u^M \rightharpoonup u \) in \( L^1(\Omega) \), we conclude that \( u \in BV(\Omega) \) and \( u^M \to u \) weakly* in \( BV(\Omega) \). Using the Closure Theorem for \( SBV \) [3, Theorem 4.7], we deduce from (67) that \( u \in SBV^2(\Omega) \). Estimate (67), as \( M \to \infty \), leads to (24).

\[ \square \]

6. Proof of the Estimate from Above in the Case \( n > 1 \). Now our purpose is to prove the \( \Gamma \)-lim sup inequality. In order to work with more regular functions and jump sets, we first introduce an approximation result. The following theorem is a small modification of a theorem due to Cortesani and Toader (see [10, Theorem 3.1]).

**Theorem 6.1.** Let \( Q \subset \mathbb{R}^n \) be an open cube, let \( 1 < p \leq 2 \), and let \( u \in SBV^p(Q) \cap L^\infty(Q) \). Then for every \( \epsilon > 0 \) there exist a function \( v \in SBV^p(Q) \) and a set \( S = \cup_{i=1}^m S_i \), with \( S_i \) closed and pairwise disjoint \((n-1)\)-simplexes contained in \( Q \), such that

1. \( \mathcal{H}^{n-1}(Q \setminus J_v) = 0; \)
2. \( v \in W^{k,\infty}(Q \setminus S) \) for every \( k; \)
3. \( ||v - u||_{L^1(Q)} < \epsilon; \)
4. \( ||\nabla v - \nabla u||_{L^1(Q, \mathbb{R}^n)} < \epsilon; \)
5. \( \mathcal{H}^{n-1}(J_v) < \mathcal{H}^{n-1}(J_u) + \epsilon; \)
6. \( \int_{J_u} ||u|| d\mathcal{H}^{n-1} < \int_{J_u} ||u|| d\mathcal{H}^{n-1} + \epsilon. \)

**Proof.** Using [10, Theorem 3.1] and [10, Remark 3.5] we can find a function \( w \in SBV^p(Q) \) and a set \( T = \cup_{i=1}^m T_i \), not necessarily contained in \( Q \), with \( T_i \) closed and pairwise disjoint \((n-1)\)-simplexes, such that conditions (a)-(f) hold for \( w \) in place of \( v \) and \( T \cap Q \) in place of \( S \). Since \( T \cap Q \) is a polyhedron, we can adapt the arguments in [10, Remark 3.5] to obtain a function \( v \) and a set \( S \subset Q \) satisfying conditions (a)-(f).

Now we can prove the estimate from above of the \( \Gamma \)-limit.

**Proof of Theorem 3.2.** Let \( u \in GSBV^2(\Omega) \cap L^1(\Omega) \). We have to construct a recovery sequence \((u_k, v_k)\) converging to \((u, 1)\) in \( L^1(\Omega) \times L^1(\Omega) \).

The case \( \alpha = \infty \) is trivial since the right-hand side in (19) is finite if and only if \( u \in H^1(\Omega) \), and in this case it is sufficient to define \((u_k, v_k) := (u, 1)\).
Let $\alpha < \infty$. We consider first the case $u \in L^\infty(\Omega)$, so that $u$ belongs to $SBV^2(\Omega) \cap L^\infty(\Omega)$. It is enough to prove (19) for a cube $Q$ and for a function $u$ satisfying properties (a) and (b) of Theorem 6.1. Indeed, if $\Omega$ is an arbitrary bounded open set $\Omega$ with Lipschitz boundary and $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$, then a local reflection argument provides an extension of $u$ to a function $\tilde{u} \in SBV^2(\Omega) \cap L^\infty(\Omega)$ such that $H^{n-1}(J_u \cap \partial \Omega) = 0$. Through this paragraph we shall write explicitly the domain of the integrals in the functionals (8), (13), (15), (16), and (17). By Theorem 6.1 for every $k$ we can find a function $w_k \in SBV^2(Q)$ satisfying properties (a)-(f). Assuming that (19) holds for $w_k$, we have $F_{\alpha,Q}''(w_k,1) \leq \Phi_{\alpha,Q}(w_k)$. Then by the lower semicontinuity of $F_{\alpha,Q}$ we obtain

$$
F_{\alpha,Q}''(\tilde{u},1) \leq \limsup_{k \to \infty} \Phi_{\alpha,Q}(w_k) \leq \limsup_{k \to \infty} \left[ \Phi_{\alpha,Q}(\tilde{u}) + \frac{1}{k^2} + \frac{2}{k} ||\nabla \tilde{u}||_{L^2(Q,\mathbb{R}^n)} + \frac{a + b_\alpha}{k} \right] = \Phi_{\alpha,Q}(\tilde{u}). \tag{68}
$$

Let us check that this implies $F_{\alpha,\Omega}(u,1) \leq \Phi_{\alpha,\Omega}(u)$. By Theorem 3.2 and inequality (68) we have

$$
F_{\alpha,Q}(\tilde{u},1) = \Phi_{\alpha,\Omega}(u) + \Phi_{\alpha,Q,\Omega}(\tilde{u}),
\Phi_{\alpha,\Omega}(u) \leq F_{\alpha,\Omega}(u,1), \quad \Phi_{\alpha,Q,\Omega}(\tilde{u}) \leq F_{\alpha,Q,\Omega}(\tilde{u}), \tag{69}
$$

so that

$$
F_{\alpha,Q}(\tilde{u},1) \leq F_{\alpha,\Omega}(u,1) + F_{\alpha,Q,\Omega}(\tilde{u}). \tag{70}
$$

Moreover [11, Proposition 6.17] implies

$$
F_{\alpha,\Omega}(u,1) + F_{\alpha,Q,\Omega}(\tilde{u}) \leq F_{\alpha,Q}(\tilde{u},1);
$$

this estimate together with (69) and (70) gives $F_{\alpha,\Omega}(u,1) = \Phi_{\alpha,\Omega}(u)$.

Therefore, in the rest of the proof we assume that $\Omega = Q$, $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$, and that properties (a) and (b) of Theorem 6.1 hold for $u$. Finally, in order to simplify the computation, we suppose that $S$ is a unique $(n-1)$-simplex and that $S \subset \{x_n = 0\}$. We write a point $x \in \mathbb{R}^n$ as $x = (\overline{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and we orient $J_u$ so that $\nu_u = (0,1)$. Let

$$
\Omega^\pm := \{x \in \Omega : \pm x_n > 0\}
$$

and let $L$ be the maximum between the Lipschitz constants of $u$ in $\Omega^+$ and $\Omega^-$. If $0 < \alpha < \infty$ we define $\delta_k^\alpha(\overline{x}) := \frac{1}{2}\epsilon_k(\frac{1}{\sqrt{\alpha}})^{1/2}||u(\overline{x}, 0)||$ for $x = (\overline{x}, x_n) \in \Omega$; whereas for $\alpha = 0$ we define $\delta_k^0$ as any sequence of constant functions such that $\eta_k/\delta_k^0 \to 0$ and $\delta_k^0/\epsilon_k \to 0$. Note that $\delta_k^\alpha$ is a Lipschitz function in $\Omega^\pm$ since $u^+$ and $u^-$ are; moreover in the case $0 < \alpha < \infty$, $\delta_k^\alpha(\overline{x}) = 0$ for $\overline{x} \in \partial S$, where $\partial S$ is the boundary of $S$ in the relative topology of $\mathbb{R}^{n-1} \times \{0\}$. 


Figure 1. The geometry for $\alpha = 0$.

We define

$$A_k := \{ x \in \mathbb{R}^n : (x, 0) \in S, |x_n| < \delta_k^\alpha (\mathcal{F}) \},$$

$$B_k := \{ x \in \mathbb{R}^n : (x, 0) \in S, \delta_k^\alpha (\mathcal{F}) \leq |x_n| \leq \delta_k^\alpha (\mathcal{F}) + \frac{\epsilon_k (1 - \eta_k)}{c_{k,\alpha}} \},$$

$$A_k' := \{ x \in \mathbb{R}^n : (x, 0) \notin S, d(x, \partial S) < \delta_k^\alpha (\mathcal{F}) \},$$

$$B_k' := \{ x \in \mathbb{R}^n : (x, 0) \notin S, \delta_k^\alpha (\mathcal{F}) \leq d(x, \partial S) \leq \delta_k^\alpha (\mathcal{F}) + \frac{\epsilon_k (1 - \eta_k)}{c_{k,\alpha}} \},$$

where $d(x, S)$ is the distance from the point $x$ to the set $S$ and $c_{k,\alpha} := 1$ for $\alpha = 0$, whereas $c_{k,\alpha} := 1 - \epsilon_k (\frac{\alpha}{\psi(0)})^{1/2} L$ for $0 < \alpha < \infty$ (see Figures 1 and 2). For $k$ large we have that the closure of $A_k \cup B_k \cup A_k' \cup B_k'$ is contained in $\Omega$.

Figure 2. The geometry for $0 < \alpha < \infty$.

Let

$$u_k(\mathcal{F}, x_n) := \begin{cases} \frac{x_n + \delta_k^\alpha (u(\mathcal{F}, \delta_k^\alpha) - u(\mathcal{F}, -\delta_k^\alpha)) + u(\mathcal{F}, -\delta_k^\alpha)}{u(x)} & \text{if } x \in A_k, \\ u(x) & \text{if } x \in \Omega \setminus (A_k \cup A_k'). \end{cases}$$
Here and henceforth $\delta_k^n$ denotes $\delta_k^n(\overline{x})$. Let us verify that $u_k \in W^{1,\infty}(\Omega \setminus A'_k)$. If $x = (\overline{x}, x_n) \in A_k$, we have

$$
|D_n u_k(\overline{x}, x_n)| = |u(\overline{x}, \delta_k^n) - u(\overline{x}, -\delta_k^n)| = \frac{u(\overline{x}, \delta_k^n) - u^-(\overline{x}, 0)}{2\delta_k^n} + \frac{u^+(\overline{x}, 0) - u^- (\overline{x}, 0)}{2\delta_k^n} + \frac{u^- (\overline{x}, 0) - u(\overline{x}, -\delta_k^n)}{2\delta_k^n} \leq L + \frac{|u(\overline{x}, 0)|}{2\delta_k^n},
$$

(71)

where the last inequality follows from the Lipschitz continuity of $u$ on $\Omega^\pm$. Using the previous estimate we also obtain

$$
\begin{align*}
&|D_j u_k(\overline{x}, x_n)| \leq \frac{x_n}{\delta_k^n} \frac{D_j u(\overline{x}, \delta_k^n) - u(\overline{x}, -\delta_k^n)}{2\delta_k^n} + \frac{|D_j u(\overline{x}, -\delta_k^n) - D_n u(\overline{x}, -\delta_k^n) D_j \delta_k^n|}{2\delta_k^n} + \frac{|D_j u(\overline{x}, \delta_k^n) + D_n u(\overline{x}, \delta_k^n) D_j \delta_k^n - D_j u(\overline{x}, -\delta_k^n) D_j \delta_k^n|}{2\delta_k^n} \\
&\leq D_j \delta_k^n \left( \frac{|u(\overline{x}, 0)|}{2\delta_k^n} + 4L + 3L, \right)
\end{align*}
$$

(72)

for $j = 1, \ldots, n - 1$ and for every $(\overline{x}, x_n) \in A_k$.

By the definition of $\delta_k^n$ and the boundness of $u$, the quotient $|u(\overline{x}, 0)|/\delta_k^n$ is bounded uniformly with respect to $\overline{x}$; since $D_j \delta_k^n \leq (\sqrt{\alpha \overline{x}})^{1/2}L \delta_k^n$, we deduce from (71) and (72) that $u_k \in W^{1,\infty}(\Omega \setminus A'_k)$, so that in the case $0 < \alpha < \infty$ we obtain $u_k \in W^{1,\infty}(\Omega)$. In the case $\alpha = 0$ inequalities (71) and (72) imply that $u_k$ is Lipschitz continuous in $\{x \in \Omega : (\overline{x}, 0) \in S\}$, with Lipschitz constant $(M/\delta_k^n) + 3nL$, where $M := ||u||_{L^\infty(\Omega)}$.

To prove that $u_k$ is Lipschitz continuous in $\Omega \setminus A'_k$ we will show that

$$
|u_k(x) - u_k(y)| \leq \left( \frac{4M}{\delta_k^n} + 12nL \right)(|\overline{x} - \overline{y}| + |x_n - y_n|) 
$$

for $x, y \in \Omega \setminus A'_k$. (73)

Let $x, y \in A_k \cup B_k \cup B'_k$. It is enough to prove (73) when $x_n$ and $y_n$ have the same sign. Indeed, if $(\overline{x}, 0) \in S$ we can write

$$
|u_k(x) - u_k(y)| \leq |u_k(\overline{x}, x_n) - u_k(\overline{x}, y_n)| + |u_k(\overline{x}, y_n) - u_k(\overline{y}, y_n)|
$$

(74)

and the estimate for the first term in the right-hand side comes from the Lipschitz continuity of $u_k$ in $\{x \in \Omega : (\overline{x}, 0) \in S\}$. If $(\overline{x}, 0) \not\in S$ and $(\overline{y}, 0) \not\in S$, then

$$
|u_k(x) - u_k(y)| = |u(x) - u(y)| \leq |u(\overline{x}, x_n) - u(\overline{x}, y_n)| + |u(\overline{x}, y_n) - u(\overline{y}, y_n)|.
$$

Since the segment with end points $(\overline{x}, x_n)$ and $(\overline{x}, y_n)$ is contained in $\Omega \setminus S$, the first term in the right-hand side is estimated by $L|x_n - y_n|$, whereas the second term is estimated by $L|\overline{x} - \overline{y}|$ due to the Lipschitz continuity of $u$ in $\Omega^\pm$.

Therefore, it is enough to prove (73) when $x_n > 0$ and $y_n > 0$. If $y_n > \delta_k^n$, then we can write (74) and the right-hand side reduces to $|u_k(\overline{x}, x_n) - u_k(\overline{x}, y_n)| + |u(\overline{x}, y_n) - u(\overline{x}, y_n)|$. The second term is estimated by $L$ as before. If $(\overline{x}, 0) \not\in S$, the first term can be written as $|u(\overline{x}, x_n) - u(\overline{y}, y_n)|$, which is estimated by $L|x_n - y_n|$, since $x, y \in \Omega^\pm$. 

It remains to consider the case \( 0 < x_n < \delta_k^0 \) and \( 0 < y_n < \delta_k^0 \). If \((\bar{x},0), (\bar{y},0) \in S\) then \( x, y \in A_k \) and the estimate has already been proved. If \((\bar{x},0), (\bar{y},0) \notin S\) then \(|u_k(x) - u_k(y)| = |u(x) - u(y)|\), which can be estimated by the Lipschitz continuity of \( u \) in \( \Omega^+ \). Assume now \((\bar{x},0) \notin S\) and \((\bar{y},0) \in S\). Let \((\bar{x},0)\) be an element of \( \partial S \) in the segment of end points \( \bar{x} \) and \( \bar{y} \), and let \( z := (\bar{x}, \delta_k^0) \). Then

\[
|u_k(x) - u_k(y)| \leq |u(x) - u(z)| + |u_k(z) - u_k(y)| \leq \left( \frac{M}{\delta_k^0} + 3nL \right) (|x-z| + |z-y|). \tag{75}
\]

We have

\[
|x-z| + |z-y| \leq |\bar{x} - \bar{z}| + |x_n - \delta_k^0| + |x - \bar{x}| + |y_n - \delta_k^0| =
|\bar{x} - \bar{z}| + |\bar{x} - \bar{y}| + 2|x_n - \delta_k^0| + |x_n - y_n|; \tag{76}
\]

so that we can estimate \((\delta_k^0 - x_n)^2\) as follows

\[
(\delta_k^0 - x_n)^2 \leq (\delta_k^0)^2 - x_n^2 \leq |\bar{x} - \bar{z}|^2. \tag{77}
\]

Inequality (73) follows from (75), (76), (77), and from \(|\bar{x} - \bar{z}| + |\bar{x} - \bar{y}| = |\bar{x} - \bar{y}|\). This concludes the proof of the Lipschitz continuity of \( u_k \) in \( \Omega \setminus A'_k \). We are now in a position to apply the McShane Theorem, so that there exists a function, still denoted \( u_k \), that extends \( u_k \) to \( A'_k \) and has the same Lipschitz constant as \( u_k \), i.e.,

\[
|u_k(x) - u_k(y)| \leq \left( \frac{4M}{\delta_k^0} + 12nL \right) (|\bar{x} - \bar{y}| + |x_n - y_n|) \quad \text{for} \quad x, y \in \Omega. \tag{78}
\]

Let us define

\[
v_k(x) := \begin{cases} 
\eta_k & \text{if } x \in A_k \cup A'_k, \\
\eta_k + \frac{c_{k,\alpha}}{\epsilon_k} (|x_n| - \delta_k^0) & \text{if } x \in B_k, \\
\eta_k + \frac{c_{k,\alpha}}{\epsilon_k} (d(x, \partial S) - \delta_k^0) & \text{if } x \in B'_k, \\
1 & \text{otherwise.}
\end{cases}
\]

Then \( u_k \to u \) in \( L^1(\Omega) \), \( \eta_k \leq v_k \leq 1 \) \( L^\infty \)-a.e. in \( \Omega \) and \( v_k \to 1 \) in \( L^1(\Omega) \); moreover \( v_k \in W^{1,\infty}(\Omega) \) and \( \nabla v_k \) \leq \( 1/\epsilon_k \) \( L^\infty \)-a.e. in \( \Omega \) by the choice of the constant \( c_{k,\alpha} \). The sequence \( F_k(u_k, v_k) \) can be written now as

\[
F_k(u_k, v_k) \quad = \quad \int_{A_k} \eta_k |\nabla u_k|^2 dx + \int_{A'_k} \eta_k |\nabla u_k|^2 dx + \int_{\Omega \setminus (A_k \cup A'_k)} v_k |\nabla u|^2 dx + 
\frac{1}{\epsilon_k} \int_{A_k} \psi(\eta_k) dx + \frac{1}{\epsilon_k} \int_{B_k} \psi(v_k) dx + \frac{1}{\epsilon_k} \int_{A'_k \cup B'_k} \psi(v_k) dx. \tag{79}
\]

Let us study each term in the previous expression. Let us start with the first one. Using (71) and (72) we obtain that there exists a constant \( K_1 \) such that

\[
\int_{A_k} \eta_k |\nabla u_k|^2 dx \quad = \quad \int_{A_k} \eta_k (D_n u_k)^2 dx + \sum_{j=1}^{n-1} \int_{A_k} \eta_k (D_j u_k)^2 dx \leq 
\leq \int_{J_n} \eta_k \frac{(u(\bar{x}, \delta_k^0) - u(\bar{x}, -\delta_k^0))^2}{2\delta_k^0} dH^{n-1} + K_1 \eta_k;
\]
if $\alpha = 0$ the right-hand side of the previous inequality tends to 0, since $u \in L^\infty(\Omega)$ and $\eta_k/\delta_k^0 \to 0$; if $0 < \alpha < \infty$, by the the Dominated Convergence Theorem it tends to
$$\frac{b_\alpha}{2} \int_{J_u} ||u|| dH^{n-1}.$$ 

Let us consider the second term in (79) in the case $\alpha = 0$, when $A'_k \neq \emptyset$. By (78) we get
$$\int_{A'_k} \eta_k |\nabla u_k|^2 dx \leq K_2 \eta_k (\delta_0^k)^2 \mathcal{L}^n(A'_k) + K_3 \eta_k,$$
where $K_2$ and $K_3$ are constants. First we note that $A'_k \subset (\partial S)_{\delta_k^0}$, where $(\partial S)_{\delta_k^0} := \{x \in \mathbb{R}^n : d(x, \partial S) < \delta_k^0\}$. From a well-known result about the Minkowski content, (see, for instance, [3, Theorem 2.106]), we can write
$$\mathcal{L}^n(A'_k) \leq O((\delta_k^0)^2),$$
so that the second term in (79) tends to 0.

The third term in (79) is estimated by
$$\int_{\Omega \setminus (A_k \cup A'_k)} |\nabla u|^2 v_k dx \leq \int_{\Omega} |\nabla u|^2 dx.$$

The fourth term in (79) is given by
$$\frac{1}{\epsilon_k} \int_{A_k} \psi(\eta_k) dx = \frac{\psi(\eta_k)}{\epsilon_k} \int_{J_u} 2\delta_k^0 dH^{n-1}.$$
It tends to 0 if $\alpha = 0$, whereas in the case $0 < \alpha < \infty$ it tends to
$$\frac{b_\alpha}{2} \int_{J_u} ||u|| dH^{n-1}.$$ 

As for the fifth term in (79), we get
$$\frac{1}{\epsilon_k} \int_{B_k} \psi(v_k) dx = \frac{2}{\epsilon_k} \int_{J_u} \left[ \int_0^{\epsilon_k \frac{1-\eta_k}{\epsilon_k \alpha}} \psi(\eta_k + \frac{\epsilon_k \alpha x_n}{\epsilon_k}) dx_n \right] dH^{n-1} =$$
$$= \frac{2}{\epsilon_k \alpha} \left( \int_0^{1-\eta_k} \psi(x_n) dx_n \right) dH^{n-1}(J_u)$$
and this term tends to $aH^{n-1}(J_u)$.

Finally, the last term in (79) can be estimated by
$$\frac{1}{\epsilon_k} \int_{A'_k \cup B'_k} \psi(v_k) dx \leq \frac{1}{\epsilon_k} \psi(\eta_k) \mathcal{L}^n(A'_k \cup B'_k);$$
arguing as above we obtain
$$\mathcal{L}^n(A'_k \cup B'_k) \leq O((d_{k,0} + \epsilon_k 1-\eta_k)^2),$$
where $d_{k,0} = \delta_k^0$ and $d_{k,\alpha} = 0$ if $\alpha > 0$; so that the last term in (79) tends to 0. Estimate (19) follows.

In the general case when $u \notin L^\infty(\Omega)$, estimate (19) follows from the previous step applied to the truncated function $u^M$, from the lower semicontinuity of $F'_\alpha$ and from the fact that $\Phi_\alpha(u^M) \leq \Phi_\alpha(u)$. \qed
7. Convergence of Minimizers. The most important result of the paper is the following theorem on the convergence of minimizers of some variational problems involving the functionals $F_k$ and $F_\alpha$.

**Theorem 7.1.** Let $q > 1$; let $(\epsilon_k)$ and $(\eta_k)$ be infinitesimal sequences of positive numbers with $\alpha := \lim_k \eta_k/\epsilon_k$. Let $V_k$ be as in (9), let $\psi \in C^1([0,1])$ be a strictly decreasing function satisfying (11), and let $g \in L^q(\Omega)$. For every $k$, let $(u_k,v_k) \in H^1(\Omega) \times V_k$ be a minimizer of the problem

$$
\min_{(u,v) \in H^1(\Omega) \times V_k} \left( \int_{\Omega} v |\nabla u|^2 dx + \frac{1}{\epsilon_k} \int_{\Omega} \psi(v) dx + \int_{\Omega} |u - g|^q dx \right). \tag{80}
$$

Then $v_k \to 1$ in $L^1(\Omega)$ and a subsequence of $(u_k)$ converges in $L^q(\Omega)$ to a minimizer $u$ of the following limit problem:

$$
\min_{u \in SBV^2(\Omega)} \left( \int_{\Omega} |\nabla u|^2 dx + a \mathcal{H}^{n-1}(J_u) + b_\alpha \int_{J_u} |[u]| d\mathcal{H}^{n-1} + \int_{\Omega} |u - g|^q dx \right),
$$

if $0 < \alpha < \infty$; whereas in the extreme cases $\alpha = 0$ and $\alpha = \infty$ the limit problems are

$$
\begin{align*}
\min_{u \in GSBV^2(\Omega)} & \left( \int_{\Omega} |\nabla u|^2 dx + \mathcal{H}^{n-1}(J_u) + \int_{\Omega} |u - g|^q dx \right), \\
\min_{u \in H^1(\Omega)} & \left( \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} |u - g|^q dx \right),
\end{align*}
$$

respectively. Moreover for $0 \leq \alpha \leq \infty$ the minimum values of (80) tend to the minimum value of the limit problem.

**Remark 4.** If $g \in L^\infty(\Omega)$, the limit problem for $\alpha = 0$ can be formulated in $SBV^2(\Omega) \cap L^\infty(\Omega)$, since the functionals considered in these problems decrease by truncation with constants larger than $\|g\|_{L^\infty(\Omega)}$. As a consequence of Proposition 2, when $n = 1$ the limit problem in the case $\alpha = 0$ can be formulated in $SBV^2(\Omega)$ even if $g \notin L^\infty(\Omega)$.

To prove Theorem 7.1 we shall consider the functionals $F_{q,k}: L^q(\Omega) \times L^1(\Omega) \to [0, +\infty]$ defined by

$$
F_{q,k}(u,v) := F_k|_{L^q(\Omega) \times L^1(\Omega)},
$$

where the functionals $(F_k)$ are defined in (8).

The first step in the proof of Theorem 7.1 is the following lemma.

**Lemma 7.2.** Under the hypotheses of Theorem 7.1, the functionals $F_{q,k}$ $\Gamma$-converge in $L^q(\Omega) \times L^1(\Omega)$ to the functional $F_{q,\alpha} := F_\alpha|_{L^q(\Omega) \times L^1(\Omega)}$, where $F_\alpha$ is defined in (16).

**Proof.** Let $F_{q,\alpha}'$ and $F_{q,\alpha}''$ be the $\Gamma$-lim inf and the $\Gamma$-lim sup of $F_{q,k}$ in $L^q(\Omega) \times L^1(\Omega)$ and let $(u,v) \in L^q(\Omega) \times L^1(\Omega)$.

Proof of the estimate from below.

The $\Gamma$-lim inf inequality follows from $F_{q,\alpha}' \geq F_\alpha'$ (see, for instance, [11, Proposition 6.3]) and from Theorem 3.1.

Proof of the estimate from above.

Let $u \in GSBV^2(\Omega) \cap L^1(\Omega)$ with $F_\alpha(u,1) < +\infty$.

First we suppose $u \in L^\infty(\Omega)$. Theorem 3.3 ensures the existence of a sequence $(u_k,v_k) \in H^1(\Omega) \times V_k(\Omega)$ such that $(u_k,v_k) \to (u,1)$ in $L^1(\Omega) \times L^1(\Omega)$ and

$$
\lim_{k \to \infty} F_k(u_k,v_k) = F_\alpha(u,1).
$$
The $\Gamma$-limsup inequality follows from this equality, from the convergence of the truncated functions $u^M_k \to u$ in $L^2(\Omega)$ with $M := ||u||_{L^\infty(\Omega)}$, and from the fact that $F_{q,k}(u^M_k, v_k) \leq F_k(u_k, v_k)$.

In the general case when $u \notin L^\infty(\Omega)$ the $\Gamma$-limsup inequality follows from the previous step applied to the truncated function $u^M$, from the lower semicontinuity of $F_{q,\alpha}$ and from the fact that $F_\alpha(u^M, 1) \leq F_\alpha(u, 1)$. 

In order to obtain Theorem 7.1 we also need a compactness result, whose proof makes use of the following theorem, due to Alberti, Bouchitté, and Seppecher (see [1]). For every set $\mathcal{F} \subset L^1(\Omega)$ we define $\mathcal{F}_y^\xi := \{u^\xi_y : u \in \mathcal{F}\}$ for every $\xi \in S^{n-1}$ and

**Theorem 7.3.** Let $\mathcal{F}$ be an equibounded subset of $L^\infty(\Omega)$. Assume that there exist $n$ linearly independent unit vectors $\xi$ which satisfy the following property: for every $\delta > 0$ there exists an equibounded subset $\mathcal{F}_y^\xi$ of $L^\infty(\Omega)$ such that $\mathcal{F}$ lies in a $\delta$-neighborhood of $\mathcal{F}_y^\xi$ with respect to the $L^1(\Omega)$ distance and $(\mathcal{F}_y^\xi)_{\mathcal{F}}$ is pre-compact in $L^1(\Omega)_{\mathcal{F}}$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega$. Then $\mathcal{F}$ is pre-compact in $L^1(\Omega)$.

The compactness result is given by the following theorem.

**Theorem 7.4.** Let $(u_k, v_k)$ be a sequence in $L^1(\Omega) \times L^1(\Omega)$ such that $(u_k)$ is bounded in $L^1(\Omega)$ and

$$\liminf_{k \to \infty} F_k(u_k, v_k) < +\infty.$$ 

Then there exists a subsequence $(u_j, v_j)$ of $(u_k, v_k)$ and a function $u \in GSBV(\Omega) \cap L^1(\Omega)$ such that $u_j \to u$ $L^1$-a.e. on $\Omega$ and $v_j \to 1$ in $L^1(\Omega)$.

**Proof.** We can suppose, up to subsequences, that there exists a constant $M < +\infty$ such that

$$F_k(u_k, v_k) \leq M.$$ 

This implies in particular that $v_k \to 1$ in $L^1(\Omega)$. We divide the proof into three steps.

The bounded case for $n = 1$. Let $n = 1$ and let $(u_k)$ be bounded in $L^\infty(\Omega)$. It is not restrictive to assume $\Omega = [0,1]$; if this is not the case we prove the statement for each connected component and then we use a diagonal argument.

Repeating step (a) of the proof of Theorem 3.2 in the case $n = 1$, we can find $m+1$ points $0 = x_0 < \cdots < x_m = 1$ such that $\nabla u_k$ is bounded in $L^2(x_i + \mu, x_{i+1} - \mu)$ uniformly with respect to $k$, $\mu > 0$, and $i = 0, \ldots, m-1$. This implies by assumption that $u_k$ is bounded in $H^1(x_i + \mu, x_{i+1} - \mu)$ uniformly with respect to $k$, $\mu$, and $i$. For every $\mu > 0$, we can find a subsequence of $(u_k)$, not relabeled, that converges in $L^2(x_i + \mu, x_{i+1} - \mu)$, for $i = 0, \ldots, m-1$. Then by a diagonal argument we extract a further subsequence $(u_j)$ of $(u_k)$ that converges in $L^1(\Omega)$ to some $u \in L^\infty(\Omega)$.

From this convergence and from Proposition 1 we also deduce $u \in SBV^2(\Omega)$.

The bounded case for $n > 1$. Let $n > 1$ and let $(u_k)$ be bounded in $L^\infty(\Omega)$.

Let $\xi \in \mathbb{R}^n$ be a unit vector and let $V_{y,k}$, $F_{y,k}$ be defined as in (52), (51). Moreover we set

$$A_k := \{y \in \Pi^\xi: F_{y,k}((u^\xi_y)_k, (v^\xi_y)_k) \leq L\},$$

where $L$ is a fixed constant, so that by the Chebyshev inequality we obtain

$$\mathcal{H}^{n-1}(A_k) \leq \frac{M}{L}.$$
Let $\delta > 0$; we can choose $L$ so that $\text{diam}(\Omega)cM/L < \delta$, with $c := \sup_k ||u_k||_{L^\infty}$. Let us define

$$(w_k)_y^x(t) := \begin{cases} (u_k)_y^x(t) & \text{if } y \in A_k, \\ 0 & \text{otherwise} \end{cases}$$

and let $w_k(y + t\xi) := (w_k)_y^x(t)$, for $y \in \Pi^x$ and $t \in \Omega_y^x$. Then

$$||w_k - u_k||_{L^1(\Omega)} \leq c \text{diam}(\Omega)\mathcal{H}^{n-1}((A_k)^c) < \delta.$$ 

Let $\mathcal{F} := (u_k)$ and $\mathcal{F}_\delta := (w_k)$, then $\mathcal{F}$ lies in a $\delta$-neighborhood of $\mathcal{F}_\delta$ with respect to the $L^1(\Omega)$ distance; moreover $\mathcal{F}_\delta$ is pre-compact by the first part of the proof. From Theorem 7.3, we deduce the existence of a function $u \in L^\infty(\Omega)$ and of a subsequence $(u_j, v_j)$ of $(u_k, v_k)$ such that $(u_j, v_j) \to (u, 1)$ in $L^1(\Omega) \times L^1(\Omega)$ and $||u||_{L^\infty(\Omega)} \leq c$. Since

$$F'_\alpha(u, 1) \leq \lim_{j \to \infty} F_j(u_j, v_j) \leq M,$$

by Theorem 3.1 we conclude $u \in \text{GSBV}^2(\Omega) \cap L^\infty(\Omega)$, i.e., $u \in \text{SBV}^2(\Omega) \cap L^\infty(\Omega)$.

The general case. For every $\mu \in \mathbb{N}$ we can consider $u_k^\mu := (-\mu \lor u_k) \land \mu$, then

$$F_k(u_k^\mu, v_k) \leq F_k(u_k, v_k)$$

and by the first part of the proof there exists a subsequence $(u_j^\mu)$ of $(u_k^\mu)$ and a function $u_\mu \in \text{SBV}^2(\Omega) \cap L^\infty(\Omega)$, with $||u_\mu||_{L^\infty(\Omega)} \leq \mu$, such that $u_j^\mu \to u_\mu$ in $L^1(\Omega)$ and $\mathcal{L}^n$-a.e. in $\Omega$. This implies that the complement of the set

$$A := \{ x \in \Omega : (u_j^\mu(x)) \text{ converges for every } \mu \in \mathbb{N} \}$$

is negligible. Let us observe that

$$(u_\mu(x))^\lambda = \lim_{j \to \infty} (u_j^\mu(x))^\lambda = \lim_{j \to \infty} u_j^\lambda(x) = u_\lambda(x) \quad \text{for every } \mu > \lambda. \quad (81)$$

We claim that the subset of $A$

$$E := \{ x \in A : |u_\lambda(x)| = \lambda \text{ for every } \lambda \in \mathbb{N} \}$$

has measure zero. Indeed, for every $\lambda \in \mathbb{N}$ and $\epsilon > 0$ we have

$$\mathcal{L}^n(E) \leq \mathcal{L}^n(\{|u_j^\lambda| > \lambda - \epsilon\}) \leq \frac{1}{\lambda - \epsilon} \int_\Omega |u_j|dx \leq \frac{c}{\lambda - \epsilon}$$

for $j$ large enough, where $c$ is the bounding constant of $(u_j)$ in $L^1(\Omega)$; as $\epsilon \to 0$ and $\lambda \to \infty$ we obtain $\mathcal{L}^n(E) = 0$. Let now $x \in A \setminus E$, so that there exists $\lambda \in \mathbb{N}$ with $|u_\lambda(x)| < \lambda$; this condition, together with the equalities (81) gives $u_\mu(x) = u_\lambda(x)$ for every $\mu > \lambda$.

Let us define for $\mathcal{L}^n$-a.e. $x \in \Omega$

$$u(x) := \lim_{\lambda \to \infty} u_\lambda(x),$$

then by (81) $u_\lambda$ coincides with the truncated function $u^\lambda \mathcal{L}^n$-a.e. in $\Omega$. This implies that $u_j \to u$ $\mathcal{L}^n$-a.e. in $\Omega$: since $(u_k)$ is contained in $\text{SBV}(\Omega)$ we deduce that $u \in \text{GSBV}(\Omega)$. Finally, since $u_j^\lambda$ is uniformly bounded in $L^1(\Omega)$ with respect to $\lambda$ and $j$, we also conclude that $u \in L^1(\Omega)$. \hfill $\square$

In the following lemma we compute the $\Gamma$-limit of the functionals introduced in (80).
Lemma 7.5. Let $1 < q < +\infty$ and let $g \in L^q(\Omega)$. Let us consider the sequence of functionals $(G_k)$ defined by

$$G_k(u, v) := F_k(u, v) + \int_\Omega |u - g|^q dx,$$

(82)

where $u, v \in L^1(\Omega)$ and $F_k$ is as in (8). Then $(G_k)$ $\Gamma$-converges in $L^1(\Omega) \times L^1(\Omega)$ to the functional $G_\alpha: L^1(\Omega) \times L^1(\Omega) \to [0, +\infty]$ defined by

$$G_\alpha(u, v) := F_\alpha(u, v) + \int_\Omega |u - g|^q dx.$$

Proof. Let $G'_\alpha$ and $G''_\alpha$ be the $\Gamma$-lim inf and the $\Gamma$-lim sup of $G_k$ in $L^1(\Omega) \times L^1(\Omega)$. First we observe that the functional $H: L^1(\Omega) \times L^1(\Omega) \to [0, +\infty]$ defined by

$$H(u, v) := \int_\Omega |u - g|^q dx$$

is lower semicontinuous.

In the case $q = 1$ the functional $H$ is continuous; since $(F_k)$ $\Gamma$-converges to $F_\alpha$ by Theorem 3.1, we can apply [11, Proposition 6.21] about the sum of $\Gamma$-limits to conclude that $G_k$ $\Gamma$-converges to $F_\alpha + H$.

Let $q > 1$. Since $H$ is not continuous, we need a different argument. To this aim we introduce $G_{q, \alpha}'$, the $\Gamma$-lim sup of $G_k$ in $L^q(\Omega) \times L^1(\Omega)$.

If $(u, v) \in (L^1(\Omega) \setminus L^q(\Omega)) \times L^1(\Omega)$ we obtain by [11, Proposition 6.17]

$$+\infty = F_\alpha(u, v) + H(u, v) \leq G'_\alpha(u, v);$$

let now $(u, v) \in L^q(\Omega) \times L^1(\Omega)$. By [11, Proposition 6.3, 6.17, and 6.21], by Theorem 3.1, and by Lemma 7.2 we can deduce that

$$F_\alpha(u, v) + H(u, v) \leq G'_\alpha(u, v) \leq G''_\alpha(u, v) \leq G''_{q, \alpha}(u, v) = F_{q, \alpha}(u, v) + H(u, v) = F_\alpha(u, v) + H(u, v),$$

so that the functionals $G_k$ $\Gamma$-converge to the functional $G_\alpha$. 

We are now in a position to prove Theorem 7.1.

Proof of Theorem 7.1. We fix $k$ and prove that each functional $G_k$, defined in (82), attains its minimum. Let $(u_j, v_j)$ be a sequence such that

$$\lim_{j \to \infty} G_k(u_j, v_j) = \inf_{L^q(\Omega) \times L^1(\Omega)} G_k.$$ 

Since $(G_k(u_j, v_j))$ is bounded, from the definition of $G_k$ we deduce $(u_j, v_j) \in H^1(\Omega) \times V_k$. In particular $(u_j)$ is bounded in $L^q(\Omega)$ and $(\nabla u_j)$ is bounded in $L^2(\Omega, \mathbb{R}^n)$; this implies that $(u_j)$ is bounded in $H^1(\Omega)$. Then we can find a function $u \in H^1(\Omega) \cap L^q(\Omega)$ and a subsequence of $(u_j)$, not relabeled, such that $u_j \rightharpoonup u$ weakly in $H^1(\Omega)$ and $L^q$-a.e. in $\Omega$. From the boundness of $(v_j)$ in $W^{1, \infty}(\Omega)$ we can deduce the existence of a function $v \in W^{1, \infty}(\Omega)$, with $\eta_k \leq v \leq 1$ and $|\nabla v| \leq 1/\epsilon_k$ $L^\infty$-a.e. on $\Omega$, and of a subsequence of $(v_j)$, not relabeled, such that $v_j \rightarrow v$ in $L^1(\Omega)$ and $L^\infty$-a.e. in $\Omega$. By [8, Theorem 2.3.1] and by the Fatou lemma, this implies that the estimates

$$\int_\Omega |\nabla u|^2 dx \leq \liminf_{j \to \infty} \int_\Omega |\nabla u_j|^2 v_j dx, \quad \int_\Omega |u - g|^q dx \leq \liminf_{j \to \infty} \int_\Omega |u_j - g|^q dx$$

(83)

hold, so that we obtain

$$G_k(u, v) \leq \lim_{j \to \infty} G_k(u_j, v_j) = \inf_{L^q(\Omega) \times L^1(\Omega)} G_k.$$
This shows that the infimum of $G_k$ is achieved.

Let now $(u_k,v_k)$ be a minimizer of $G_k$, which obviously belongs to $H^1(\Omega) \times V_k$. Since the sequence $(F_k(u_k,v_k))$ is bounded, by the Compactness Theorem 7.4 there exists a function $u \in GSBV(\Omega) \cap L^q(\Omega)$ and a subsequence of $(u_k,v_k)$, not relabeled, such that $u_k \rightharpoonup u$ $L^q$-a.e. in $\Omega$ and $v_k \to 1$ in $L^1(\Omega)$. Let us prove that $u_k \to u$ in $L^1(\Omega)$. By the the Dominated Convergence Theorem we get

$$\int_{B_k} |u_k-u| \, dx \leq \left( \|u_k-g\|_{L^q(\Omega)} + \|u-g\|_{L^q(\Omega)} \right) L^n(B_k)^{1-\frac{q}{2}} \leq 2 \|g\|_{L^q(\Omega)} L^n(B_k)^{1-\frac{q}{2}},$$

where the last inequality follows from the estimate $G_k(u_k, v_k) \leq G_k(0,1) = \|g\|_{L^q(\Omega)}^q$ and from (83). Since $u_k \to u$ in measure we conclude that $L^n(B) \to 0$ and the convergence $u_k \to u$ in $L^1(\Omega)$ follows.

By the $\Gamma$-convergence of $G_k$ to $G_\alpha$ (Lemma 7.5) and by a general property of $\Gamma$-convergence (see [11, Corollary 7.20]), we find that $(u,1)$ is a minimizer for $G_\alpha$, so that $u \in GSBV^2(\Omega) \cap L^q(\Omega)$. Moreover we have the convergence of minimum values and the convergence of minimizer in $L^1(\Omega) \times L^1(\Omega)$.

Let us prove now that $u_k \to u$ in $L^q(\Omega)$, up to subsequences. Since

$$F_\alpha(u,1) + \int_\Omega |u-g|^q \, dx = \lim_{k \to \infty} \left( F_k(u_k,v_k) + \int_\Omega |u_k-g|^q \, dx \right),$$

$$F_\alpha(u,1) \leq \lim inf_{k \to \infty} F_k(u_k,v_k), \quad \text{and} \quad \int_\Omega |u-g|^q \, dx \leq \lim inf_{k \to \infty} \int_\Omega |u_k-g|^q \, dx,$$

we obtain

$$\int_\Omega |u-g|^q \, dx = \lim_{k \to \infty} \int_\Omega |u_k-g|^q \, dx. \quad \text{(84)}$$

This fact, together with the $L^n$-a.e. convergence in $\Omega$ of $u_k - g$ to $u - g$, implies that $u_k \to u$ in $L^q(\Omega)$ by the Generalized Dominated Convergence Theorem. □

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