Non-local variational limits of discrete systems

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Abstract We show that the limit of a class of discrete energies defined on 1-dimensional lattices of step size ε when $\varepsilon \to 0$ define a continuum energy with a local and a non-local term with domain a subspace of the special functions of bounded variation.

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1 Introduction

Limits of discrete systems of interactions on a lattice as the mesh size tends to zero intervene both in applied settings and as a theoretical justification of continuum theories (for some recent applications to continuum mechanics see e.g. [4] and [19]). A general form of an energy for a discrete system defined on a (subset of a) lattice of mesh size ε is

$$\sum_{\substack{x,y\in\Omega\cap\varepsilon\mathbb{Z}^n\\ x\neq y}} \Psi_\varepsilon(u(x)-u(y), x-y), \tag{1.1}$$

where $u : \Omega \cap \varepsilon\mathbb{Z}^n \to \mathbb{R}^m$ is an unknown function. After identifying the set of such discrete functions as a subset of $L^1(\Omega;\mathbb{R}^m)$ the problem of the variational limit of such energies as $\varepsilon > 0$ can be set in the framework of the Γ-convergence of functionals defined on $L^1(\Omega;\mathbb{R}^m)$ (see [12], [10]). This issue has been studied under several hypotheses on the functions Ψ_ε that ensure the convergence to a *local* limit functional (see [9], [7], [8], [19], etc.). A key step is the description of the limit in the case $n = 1$, since it is frequently possible to reduce the study to a one-dimensional problem by a 'slicing' procedure. An important model class for such monodimensional energies is those of the form

$$\sum_{\substack{x,y\in\varepsilon\mathbb{Z}\\ x\neq y}} \rho_\varepsilon(x-y)\,\Psi_\varepsilon\Big(\frac{u(x)-u(y)}{x-y}\Big) \tag{1.2}$$

(for simplicity of notation we drop the dependence on Ω). The typical shape for the function Ψ_ε is convex for small values of its argument and concave otherwise, and such that $\Psi_\varepsilon/\varepsilon$ locally converges to some convex function as the lattice parameter ε tends to zero. Assumptions of this type are derived from modelling considerations (e.g. in Computer Vision, see [14], or Fracture Mechanics, see [2]). In particular, the case when

$$\Psi_\varepsilon(z) = \min\{\varepsilon|z|^2, 1\}. \tag{1.3}$$

has been widely considered. If only 'nearest-neighbour interactions' are considered (i.e. $\rho_\varepsilon = 0$ if $|x-y| > \varepsilon$) then the discrete energies take the form

$$\sum_{i\in\mathbb{Z}} \Psi_\varepsilon\Big(\frac{u(i\varepsilon) - u((i-1)\varepsilon)}{\varepsilon}\Big). \tag{1.4}$$

The continuum limit of these energies when Ψ_ε is as in (1.3) has been shown by Chambolle [9] to be the 1-dimensional version of the so called Mumford-Shah functional [17],

$$\int |u'|^2\,dt + \#(S(u)), \tag{1.5}$$

whose domain consists of piecewise locally $W^{1,2}$ functions, and where $S(u)$ denotes the set of (essential) discontinuity points of u. For more general Ψ_ε, one can obtain non-convex limit functionals whose domain is $BV(\Omega)$ (a complete 1-dimensional study can be easily deduced from the work of Braides, Dal Maso and Garroni [4]), or a subclass of functionals defined on De Giorgi and Ambrosio's space $SBV(\Omega)$ (see Braides and Gelli [7, 8]).

The case of 'finite-order neighbourhood interactions' has been considered in [7]. In particular, the nearest-neighbour result can be extended to deal with discrete energies of the form (1.2). If $\rho_\varepsilon(z) = \rho(z/\varepsilon)$ with $\rho : \mathbb{Z} \to [0, +\infty)$ and Ψ_ε is as in (1.3), then the limit of these energies is still of the form

$$c_1 \int |u'|^2\,dt + c_2\#(S(u)), \tag{1.6}$$

where $c_1 = \sum_{k\neq 0}\rho(k)$ and $c_2 = \sum_{k\neq 0} k\rho(k)$ can be explicitly computed from ρ. However, this is not the general form of a limit of discrete energies of the form (1.2). Take for example $\rho_\varepsilon(\varepsilon) = \rho_\varepsilon(\varepsilon[1/\varepsilon]) = 1$ and $\rho_\varepsilon = 0$ elsewhere ($[t]$ stands for the integer part of t). Then, it is easily seen that the limit of the energies in (1.2) with this choice of ρ_ε is

$$\int |u'|^2\,dt + \#(S(u)) + \int (u(t+1) - u(t))^2\,dt. \tag{1.7}$$

We therefore may have a new *non local term* added to the energy in (1.5).

The aim of this paper is to present a complete characterization of all the possible limits of the model energies (1.2) with Ψ_ε as in (1.3) for arbitrary choice of ρ_ε. We will show that, under the only hypotheses that $\inf_\varepsilon \rho_\varepsilon(\varepsilon) > 0$ (equi-coerciveness of nearest-neighbour interactions) and that ρ_ε are positive and locally uniformly summable, and upon passing to a subsequence, the limit is an energy defined on piecewise-$W^{1,2}$ functions of the form

$$c_1 \int |u'|^2\, dt + \sum_{S(u)} \varphi([u]) + \int\!\!\int (u(t+s) - u(t))^2\, d\lambda(s)\, dt. \tag{1.8}$$

The Radon measure λ and the constant c_1 are determined by the (local) weak limit of the measures

$$\lambda_\varepsilon = \sum_{z\in\varepsilon\mathbb{Z}} \rho_\varepsilon(z)\delta_z \tag{1.9}$$

(δ_z is the Dirac mass at z), while the function φ is completely characterized by a *non-local discrete phase-transition formula*, which is reminiscent of a non-local continuous formula by Braides and Garroni [6] (see also [3]). Note that due to the generality of (ρ_ε) we may obtain that φ is not constant even in the 'local' case $\lambda = 0$. Note moreover that (1.8) gives an interesting extension of a Dirichlet form (see e.g. [13], [16]) to discontinuous functions, and the form of the additional non-local term agrees with that of the energies that can be deduced from the non local formulation of elasticity theory (see e.g. [18]) and with that proposed by De Giorgi to approximate some free-discontinuity problems (see [15], [2]).

2 Statement of the main result

For all $\varepsilon > 0$ let $\rho_\varepsilon : \varepsilon\mathbb{Z} \to [0, +\infty)$. With fixed a bouded open interval (a, b), consider the discrete energies

$$\sum_{\substack{x,y\in\varepsilon\mathbb{Z}\cap(a,b)\\ x\neq y}} \rho_\varepsilon(x-y)\, \Psi_\varepsilon\Big(\frac{u(x)-u(y)}{x-y}\Big) \tag{2.1}$$

defined for $u : \varepsilon\mathbb{Z} \to \mathbb{R}$, where

$$\Psi_\varepsilon(z) = \min\{\varepsilon z^2, 1\}.$$

Note that we may assume that ρ_ε is an even function, upon replacing $\rho_\varepsilon(z)$ by $\tilde\rho_\varepsilon(z) = (1/2)(\rho_\varepsilon(z) + \rho_\varepsilon(-z))$. We will tacitly make this simplifying assumption in the sequel.

We will consider the following hypotheses on ρ_ε:

(H1) (*equi-coerciveness of nearest-neighbour interactions*) $\inf_\varepsilon \rho_\varepsilon(\varepsilon) > 0$;

(H2) (*local uniform summability of ρ_ε*) for all $T > 0$ we have

$$\sup_\varepsilon \sum_{x\in\varepsilon\mathbb{Z}\cap(0,T)} \rho_\varepsilon(x) < +\infty.$$

Remark 2.1 Note that (H2) can be rephrased as a local uniform integrability property for $\varepsilon\rho_\varepsilon$ on $\mathbb{R}^2$: for all $T > 0$

$$\sup_\varepsilon \sum_{\substack{x,y\in\varepsilon\mathbb{Z} \\ x\neq y, |x|,|y|\leq T}} \varepsilon\rho_\varepsilon(x-y) < +\infty.$$

As a consequence, if (H2) holds then, up to a subsequence, we can assume that the Radon measures

$$\mu_\varepsilon = \sum_{x,y\in\varepsilon\mathbb{Z},\ x\neq y} \varepsilon\rho_\varepsilon(x-y)\delta_{(x,y)}$$

(δ_z denotes the Dirac mass at z) locally converge weakly in $\mathbb{R}^2$ to a Radon measure μ_0, and that the Radon measures

$$\lambda_\varepsilon = \sum_{z\in\varepsilon\mathbb{Z}} \rho_\varepsilon(z)\delta_z$$

locally converge weakly in $\mathbb{R}$ to a Radon measure λ_0. These two limit measures are linked by the relation

$$\mu_0(A) = \frac{1}{\sqrt{2}}\int_{\mathbb{R}} |A_s| d\lambda_0(s), \tag{2.2}$$

where $|A_s|$ is the Lebesgue measure of the set

$$A_s = \{t\in\mathbb{R} : (s(e_1-e_2)+t(e_1+e_2))/\sqrt{2}\in A\}.$$

If (H1) holds then we have the orthogonal decomposition

$$\lambda_0 = \lambda_1 + c_1\delta_0, \tag{2.3}$$

for some $c_1 > 0$ and a Radon measure λ_1 on $\mathbb{R}$. We also denote

$$\mu = \mu_0 \llcorner (\mathbb{R}^2\setminus\Delta) \tag{2.4}$$

(the restriction of μ_0 to $\mathbb{R}^2\setminus\Delta$), where $\Delta = \{(x,x) : x\in\mathbb{R}\}$. By the decomposition above, we have

$$\mu_0 = \mu + \frac{1}{\sqrt{2}}c_1\mathcal{H}^1\llcorner\Delta,$$

where $\mathcal{H}^1$ stands for the 1-dimensional Hausdorff measure.

Each function $u : \varepsilon\mathbb{Z}\to\mathbb{R}$ will be identified, upon slightly abusing notation, with its extension to a function $u\in L^1_{\rm loc}(\mathbb{R})$ which is continuous on $\mathbb{R}$ and affine on each interval $(i\varepsilon, (i+1)\varepsilon)$. We denote by $\mathcal{A}_\varepsilon$ the set of such functions. The energy (2.1) is extended to an equivalent functional defined on $L^1(a,b)$ by setting

$$F_\varepsilon(u) = \begin{cases} \displaystyle\sum_{x,y\in\varepsilon\mathbb{Z}\cap(a,b),\ x\neq y} \rho_\varepsilon(x-y)\,\Psi_\varepsilon\Big(\frac{u(x)-u(y)}{x-y}\Big) & \text{if } u\in\mathcal{A}_\varepsilon \\ +\infty & \text{otherwise.} \end{cases} \tag{2.5}$$

We will investigate the variational limit of F_ε as $\varepsilon\to 0$ in the sense of De Giorgi's Γ-convergence [12].

We recall that the Γ-limit (with respect to the $L^1(a,b)$-convergence) of a family F_ε is the functional $F : L^1(a,b)\to[0,+\infty]$ satisfying the two conditions:

(i) (*liminf inequality*) for all $u_\varepsilon\to u$ in $L^1(a,b)$ we have $F(u)\leq\liminf_\varepsilon F_\varepsilon(u_\varepsilon)$;

(ii) (*limsup inequality*) for all $u\in L^1(a,b)$ and $\eta>0$ there exists $u_\varepsilon\to u$ in $L^1(a,b)$ such that $\limsup_\varepsilon F_\varepsilon(u_\varepsilon)\leq F(u)+\eta$.

The importance of Γ-convergence in the calculus of variations is well known since it ensures and describes the convergence of equi-coercive minimum problems (see [10] and [5] Part II for a general introduction to Γ-convergence, and [4, 7] for a complete statement of the convergence of minimum problems in a discrete setting).

The main result of this paper is the following.

Theorem 2.2 (Compactness and representation) *If conditions* (H1) *and* (H2) *hold, then there exist a subsequence (not relabelled) of $\{\varepsilon\}$ converging to 0, a Radon measure μ on $\mathbb{R}^2$, a constant $c_1>0$ and an even subadditive and lower semicontinuous function $\varphi:\mathbb{R}\to[0,+\infty]$ such that the energies F_ε Γ-converge to the energy F defined on $L^1(a,b)$ by*

$$F(u) = \begin{cases} c_1\displaystyle\int_{(a,b)}|u'|^2\,dt + \sum_{S(u)}\varphi([u]) + \int_{(a,b)^2}\Big(\frac{u(x)-u(y)}{x-y}\Big)^2 d\mu(x,y) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } u \text{ is piecewise } W^{1,2} \text{ on } (a,b) \\ +\infty \qquad\qquad\qquad\qquad\qquad\quad \text{otherwise,} \end{cases} \tag{2.6}$$

where $S(u)$ denotes the set of discontinuity points for u and $[u](t)=u(t+)-u(t-)$ is the jump of u at t. The measure μ and c_1 are given by (2.4) *and* (2.3)*, respectively, and the function φ is given by the* discrete phase-transition energy density formula

$$\varphi(z) = \liminf_{m\to+\infty}\inf_{|w|<|z|}\lim_{\varepsilon\to0}\min\Big\{\sum_{\substack{j,k\in\mathbb{Z},\ j\neq k\\ -2/m\varepsilon\leq j,k\leq 2/m\varepsilon}}\rho_\varepsilon(\varepsilon(j-k))\Psi_\varepsilon\Big(\frac{u(j)-u(k)}{\varepsilon(j-k)}\Big)\ : \\ u:\mathbb{Z}\to\mathbb{R},\ u(j)=0 \text{ if } j<-\frac{1}{m\varepsilon},\ u(j)=w \text{ if } j>\frac{1}{m\varepsilon}\Big\} \tag{2.7}$$

for $z\in\mathbb{R}$.

Remark 2.3 (i) Since φ is subadditive, and it is also non decreasing on $[0,+\infty)$ and even, we have that either it is finite everywhere or $\varphi(z)=+\infty$ for all $z\neq 0$ (see [2] Chapter 2). In the latter case jumps are prohibited and the domain of F is indeed $W^{1,2}(a,b)$.

(ii) We will show in Section 4 that the function φ may be not constant, in contrast with the case when $\rho_\varepsilon(z)=\rho(z/\varepsilon)$ for a fixed ρ (which is a particular case of the discrete functionals treated in [7]).

(iii) Note that, by taking (2.2) into account, we can also write (2.6) in the form (1.8), with $\lambda=\sqrt{2}\,s^2\lambda_1$ and λ_1 given by (2.3).

3 Proof of the result

With fixed $m\in\mathbb{N}$ and $\varepsilon>0$ the minimum value in (2.7) defines an even function of w which is non-decreasing on $[0,+\infty)$; hence, by Helly's Theorem there exists a sequence $\{\varepsilon_j\}$ of positive numbers converging to 0 such that these minimum values converge for all w and for all m. Hence, we can assume, upon passing to this subsequence $\{\varepsilon_j\}$, that the function φ is well defined. Upon passing to a further subsequence we may also assume that the measures μ_ε in Remark 2.1 converge to μ_0. Then, μ and c_1 given by (2.4) and (2.3) are well defined as well. Hence, it suffices to prove the representation for the Γ-limit along this sequence, since the subadditivity and lower semicontinuity of φ are necessary conditions for the lower semicontinuity of F (see [2] Chapter 2).

We begin by proving the liminf inequality. Let $u_\varepsilon\to u$ in $L^1(a,b)$ be such that $\sup_\varepsilon F_\varepsilon(u_\varepsilon)<+\infty$. By hypothesis (H1), if we set

$$S^\varepsilon=\{x\in\varepsilon\mathbb{Z}:\ |u(x+\varepsilon)-u(x)|^2>1/\varepsilon\},$$

then $\#S^\varepsilon$ is equibounded, and, upon extracting a subsequence, we can suppose that $S^\varepsilon=\{x^\varepsilon_j:\ j=1,\dots,N\}$ with N independent of ε $x^\varepsilon_1<x^\varepsilon_2<\dots<x^\varepsilon_N$ and $x^\varepsilon_j\to t_j$ for all j. Set $S=\{t_j\}\subset[a,b]$. If $\{x^\varepsilon_{M_1}\},\dots,\{x^\varepsilon_{M_2}\}$ are the sequences converging to $t\in S$ then $u_\varepsilon(x^\varepsilon_{M_1})\to u(t-)$ and $u_\varepsilon(x^\varepsilon_{M_2}+\varepsilon)\to u(t+)$. Furthermore, the sequence u_ε converges locally weakly in $W^{1,2}((a,b)\setminus S)$ (see e.g. [8] for details).

For all $\eta>0$ let $S_\eta=\{t\in\mathbb{R}:\ \mathrm{dist}\,(t,S)<\eta\}$; set also $\Delta_\eta=\{(x,y)\in\mathbb{R}^2:\ |x-y|>\eta\}$. Note that the convergence

$$\frac{u_\varepsilon(x)-u_\varepsilon(y)}{x-y}\ \longrightarrow\ \frac{u(x)-u(y)}{x-y}$$

as $\varepsilon\to 0$ is uniform on $(a,b)^2\setminus(S^2_\eta\cup\Delta_\eta)$.

With fixed $m\in\mathbb{N}$, we have the inequality

$$F_\varepsilon(u_\varepsilon)\quad\geq\quad\sum_{\substack{x,y\in\varepsilon\mathbb{Z}\cap(a,b)\cap S_{4/m}\\ |x-y|\leq 4/m,\ x\neq y}}\rho_\varepsilon(x-y)\Psi_\varepsilon\Big(\frac{u_\varepsilon(x)-u_\varepsilon(y)}{x-y}\Big)$$

$$
\begin{aligned}
&+ \sum_{\substack{x,y\in\varepsilon\mathbb{Z}\cap(a,b)\setminus S_{4/m}\\ |x-y|\le 4/m,\, x\neq y}} \rho_\varepsilon(x-y)\Psi_\varepsilon\Big(\frac{u_\varepsilon(x)-u_\varepsilon(y)}{x-y}\Big)\\
&+ \sum_{\substack{x,y\in\varepsilon\mathbb{Z}\cap(a,b)\\ |x-y|>4/m}} \rho_\varepsilon(x-y)\Psi_\varepsilon\Big(\frac{u_\varepsilon(x)-u_\varepsilon(y)}{x-y}\Big)\\
=:\ & I^1_\varepsilon(u_\varepsilon)+I^2_\varepsilon(u_\varepsilon)+I^3_\varepsilon(u_\varepsilon). \qquad (3.1)
\end{aligned}
$$

We now estimate these three terms separately.

We first note that

$$
I^1_\varepsilon(u_\varepsilon) \ \ge \ \sum_{t\in S(u)} \sum_{\substack{x,y\in\varepsilon\mathbb{Z}\cap[t-(2/m),t+(2/m)]\\ x\neq y}} \rho_\varepsilon(x-y)\Psi_\varepsilon\Big(\frac{u_\varepsilon(x)-u_\varepsilon(y)}{x-y}\Big). \qquad (3.2)
$$

We use the notation introduced above for the sets S^ε and S: let $t_j\in S(u)$ with corresponding sequences $\{x^\varepsilon_{M_1}\},\dots,\{x^\varepsilon_{M_2}\}$ converging to t_j. We can suppose, up to a translation and reflection argument, that $[u](t_j)>0$, that

$$
\max\{u_\varepsilon(x):\ x\in\varepsilon\mathbb{Z},\ t_j-(2/m)\le x\le x^\varepsilon_{M_1}\}=0
$$

and that

$$
\min\{u_\varepsilon(x):\ x\in\varepsilon\mathbb{Z},\ x^\varepsilon_{M_2}+\varepsilon\le x\le t_j+(2/m)\}=z_\varepsilon,
$$

with $z_\varepsilon\to[u](t_j)$. We then have

$$
\begin{aligned}
&\sum_{\substack{x,y\in\varepsilon\mathbb{Z}\cap[t_j-(2/m),t_j+(2/m)]\\ x\neq y}} \rho_\varepsilon(x-y)\Psi_\varepsilon\Big(\frac{u_\varepsilon(x)-u_\varepsilon(y)}{x-y}\Big)\\
\ge\ & \min\Big\{\sum_{\substack{x,y\in\varepsilon\mathbb{Z}\cap[t_j-(2/m),t_j+(2/m)]\\ x\neq y}} \rho_\varepsilon(x-y)\Psi_\varepsilon\Big(\frac{v(x)-v(y)}{x-y}\Big):\\
& v(x^\varepsilon_{M_1})=u_\varepsilon(x^\varepsilon_{M_1}),\ v(x^\varepsilon_{M_2}+\varepsilon)=u_\varepsilon(x^\varepsilon_{M_2}+\varepsilon)\Big\}\\
\ge\ & \min\Big\{\sum_{\substack{x,y\in\varepsilon\mathbb{Z}\cap[t_j-(2/m),t_j+(2/m)]\\ x\neq y}} \rho_\varepsilon(x-y)\Psi_\varepsilon\Big(\frac{v(x)-v(y)}{x-y}\Big):\\
& v(x)=0 \text{ if } x\le x^\varepsilon_{M_1},\ v(x)=z_\varepsilon \text{ if } x\ge x^\varepsilon_{M_2}+\varepsilon\Big\}\\
\ge\ & \min\Big\{\sum_{\substack{x,y\in\varepsilon\mathbb{Z}\cap[t_j-(2/m),t_j+(2/m)]\\ x\neq y}} \rho_\varepsilon(x-y)\Psi_\varepsilon\Big(\frac{v(x)-v(y)}{x-y}\Big):\\
& v(x)=0 \text{ if } t_j-\frac{2}{m}\le x\le t_j-\frac{1}{m},\ v(x)=z_\varepsilon \text{ if } t_j+\frac{1}{m}\le x\le t_j+\frac{2}{m}\Big\}
\end{aligned}
$$

$$= \min\Big\{ \sum_{\substack{j,k\in\mathbb{Z}\cap[-2/(m\varepsilon),2/(m\varepsilon)]\\ j\neq k}} \rho_\varepsilon(\varepsilon(j-k))\Psi_\varepsilon\Big(\frac{v(j)-v(k)}{\varepsilon(j-k)}\Big) :$$

$$v(j)=0 \text{ if } -\frac{2}{m\varepsilon}\le j\le -\frac{1}{m\varepsilon},\ v(j)=z_\varepsilon \text{ if } \frac{1}{m\varepsilon}\le j\le\frac{2}{m\varepsilon}\Big\}. \tag{3.3}$$

Note that we have used the fact that Ψ_ε in non decreasing on $(0,+\infty)$ so that our functionals decrease by truncation (namely, when we substitute v by $(v\vee 0)\wedge z_\varepsilon$). By taking (2.7) into account and summing up for $t_j \in S(u)$, we obtain

$$\liminf_{\varepsilon\to 0} I^1_\varepsilon(u_\varepsilon) \ge \sum_{t\in S(u)} \varphi([u](t)) + o(1) \tag{3.4}$$

as $m\to+\infty$.

As for the second term, there exist positive α_ε converging to 0 as $\varepsilon\to 0$ such that

$$\lim_\varepsilon 2\sum_{k=1}^{[\alpha_\varepsilon/\varepsilon]} \rho_\varepsilon(\varepsilon k) \ge c_1 - \frac{1}{m}.$$

Let $(a',b')\subset(a,b)\setminus S_{4/m}$. For all $N\in\mathbb{N}$ and ε small enough we then have

$$\begin{aligned}
&\sum_{\substack{x,y\in\varepsilon\mathbb{Z}\cap(a',b')\\ |x-y|\le\alpha_\varepsilon,\ x\neq y}} \rho_\varepsilon(x-y)\Psi_\varepsilon\Big(\frac{u_\varepsilon(x)-u_\varepsilon(y)}{x-y}\Big)\\
&\quad= \sum_{\substack{x,y\in\varepsilon\mathbb{Z}\cap(a',b')\\ |x-y|\le\alpha_\varepsilon,\ x\neq y}} \rho_\varepsilon(x-y)\varepsilon\Big(\frac{u_\varepsilon(x)-u_\varepsilon(y)}{x-y}\Big)^2\\
&\quad\ge \sum_{i=1}^N 2\sum_{k=1}^{[\alpha_\varepsilon/\varepsilon]} \sum_{\substack{x,y\in\varepsilon\mathbb{Z}\cap(y_{i-1},y_i)\\ |x-y|=\varepsilon k}} \varepsilon\rho(\varepsilon k)\Big(\frac{u(x)-u(y)}{x-y}\Big)^2\\
&\quad\ge \sum_{i=1}^N 2\sum_{k=1}^{[\alpha_\varepsilon/\varepsilon]} \frac{(b'-a')}{N}\rho(\varepsilon k)\Big(\frac{u(y_i)-u(y_{i-1})}{y_i-y_{i-1}}\Big)^2 + o(1)
\end{aligned}$$

as $\varepsilon\to 0$, where we have set

$$y_i = a' + \frac{i}{N}(b'-a'),$$

we have used the fact that $u_\varepsilon\to u$ uniformly and the convexity of $z\mapsto z^2$. The same reasoning applied to a set $I\subset(a,b)\setminus S_{4/m}$ which can be written as a finite union of open intervals shows that

$$\liminf_\varepsilon I^2_\varepsilon(u_\varepsilon) \ge \Big(c_1-\frac{1}{m}\Big)\int_I |u'|^2\,dt.$$

¿From this inequality and the arbitrariness of I, we easily obtain that

$$\liminf_{\varepsilon} I^2_\varepsilon(u_\varepsilon) \geq \Big(c_1 - \frac{1}{m}\Big) \int_{(a,b)\setminus S_{4/m}} |u'|^2\, dt.$$

As for the third term, it suffices to remark that for all $\eta > 0$

$$\lim_{\varepsilon} \frac{1}{\varepsilon} \Psi_\varepsilon\Big(\frac{u_\varepsilon(x) - u_\varepsilon(y)}{x-y}\Big) = \Big(\frac{u(x)-u(y)}{x-y}\Big)^2$$

uniformly on $(a,b)^2 \setminus (S^2_\eta \cup \Delta_\eta)$ as $\varepsilon \to 0$, so that, by the weak convergence of μ_ε we have

$$\liminf_{\varepsilon} I^3_\varepsilon(u_\varepsilon) \geq \int_{(a,b)^2\setminus(S^2_{4/m}\cup\Delta_{4/m})} \Big(\frac{u(x)-u(y)}{x-y}\Big)^2 d\mu(x,y).$$

By summing up all these inequalities and letting $m \to +\infty$ we eventually get

$$\begin{aligned}\liminf_{\varepsilon} F_\varepsilon(u_\varepsilon) \quad &\geq \quad c_1 \int_{(a,b)} |u'|^2\, dt + \sum_{S(u)} \varphi([u]) \\ &\quad + \int_{(a,b)^2} \Big(\frac{u(x)-u(y)}{x-y}\Big)^2 d\mu(x,y).\end{aligned}$$

We now prove the limsup inequality. It suffices to show it for piecewise-affine functions, since this set is strongly dense in the space of piecewise $W^{1,2}$ functions (this density argument is customary in the theory of Γ-convergence; see [2] Chapter 4.2). We explicitly treat the case $(a,b) = (-1,1)$ and

$$u(t) = \begin{cases} \alpha t & \text{if } t < 0 \\ \beta t + z & \text{if } t > 0 \end{cases}$$

only, as the general case easily follows by repeating the construction we propose locally in the neighbourhood of each point in $S(u)$. It is not restrictive to suppose that $z > 0$, by a reflection argument, and that $\varphi(z) < +\infty$, otherwise there is nothing to prove.

Let $\eta > 0$, let $m \in \mathbb{N}$ with $0 < 1/m < \eta$ and let $z - (1/m) < z_m < z$ be such that

$$\begin{aligned}\varphi(z) \geq \lim_{\varepsilon\to 0} \min\Big\{ &\sum_{x,y\in\mathbb{Z},-2/(m\varepsilon)\leq j,k\leq 2/(m\varepsilon)} \rho_\varepsilon(\varepsilon(j-k))\Psi_\varepsilon\Big(\frac{u(j)-u(k)}{\varepsilon(j-k)}\Big) \; : \\ &u : \mathbb{Z} \to \mathbb{R},\; u(j) = 0 \text{ if } j < -\frac{1}{m\varepsilon},\; u(j) = z_m \text{ if } j > \frac{1}{m\varepsilon}\Big\} - \eta. \qquad (3.5)\end{aligned}$$

Then there exist functions $v_\varepsilon^m : \varepsilon\mathbb{Z} \to \mathbb{R}$ such that $v_\varepsilon^m(x) = 0$ for $x < -1/m$, $v_\varepsilon^m(x) = z_m$ for $x > T$, $0 \le v_\varepsilon^m \le z_m$ and

$$\lim_\varepsilon \sum_{\substack{x,y\in\varepsilon\mathbb{Z} \\ -(2/m)\le x,y\le(2/m)}} \rho_\varepsilon(x-y)\,\Psi_\varepsilon\Big(\frac{v_\varepsilon^m(x)-v_\varepsilon^m(y)}{x-y}\Big) \le \varphi(z)+\eta.$$

We set

$$u_\varepsilon^m(t) = \begin{cases} u(t+(2/m)) & \text{if } t < -2/m \\ v_\varepsilon^m(t) & \text{if } -2/m \le t \le 2/m \\ u(t-(2/m)) & \text{if } t > 2/m. \end{cases}$$

Note that $u_\varepsilon^m \to u^m$ in $L^1((-1,1)\setminus[-1/m,1/m])$ as $\varepsilon \to 0$, where

$$u^m(t) = \begin{cases} u(t+(2/m)) & \text{if } t < -2/m \\ 0 & \text{if } -2/m \le -1/m \\ z & \text{if } 1/m < t \le 2/m \\ u(t-(2/m)) & \text{if } t > 2/m. \end{cases}$$

We can then easily estimate

$$\begin{aligned} & \limsup_\varepsilon F_\varepsilon(u_\varepsilon^m) \\ \le\ & \limsup_\varepsilon \sum_{\substack{x,y\in\varepsilon\mathbb{Z},x\neq y \\ -2/m\le x,y\le 2/m}} \rho_\varepsilon(x-y)\,\Psi_\varepsilon\Big(\frac{v_\varepsilon^m(x)-v_\varepsilon^m(y)}{x-y}\Big) \\ & + \limsup_\varepsilon \int_{(a,b)^2\setminus\Delta_{2/m}} \rho_\varepsilon(x-y)\,\frac{1}{\varepsilon}\Psi_\varepsilon\Big(\frac{u_\varepsilon^m(x)-u_\varepsilon^m(y)}{x-y}\Big)\,d\mu_\varepsilon \\ & + \limsup_\varepsilon \sum_{x,y\in\varepsilon\mathbb{Z}\cap(a,b),\ x,y<-1/m,\ |x-y|\le 2/m} \rho_\varepsilon(x-y)\,\Psi_\varepsilon\Big(\frac{u_\varepsilon^m(x)-u_\varepsilon^m(y)}{x-y}\Big) \\ & + \limsup_\varepsilon \sum_{x,y\in\varepsilon\mathbb{Z}\cap(a,b),\ x,y>1/m,\ |x-y|\le 2/m} \rho_\varepsilon(x-y)\,\Psi_\varepsilon\Big(\frac{u_\varepsilon^m(x)-u_\varepsilon^m(y)}{x-y}\Big) \\ \le\ & \varphi(z)+\eta+\int_{(a,b)^2}\Big(\frac{u^m(x)-u^m(y)}{x-y}\Big)^2 d\mu + c_1\int_{(a,b)}|u'|^2\,dt + o(1) \end{aligned}$$

as $m \to +\infty$. Note that we have used the fact that by (2.2) the limit measure μ does not charge $\partial(a,b)^2$. By choosing $m = m(\varepsilon)$ with $m(\varepsilon) \to +\infty$ as $\varepsilon \to 0$, and setting $u_\varepsilon = u_\varepsilon^{m(\varepsilon)}$ we obtain the desired inequality. $\square$

4 Remarks and examples

Example 4.1 If $\rho_\varepsilon(z) = \rho(z/\varepsilon)$ with ρ summable then we have $\mu = 0$, $c_1 = 2\sum_{k=1}^\infty \rho(k)$ and $\varphi(z) = c_2 = 2\sum_{k=1}^\infty k\rho(k)$. In particular, the limit is local, and it is of the form of the one-dimensional Mumford Shah functional (see [7]).

In the following examples we drop the hypothesis that ρ_ε is even.

Example 4.2 The function φ is not always constant. As an example, take

$$\rho_\varepsilon(z) = \begin{cases} 1 & \text{if } z = \varepsilon \\ \sqrt{\varepsilon} & \text{if } z = \varepsilon[1/\sqrt{\varepsilon}] \\ 0 & \text{otherwise.} \end{cases}$$

Then it can be easily seen that the minimum for the problem defining φ is achieved on the function $v = z\chi_{(0,+\infty)}$, which gives

$$\varphi(z) = \min\{1 + z^2, 2\}.$$

Note that in this case the Γ-limit is

$$\int_{(a,b)} |u'|^2\, dt + \sum_{S(u)} \varphi([u]),$$

which is local, but not of the Mumford Shah type.

Example 4.3 If we take

$$\rho_\varepsilon(z) = \begin{cases} 1 & \text{if } z = \varepsilon \\ 4\sqrt{\varepsilon} & \text{if } z = \varepsilon[1/\sqrt{\varepsilon}] \\ 0 & \text{otherwise} \end{cases}$$

then by using the (dicretization of) $v = z\chi_{(0,+\infty)}$ as a test function we deduce the estimate

$$\varphi(z) \le \min\{1 + 4z^2, 5\}.$$

Since the right hand side is not subadditive, which is a necessary condition for lower semicontinuity, we deduce that the minimum in the definition of φ is obtained by using more than one 'discontinuity'.

Remark 4.4 By the density of the sums of Dirac deltas in the space of Radon measures on the real line, in the limit functional we may obtain any measure μ satisfying the invariance property

$$\mu(A) = \mu(A + t(e_1 + e_2))$$

for all Borel set A and $t \in \mathbb{R}$.

Remark 4.5 In the formula defining φ we cannot substitute the limit of minimum problems on $[-2/(m\varepsilon), 2/(m\varepsilon)]$ by a transition problem on the whole discrete line. In fact, if we take, as in the Introduction,

$$\rho_\varepsilon(x) = \begin{cases} 1 & \text{if } x = \varepsilon \\ 1 & \text{if } x = \varepsilon[1/\varepsilon] \\ 0 & \text{otherwise,} \end{cases}$$

then the two results are different.

Example 4.6 By again taking ρ_ε as in the previous remark, we check that in this case $\mu = (1/\sqrt{2})\mathcal{H}^1 \llcorner (r_1 \cup r_{-1})$, where $r_i = \{x - y = i\}$.

Remark 4.7 If we replace $\Psi_\varepsilon(z)$ simply by εz^2 then the domain of the limit functional is $W^{1,2}(a,b)$ and Theorem 2.2 describes the non-local limit of a class of discrete quadratic forms. In this case the compactness result is related to the study of asymptotic Dirichlet forms (see [16]).

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