Concentration phenomena for singularly perturbed elliptic problems and related topics

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Part I

INTRODUCTION

1 OUTLINE OF THE THESIS

1.1 SOME MOTIVATIONS

In this thesis we consider the following singular perturbation problem

$$-\epsilon^2 \Delta u + u = u^p \quad \text{in } \Omega, \tag{1.1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $p \in \left(1, \frac{n+2}{n-2}\right)$ is subcritical with respect to the Sobolev embedding and $\epsilon > 0$ is a small parameter.

We analyze the concentration phenomena of the solutions to the problem (1.1.1) in a bounded domain $\Omega \subset \mathbb{R}^n$ whose boundary is non-smooth.

Problem (1.1.1) or some of its variants arise in several physical and biological models. Consider, for example, the Nonlinear Schrödinger Equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + V\psi - \gamma |\psi|^{p-2} \psi, \tag{1.1.2}$$

where $\hbar$ is the Planck constant, V is the potential, and γ and m are positive constants. Then standing waves of (1.1.2) can be found setting $\psi(x,t) = e^{-iEt/\hbar} v(x)$, where E is a constant and the real function v satisfies the elliptic equation

$$-\hbar^2 \Delta v + \tilde{V} v = |v|^{p-2} v$$

for some modified potential $\tilde{V}$. In particular, when one considers the semiclassical limit $\hbar \to 0$, the last equation becomes a singularly perturbed one; see for example [4], [29], and references therein.

Concerning reaction-diffusion systems, this phenomenon is related to the so-called Turing's instability. More precisely, it is known that scalar reaction-diffusion equations in a convex domain admit only constant stable steady state solutions; see [16], [54]. On the other hand, as noticed in [77], reaction-diffusion systems with different diffusivities might generate non-homogeneous stable steady states. A well-known example is the Gierer-Meinhardt system, introduced in [33] to describe some biological experiment. The system is the following:

$$\begin{cases} u_t = d_1 \Delta u - u + \frac{u^p}{v^p} & \text{in } \Omega \times (0, +\infty), \\ v_t = d_2 \Delta v - v + \frac{u^r}{v^s} & \text{in } \Omega \times (0, +\infty), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, +\infty). \end{cases} \tag{1.1.3}$$

The functions u and v represent the densities of some chemical substances, the numbers p, q, r, s are non-negative and such that $0 < \frac{p-1}{q} < \frac{r}{s+1}$, and it is assumed that the diffusivities d_1 and d_2 satisfy $d_1 \ll 1 \ll d_2$. In the stationary case of (1.1.3), when $d_2 \to +\infty$ the function v is close to a constant (being nearly harmonic and with zero normal derivative at the boundary), and therefore the equation satisfied by u is similar to (1.1.1), with $\epsilon^2 = d_1$. We refer to [57], [61] for more details.

Finally, one can consider the problem (1.1.1) with mixed boundary conditions in the study of the *population dynamics*: suppose that a species lives in a bounded region Ω whose boundary has two parts, the Neumann part $\partial_N \Omega$, which is an obstacle that blocks the pass across, and the Dirichlet part $\partial_D\Omega$, which is a killing zone for the population. Moreover (1.1.1) with mixed boundary conditions is a model of the *heat conduction* for small conductivity, when there is a nonlinear source in the interior of the domain, with combined isothermal and isolated regions at the boundary.

1.2 THE CASE OF Ω SMOOTH

The study of the concentration phenomena at points for smooth domains is very rich and has been intensively developed in recent years. The search for such condensing solutions is essentially carried out by two methods. The first approach is variational and uses tools of the critical point theory or topological methods. A second way is to reduce the problem to a finite-dimensional one by means of Lyapunov-Schmidt reduction.

The typical concentration behavior of solution $u_{Q,\epsilon}$ to (1.1.1) is via a scaling of the variables in the form

$$u_{Q,\epsilon}(x) \sim U\left(\frac{x-Q}{\epsilon}\right), \tag{1.2.1}$$

where Q is some point of $\bar{\Omega}$, and U is a solution of the problem

$$-\Delta U + U = U^p \quad \text{in } \mathbb{R}^n \quad (\text{or in } \mathbb{R}^n_+ = \{(x_1,\dots,x_n) \in \mathbb{R}^n : x_n > 0\}), \tag{1.2.2}$$

the domain depending on whether Q lies in the interior of Ω or at the boundary. When $p < \frac{n+2}{n-2}$ (and indeed only if this inequality is satisfied), problem (1.2.2) admits positive radial solutions which decay to zero at infinity; see [9, 10, 75]. Solutions of (1.1.1) with this profile are called *spike-layers*, since they are highly concentrated near some point of $\bar{\Omega}$.

Let us now describe some results which concern singularly perturbed problems with Neumann or Dirichlet boundary conditions, and specifically

$$\begin{cases} -\epsilon^2\Delta u + u = u^p & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{1.2.3}$$

and

$$\begin{cases} -\epsilon^2\Delta u + u = u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega. \end{cases} \tag{1.2.4}$$

Consider first the problem with pure Neumann boundary conditions. Solutions of (1.2.3) with a concentration at one or more points of the boundary $\partial\Omega$ as $\epsilon \to 0$ are called *boundary-spike layers*. They are peaked near critical points of the mean curvature. In particular, it was shown in [59], [60] that mountain-pass solutions of (1.2.3) concentrate at $\partial\Omega$ near global maxima of the mean curvature. One can see this fact considering the variational structure of the problem. In fact, solutions of (1.2.3) can be found as critical points of the following Euler-Lagrange functional

$$I_{\epsilon,N}(u) = \frac{1}{2}\int_\Omega \left(\epsilon^2|\nabla u|^2 + u^2\right) dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx, \quad u \in H^1(\Omega).$$

Plugging into $I_{\epsilon,N}$ a function of the form (1.2.1) with $Q \in \partial\Omega$ one sees that

$$I_{\epsilon,N}(u_{Q,\epsilon}) = C_0\epsilon^n - C_1\epsilon^{n+1}H(Q) + o\left(\epsilon^{n+1}\right), \tag{1.2.5}$$

where C_0, C_1 are positive constants depending only on n and p, and H is the mean curvature; see for instance [4], Lemma 9.7. To obtain this expansion one can use the radial symmetry of U and parametrize $\partial\Omega$ as a normal graph near Q. From the above formula one can see that the bigger is the mean curvature the lower is the energy of this function: roughly speaking, boundary spike layers would tend to move along the gradient of H in order to minimize their energy. Moreover one can say that the energy of spike-layers is of order ϵ^n, which is proportional to the volume of their *support*, heuristically identified with a ball of radius ϵ centered at the peak. There is an extensive literature regarding the search of more general solutions of (1.2.3) concentrating at critical points of H; see [21], [36], [37], [38], [45], [47], [58], [80].

Consider now the problem with pure Dirichlet boundary conditions. In this case spike layers with minimal energy concentrate at the interior of the domain, at points which maximize the distance from the boundary; see [46], [62]. The intuitive reason for this is that, if Q is in the interior of Ω and if we want to adapt a function like (1.2.1) to the Dirichlet conditions, the adjustment needs an energy which increases as Q becomes closer and closer to $\partial\Omega$. Following the above heuristic argument, we could say that spike layers are *repelled* from the regions where Dirichlet conditions are imposed.

There are other types of solutions of (1.1.1) with interior and/or boundary peaks, possible multiple, which are constructed by using gluing techniques or topological methods; see [18], [19], [39], [40], [41], [79]. For *interior spike* solutions the distance function d from the boundary $\partial\Omega$ plays a role similar to that of the mean curvature H. In fact, solutions with interior peaks, as for the problem with the Dirichlet boundary condition, concentrate at critical points of d, in a generalized sense; see [46], [62], [81].

Concerning problem (1.1.1) with mixed Neumann and Dirichlet boundary conditions, in two recent papers [31], [32] it was proved that, under suitable geometric conditions on the boundary of a smooth domain, there exist solutions which approach the intersection of the Neumann and the Dirichlet parts as the singular perturbation parameter tends to zero. In particular, this concentration phenomenon on the interface between the Neumann and the Dirichlet parts occurs for the mountain pass type solutions. In fact, denoting by $u_{\epsilon,Q}$ an approximate solution peaked at Q and by d_ϵ the distance of Q from the interface between the Neumann part and the Dirichlet part, then its energy turns out to be the following

$$I_\epsilon\left(u_{Q,\epsilon}\right) = C_0\epsilon^n - C_1\epsilon^{n+1}H(Q) + \epsilon^n e^{-2\frac{d_\epsilon}{\epsilon}(1+o(1))} + o\left(\epsilon^{n+2}\right), \tag{1.2.6}$$

where I_ϵ is the functional associated to the mixed problem. Note that the first two terms in (1.2.6) are as in the expansion (1.2.5), while the third one represents a sort of *potential energy* which decreases with the distance of Q from the interface, consistently with the *repulsive effect* which was described before for (1.2.4).

There is an extensive literature regarding this type of problems, but in almost all cases the domain Ω was assumed to be smooth.

Concerning the case Ω non smooth, at our knowledge there is only a bifurcation result for the equation

$$\begin{cases} \Delta u + \lambda f(u) = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

obtained by Shi in [74] when Ω is a rectangle $(0,a)\times(0,b)$ in $\mathbb{R}^2$.

First we study the problem (1.1.1) with Neumann boundary conditions; this problem has been studied in [23]. Assuming for simplicity that $\Omega \subset \mathbb{R}^3$ is a piecewise smooth bounded domain whose boundary $\partial\Omega$ has a finite number of smooth edges, one can fix an edge Γ on the boundary and consider the function $\alpha : \Gamma \to \mathbb{R}$ which associates to every $Q \in \Gamma$ the opening angle at Q, $\alpha(Q)$. As in the smooth case, we can expect that the function α plays the same role as the mean curvature H for a smooth domain. In fact, plugging into $I_{\epsilon,N}$ a function of the form (1.2.1) with $Q \in \Gamma$ one obtains the analogous expression to (1.2.5) for this kind of domains, with $C_0\alpha(Q)$ instead of C_0; see Lemma 3.2.3. Roughly speaking, we can say that the energy of solutions is of order ϵ^3, which is proportional to the volume of their *support*, heuristically identified with a ball of radius ϵ centered at the peak $Q \in \Gamma$; then, when we intersect this ball with the domain we obtain the dependence on the angle $\alpha(Q)$.

The first result of this thesis is the following

Theorem (Theorem 3.0.2)**.** *Let $\Omega \subset \mathbb{R}^3$ be a piecewise smooth bounded domain whose boundary $\partial\Omega$ has a finite number of smooth edges, and $1 < p < 5$. Fix an edge Γ, and suppose $Q \in \Gamma$ is a local strict maximum or minimum of the function α, with $\alpha(Q) \neq \pi$. Then for $\epsilon > 0$ sufficiently small problem* (1.2.3) *admits a solution concentrating at Q.*

Remark 1.3.1. The condition that Q is a local strict maximum or minimum of α can be replaced by the fact that there exists an open set V of Γ containing Q such that $\alpha(Q) > \sup_{\partial V} \alpha$ or $\alpha(Q) < \inf_{\partial V} \alpha$.

Remark 1.3.2. The condition $\alpha(Q) \neq \pi$ is natural since it is needed to ensure that $\partial\Omega$ is not flat at Q.

Remark 1.3.3. We expect a similar result to hold in higher dimension, with substantially the same proof. For simplicity we only treat the 3-dimensional case.

The general strategy for proving Theorem 3.0.2 relies on a finite-dimensional reduction; see for example the book [4].

By the change of variables $x \mapsto \epsilon x$, problem (1.2.3) can be transformed into

$$\begin{cases} -\Delta u + u = u^p & \text{in } \Omega_\epsilon, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega_\epsilon, \end{cases} \tag{1.3.1}$$

where $\Omega_\epsilon := \frac{1}{\epsilon}\Omega$. Solutions of (1.3.1) can be found as critical points of the Euler-Lagrange functional

$$I_{\epsilon,N}(u) = \frac{1}{2}\int_{\Omega_\epsilon} \left(|\nabla u|^2 + u^2\right) dx - \frac{1}{p+1}\int_{\Omega_\epsilon} |u|^{p+1} dx, \quad u \in H^1(\Omega_\epsilon). \tag{1.3.2}$$

Now, first of all, one finds a manifold Z_ϵ of approximate solutions to the given problem, which are of the form $U_{Q,\epsilon}(x) = \varphi_\mu(\epsilon x) U(x - Q)$, where φ_μ is a suitable cut-off function defined in a neighborhood of $Q \in \Gamma$; see the beginning of Section 3.2, Lemma 3.2.1.

To apply the method described in Section 2.1 one needs the condition that the critical manifold Z_ϵ is non-degenerate, in the sense that it satisfies property ii) in Section 2.1. The result of non-degeneracy in Ω_ϵ, obtained in Lemma 3.2.2, follows from the non-degeneracy of a manifold Z of critical points of the unperturbed problem in $K = \tilde{K} \times \mathbb{R} \subset \mathbb{R}^3$, where $\tilde{K} \subset \mathbb{R}^2$ is a cone of opening angle $\alpha(Q)$. In fact, one sees that Ω_ϵ tends to K as $\epsilon \to 0$. To show the non-degeneracy of the unperturbed manifold Z we follow the line of Lemma 4.1 in the book

[4] or Lemma 3.1 in [52]. We prove that $\lambda = 0$ is a simple eigenvalue of the linearization of the unperturbed problem at $U \in Z$; see Lemma 3.1.1. Moreover, if $\alpha(Q) < \pi$, it has only one negative simple eigenvalue; whereas, if $\alpha(Q) > \pi$, it has two negative simple eigenvalues; see Corollary 3.1.4. We note that in the case $\alpha(Q) = \pi$, that is when $\partial\Omega$ is flat at Q, $\lambda = 0$ is an eigenvalue of multiplicity 2. The proof relies on Fourier analysis, but in this case one needs spherical functions defined on a portion of the sphere instead of the whole S^2.

Then one solves the equation up to a vector parallel to the tangent plane of the manifold Z_ϵ, and generates a new manifold $\tilde{Z}_\epsilon$ close to Z_ϵ which represents a natural constraint for the Euler functional (1.3.2); see the proof of Proposition 3.2.5. By *natural constraint* we mean a set for which constrained critical points of $I_{\epsilon,N}$ are true critical points.

We can finally apply the above mentioned perturbation method to reduce the problem to a finite dimensional one, and study the functional constrained on $\tilde{Z}_\epsilon$. Lemma 3.2.3 provides an expansion of the energy of the approximate solution peaked at Q and allows us to see that the dominant term in the expression of the reduced functional at Q is $\alpha(Q)$. This implies Theorem 3.0.2.

1.4 MIXED PROBLEMS IN NON-SMOOTH DOMAINS

The second goal of this thesis is studying the concentration of solutions for the singular perturbation problem with mixed Dirichlet and Neumann boundary conditions:

$$\begin{cases} -\epsilon^2\Delta u + u = u^p & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial_N\Omega, & u = 0 \quad \text{on } \partial_D\Omega, \\ u > 0 & \text{in } \Omega. \end{cases} \tag{1.4.1}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded domain whose bundary has an $(n-2)$-dimensional smooth singularity Γ, $p \in \left(1, \frac{n+2}{n-2}\right)$ is subcritical, ν denotes the outer unit normal at $\partial\Omega$ and $\epsilon > 0$ is a small parameter. Moreover $\partial_N\Omega$, $\partial_D\Omega$ are two subsets of the boundary of Ω such that the union of their closures coincides with the whole $\partial\Omega$, and their intersection is the singularity. This problem has been studied in [24].

We denote by H the mean curvature of $\partial\Omega$ restricted to the closure of $\partial_N\Omega$, that is $H : \overline{\partial_N\Omega} \to \mathbb{R}$. The result we prove is the following:

Theorem (Theorem 4.0.6)**.** *Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be a bounded domain whose boundary $\partial\Omega$ has an $(n-2)$-dimensional smooth singularity, and $1 < p < \frac{n+2}{n-2}$ ($1 < p < +\infty$ if $n = 2$). Suppose that $\partial_N\Omega$, $\partial_D\Omega$ are disjoint open sets of $\partial\Omega$ such that the union of the closures is the whole boundary of Ω and such that their intersection Γ is the singularity. Suppose $Q \in \Gamma$ is such that $\alpha(Q) \neq 0$ and $H|_\Gamma$ is critical and non degenerate at Q, and that $\nabla H(Q) \neq 0$ points toward $\partial_D\Omega$. Then for $\epsilon > 0$ sufficiently small problem* (1.4.1) *admits a solution concentrating at Q.*

Remark 1.4.1. (a) The non degeneracy condition in Theorem 4.0.6 can be replaced by the condition that Q is a strict local maximum or minimum of $H|_\Gamma$, or by the fact that there exists an open set V of Γ containing Q such that $H(Q) > \sup_{\partial V} H$ or $H(Q) < \inf_{\partial V} H$.

(b) With more precision, as $\epsilon \to 0$, the above solution possesses a unique global maximum point $Q_\epsilon \in \partial_N\Omega$, and $\operatorname{dist}(Q_\epsilon, \Gamma)$ is of order $\epsilon \log \frac{1}{\epsilon}$.

The general strategy for proving Theorem 4.0.6 relies on a finite-dimensional reduction as described below Remark 1.3.3. The main difference here is that one needs to adjust the solution in such a way that it vanishes on the Dirichlet part, so we explain how the strategy described before has to be adapted to this case. First, as before, one finds a manifold Z of approximate solutions to the given problem, which in our case are of the form (1.2.1), and solve the equation up to a vector parallel to the tangent plane of this manifold. To do this one can use the spectral properties of the linearization of (1.2.2), see Lemma 4.3.3. Then, see Theorem 2.1.6, one generates a new manifold $\tilde{Z}$ close to Z which represents a natural constraint for the Euler functional of (1.4.1), which is

$$\tilde{I}_\epsilon(u) = \frac{1}{2}\int_\Omega \left(\epsilon^2|\nabla u|^2 + u^2\right) dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx, \quad u \in H^1_D(\Omega),$$

where $H^1_D(\Omega)$ is the space of functions $H^1(\Omega)$ which have zero trace on $\partial_D\Omega$.

Now, we want to have a good control of the functional $\tilde{I}_\epsilon|_{\tilde{Z}}$. Improving the accuracy of the functions in the original manifold Z, we make $\tilde{Z}$ closer to Z; in this way the main term in the constrained functional will be given by $\tilde{I}_\epsilon|_Z$, see Propositions 4.2.12, 4.2.14, 4.2.15. To find sufficiently good approximate solutions we start with those constructed in literature for the Neumann problem (1.2.3) (see Subsection 4.1.1) which reveal the role of the mean curvature. The problem is that these functions are non zero on $\partial_D\Omega$, and even if one use cut-off functions to annihilate them the corresponding error turns out to be too large. Following the line of [31] and [62], we will use the *projection operator* in $H^1(\Omega)$, which associates to every function in this space its closest element in $H^1_D(\Omega)$. To study the asymptotic behavior of this projection we will use the limit behavior of the solution U to (1.2.2):

$$\lim_{r\to+\infty} e^r r^{\frac{n-2}{2}} U(r) = c_{n,p}, \tag{1.4.2}$$

where $r = |x|$ and $c_{n,p}$ is a positive constant depending only on the dimension n and p, together with

$$\lim_{r\to+\infty} \frac{U'(r)}{U(r)} = -\lim_{r\to+\infty} \frac{U''(r)}{U(r)} = -1, \tag{1.4.3}$$

as it was done in some previous works, see for instance [46] and [82]. Moreover, we will work at a scale $d \simeq \epsilon|\log\epsilon|$, which is the order of the distance of the peak from Γ, see Remark 1.4.1 (b). At this scale both $\partial_N\Omega$ and $\partial_D\Omega$ look flat; so we can identify them with the hypersurfaces of equations $x_n = 0$ and $x_1 \tan\alpha + x_n = 0$, and their intersection with the set $\{x_1 = x_n = 0\}$. Note that $\alpha = \alpha(Q)$ is the angle between x_1 and x_n at a fixed point $Q \in \Gamma$. Then we can replace Ω with a suitable domain Σ_D, which in particular for $0 < \alpha \leqslant \pi$ is even with respect to the coordinate x_n, see the beginning of Subsections 4.2.1 and 4.2.2. Now, studying the projections in this domain, we will find functions which have zero x_n-derivative on $\{x_n = 0\} \setminus \partial\Sigma_D$, which mimics the Neumann boundary condition on $\partial_N\Omega$. After analyzing carefully the projection in Subsections 4.2.1, 4.2.2, we will be able to define a family of suitable approximate solutions to (1.4.1) which have sufficient accuracy for our analysis, estimated in Propositions 4.2.12, 4.2.14, 4.2.15.

We can finally apply the above mentioned perturbation method to reduce the problem to a finite-dimensional one, and study the functional constrained on $\tilde{Z}$. We obtain an expansion of the energy of the approximate solutions, which turns out to be

$$\begin{aligned}\tilde{I}_\epsilon(u_{\epsilon,Q}) &= \tilde{C}_0\epsilon^n - \tilde{C}_1\epsilon^{n+1}H(Q) + \epsilon^n e^{-2\frac{d_\epsilon}{\epsilon}(1+o(1))} \\ &\quad + \epsilon^n e^{-\frac{d_\epsilon}{\epsilon}\left(1+\frac{\sqrt{2}\tan\alpha(Q)}{\sqrt{\tan^2\alpha(Q)+1}}\right)(1+o(1))} + o\left(\epsilon^{n+2}\right),\end{aligned}$$

in the case $0 < \alpha < \frac{\pi}{2}$, and

$$\tilde{I}_\epsilon(u_{\epsilon,Q}) = \tilde{C}_0\epsilon^n - \tilde{C}_1\epsilon^{n+1}H(Q) + \epsilon^n e^{-2\frac{d_\epsilon}{\epsilon}(1+o(1))} + o\left(\epsilon^{n+2}\right), \tag{1.4.4}$$

in the case $\frac{\pi}{2} \leqslant \alpha < 2\pi$. As for (1.2.6), we have that the first two terms come from the Neumann condition, while the others are related to the repulsive effect due to the Dirichlet condition. Let us notice that, in the first case, in the terms related to the Dirichlet condition appears the opening angle α, whereas in the second case it does not; this phenomenon comes from the fact that the distance of the point Q from the Dirichlet part $\partial_D\Omega$ depends on α only if $0 < \alpha < \frac{\pi}{2}$.

We remark that the expansion given in (1.4.4) is coherent with the case of smooth domains in which $\alpha = \pi$ (compare (1.4.4) with (1.2.6)).

Concerning the regularity of the solution, following the ideas in [34], it is possible to say that it is influenced by the presence of the angle. In fact, the solution is at least C^2 in the interior of the domain, far from the angle; whereas, near the angle, one can split the solution into a regular part and a singular one, whose regularity depends on the value of α. For more details about the regularity of solutions in non-smooth domains we refer the reader to the book [34].

The fact that the solution u is C^2 in the interior of the domain allows to say also that it is strictly positive, by using the strong Maximum Principle. In fact, we have that $u \geqslant 0$ in the domain. Moreover, if there exists a point x_0 in the interior of the domain such that $u(x_0) = 0$, we can consider a ball centered at x_0 of small radius suct that it is contained in the domain; since in the ball u is C^2 we can conclude that u cannot be zero in x_0.

1.5 EXISTENCE PROBLEMS INVOLVING THE FRACTIONAL LAPLACIAN

We would like to investigate the concentration phenomena of elliptic equations driven by the fractional Laplacian. For instance, a natural question is whether the technique developed in this thesis for equations involving classical Laplacian may be adapted to the construction of solutions concentrating either in the interior or along the boundary of the domain for an equation of the type

$$\epsilon^{2s}(-\Delta)^s u + u = u^p, \qquad \text{for } s \in (0,1).$$

To do this, a first step is constructing solutions in the whole of the space which might be used as the leading order of a perturbation argument; see [26].

For this scope, let us first briefly review what happens in the classical setting $s = 1$. In this case, one considers the problem

$$\begin{cases} -\Delta u + \eta u = \lambda |u|^{p-1} u & \text{in } \mathbb{R}^n, \\ u \in H^1(\mathbb{R}^n), \ u \not\equiv 0, \end{cases} \tag{1.5.1}$$

where λ and η are fixed positive constants and $p > 1$. Notice that, up to scaling the space variables and up to a multiplicative normalization on the solution, one may reduce himself to the case $\lambda = \eta = 1$, and so the case of positive solutions reduces to studying the equation $-\Delta u + u = u^p$.

The equation in (1.5.1) has been widely studied in the last decades, since it is the basic version of some fundamental models arising in various applications (e.g., stationary states in nonlinear equations of Schrödinger type). One of the first contributions to the analysis of problem (1.5.1) was given by Pohozaev in [67], where he proved that there exists a solution u of (1.5.1) if and only if $1 < p < 2^* - 1$, being $2^* = 2n/(n-2)$ the so-called critical Sobolev exponent. In [67] also a by-now classical "identity" appears, in order to prove that there are no solutions to (1.5.1) when p is greater or equal to $2^* - 1$.

Another important contribution to the analysis of problem (1.5.1) has been given in [9] (see also [10]), in which the authors consider an extension of the equation in (1.5.1) by replacing the nonlinearity $-\eta u + \lambda |u|^{p-1} u$ by a wider class of odd continuous functions $g = g(u)$ satisfying $g(0) = 0$ and some superlinearity and growth assumptions. Among other results, in [9] it has been shown the existence of a solution u to (1.5.1), with some properties of symmetry and a precise decay at infinity. It is worth pointing out that the method to prove the existence of solutions to (1.5.1) relies on a variational approach (the *constrained minimization method*, see [9, Section 3]), by working directly with the energy functional related to (1.5.1).

A natural question could be whether or not this method can be adapted to deal with a nonlocal version of the problem above. In this respect, we would like to extend the existence and symmetry results in [9] for the nonlocal analogue of problem (1.5.1) by replacing the standard Laplacian operator by the fractional Laplacian operator $(-\Delta)^s$, where, as usual, for any $s \in (0,1)$, $(-\Delta)^s$ denotes the s-power of the Laplacian operator and, omitting a multiplicative constant $C = C(n,s)$, we have

$$(-\Delta)^s u(x) \;=\; \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy \;=\; \lim_{\epsilon \to 0} \int_{\mathcal{C}B_\epsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy. \tag{1.5.2}$$

Here $B_\epsilon(x)$ denotes the n-dimensional ball of radius ϵ, centered at $x \in \mathbb{R}^n$, $\mathcal{C}$ denotes the complementary set, and "P.V." is a commonly used abbreviation for "in the principal value sense".

Recently, a great attention has been focused on the study of problems involving the fractional Laplacian, from a pure mathematical point of view as well as from concrete applications, since this operator naturally arises in many different contexts, such as, among the others, obstacle problems, financial market, phase transitions, anomalous diffusions, crystal dislocations, soft thin films, semipermeable membranes, flame propagations, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, minimal surfaces, materials science, water waves, etc... The literature is really too wide to attempt any reasonable comprehensive treatment here[1]. We would just cite some very recent papers which analyze fractional elliptic equations involving the critical Sobolev exponent, [73, 76, 30, 12, 7, 66, 72].

Let us come back to the present thesis. In the last part of it, we will deal with the following problem

$$\begin{cases} (-\Delta)^s u + u = |u|^{p-1} u \ \text{ in } \mathbb{R}^n, \\ u \in H^s(\mathbb{R}^n), \ u \not\equiv 0, \end{cases} \tag{1.5.3}$$

where $H^s(\mathbb{R}^n)$ denotes the fractional Sobolev space; we immediately refer to Section 5.1.2 for the definitions of the space $H^s(\mathbb{R}^n)$ and of variational solutions to (1.5.3).

Precisely, we are interested in existence and symmetry properties of the variational solutions u to (1.5.3), as stated in the following

Theorem (Theorem 5.0.7)**.** *Let $s \in (0,1)$ and $p \in (1, (n+2s)/(n-2s))$, with $n > 2s$. There exists a solution $u \in H^s(\mathbb{R}^n)$ to problem (1.5.3) which is positive and spherically symmetric.*

Note that the upper bound on the exponent p is exactly $2^*_s + 1$, where $2^*_s = 2n/(n-2s)$ is the critical Sobolev exponent of the embedding $H^s \hookrightarrow L^p$. This fractional Sobolev exponent also plays a role for the nonlinear analysis methods for equations in bounded domains; see [72]. As in the classical case, the threshold given by this exponent is essentially optimal,

1 For an elementary introduction to this topic and a wide, but still not fully comprehensive, list of related references, we refer to [22].

since non-existence results may be obtained from a fractional Pohozaev identity (see, e.g., Lemma 5.1 in [30]).

The proof of Theorem 5.0.7 extends part of that of Theorem 2 in [9]; in particular, we will apply the variational approach by the constrained method mentioned above, for the energy functional related to (1.5.3), that is

$$\mathcal{E}(u) := \frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_{\mathbb{R}^n}\left(\frac{1}{2}|u(x)|^2 - \frac{1}{p+1}|u(x)|^{p+1}\right)dx.$$

It is worth mentioning that the results in Theorem 5.0.7 for $n = 1$ have been obtained in [83], where modulation stability of ground states solitary wave solutions of nonlinear Schrödinger equations has been studied, via an *unconstrained* variational approach within the "concentration-compactness" framework of P. L. Lions ([48, 49]). Also, in the more recent papers [50] and [51], an alternative approach has been presented, which permits to handle a very general context, also including the equations we are dealing with (see, in addition, [53], where the decay of solutions is analyzed in the case $s = 1/2$).

Here, we will present a very simple proof, whose general strategy will follow the original argument in [9]. The method used here (and in [9]) relies on the selection of a specific minimizing sequence composed of radial functions: though this idea is now classical, we thought it was interesting to point out that this argument also works in the case of the fractional Laplacian. Clearly, we need to operate various technical modifications due to the non-locality of the fractional Laplacian operator (and of the correspondent norm $H^s(\mathbb{R}^n)$). Moreover, we will need some energy estimates and preliminary results, also including the analogue of the classical Polya-Szegö inequality, as given in forthcoming Section 5.1.3.

As for the decay of the solution, a precise bound may be obtained via the construction of exact barriers (see Lemma 3.1 in [71] and, also, Lemma 8 in [64]). Remarkably, the decay of the solutions in the fractional case is only polynomial, and not exponential as it happens in the classical case and this feature is, of course, the source of major complications. As a matter of fact, a slow decay of the solutions in the entire space may reflect an additional difficulty in localizing possible interior concentration points.

Also, it could be taken into account to extend all the results above in order to investigate a problem of type (1.5.3) by substituting the nonlinearity with an odd continuous function satisfying standard growth assumptions, in the same spirit of [9].[2]

The thesis is organized as follows. In the first part we deal with some concentration phenomena for the singularly perturbed equation (1.1.1) in non-smooth domains. In Chapter 2 we introduce the abstract perturbation method that we use in the subsequent chapters and we make some some geometric preliminaries. In Chapters 3 and 4 we construct the solutions to the problem (1.1.1) with both Dirichlet and mixed Dirichlet and Neumann boundary conditions. In the second part we study the problem (1.5.3). Finally, in the third part, we describe some future projects.

Notation

Generic fixed constant will be denoted by C, and will be allowed to vary within a single line or formula. The symbols $O(t)$ (respectively $o(t)$) will denote quantities for which $\frac{O(t)}{|t|}$ stays bounded (respectively $\frac{o(t)}{|t|}$ tends to zero) as the argument t goes to zero or to infinity.

2 After completing this project, we have heard of an interesting work, where related results have been presented by using different techniques (see [28]).

The symbols $o_\epsilon(1)$, $o_R(1)$ $o_{\epsilon,R}(1)$ will denote respectively a function depending on ϵ that tends to 0 as $\epsilon \to 0$, a function depending on R that tends to 0 as $R \to +\infty$ and a function depending on both ϵ and R that tends to 0 as $\epsilon \to 0$ and $R \to +\infty$. We will often use the notation $d(1+o(1))$, where $o(1)$ stands for a quantity which tends to zero as $d \to +\infty$. We will work in the space $H^1(\Omega_\epsilon)$, endowed with the norm $\|u\|^2 = \int_{\Omega_\epsilon} \left(|\nabla u|^2 + u^2\right) dx$, which we denote simply by $\|u\|$, without any subscript.

Part II

CONCENTRATION OF SOLUTIONS FOR A SINGULARLY PERTURBED ELLIPTIC PDE PROBLEM IN NON-SMOOTH DOMAINS

2 SOME PRELIMINARIES

In this chapter we introduce the abstract perturbation method which takes advantage of the variational structure of the problems we consider, and allows us to reduce them to finite dimensional ones. We refer the reader mainly to [4], [52] and the bibliography therein.

In the second part we make some computations concerning the parametrization of the boundary $\partial\Omega$ and $\partial\Omega_\epsilon$, and in particular of the edge.

2.1 PERTURBATION IN CRITICAL POINT THEORY

In this section we recall some results about the existence of critical points for a class of functionals which are perturbative in nature. We refer the reader mainly to [4], [52] and the bibliography therein for the abstract method. Given an Hilbert space H, which might depend on the perturbation parameter ϵ, let $I_\epsilon : H \to \mathbb{R}$ be a functional of class C^2 which satisfies the following properties

i) there exists a smooth finite-dimensional manifold, compact or not, $Z_\epsilon \subseteq H$ such that $\|I'_\epsilon(z)\| \leqslant C\epsilon$ for every $z \in Z_\epsilon$ and for some fixed constant C, independent of z and ϵ; moreover $\|I''_\epsilon(z)[q]\| \leqslant C\epsilon \|q\|$ for every $z \in Z_\epsilon$ and every $q \in T_z Z_\epsilon$;

ii) letting $P_z : H \to (T_z Z_\epsilon)^\perp$, for every $z \in Z_\epsilon$, be the projection onto the orthogonal complement of $T_z Z_\epsilon$, there exists $C > 0$, independent of z and ϵ, such that $P_z I''_\epsilon(z)$, restricted to $(T_z Z_\epsilon)^\perp$, is invertible from $(T_z Z_\epsilon)^\perp$ into itself, and the inverse operator satisfies $\left\|(P_z I''_\epsilon(z))^{-1}\right\| \leqslant C$.

We assume that Z_ϵ has a local C^2 parametric representation $z = z_\xi$, $\xi \in \mathbb{R}^d$. If we set $W = (T_z Z_\epsilon)^\perp$, we look for critical points of I_ϵ in the form $u = z + w$ with $z \in Z_\epsilon$ and $w \in W$. If $P_z : H \to W$ is as in ii), the equation $I'_\epsilon(z+w) = 0$ is equivalent to the following system

$$\begin{cases} P_z I'_\epsilon(z+w) = 0 & (\textit{the auxiliary equation}), \\ (Id - P_z) I'_\epsilon(z+w) = 0 & (\textit{the bifurcation equation}). \end{cases} \tag{2.1.1}$$

First we solve the auxiliary equation by means of the Implicit Function Theorem. In fact, the following result holds:

Proposition 2.1.1. *(See Proposition 2.2 in [52]) Let* i), ii) *hold. Then there exists $\epsilon_0 > 0$ with the following property: for all $|\epsilon| < \epsilon_0$ and for all $z \in Z_\epsilon$, the auxiliary equation in (2.1.1) has a unique solution $w = w_\epsilon(z)$ such that:*

j) $w_\epsilon(z) \in W$ is of class C^1 with respect to $z \in Z_\epsilon$ and $w_\epsilon(z) \to 0$ as $|\epsilon| \to 0$, uniformly with respect to $z \in Z_\epsilon$, together with its derivative with respect to z, w'_ϵ;

jj) more precisely one has that $\|w_\epsilon(z)\| = O(\epsilon)$ as $\epsilon \to 0$, for all $z \in Z_\epsilon$.

We shall now solve the bifurcation equation in (2.1.1). In order to do this, let us define the *reduced functional* $\mathbf{I}_\epsilon : Z_\epsilon \to \mathbb{R}$ by setting $\mathbf{I}_\epsilon(z) = I_\epsilon(z + w_\epsilon(z))$.

Theorem 2.1.2. *(See Theorem 2.3 in [52]) Suppose we are in the situation of Proposition 2.1.1, and let us assume that* $\mathbf{I}_\epsilon$ *has, for* $|\epsilon|$ *sufficiently small, a critical point* z_ϵ. *Then* $u_\epsilon = z_\epsilon + w(z_\epsilon)$ *is a critical point of* I_ϵ.

From a geometric point of view the argument can be outlined as follows. Consider the manifold $\tilde{Z}_\epsilon = \{z + w_\epsilon(z) : z \in Z_\epsilon\}$. If z_ϵ is a critical point of $\mathbf{I}_\epsilon$, it follows that $u_\epsilon = z_\epsilon + w(z_\epsilon) \in \tilde{Z}_\epsilon$ is a critical point of I_ϵ constrained on $\tilde{Z}_\epsilon$ and thus u_ϵ satisfies $I'_\epsilon(u_\epsilon) \perp T_{u_\epsilon}\tilde{Z}_\epsilon$. Moreover, the definition of w_ϵ, see Proposition 2.1.1, implies that $I'_\epsilon(z + w_\epsilon(z)) \in T_z Z_\epsilon$. In particular, $I'_\epsilon(u_\epsilon) \in T_{z_\epsilon} Z_\epsilon$. Since, for $|\epsilon|$ small, $T_{u_\epsilon}\tilde{Z}_\epsilon$ and $T_{z_\epsilon} Z_\epsilon$ are close, see j) in Proposition 2.1.1, which is a consequence of the smallness of w'_ϵ, it follows that $I'_\epsilon(u_\epsilon) = 0$. A manifold with these properties is called a *natural constraint* for I_ϵ.

The next result is a useful criterion for applying Theorem 2.1.2, based on expanding I_ϵ on Z_ϵ in powers of ϵ.

Theorem 2.1.3. *(See Theorem 2.4 in [52]) Suppose the assumptions of Proposition 2.1.1 hold, and that for* ϵ *small there is a local parametrization* $\xi \in \frac{1}{\epsilon}U \subseteq \mathbb{R}^d$ *of* Z_ϵ *such that, as* $\epsilon \to 0$, I_ϵ *admits the expansion* $I_\epsilon(z_\xi) = C_0 + \epsilon G(\epsilon\xi) + o(\epsilon)$, *for* $\xi \in \frac{1}{\epsilon}U$, *for some function* $G : U \to \mathbb{R}$. *Then we still have the expansion* $\mathbf{I}_\epsilon(z_\xi) = C_0 + \epsilon G(\epsilon\xi) + o(\epsilon)$, *as* $\epsilon \to 0$. *Moreover, if* $\bar{\xi} \in U$ *is a strict local maximum or minimum of* G, *then for* $|\epsilon|$ *small the functional* I_ϵ *has a critical point* u_ϵ. *Furthermore, if* $\bar{\xi}$ *is isolated, we can take* $u_\epsilon - z_{\bar{\xi}/\epsilon} = o(1/\epsilon)$ *as* $\epsilon \to 0$.

Remark 2.1.4. The last statement asserts that, once we scale back in ϵ, the solution concentrates near $\bar{\xi}$.

To study the concentration phenomena for the solutions to (1.4.1) we will use some small modifications of the arguments in [4] and [52], which can be found in Subsection 2.1 of [31].

Given an Hilbert space H, which might depend on the perturbation parameter ϵ, we consider manifolds embedded smoothly in H, for which

- i)′ there exists a smooth finite-dimensional manifold $Z_\epsilon \subseteq H$ and $C, r > 0$ such that for any $z \in Z_\epsilon$, the set $Z_\epsilon \cap B_r(z)$ can be parametrized by a map on $B_1^{\mathbb{R}^d}$ whose C^3 norm is bounded by C.

Moreover we are interested in functionals $I_\epsilon : H \to \mathbb{R}$ of class $C^{2,\gamma}$ which satisfy the following properties:

- ii)′ there exists a continuous function $f : (0, \epsilon_0) \to \mathbb{R}$ with $\lim_{\epsilon\to 0} f(\epsilon) = 0$ such that $\|I'_\epsilon(z)\| \leqslant f(\epsilon)$ for every $z \in Z_\epsilon$; moreover $\|I''_\epsilon(z)[q]\| \leqslant f(\epsilon)\|q\|$ for every $z \in Z_\epsilon$ and every $q \in T_z Z_\epsilon$;
- iii)′ there exist $C, \gamma \in (0, 1]$, $r_0 > 0$ such that $\|I''_\epsilon\|_{C^\gamma} \leqslant C$ in the subset $\{u : \mathrm{dist}(u, Z_\epsilon) < r_0\}$;
- iv)′ letting $P_z : H \to (T_z Z_\epsilon)^\perp$, for every $z \in Z_\epsilon$, be the projection onto the orthogonal complement of $T_z Z_\epsilon$, there exists $C > 0$, independent of z and ϵ, such that $P_z I''_\epsilon(z)$, restricted to $(T_z Z_\epsilon)^\perp$, is invertible from $(T_z Z_\epsilon)^\perp$ into itself, and the inverse operator satisfies $\left\|(P_z I''_\epsilon(z))^{-1}\right\| \leqslant C$.

As before, we set $W = (T_z Z_\epsilon)^\perp$, and we look for critical points of I_ϵ in the form $u = z + w$ with $z \in Z_\epsilon$ and $w \in W$. If $P_z : H \to W$ is as in iv)′, the equation $I'_\epsilon(z + w) = 0$ is equivalent to the system in (2.1.1).

Now, we solve the auxiliary equation by means of the Implicit Function Theorem.

Proposition 2.1.5. *(See Proposition* 2.1 *in [31]) Let* $i)' - iv)'$ *hold true. Then there exists* $\epsilon_0 > 0$ *with the following property: for all* $|\epsilon| < \epsilon_0$ *and for all* $z \in Z_\epsilon$*, the auxiliary equation in* (2.1.1) *has a unique solution* $w = w_\epsilon(z) \in W$*, which is of class* C^1 *with respect to* $z \in Z_\epsilon$ *and such that* $\|w_\epsilon(z)\| \leqslant C_1 f(\epsilon)$ *as* $|\epsilon| \to 0$*, uniformly with respect to* $z \in Z_\epsilon$*. Moreover the derivative of* w *with respect to* z*,* w'_ϵ *satisfies the bound* $\|w'_\epsilon(z)\| \leqslant CC_1 f(\epsilon)^\gamma$*.*

We shall now solve the bifurcation equation in (2.1.1).

Theorem 2.1.6. *(See Proposition* 2.3 *in [31]) Suppose we are in the situation of Proposition 2.1.5, and let us assume that* $\mathbf{I}_\epsilon$ *has, for* $|\epsilon|$ *sufficiently small, a stationary point* z_ϵ*. Then* $u_\epsilon = z_\epsilon + w(z_\epsilon)$ *is a critical point of* I_ϵ*. Furthermore, there exist* $\tilde{c}, \tilde{r} > 0$ *such that if* u *is a critical point of* I_ϵ *with* $\operatorname{dist}(u, Z_{\epsilon,\tilde{c}}) < \tilde{r}$*, where* $Z_{\epsilon,\tilde{c}} = \{z \in Z_\epsilon : \operatorname{dist}(z, \partial Z_\epsilon) > \tilde{c}\}$*, then* u *has to be of the form* $z_\epsilon + w(z_\epsilon)$ *for some* $z_\epsilon \in Z_\epsilon$*.*

2.2 GEOMETRIC PRELIMINARIES

In this thesis we deal with non-smooth domains, so it is useful to introduce a new set of coordinates in order to stretch the non-smooth parts of the boundary. Moreover, for our purposes, we make the change of variables $x \mapsto \epsilon x$, and we consider the scaled domain $\Omega_\epsilon := \frac{1}{\epsilon}\Omega$.

More precisely, in chapter 3, we consider a piecewise smooth bounded domain $\Omega \subset \mathbb{R}^n$ whose boundary $\partial\Omega$ has a finite number of smooth edges. We fix an edge Γ of the boundary and we denote by Γ_ϵ the scaled edge.

Let us describe $\partial\Omega_\epsilon$ near a generic point Q on the edge Γ_ϵ of $\partial\Omega_\epsilon$. Without loss of generality, we can assume that $Q = 0 \in \mathbb{R}^n$, that x_1-axis is the tangent line at Q to Γ in $\partial\Omega_\epsilon$, or $\partial\Omega$. In a neighborhood of Q, let $\gamma : (-\mu_0, \mu_0) \to \mathbb{R}^2$ be a local parametrization of Γ, that is $(x_2, x_3) = \gamma(x_1) = (\gamma_1(x_1), \dots, \gamma_2(x_1))$. Then one has, for $|x_1| < \mu_0$,

$$\begin{aligned}(x_2, x_3) &= \gamma(x_1) \\ &= \gamma(0) + \gamma'(0)\,x_1 + \frac{1}{2}\gamma''(0)\,x_1^2 + O\left(|x_1|^3\right) \\ &= \frac{1}{2}\gamma''(0)\,x_1^2 + O\left(|x_1|^3\right).\end{aligned}$$

On the other hand, Γ is parametrized by $(x_2, x_3) = \gamma_\epsilon(x_1) := \frac{1}{\epsilon}\gamma(\epsilon x_1)$, for which the following expansions hold

$$\begin{aligned}\gamma_\epsilon(x_1) &= \frac{\epsilon}{2}\gamma''(0)\,x_1^2 + O\left(\epsilon^2|x_1|^3\right), \\ \frac{\partial\gamma_\epsilon}{\partial x_1} &= \epsilon\gamma''(0)\,x_1 + O\left(\epsilon^2|x_1|^2\right).\end{aligned} \tag{2.2.1}$$

Now we introduce a new set of coordinates on $B_{\frac{\mu_0}{\epsilon}}(Q) \bigcap \Omega_\epsilon$:

$$y_1 = x_1, \qquad (y_2, y_3) = (x_2, x_3) - \gamma_\epsilon(x_1).$$

The advantage of these coordinates is that the edge identifies with y_1-axis, but the corresponding metric $g = (g_{ij})_{ij}$ will not be flat anymore. If $\gamma_\epsilon(x_1) = (\gamma_{\epsilon 1}(x_1), \gamma_{\epsilon 2}(x_1))$, the coefficients of g are given by

$$(g_{ij}) = \left(\frac{\partial x}{\partial y_i} \cdot \frac{\partial x}{\partial y_j}\right) = \begin{pmatrix} 1 + \frac{\partial\gamma_{\epsilon 1}}{\partial y_1}\frac{\partial\gamma_{\epsilon 1}}{\partial y_1} + \frac{\partial\gamma_{\epsilon 2}}{\partial y_1}\frac{\partial\gamma_{\epsilon 2}}{\partial y_1} & \frac{\partial\gamma_{\epsilon 1}}{\partial y_1} & \frac{\partial\gamma_{\epsilon 2}}{\partial y_1} \\ \frac{\partial\gamma_{\epsilon 1}}{\partial y_1} & 1 & 0 \\ \frac{\partial\gamma_{\epsilon 2}}{\partial y_1} & 0 & 1 \end{pmatrix}.$$

From the estimates in (2.2.1) it follows that

$$g_{ij} = \mathrm{Id} + \epsilon A + O\left(\epsilon^2 |x_1|^2\right), \tag{2.2.2}$$

where

$$A = \begin{pmatrix} 0 & \gamma''(0)\, x_1 \\ \gamma''(0)^T x_1 & 0 \end{pmatrix}.$$

It is also easy to check that the inverse matrix $\left(g^{ij}\right)$ is of the form $g^{ij} = \mathrm{Id} - \epsilon A + O\left(\epsilon^2 |x_1|^2\right)$. Furthermore one has $\det g = 1$. Therefore, by (2.2.2), for any smooth function u there holds

$$\begin{aligned} \Delta_g u &= \Delta u - \epsilon \left[2 \left(\gamma''(0)\, y_1 \cdot \nabla_{(y_2, y_3)} \frac{\partial u}{\partial y_1} \right) + \left(\gamma''(0) \cdot \nabla_{(y_2,y_3)} u \right) \right] \\ &\quad + O\left(\epsilon^2 |x_1|^2\right) |\nabla^2 u| + O\left(\epsilon^2 |x_1|^2\right) |\nabla u|. \end{aligned} \tag{2.2.3}$$

Now, let us consider a smooth domain $\tilde{\Omega} \subset \mathbb{R}^3$ and $\tilde{\Omega}_\epsilon = \frac{1}{\epsilon}\tilde{\Omega}$. In the same way we can describe $\partial\tilde{\Omega}_\epsilon$ near a generic point $Q \in \partial\tilde{\Omega}_\epsilon$. Without loss of generality, we can assume that $Q = 0 \in \mathbb{R}^3$, that $\{x_3 = 0\}$ is the tangent plane of $\partial\tilde{\Omega}_\epsilon$, or $\partial\tilde{\Omega}$, at Q, and that the outer normal $\nu(Q) = (0, 0, -1)$. In a neighborhood of Q, let $x_3 = \psi(x_1, x_2)$ be a local parametrization of $\partial\tilde{\Omega}$. Then one has, for $|(x_1, x_2)| < \mu_1$,

$$\begin{aligned} x_3 &= \psi(x_1, x_2) \\ &= \frac{1}{2}\left(A_Q(x_1, x_2) \cdot (x_1, x_2)\right) + C_Q(x_1, x_2) + O\left(|(x_1, x_2)|^4\right), \end{aligned}$$

where A_Q is the Hessian of ψ at $(0, 0)$ and C_Q is a cubic polynomial, which is given precisely by

$$C_Q(x_1, x_2) = \frac{1}{6} \sum_{i,j,k=1}^{2} \frac{\partial^3 \psi}{\partial x_i \partial x_j \partial x_k}(0,0)\, x_i x_j x_k.$$

On the other hand, $\partial\tilde{\Omega}_\epsilon$ is parametrized by $x_3 = \psi_\epsilon(x_1, x_2) := \frac{1}{\epsilon}\psi(\epsilon x_1, \epsilon x_2)$, for which the following expansions hold

$$\begin{aligned} \psi_\epsilon(x_1, x_2) &= \frac{\epsilon}{2}\left(A_Q(x_1, x_2) \cdot (x_1, x_2)\right) + \epsilon^2 C_Q(x_1, x_2) + O\left(\epsilon^3 |(x_1, x_2)|^4\right), \\ \frac{\partial \psi_\epsilon}{\partial x_i}(x_1, x_2) &= \epsilon\left(A_Q(x_1, x_2)\right)_i + \epsilon^2 D_Q^i(x_1, x_2) + O\left(\epsilon^3 |(x_1, x_2)|^3\right), \end{aligned} \tag{2.2.4}$$

where D_Q^i are quadratic forms in (x_1, x_2) given by

$$D_Q^i(x_1, x_2) = \frac{1}{2} \sum_{j,k=1}^{2} \frac{\partial^3 \psi}{\partial x_i \partial x_j \partial x_k}(0,0)\, x_j x_k.$$

Concerning the outer normal ν, we have also

$$\begin{aligned} \nu = \frac{\left(\frac{\partial \psi_\epsilon}{\partial x_1}, \frac{\partial \psi_\epsilon}{\partial x_2}, -1\right)}{\sqrt{1 + |\nabla \psi_\epsilon|^2}} &= \left(\epsilon\left(A_Q(x_1, x_2)\right) + \epsilon^2 D_Q(x_1, x_2), -1 + \frac{1}{2}\epsilon^2 |A_Q(x_1, x_2)|^2 \right) \\ &\quad + O\left(\epsilon^3 |(x_1, x_2)|^3\right). \end{aligned} \tag{2.2.5}$$

Now we introduce a new set of coordinates on $B_{\frac{\mu_1}{\epsilon}}(Q)\bigcap\tilde{\Omega}_\epsilon$:

$$z_1 = x_1, \qquad z_2 = x_2, \qquad z_3 = x_3 - \psi_\epsilon(x_1, x_2).$$

The advantage of these coordinates is that $\partial\tilde{\Omega}_\epsilon$ identifies with $\{z_3 = 0\}$, but, as before, the corresponding metric $\tilde{g} = (\tilde{g}_{ij})_{ij}$ will not be flat anymore. Its coefficients are given by

$$(\tilde{g}_{ij}) = \left(\frac{\partial x}{\partial z_i}\cdot\frac{\partial x}{\partial z_j}\right) = \begin{pmatrix} 1+\frac{\partial\psi_\epsilon}{\partial z_1}\frac{\partial\psi_\epsilon}{\partial z_1} & \frac{\partial\psi_\epsilon}{\partial z_1}\frac{\partial\psi_\epsilon}{\partial z_2} & \frac{\partial\psi_\epsilon}{\partial z_1} \\ \frac{\partial\psi_\epsilon}{\partial z_2}\frac{\partial\psi_\epsilon}{\partial z_1} & 1+\frac{\partial\psi_\epsilon}{\partial z_2}\frac{\partial\psi_\epsilon}{\partial z_2} & \frac{\partial\psi_\epsilon}{\partial z_2} \\ \frac{\partial\psi_\epsilon}{\partial z_1} & \frac{\partial\psi_\epsilon}{\partial z_2} & 1 \end{pmatrix}.$$

From the estimates in (2.2.4) it follows that

$$\tilde{g}_{ij} = \mathrm{Id} + \epsilon A + \epsilon^2 B + O\left(\epsilon^3|(z_1,z_2)|^3\right), \tag{2.2.6}$$

where

$$A = \begin{pmatrix} 0 & A_Q(z_1,z_2) \\ (A_Q(z_1,z_2))^T & 0 \end{pmatrix},$$

and

$$B = \begin{pmatrix} A_Q(z_1,z_2)\otimes A_Q(z_1,z_2) & D_Q(z_1,z_2) \\ (D_Q(z_1,z_2))^T & 0 \end{pmatrix}.$$[1]

It is also easy to check that the inverse matrix $(\tilde{g}^{ij})$ is of the form $\tilde{g}^{ij} = \mathrm{Id} - \epsilon A + \epsilon^2 C + O\left(\epsilon^3|(z_1,z_2)|^3\right)$, where

$$C = \begin{pmatrix} 0 & -D_Q(z_1,z_2) \\ -(D_Q(z_1,z_2))^T & |A_Q(z_1,z_2)|^2 \end{pmatrix}.$$

Furthermore one has $\det\tilde{g} = 1$. Therefore, by (2.2.6), for any smooth function u there holds

$$\begin{aligned} \Delta_{\tilde{g}} u \;=\; & \Delta u - \epsilon\left[2\left(A_Q(z_1,z_2)\cdot\nabla_{(z_1,z_2)}\frac{\partial u}{\partial z_3}\right) + \mathrm{tr}A_Q\frac{\partial u}{\partial z_3}\right] \\ & +\epsilon^2\left[-2\left(D_Q\cdot\nabla_{(z_1,z_2)}\frac{\partial u}{\partial z_3}\right) + |A_Q(z_1,z_2)|^2\frac{\partial^2 u}{\partial z_3\partial z_3} - \mathrm{div}D_Q\frac{\partial u}{\partial z_3}\right] \\ & +O\left(\epsilon^3|(z_1,z_2)|^3\right)|\nabla^2 u| + O\left(\epsilon^3|(z_1,z_2)|^3\right)|\nabla u|. \end{aligned}$$

Moreover, from (2.2.5), we obtain the expression of the unit outer normal to $\partial\tilde{\Omega}_\epsilon$, $\tilde{\nu}$, in the new coordinates z:

$$\begin{aligned} \tilde{\nu} \;=\; & \left(\epsilon(A_Q(z_1,z_2)) + \epsilon^2 D_Q(z_1,z_2), -1 + \frac{3}{2}\epsilon^2|A_Q(z_1,z_2)|^2\right) \\ & +O\left(\epsilon^3|(z_1,z_2)|^3\right). \end{aligned}$$

Finally the area-element of $\partial\tilde{\Omega}_\epsilon$ can be estimated as

$$d\sigma = \left(1 + O\left(\epsilon^2|(z_1,z_2)|^2\right)\right)dz_1\,dz_2.$$

Now, locally, in a suitable neighborhood of $Q \in \Gamma$, we can consider Ω as the intersection of two smooth domains $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ if the opening angle at Q is less than π, or as the union of them if the opening angle is greater than π. In the first case one has $\partial\Omega = \left(\partial\tilde{\Omega}_1 \cap \tilde{\Omega}_2\right) \cup \left(\partial\tilde{\Omega}_2 \cap \tilde{\Omega}_1\right)$, whereas in the second case $\partial\Omega = \left(\partial\tilde{\Omega}_1 \cap \tilde{\Omega}_2^c\right) \cup \left(\partial\tilde{\Omega}_2 \cap \tilde{\Omega}_1^c\right)$. Then, locally, one can straighten Γ and stretch the two parts of the boundary using the coordinates z for the smooth domains $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$.

In chapter 4 we consider a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, whose boundary $\partial\Omega$ has an $(n-2)$-dimensional smooth singularity. Moreover, we suppose that there are two disjoint open sets of the boundary $\partial\Omega$, which will be denoted by $\partial_N\Omega$ and $\partial_D\Omega$, such that the union of their closures is the whole boundary of Ω and such that their intersection, that we will call Γ, is the singularity.

As in the previous case, it can be shown that near a generic point $Q \in \Gamma$ the boundary of Ω can be described by a coordinate system $y = (y_1, \dots, y_n)$ such that

(a) $\partial_N\Omega$ coincides with $\{y_n = 0\}$,

(b) $\partial_D\Omega$ coincides with $\{y_1 \tan\alpha + y_n = 0\}$, where $\alpha = \alpha(Q)$ is the opening angle of Γ at Q,

(c) the corresponding metric coefficients are given by $g_{ij} = \delta_{ij} + O(\epsilon)$.

Remark 2.2.1. (i) We stress that, in the new coordinates y, the origin parametrizes the point Q, and those functions decaying as $|y| \to +\infty$ will *concentrate* near Q.

(ii) It is also useful to understand how the metric coefficients g_{ij} vary with Q. Notice that condition (c) says that the deviation from the Kronecker symbols is of order ϵ, and we are working in a domain scaled of $\frac{1}{\epsilon}$; hence a variation of order 1 of Q corresponds to a variation of order ϵ in the original domain. Therefore, a variation of order 1 in Q yields a difference of order ϵ^2 in g_{ij}, and precisely

$$\frac{\partial g_{ij}}{\partial Q} = O\left(\epsilon^2 |y|^2\right),$$

with a similar estimate for the derivatives of the inverse coefficients g^{ij}. For more details see the end of Subsection 9.2 in [4].

3 CONCENTRATION OF SOLUTIONS FOR A SINGULARLY PERTURBED NEUMANN PROBLEM IN NON-SMOOTH DOMAINS

INTRODUCTION

In this chapter we study the following singular perturbation problem with Neumann boundary condition in a bounded domain $\Omega \subset \mathbb{R}^3$ whose boundary $\partial\Omega$ is non smooth, in the sense that is has smooth edges:

$$\begin{cases} -\epsilon^2 \Delta u + u = u^p & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.0.1}$$

Here $p \in (1,5)$ is subcritical and ν denotes the outer unit normal at $\partial\Omega$.

If we denote by Γ an edge of $\partial\Omega$, we can consider the function $\alpha : \Gamma \to \mathbb{R}$ which associates to every $Q \in \Gamma$ the opening angle at Q, $\alpha(Q)$.

By the change of variables $x \mapsto \epsilon x$, problem (3.0.1) can be transformed into

$$\begin{cases} -\Delta u + u = u^p & \text{in } \Omega_\epsilon, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega_\epsilon, \end{cases} \tag{3.0.2}$$

where $\Omega_\epsilon := \frac{1}{\epsilon}\Omega$. Solutions of (3.0.2) can be found as critical points of the Euler-Lagrange functional

$$I_{\epsilon,N}(u) = \frac{1}{2}\int_{\Omega_\epsilon} \left(|\nabla u|^2 + u^2\right) dx - \frac{1}{p+1}\int_{\Omega_\epsilon} |u|^{p+1} dx, \quad u \in H^1(\Omega_\epsilon). \tag{3.0.3}$$

As in the smooth case, we can expect that the function α plays the same role as the mean curvature H for a smooth domain (see Section 1.2). In fact, plugging into $I_{\epsilon,N}$ a function of the form (1.2.1) with $Q \in \Gamma$ one obtains an expression similar to (1.2.5), with $C_0\alpha(Q)$ instead of C_0; see Lemma 3.2.3. Roughly speaking, we can say that the energy of solutions is of order ϵ^3, which is proportional to the volume of their *support*, heuristically identified with a ball of radius ϵ centered at the peak $Q \in \Gamma$; then, when we intersect this ball with the domain we obtain the dependence on the angle $\alpha(Q)$.

The main result of this chapter is the following

Theorem 3.0.2. *Let $\Omega \subset \mathbb{R}^3$ be a piecewise smooth bounded domain whose boundary $\partial\Omega$ has a finite number of smooth edges, and $1 < p < 5$. Fix an edge Γ, and suppose $Q \in \Gamma$ is a local strict maximum or minimum of the function α, with $\alpha(Q) \neq \pi$. Then for $\epsilon > 0$ sufficiently small problem (3.0.1) admits a solution concentrating at Q.*

Remark 3.0.3. The condition that Q is a local strict maximum or minimum of α can be replaced by the fact that there exists an open set V of Γ containing Q such that $\alpha(Q) > \sup_{\partial V} \alpha$ or $\alpha(Q) < \inf_{\partial V} \alpha$.

Remark 3.0.4. The condition $\alpha(Q) \neq \pi$ is natural since it is needed to ensure that $\partial\Omega$ is not flat at Q.

Remark 3.0.5. We expect a similar result to hold in higher dimension, with substantially the same proof. For simplicity we only treat the 3-dimensional case.

The general strategy for proving Theorem 3.0.2 relies on a finite-dimensional reduction; see Section 2.1.

Now, first of all, one finds a manifold Z_ϵ of approximate solutions to the given problem, which are of the form $U_{Q,\epsilon}(x) = \varphi_\mu(\epsilon x) U(x - Q)$, where φ_μ is a suitable cut-off function defined in a neighborhood of $Q \in \Gamma$; see the beginning of Section 4, Lemma 3.2.1.

To apply the method described in Section 2.1 one needs the condition that the critical manifold Z_ϵ is non-degenerate, in the sense that it satisfies property ii) in Section 2.1. The result of non-degeneracy in Ω_ϵ, obtained in Lemma 3.2.2, follows from the non-degeneracy of a manifold Z of critical points of the unperturbed problem in $K = \tilde{K} \times \mathbb{R} \subset \mathbb{R}^3$, where $\tilde{K} \subset \mathbb{R}^2$ is a cone of opening angle $\alpha(Q)$. In fact, one sees that Ω_ϵ tends to K as $\epsilon \to 0$. To show the non-degeneracy of the unperturbed manifold Z we follow the line of Lemma 4.1 in the book [4] or Lemma 3.1 in [52]. We prove that $\lambda = 0$ is a simple eigenvalue of the linearized of the unperturbed problem at $U \in Z$; see Lemma 3.1.1. Moreover, if $\alpha(Q) < \pi$, it has only one negative simple eigenvalue; whereas, if $\alpha(Q) > \pi$, it has two negative simple eigenvalues; see Corollary 3.1.4. We note that in the case $\alpha(Q) = \pi$, that is when $\partial\Omega$ is flat at Q, $\lambda = 0$ is an eigenvalue of multiplicity 2. The proof relies on Fourier analysis, but in this case one needs spherical functions defined on a portion of the sphere instead of the whole S^2.

Then one solves the equation up to a vector parallel to the tangent plane of the manifold Z_ϵ, and generates a new manifold $\tilde{Z}_\epsilon$ close to Z_ϵ which represents a natural constraint for the Euler functional (3.0.3); see the proof of Proposition 3.2.5. By *natural constraint* we mean a set for which constrained critical points of I_ϵ are true critical points.

We can finally apply the above mentioned perturbation method to reduce the problem to a finite dimensional one, and study the functional constrained on $\tilde{Z}_\epsilon$. Lemma 3.2.3 provides an expansion of the energy of the approximate solution peaked at Q and allows us to see that the dominant term in the expression of the reduced functional at Q is $\alpha(Q)$. This implies Theorem 3.0.2.

The chapter is organized in the following way. In Section 3.1 we prove the non-degeneracy of the critical manifold for the unperturbed problem in the cone K. In Section 3.2 we construct the manifold of approximate solutions, showing that it is a non-degenerate pseudo-critical manifold, expand the functional on the natural constraint and deduce Theorem 3.0.2.

3.1 STUDY OF THE NON DEGENERACY FOR THE UNPERTURBED PROBLEM IN THE CONE

Let us consider $K = \tilde{K} \times \mathbb{R} \subset \mathbb{R}^3$, where $\tilde{K} \subset \mathbb{R}^2$ is a cone of opening angle α, and the problem

$$\begin{cases} -\Delta u + u = u^p & \text{in } K, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial K, \end{cases} \tag{3.1.1}$$

where $p > 1$.

If $p < 5$ and if $u \in H^1(K)$, solutions of (3.1.1) can be found as critical points of the functional $I_K : H^1(K) \to \mathbb{R}$ defined as

$$I_K(u) = \frac{1}{2}\int_K \left(|\nabla u|^2 + u^2\right) dx - \frac{1}{p+1}\int_K |u|^{p+1} dx. \tag{3.1.2}$$

Note that I_k is well defined on $H^1(K)$; in fact, since K is Lipschitz, the Sobolev embeddings hold for $p \leqslant 5$; see for instance [1], [34].

Let us consider also the elliptic equation in $\mathbb{R}^3$

$$-\Delta u + u = u^p, \quad u \in H^1\left(\mathbb{R}^3\right), \quad u > 0, \tag{3.1.3}$$

which has a positive radial solution U; see for instance [4, 9, 10, 52, 75]. It has been shown in [44] that such a solution is unique. Moreover U and its radial derivatives decay to zero exponentially: more precisely satisfy the properties

$$\lim_{r\to+\infty} e^r rU(r) = c_{3,p}, \qquad \lim_{r\to+\infty} \frac{U'(r)}{U(r)} = -\lim_{r\to+\infty} \frac{U''(r)}{U(r)} = -1,$$

where $r = |x|$ and $c_{3,p}$ is a positive constant depending only on the dimension $n = 3$ and p; see [9].

Now, if p is subcritical, the function U is also a solution of problem (3.1.1). Moreover, if we consider a coordinate system with the x_1-axis coinciding with the edge of K, the problem (3.1.1) is invariant under a translation along the x_1-axis. This means that any

$$U_{x_1}(x) = U(x - (x_1, 0, 0))$$

is also a solution of (3.1.1). Then the functional I_k has a non-compact critical manifold given by

$$Z = \left\{U_{x_1}(x) : x_1 \in \mathbb{R}\right\} \simeq \mathbb{R}.$$

Now, to apply the results of Section 2.1, we have to characterize the spectrum and some eigenfunctions of $I''_K(U_{x_1})$. More precisely we have to show the following

Lemma 3.1.1. *Suppose $\alpha \in (0, 2\pi) \setminus \{\pi\}$. Then the following properties are true:*

a) $T_{U_{x_1}} Z = \mathrm{Ker}\left[I''_K(U_{x_1})\right]$*, for all* $x_1 \in \mathbb{R}$*;*

b) $I''_K(U_{x_1})$ *is an index 0 Fredholm map* [1] *, for all* $x_1 \in \mathbb{R}$*.*

Remark 3.1.2. The properties a) and b) imply that Z satisfies condition ii) in Section 2.1 and then it is non-degenerate for I_K.

Proof. We will prove the lemma by taking $x_1 = 0$, hence $U_0 = U$. The case of a general x_1 will follow immediately.

Let us show a). It is known that there holds the inclusion $T_U Z \subset \mathrm{Ker}\left[I''_K(U)\right]$; see for instance [4], Section 2.2. Then it is sufficient to prove that $\mathrm{Ker}\left[I''_K(U)\right] \subset T_U Z$. Now, $v \in H^1(K)$ belongs to $\mathrm{Ker}\left[I''_K(U)\right]$ if and only if

$$\begin{cases} -\Delta v + v = pU^{p-1}v & \text{in } K, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial K. \end{cases} \tag{3.1.4}$$

We use the polar coordinates in K, r, θ, φ, where $r \geqslant 0$, $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \varphi \leqslant \alpha$. Then we write $v \in H^1(K)$ in the form

$$v(x_1, x_2, x_3) = \sum_{k=0}^{\infty} v_k(r) Y_k(\theta, \varphi), \tag{3.1.5}$$

where the $Y_k(\theta, \varphi)$ are the spherical functions satisfying

$$\begin{cases} -\Delta_{S^2} Y_k = \lambda_k Y_k & \text{in } K, \\ \frac{\partial Y_k}{\partial \varphi} = 0 & \varphi = 0, \alpha. \end{cases} \tag{3.1.6}$$

1 A linear map $T \in L(H, H)$ is Fredholm if the kernel is finite-dimensional and the image is closed and has finite codimension. The index of T is $\dim(\mathrm{Ker}[T]) - \mathrm{codim}(\mathrm{Im}[T])$.

Here Δ_{S^2} denotes the Laplace-Beltrami operator on S^2 (acting on the variables θ, φ). To determine λ_k and the expression of Y_k, let us split Y_k as

$$Y_k(\theta,\varphi) = \sum_{m=0}^{\infty} \Theta_{k,m}(\theta)\,\Phi_{k,m}(\varphi)$$

so that

$$\begin{aligned}\Delta_{S^2} Y_k &= \sum_{m=0}^{\infty} \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\Theta_{k,m}\Phi_{k,m} \\ &= \sum_{m=0}^{\infty} \left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\Theta'_{k,m}\right)\Phi_{k,m} + \frac{1}{\sin^2\theta}\Theta_{k,m}\Phi''_{k,m}\right].\end{aligned}$$

Then (3.1.6) becomes

$$\begin{cases} -\sum_{m=0}^{\infty}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\Theta'_{k,m}\right)\Phi_{k,m} + \frac{1}{\sin^2\theta}\Theta_{k,m}\Phi''_{k,m}\right] & \\ \qquad = \sum_{m=0}^{\infty}\lambda_{k,m}\Theta_{k,m}\Phi_{k,m} & \text{in } K, \\ \Phi'_{k,m}(0) = \Phi'_{k,m}(\alpha) = 0. & \end{cases} \tag{3.1.7}$$

If we require that for all m

$$\begin{cases} -\Phi''_{k,m} = \mu_m \Phi_{k,m} & \text{in } [0,\alpha], \\ \Phi'_{k,m}(0) = \Phi'_{k,m}(\alpha) = 0, & \end{cases} \tag{3.1.8}$$

we obtain that $\Phi_{k,m}(\varphi) = a_{k,m}\cos\left(\frac{\pi m}{\alpha}\varphi\right)$ satisfies (3.1.8) with $\mu_m = \frac{\pi^2 m^2}{\alpha^2}$. Replacing this expression in (3.1.7) we have

$$\begin{cases} \sum_{m=0}^{\infty}\left[-\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\Theta'_{k,m}\right) + \frac{1}{\sin^2\theta}\frac{\pi^2 m^2}{\alpha^2}\Theta_{k,m}\right]\Phi_{k,m} & \\ \qquad = \sum_{m=0}^{\infty}\lambda_{k,m}\Theta_{k,m}\Phi_{k,m} & \text{in } K, \\ \Phi'_{k,m}(0) = \Phi'_{k,m}(\alpha) = 0. & \end{cases}$$

Since the $\Phi_{k,m}$ are independent, we have to solve, for every m, the Sturm-Liouville equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\Theta'_{k,m}\right) + \left[\lambda_{k,m} - \frac{1}{\sin^2\theta}\frac{\pi^2 m^2}{\alpha^2}\right]\Theta_{k,m} = 0. \tag{3.1.9}$$

Let us rewrite (3.1.9) in the following form

$$-\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\Theta'_{k,m}\right) + \frac{1}{\sin^2\theta}\frac{\pi^2 m^2}{\alpha^2}\Theta_{k,m} = \lambda_k\Theta_{k,m}, \tag{3.1.10}$$

so that we have to determine the eigenvalues $\lambda_{k,m}$ and the eigenfunctions of the operator

$$-\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\Theta'(\theta)\right) + \frac{1}{\sin^2\theta}\frac{\pi^2 m^2}{\alpha^2}\Theta(\theta).$$

In order to do this, let us consider the case $\alpha = \pi$, that is the following equation

$$-\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\Theta'_{k,m}\right) + \frac{1}{\sin^2\theta}m^2\Theta_{k,m} = \lambda_{k,m}\Theta_{k,m}. \tag{3.1.11}$$

Now, for every m, (3.1.11) has solution if $\lambda_{k,m} = k(k+1)$, with $k \geqslant |m|$, and the solutions are the Legendre polynomials $\Theta_{k,m}(\theta) = P_{k,m}(\cos\theta)$; see for instance [35], [42], [55], [56]. Then, for a given value of k, there are $2k+1$ independent solutions of the form $\Theta_{k,m}(\theta)\,\Phi_{k,m}(\varphi)$, one for each integer m with $-k \leqslant m \leqslant k$. Now, by the classical comparison principle, if we decrease α the corresponding eigenvalues $\lambda_{k,m}$, given by (3.1.10), should increase, whereas if we increase α they should decrease; see for instance [17]. More precisely, if $m = 0$ the equations (3.1.10) and (3.1.11) are the same, therefore the eigenvalues do not change (and they are $0, 2, 6, \ldots$). If $m \geqslant 1$ we cannot give an explicit expression for the $\lambda_{k,m}$ for general α, but we can use the comparison principle. In conclusion, we obtain that each $Y_k = \sum_{m=0}^{\infty} \Theta_{k,m}\Phi_{k,m}$ satisfies

$$-\Delta_{S^2} Y_k = \lambda_{k,m} Y_k. \tag{3.1.12}$$

Now, one has that

$$\Delta(v_k Y_k) = \Delta_r(v_k)\,Y_k + \frac{1}{r^2} v_k \Delta_{S^2} Y_k, \tag{3.1.13}$$

where Δ_r denotes the Laplace operator in radial coordinates, that is $\Delta_r = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$. Then, using (3.1.5), (3.1.12) and (3.1.13), the condition (3.1.4) becomes

$$\sum_{k=0}^{\infty}\left[-v_k'' - \frac{2}{r}v_k' + v_k + \frac{\lambda_{k,m}}{r^2}v_k - pU^{p-1}v_k\right] Y_k = 0.$$

Since the Y_k are independent, we get the following equations for v_k:

$$A_{k,m}(v_k) := -v_k'' - \frac{2}{r}v_k' + v_k + \frac{\lambda_{k,m}}{r^2}v_k - pU^{p-1}v_k = 0, \quad m = 0, 1, 2 \ldots, \; k \geqslant m.$$

Let us first consider the case $m = 0$. If $k = 0$, we have to find a v_0 such that

$$A_{0,0}(v_0) = -v_0'' - \frac{2}{r}v_0' + v_0 - pU^{p-1}v_0 = 0.$$

It has been shown in [44], Lemma 6, that all the solutions of $A_{0,0}(v) = 0$ are unbounded. Since we are looking for solutions $v_0 \in H^1(\mathbb{R})$, it follows that $v_0 = 0$.

For $k = 1$ we have to solve

$$A_{1,0}(v_1) = -v_1'' - \frac{2}{r}v_1' + v_1 + \frac{2}{r^2}v_1 - pU^{p-1}v_1 = 0.$$

Let $\hat{U}(r)$ denote the function such that $U(x) = \hat{U}(|x|)$, where $U(x)$ is the solution of (3.1.3).

Reasoning as in the proof of Lemma 4.1 in [4], we obtain that the family of solutions of $A_{1,0}(v_1) = 0$, with $v_1 \in H^1(\mathbb{R})$, is given by $v_1(r) = c\hat{U}'(r)$, for some $c \in \mathbb{R}$.

Now, let us show that the equation $A_{k,0}(v_k) = 0$ has only the trivial solution in $H^1(\mathbb{R})$, provided that $k \geqslant 2$. First of all, note that the operator $A_{1,0}$ has the solution $\hat{U}'$ which does not change sign in $(0, \infty)$ and therefore is a non-negative operator. In fact, if σ denotes its smallest eigenvalue, any corresponding eigenfunction ψ_σ does not change sign. If $\sigma < 0$, then ψ_σ should be orthogonal to $\hat{U}'$ and this is a contradiction. Thus $\sigma \geqslant 0$ and $A_{1,0}$ is non-negative. Now, we can write

$$A_{k,0} = A_{1,0} + \frac{\lambda_{k,0} - 2}{r^2}.$$

Since $\lambda_{k,0} - 2 > 0$ whenever $k \geqslant 2$, it follows that $A_{k,0}$ is a positive operator. Thus $A_{k,0}(v_k) = 0$ implies that $v_k = 0$.

If $m \geqslant 1$ and $\alpha < \pi$, using the comparison principle, we obtain that each $\lambda_{k,m}$ is greater than 2. Then, reasoning as above, we have that each $v_k = 0$.

Let us consider the case $\alpha > \pi$. If $m = 1$ and $k = 1$, using again the comparison principle, we have that $0 < \lambda_{1,1} < 2$; whereas for $m = 1$, $k \geqslant 2$, and for $m \geqslant 2$, $k \geqslant m$, we have that each $\lambda_{k,m} > 2$. Then in the last two cases we can use the non-negativity of the operator $A_{1,0}$ and conclude that $v_k = 0$. In the case $m = 1$ and $k = 1$ we note that the operator

$$A_{1,1}(v_1) := -v_1'' - \frac{2}{r}v_1' + v_1 + \frac{\lambda_{1,1}}{r^2}v_1 - pU^{p-1}v_1$$

has a negative eigenvalue, instead of the eigenvalue 0, since $\lambda_{1,1} < 2$. Then also $v_1 = 0$.

Putting together all the previous information, we deduce that any $v \in \mathrm{Ker}\,[I''(U)]$ has to be of the form

$$v(x_1, x_2, x_3) = c\hat{U}'(r)\,Y_1(\theta, \varphi).$$

Now, Y_1 is such that $-\Delta_{S^2}Y_1 = \lambda_{1,m}Y_1$, namely it belongs to the kernel of the operator $-\Delta_{S^2} - \lambda_{1,m}\mathrm{Id}$, and such a kernel is 1-dimensional. In conclusion, we find that

$$v \in \mathrm{span}\left\{\hat{U}'Y_1\right\} = \mathrm{span}\left\{\frac{\partial U}{\partial x_1}\right\} = T_U Z.$$

This proves that a) holds. It is also easy to check that the operator $I_K''(U)$ is a compact perturbation of the identity, showing that b) holds true, too. This complete the proof of Lemma 3.1.1. □

Remark 3.1.3. Since U is a Mountain-Pass solution of (3.1.3), the spectrum of $I_K''(U)$ has one negative simple eigenvalue, $1 - p$, with eigenspace spanned by U itself. Moreover, we have shown in the preceding lemma that $\lambda = 0$ is an eigenvalue with multiplicity 1 and eigenspace spanned by $\frac{\partial U}{\partial x_1}$. If $\alpha < \pi$ the rest of the spectrum is positive. Whereas if $\alpha > \pi$ there is an other negative simple eigenvalue, corresponding to an eigenfunction $\tilde{U}$ given by

$$\tilde{U}(r, \theta, \varphi) = \tilde{u}(r)\cos\left(\frac{\pi}{\alpha}\varphi\right)\tilde{\Theta}(\theta),$$

where $\tilde{\Theta}$ satisfies (3.1.9) with $m = 1$ and $k = 1$, and $\tilde{u}$ satisfies the equation

$$-v'' - \frac{2}{r}v' + v + \frac{\lambda_{1,1}}{r^2}v - pU^{p-1}v = 0. \tag{3.1.14}$$

From (3.1.14) one has that there exists a positive constant C such that, for r sufficiently large, $\tilde{u}(r) \leqslant Ce^{-r/C}$. In conclusion, one has the following result:

Corollary 3.1.4. *Let U and $\tilde{U}$ be as above and consider the functional I_K given in (3.1.2). Then for every $x_1 \in \mathbb{R}$, $U_{x_1}(x) = U(x - (x_1, 0, 0))$ is a critical point of I_K. Moreover, the kernel of $I_K''(U)$ is generated by $\frac{\partial U}{\partial x_1}$. If $\alpha < \pi$ the operator has only one negative eigenvalue, and therefore there exists $\delta > 0$ such that*

$$I_K''(U)[v, v] \geqslant \delta\|v\|^2, \quad \text{for every } v \in H^1(K), v \perp U, \frac{\partial U}{\partial x_1}.$$

If $\alpha > \pi$ the operator has two negative eigenvalues, and therefore there exists $\delta > 0$ such that

$$I_K''(U)[v, v] \geqslant \delta\|v\|^2, \quad \text{for every } v \in H^1(K), v \perp U, \tilde{U}, \frac{\partial U}{\partial x_1}.$$

For every Q on the edge Γ of $\partial\Omega_\epsilon$, let $\mu = \min\{\mu_i\}$, so that in $B_{\frac{\mu}{\epsilon}}(Q)\bigcap\Omega_\epsilon$ we can use the new set of coordinates z. Now we choose a cut-off function φ_μ with the following properties

$$\begin{cases} \varphi_\mu(x) = 1 & \text{in } B_{\frac{\mu}{4}}(Q), \\ \varphi_\mu(x) = 0 & \text{in } \mathbb{R}^3 \setminus B_{\frac{\mu}{2}}(Q), \\ |\nabla\varphi_\mu| + |\nabla^2\varphi_\mu| \leqslant C & \text{in } B_{\frac{\mu}{2}}(Q) \setminus B_{\frac{\mu}{4}}(Q). \end{cases} \tag{3.2.1}$$

For any $Q \in \Gamma$, we define the following function, in the coordinates (z_1, z_2, z_3),

$$U_{Q,\epsilon}(z) := \varphi_\mu(\epsilon z)\, U_Q(z), \tag{3.2.2}$$

where $U_Q(z) = U(z - Q)$. Then we consider the manifold

$$Z_\epsilon = \left\{U_{Q,\epsilon} : Q \in \Gamma\right\}.$$

Now, we estimate the gradient of $I_{\epsilon,N}$ at $U_{Q,\epsilon}$, showing that Z_ϵ constitute a manifold of pseudo-critical points of $I_{\epsilon,N}$.

Lemma 3.2.1. *There exists $C > 0$ such that for ϵ small there holds*

$$\left\|I'_{\epsilon,N}\left(U_{Q,\epsilon}\right)\right\| \leqslant C\epsilon, \qquad \text{for all } Q \in \Gamma.$$

Proof. Let $v \in H^1(\Omega_\epsilon)$. Since the function $U_{Q,\epsilon}$ is supported in $B := B_{\frac{\mu}{2\epsilon}}(Q)$, see (3.2.2), we can use the coordinate z in this set, and we obtain

$$\begin{aligned} I'_{\epsilon,N}\left(U_{Q,\epsilon}\right)[v] &= \int_{\partial\Omega_\epsilon} \frac{\partial U_{Q,\epsilon}}{\partial\tilde{\nu}} v\, d\tilde{\sigma} + \int_{\Omega_\epsilon} \left(-\Delta_{\tilde{g}} U_{Q,\epsilon} + U_{Q,\epsilon} - |U_{Q,\epsilon}|^p\right) v\, dV_{\tilde{g}}(z) \\ &\doteqdot I + II. \end{aligned}$$

Let us now estimate I:

$$I = \int_{\partial\Omega_{\epsilon 1}} \frac{\partial U_{Q,\epsilon}}{\partial\tilde{\nu}_1} v\, d\tilde{\sigma}_1 + \int_{\partial\Omega_{\epsilon 2}} \frac{\partial U_{Q,\epsilon}}{\partial\tilde{\nu}_2} v\, d\tilde{\sigma}_2 \doteqdot I_1 + I_2.$$

If $K = K_{\alpha(Q)}$ denotes the cone of angle equal to the angle of the edge in Q, we have

$$\begin{aligned} I_1 &= \int_{\partial K} \left(U_Q(z)\nabla\varphi_\mu(\epsilon z)\cdot\tilde{\nu}_1 + \varphi_\mu(\epsilon z)\nabla U_Q(z)\cdot\tilde{\nu}_1\right) v\, d\tilde{\sigma}_1 \\ &= \int_{\partial K} \Big[U_Q(z)\nabla\varphi_\mu(\epsilon z)\cdot\left(\epsilon\left(A_Q(z_1,z_2)\right) + \epsilon^2 D_Q(z_1,z_2), -1 + \frac{3}{2}\epsilon^2|A_Q(z_1,z_2)|^2\right) \\ &\qquad + \varphi_\mu(\epsilon z)\nabla U_Q(z)\cdot\left(\epsilon\left(A_Q(z_1,z_2)\right) + \epsilon^2 D_Q(z_1,z_2), -1 + \frac{3}{2}\epsilon^2|A_Q(z_1,z_2)|^2\right)\Big] \\ &\qquad \cdot v\left(1 + O\left(\epsilon^2|(z_1,z_2)|^2\right)\right) dz_1\, dz_2 \\ &\doteqdot a + b. \end{aligned}$$

Since $\nabla\varphi_\mu(\epsilon\cdot)$ is supported in $\mathbb{R}^3 \setminus B_{\frac{\mu}{4\epsilon}}(Q)$ and U_Q has an exponential decay, we have that, for ϵ small,

$$|a| \leqslant C\epsilon e^{-\frac{\mu}{4\epsilon}} \int_{\partial K} |v|\, dz_1\, dz_2. \tag{3.2.3}$$

On the other hand

$$
\begin{aligned}
b &= \int_{\frac{\mu}{4\epsilon}\leqslant|z-Q|\leqslant\frac{\mu}{2\epsilon}} \varphi_\mu(\epsilon z)\nabla U_Q(z) \\
&\qquad \cdot\left(\epsilon\left(A_Q(z_1,z_2)\right)+\epsilon^2 D_Q(z_1,z_2),-1+\frac{3}{2}\epsilon^2|A_Q(z_1,z_2)|^2\right) \\
&\qquad \cdot\nu\left(1+O\left(\epsilon^2|(z_1,z_2)|^2\right)\right)dz_1\,dz_2 \\
&\quad +\int_{|z-Q|\leqslant\frac{\mu}{4\epsilon}} \varphi_\mu(\epsilon z)\nabla U_Q(z) \\
&\qquad \cdot\left(\epsilon\left(A_Q(z_1,z_2)\right)+\epsilon^2 D_Q(z_1,z_2),-1+\frac{3}{2}\epsilon^2|A_Q(z_1,z_2)|^2\right) \\
&\qquad \cdot\nu\left(1+O\left(\epsilon^2|(y_1,y_2)|^2\right)\right)dy_1\,dy_2 \\
&\leqslant C\epsilon e^{-\frac{\mu}{4\epsilon}}\int_{\partial K}|\nu|dz_1\,dz_2+C\epsilon\int_{\partial K}|\nabla U_Q|\cdot|\nu|dz_1\,dz_2. \qquad (3.2.4)
\end{aligned}
$$

The estimates (3.2.3) and (3.2.4), and the trace Sobolev inequalities imply $|I_1|\leqslant C\epsilon\|\nu\|$. In the same way we can estimate I_2, getting

$$|I|\leqslant C\epsilon\|\nu\|. \qquad (3.2.5)$$

Now let's evaluate II. Using (2.2.3) one has

$$
\begin{aligned}
II &= \int_K\left(-\Delta U_{Q,\epsilon}+U_{Q,\epsilon}-|U_{Q,\epsilon}|^p\right)\nu dV_{\tilde g}(z) \\
&\quad +\epsilon\int_K\left[2\left(\gamma''(0)z_1\cdot\nabla_{(z_2,z_3)}\frac{\partial U_{Q,\epsilon}}{\partial z_1}\right)+\left(\gamma''(0)\cdot\nabla_{(z_2,z_3)}U_{Q,\epsilon}\right)\right]\nu dV_{\tilde g}(z) \\
&\quad +O\left(\epsilon^2\right)\int_K\left(|z_1|^2|\nabla^2 U_{Q,\epsilon}|+|z_1|^2|\nabla U_{Q,\epsilon}|\right)\nu dV_{\tilde g}(z) \\
&\doteqdot II_1+\epsilon II_2+O\left(\epsilon^2\right)II_3.
\end{aligned}
$$

Since $\Delta U_{Q,\epsilon}=U_Q\Delta\varphi_\mu(\epsilon z)+2\nabla U_Q\cdot\nabla\varphi_\mu(\epsilon z)+\varphi_\mu(\epsilon z)\Delta U_Q$ and both $\Delta\varphi_\mu(\epsilon\cdot)$ and $\nabla\varphi_\mu(\epsilon\cdot)$ are supported in $\mathbb{R}^3\setminus B_{\frac{\mu}{4\epsilon}}(Q)$, we get

$$
\begin{aligned}
II_1 &= \int_{\frac{\mu}{4\epsilon}\leqslant|z-Q|\leqslant\frac{\mu}{2\epsilon}}\left(-U_Q\Delta\varphi_\mu(\epsilon z)-2\nabla U_Q\cdot\nabla\varphi_\mu(\epsilon z)\right)\nu\left(1+O(\epsilon|z|)\right)dz \\
&\quad +\int_{\frac{\mu}{4\epsilon}\leqslant|z-Q|\leqslant\frac{\mu}{2\epsilon}}\left(-\varphi_\mu(\epsilon z)\Delta U_Q+U_{Q,\epsilon}-|U_{Q,\epsilon}|^p\right)\nu\left(1+O(\epsilon|z|)\right)dz \\
&\quad +\int_{|z-Q|\leqslant\frac{\mu}{4\epsilon}}\left(-\Delta U_Q+U_Q-|U_Q|^p\right)\nu\left(1+O(\epsilon|z|)\right)dz. \qquad (3.2.6)
\end{aligned}
$$

Since U_Q is a solution in $\mathbb{R}^3$ the last term in (3.2.6) vanishes, and using the exponential decay of U_Q at infinity and the properties of the cut-off function, see (3.2.1), one has

$$|II_1|\leqslant Ce^{-\frac{\mu}{4\epsilon}}\int_K|\nu|dz.$$

By (3.2.2) we can compute also $\nabla_{(z_2,z_3)}\frac{\partial U_{Q,\epsilon}}{\partial z_1}$ and $\nabla_{(z_2,z_3)}U_{Q,\epsilon}$ and we have

$$\begin{aligned}
II_2 &= \int_K 2\gamma''(0)\, z_1 \cdot \left[\nabla_{(z_2,z_3)}\frac{\partial\varphi_\mu(\epsilon z)}{\partial z_1}U_Q + \nabla_{(z_2,z_3)}\varphi_\mu(\epsilon z)\frac{\partial U_Q}{\partial z_1}\right]\\
&\quad +2\gamma''(0)\, z_1 \cdot \left[\frac{\partial\varphi_\mu(\epsilon z)}{\partial z_1}\nabla_{(z_2,z_3)}U_Q + \varphi_\mu(\epsilon z)\nabla_{(z_2,z_3)}\frac{\partial U_Q}{\partial z_1}\right]\\
&\quad +\gamma''(0)\cdot\left[\nabla_{(z_2,z_3)}\varphi_\mu(\epsilon z)U_Q + \varphi_\mu(\epsilon z)\nabla_{(z_2,z_3)}U_Q\right] v dV_{\tilde g}(z)\\
&= \int_{\frac{\mu}{4\epsilon}\leqslant|z-Q|\leqslant\frac{\mu}{2\epsilon}} 2\gamma''(0)\, z_1\cdot\left[\nabla_{(z_2,z_3)}\frac{\partial\varphi_\mu(\epsilon z)}{\partial z_1}U_Q + \nabla_{(z_2,z_3)}\varphi_\mu(\epsilon z)\frac{\partial U_Q}{\partial z_1}\right.\\
&\quad \left.+\frac{\partial\varphi_\mu(\epsilon z)}{\partial z_1}\nabla_{(z_2,z_3)}U_Q\right] + \gamma''(0)\cdot\nabla_{(z_2,z_3)}\varphi_\mu(\epsilon z)U_Q v dV_{\tilde g}(z)\\
&\quad +\int_{|z-Q|\leqslant\frac{\mu}{2\epsilon}}\varphi_\mu(\epsilon z)\left[2\gamma''(0)\, z_1\cdot\nabla_{(z_2,z_3)}\frac{\partial U_Q}{\partial z_1} + \gamma''(0)\cdot\nabla_{(z_2,z_3)}U_Q\right] v dV_{\tilde g}(z).
\end{aligned}$$

Hence

$$\begin{aligned}
|II_2| &\leqslant C\int_{\frac{\mu}{4\epsilon}\leqslant|z-Q|\leqslant\frac{\mu}{2\epsilon}}\left[2|\gamma''(0)|\cdot|z_1|\left(|U_Q| + |\frac{\partial U_Q}{\partial z_1}| + |\nabla_{(z_2,z_3)}U_Q|\right)\right.\\
&\qquad \left.+|\gamma''(0)|\cdot|U_Q|\right]|v|dV_{\tilde g}(z)\\
&\quad +\int_{|z-Q|\leqslant\frac{\mu}{2\epsilon}} 2|\varphi_\mu(\epsilon z)|\cdot\sup_Q|\gamma''(0)|\\
&\qquad \left(|z_1|\cdot|\nabla_{(z_2,z_3)}\frac{\partial U_Q}{\partial z_1}| + |\nabla_{(z_2,z_3)}U_Q|\right)|v|dV_{\tilde g}(z).
\end{aligned}$$

Using again the exponential decay of U_Q at infinity one can estimate the first term by $Ce^{-\frac{\mu}{4\epsilon}}\int_K|v|dz$ and conclude that the second term is bounded. In the same way we can estimate II_3, getting

$$|II| \leqslant C\epsilon\,\|v\|. \tag{3.2.7}$$

From (3.2.5) and (3.2.7) we obtain the conclusion. □

Now, we need a result of non-degeneracy, which allows us to say that the operator $I''_{\epsilon,N}(U_{Q,\epsilon})$ is invertible on the orthogonal complement of $T_{U_{Q,\epsilon}}Z_\epsilon$.

Lemma 3.2.2. *There exists $\bar\delta > 0$ such that for ϵ small, if $\alpha < \pi$, there holds*

$$I''_{\epsilon,N}(U_{Q,\epsilon})[v,v] \geqslant \bar\delta\,\|v\|^2, \qquad \text{for every } v\in H^1(\Omega_\epsilon), v\perp U_{Q,\epsilon}, \frac{\partial U_{Q,\epsilon}}{\partial Q},$$

and, if $\alpha > \pi$, there holds

$$I''_{\epsilon,N}(U_{Q,\epsilon})[v,v] \geqslant \bar\delta\,\|v\|^2, \qquad \text{for every } v\in H^1(\Omega_\epsilon), v\perp U_{Q,\epsilon}, \tilde U_{Q,\epsilon}\frac{\partial U_{Q,\epsilon}}{\partial Q},$$

where $\tilde U_{Q,\epsilon}$ is defined as $U_{Q,\epsilon}$ in (3.2.2).

Proof. Let us consider the case $\alpha < \pi$. Let $R \gg 1$; consider a radial smooth function $\chi_R : \mathbb{R}^3 \to \mathbb{R}$ such that

$$\begin{cases} \chi_R(x) = 1 & \text{in } B_R(0),\\ \chi_R(x) = 0 & \text{in } \mathbb{R}^3\setminus B_{2R}(0),\\ |\nabla\chi_R| \leqslant \frac{2}{R} & \text{in } B_{2R}(0)\setminus B_R(0), \end{cases} \tag{3.2.8}$$

and set

$$v_1(x) = \chi_R(x-Q)\,v(x), \qquad v_2(x) = (1-\chi_R(x-Q))\,v(x).$$

A straight computation yields

$$\|v\|^2 = \|v_1\|^2 + \|v_2\|^2 + 2\int_{\Omega_\epsilon} (\nabla v_1 \cdot \nabla v_2 + v_1 v_2)\,dx.$$

We write $\int_{\Omega_\epsilon} (\nabla v_1 \cdot \nabla v_2 + v_1 v_2)\,dx = \gamma_1 + \gamma_2$, where

$$\begin{aligned}\gamma_1 &= \int_{\Omega_\epsilon} \chi_R(1-\chi_R)\left(v^2 + |\nabla v|^2\right)dx,\\ \gamma_2 &= \int_{\Omega_\epsilon} \left(v_2 \nabla v \cdot \nabla \chi_R - v_1 \nabla v \cdot \nabla \chi_R - v^2 |\nabla \chi_R|^2\right)dx.\end{aligned}$$

Since the integrand in γ_2 is supported in $B_{2R}(Q) \setminus B_R(Q)$, using (3.2.8) and the Young's inequality we obtain that $|\gamma_2| = o_R(1)\,\|v\|^2$. As a consequence we have

$$\|v\|^2 = \|v_1\|^2 + \|v_2\|^2 + 2\gamma_1 + o_R(1)\,\|v\|^2.$$

Now let us evaluate $I''_\epsilon(U_{Q,\epsilon})[v,v] = \sigma_1 + \sigma_2 + \sigma_3$, where

$$\sigma_1 = I''_\epsilon(U_{Q,\epsilon})[v_1,v_1], \quad \sigma_2 = I''_\epsilon(U_{Q,\epsilon})[v_2,v_2], \quad \sigma_3 = 2I''_\epsilon(U_{Q,\epsilon})[v_1,v_2].$$

Similarly to the previous estimates, since U_Q decays exponentially away from Q, we get

$$\begin{aligned}\sigma_2 &\geqslant C^{-1}\|v_2\|^2 + o_{\epsilon,R}(1)\,\|v\|^2,\\ \sigma_3 &\geqslant C^{-1}\gamma_1 + o_{\epsilon,R}(1)\,\|v\|^2.\end{aligned} \tag{3.2.9}$$

Hence it is sufficient to estimate the term σ_1. From the exponential decay of U_Q and the fact that $v \perp U_{Q,\epsilon}, \frac{\partial U_{Q,\epsilon}}{\partial Q}$ it follows that

$$\begin{aligned}(v_1, U_{Q,\epsilon})_{H^1(\Omega_\epsilon)} &= -(v_2, U_{Q,\epsilon})_{H^1(\Omega_\epsilon)} = o_{\epsilon,R}(1)\,\|v\|^2,\\ \left(v_1, \frac{\partial U_{Q,\epsilon}}{\partial Q}\right)_{H^1(\Omega_\epsilon)} &= -\left(v_2, \frac{\partial U_{Q,\epsilon}}{\partial Q}\right)_{H^1(\Omega_\epsilon)} = o_{\epsilon,R}(1)\,\|v\|^2.\end{aligned} \tag{3.2.10}$$

Moreover, since $U_{Q,\epsilon}$ is supported in $B := B_{\frac{\mu}{2\epsilon}}(Q)$, see (3.2.2), we can use the coordinate z in this set, and we obtain

$$\begin{aligned}(v_1, U_{Q,\epsilon})_{H^1(\Omega_\epsilon)} &= \int_{\partial\Omega_\epsilon} v_1 \frac{\partial U_{Q,\epsilon}}{\partial \tilde{\nu}} v\,d\tilde{\sigma} + \int_{\Omega_\epsilon} v_1\left(-\Delta_{\tilde{g}} U_{Q,\epsilon} + U_{Q,\epsilon}\right)dV_{\tilde{g}}(z)\\ &= (v_1, U_Q)_{H^1(K)} + o_\epsilon(1)\,\|v_1\|,\end{aligned} \tag{3.2.11}$$

where $K = K_\alpha$ is the cone of opening angle equal to the angle of Γ in Q. In the same way we can obtain that

$$\left(v_1, \frac{\partial U_{Q,\epsilon}}{\partial Q}\right)_{H^1(\Omega_\epsilon)} = \left(v_1, \frac{\partial U_Q}{\partial Q}\right)_{H^1(K)} + o_\epsilon(1)\,\|v_1\|. \tag{3.2.12}$$

From the estimates (3.2.10), (3.2.11) and (3.2.12), we deduce that for R sufficiently large and ϵ sufficiently small

$$\begin{aligned}(v_1, U_Q)_{H^1(K)} &= o_{\epsilon,R}(1)\,\|v_1\|,\\ \left(v_1, \frac{\partial U_Q}{\partial Q}\right)_{H^1(K)} &= o_{\epsilon,R}(1)\,\|v_1\|.\end{aligned}$$

Now we can apply Lemma 3.1.1, getting

$$I''\left(U_Q\right)[v_1,v_1] \geqslant \delta \|v_1\|_{H^1(K)} + o_{\epsilon,R}(1).$$

Then the following estimate holds

$$\begin{aligned} \sigma_1 &= I''\left(U_Q\right)[v_1,v_1] + o_\epsilon(1)\|v_1\| \geqslant \delta\|v_1\|_{H^1(K)} + o_{\epsilon,R}(1)\|v\| \\ &\geqslant \delta\|v_1\| + o_{\epsilon,R}(1)\|v\|. \end{aligned} \tag{3.2.13}$$

In conclusion, from (3.2.9) and (3.2.13) we deduce

$$I''_{\epsilon,N}\left(U_{Q,\epsilon}\right)[v,v] \geqslant \delta\|v\| + o_{\epsilon,R}(1)\|v\| \geqslant \frac{\delta}{2}\|v\|,$$

provided R is taken large and ϵ sufficiently small. This concludes the proof.

The case $\alpha > \pi$ has substantially the same proof, but we have to consider also the function $\tilde{U}$ and use the exponential decay of $\tilde{u}$ at infinity, see Remark 3.1.3. □

The following lemma provides an expansion of the functional $I_{\epsilon,N}\left(U_{Q,\epsilon}\right)$ with respect to Q.

Lemma 3.2.3. *For ϵ small the following expansion holds*

$$I_{\epsilon,N}\left(U_{Q,\epsilon}\right) = C_0\alpha(Q) + O(\epsilon), \tag{3.2.14}$$

where

$$C_0 = \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_0^\infty\int_0^\pi |U_Q(r)|^{p+1} r\sin^2\theta \, dr d\theta.$$

Proof. Since the function $U_{Q,\epsilon}$ is supported in $B := B_{\frac{\mu}{2\epsilon}}(Q)$, see (3.2.2), we can use the coordinate z in this set, and we obtain

$$I_{\epsilon,N}\left(U_{Q,\epsilon}\right) = \frac{1}{2}\int_{B\cap\Omega_\epsilon}\left(|\nabla_{\tilde{g}}U_{Q,\epsilon}|^2 + U_{Q,\epsilon}^2\right)dV_{\tilde{g}}(z) - \frac{1}{p+1}\int_{B\cap\Omega_\epsilon}|U_{Q,\epsilon}|^{p+1}dV_{\tilde{g}}(z).$$

Integrating by parts, we get

$$\begin{aligned} I_{\epsilon,N}\left(U_{Q,\epsilon}\right) &= \frac{1}{2}\int_{B\cap\partial\Omega_\epsilon} U_{Q,\epsilon}\frac{\partial U_{Q,\epsilon}}{\partial\tilde{\nu}}d\tilde{\sigma} + \frac{1}{2}\int_{B\cap\Omega_\epsilon}U_{Q,\epsilon}\left(-\Delta_{\tilde{g}}U_{Q,\epsilon} + U_{Q,\epsilon}\right)dV_{\tilde{g}}(z) \\ &\quad - \frac{1}{p+1}\int_{B\cap\Omega_\epsilon}|U_{Q,\epsilon}|^{p+1}dV_{\tilde{g}}(z) \\ &\doteqdot I + II, \end{aligned}$$

where I is the surface integral over the boundary and II refers to the last two terms. Now, I can be split in two terms which correspond to the surface integrals on the "faces" of the edge Γ:

$$I = \frac{1}{2}\int_{B\cap\partial\Omega_{\epsilon 1}} U_{Q,\epsilon}\frac{\partial U_{Q,\epsilon}}{\partial\tilde{\nu}_1}d\tilde{\sigma}_1 + \frac{1}{2}\int_{B\cap\partial\Omega_{\epsilon 2}} U_{Q,\epsilon}\frac{\partial U_{Q,\epsilon}}{\partial\tilde{\nu}_2}d\tilde{\sigma}_2 \doteqdot I_1 + I_2.$$

It is sufficient to evaluate I_1, since the estimate of I_2 is similar. Using the expression of $U_{Q,\epsilon}$, see (3.2.2), we get

$$\begin{aligned} I_1 &= \frac{1}{2}\int_{B\cap\partial\Omega_{\epsilon 1}} U_{Q,\epsilon}\left(U_Q\nabla\varphi_\mu(\epsilon z)+\varphi_\mu(\epsilon z)\nabla U_Q\right) \\ &\qquad\cdot\left(\epsilon\left(A_Q(z_1,z_2)\right)+\epsilon^2 D_Q(z_1,z_2),-1+\frac{3}{2}\epsilon^2|A_Q(z_1,z_2)|^2\right) \\ &\qquad\cdot\left(1+O\left(\epsilon^2|(z_1,z_2)|^2\right)\right)dz_1\,dz_2 \\ &= \frac{1}{2}\int_{\frac{\mu}{4\epsilon}\leqslant|z-Q|\leqslant\frac{\mu}{2\epsilon}} \varphi_\mu(\epsilon z)U_Q^2\nabla\varphi_\mu(\epsilon z) \\ &\qquad\cdot\left(\epsilon\left(A_Q(z_1,z_2)\right)+\epsilon^2 D_Q(z_1,z_2),-1+\frac{3}{2}\epsilon^2|A_Q(z_1,z_2)|^2\right) \\ &\qquad\cdot\left(1+O\left(\epsilon^2|(z_1,z_2)|^2\right)\right)dz_1\,dz_2 \\ &\quad+\frac{1}{2}\int_{|z-Q|\leqslant\frac{\mu}{2\epsilon}} \varphi_\mu^2(\epsilon z)U_Q\nabla U_Q \\ &\qquad\cdot\left(\epsilon\left(A_Q(z_1,z_2)\right)+\epsilon^2 D_Q(z_1,z_2),-1+\frac{3}{2}\epsilon^2|A_Q(z_1,z_2)|^2\right) \\ &\qquad\cdot\left(1+O\left(\epsilon^2|(z_1,z_2)|^2\right)\right)dz_1\,dz_2. \end{aligned}$$

Similarly to the previous estimates, we get $I_1=O\left(e^{-\frac{\mu}{2\epsilon}}\right)+O(\epsilon)$. Then we obtain that

$$I=O(\epsilon). \tag{3.2.15}$$

Now, we have to evaluate II:

$$\begin{aligned} II &= \frac{1}{2}\int_{B\cap\Omega_\epsilon} U_{Q,\epsilon}\left(-\Delta U_{Q,\epsilon}+U_{Q,\epsilon}\right)(1+O(\epsilon|z|))\,dz \\ &\quad+\frac{\epsilon}{2}\int_{B\cap\Omega_\epsilon} U_{Q,\epsilon}\left[2\gamma''(0)z_1\cdot\nabla_{(z_2,z_3)}\frac{\partial U_{Q,\epsilon}}{\partial z_1}+\gamma''(0)\cdot\nabla_{(z_2,z_3)}U_{Q,\epsilon}\right] \\ &\quad\cdot(1+O(\epsilon|z|))\,dz \\ &\quad+O\left(\epsilon^2|z_1|^2\right)-\frac{1}{p+1}\int_{B\cap\Omega_\epsilon}|U_{Q,\epsilon}|^{p+1}(1+O(\epsilon|z|))\,dz. \end{aligned}$$

We have

$$II=\left(\frac{1}{2}-\frac{1}{p+1}\right)\alpha(Q)\int_0^\infty\int_0^\pi|U_Q(r)|^{p+1}r\sin^2\theta\,dr\,d\theta+O(\epsilon). \tag{3.2.16}$$

Putting together (3.2.15) and (3.2.16), we obtain (3.2.14) and this concludes the proof. □

Let $P_Q: H^1(\Omega_\epsilon)\longrightarrow\left(T_{U_{Q,\epsilon}}Z_\epsilon\right)^\perp$ be the projection onto the orthogonal complement of $T_{U_{Q,\epsilon}}Z_\epsilon$, for all Q on the edge Γ of $\partial\Omega_\epsilon$. According to the Lemma 3.2.2, we have that for ϵ sufficiently small the operator $L_Q=P_Q\circ I''_{\epsilon,N}\left(U_{Q,\epsilon}\right)\circ P_Q$ is invertible and there exists $C>0$ such that

$$\left\|L_Q^{-1}\right\|\leqslant C.$$

Now, using the fact that $I''_{\epsilon,N}\left(U_{Q,\epsilon}\right)$ is invertible on the orthogonal complement of $T_{U_{Q,\epsilon}}Z_\epsilon$, we will solve the auxiliary equation.

Proposition 3.2.4. *Let $I_{\epsilon,N}$ be the functional defined in (3.0.3). Then for $\epsilon > 0$ small there exists a unique $w = w(\epsilon, Q) \in \left(T_{U_{Q,\epsilon}} Z_\epsilon\right)^\perp$ such that $I'_{\epsilon,N}\left(U_{Q,\epsilon} + w(\epsilon, Q)\right) \in T_{U_{Q,\epsilon}} Z_\epsilon$. Moreover the function $w(\epsilon, Q)$ is of class C^1 with respect to Q and there holds*

$$\|w(\epsilon, Q)\| \leqslant C\epsilon, \qquad \left\|\frac{\partial w(\epsilon, Q)}{\partial Q}\right\| \leqslant C\epsilon. \tag{3.2.17}$$

Proof. We want to find a solution $w \in \left(T_{U_{Q,\epsilon}} Z_\epsilon\right)^\perp$ of $P_Q I'_{\epsilon,N}\left(U_{Q,\epsilon} + w\right) = 0$. For every $w \in \left(T_{U_{Q,\epsilon}} Z_\epsilon\right)^\perp$ we can write

$$I'_{\epsilon,N}\left(U_{Q,\epsilon} + w\right) = I'_{\epsilon,N}\left(U_{Q,\epsilon}\right) + I''_{\epsilon,N}\left(U_{Q,\epsilon}\right)[w] + R_{Q,\epsilon}(w),$$

where $R_{Q,\epsilon}(w)$ is given by

$$R_{Q,\epsilon}(w) = I'_{\epsilon,N}\left(U_{Q,\epsilon} + w\right) - I'_{\epsilon,N}\left(U_{Q,\epsilon}\right) - I''_{\epsilon,N}\left(U_{Q,\epsilon}\right)[w].$$

Given $v \in H^1(\Omega_\epsilon)$ there holds

$$R_{Q,\epsilon}(w)[v] = -\int_{\Omega_\epsilon} \left(|U_{Q,\epsilon} + w|^p - |U_{Q,\epsilon}|^p - p|U_{Q,\epsilon}|^{p-1} w\right) v dx.$$

Using the following inequality

$$|(a+b)^p - a^p - pa^{p-1}b| \leqslant \begin{cases} C(p)\,|b|^p & \text{for } p \leqslant 2, \\ C(p)\left(|b|^2 + |b|^p\right) & \text{for } p > 2, \end{cases}$$

for $a, b \in \mathbb{R}$, $|a| \leqslant 1$, the Hölder's inequality and the Sobolev embeddings we obtain

$$\left\|R_{Q,\epsilon}(w)[v]\right\| \leqslant C \int_{\Omega_\epsilon} \left(|w|^2 + |w|^p\right)|v| dx \leqslant C\left(\|w\|^2 + \|w\|^p\right)\|v\|. \tag{3.2.18}$$

Similarly, from the inequality

$$\begin{aligned} &|(a+b_1)^p - (a+b_2)^p - pa^{p-1}(b_1 - b_2)| \\ &\qquad \leqslant \begin{cases} C(p)\left(|b_1|^{p-1} + |b_2|^{p-1}\right)|b_1 - b_2| & \text{for } p \leqslant 2, \\ C(p)\left(|b_1| + |b_2| + |b_1|^{p-1} + |b_2|^{p-1}\right)|b_1 - b_2| & \text{for } p > 2, \end{cases} \end{aligned}$$

for $a, b_1, b_2 \in \mathbb{R}$, $|a| \leqslant 1$, we get

$$\begin{aligned} &\left\|R_{Q,\epsilon}(w_1)[v] - R_{Q,\epsilon}(w_2)[v]\right\| \\ &\quad \leqslant C \int_{\Omega_\epsilon} \left(|w_1| + |w_2| + |w_1|^{p-1} + |w_2|^{p-1}\right)|w_1 - w_2| \cdot |v| dx \\ &\quad \leqslant C\left(\|w_1\| + \|w_2\| + \|w_1\|^{p-1} + \|w_2\|^{p-1}\right) \cdot \|w_1 - w_2\| \cdot \|v\|. \end{aligned} \tag{3.2.19}$$

Now, by the invertibility of the operator $L_Q = P_Q \circ I''_{\epsilon,N}\left(U_{Q,\epsilon}\right) \circ P_Q$, we have that the function w solves $P_Q I'_{\epsilon,N}\left(U_{Q,\epsilon} + w\right) = 0$ if and only if

$$w = -\left(L_Q\right)^{-1}\left[P_Q I'_{\epsilon,N}\left(U_{Q,\epsilon}\right) + P_Q R_{Q,\epsilon}(w)\right].$$

Setting

$$N_{Q,\epsilon}(w) = -\left(L_Q\right)^{-1}\left[P_Q I'_{\epsilon,N}\left(U_{Q,\epsilon}\right) + P_Q R_{Q,\epsilon}(w)\right],$$

we have to solve

$$w = N_{Q,\epsilon}(w).$$

The norm of $I'_{\epsilon,N}\left(U_{Q,\epsilon}\right)$ has been estimated in Lemma 3.2.1. Then from (3.2.18) and (3.2.19) we obtain the two relations

$$\begin{aligned} \left\|N_{Q,\epsilon}(w)\right\| &\leqslant C_1\epsilon + C_2\left(\|w\|^2 + \|w\|^p\right), & (3.2.20)\\ \left\|N_{Q,\epsilon}(w_1) - N_{Q,\epsilon}(w_2)\right\| &\leqslant C\left(\|w_1\| + \|w_2\| + \|w_1\|^{p-1} + \|w_2\|^{p-1}\right) \\ &\quad \cdot \|w_1 - w_2\|. & (3.2.21) \end{aligned}$$

Now, for $\bar{C} > 0$, we define the set

$$W_{\bar{C}} = \left\{w \in \left(T_{U_{Q,\epsilon}} Z_\epsilon\right)^\perp : \|w\| \leqslant \bar{C}\epsilon\right\}.$$

We show that $N_{Q,\epsilon}$ is a contraction in $W_{\bar{C}}$ for $\bar{C}$ sufficiently large and for ϵ small. Clearly, by (3.2.20), if $\bar{C} > 2C_1$ the set $W_{\bar{C}}$ is mapped into itself if ϵ is sufficiently small. Then, if $w_1, w_2 \in W_{\bar{C}}$, by (3.2.21) there holds

$$\left\|N_{Q,\epsilon}(w_1) - N_{Q,\epsilon}(w_2)\right\| \leqslant 2C\left(\bar{C}\epsilon + \bar{C}^{p-1}\epsilon^{p-1}\right)\|w_1 - w_2\|.$$

Therefore, again if ϵ is sufficiently small, the coefficient of $\|w_1 - w_2\|$ in the last formula is less than 1. Hence the Contraction Mapping Theorem applies, yielding the existence of a solution w satisfying the condition

$$\|w\| \leqslant \bar{C}\epsilon. \qquad (3.2.22)$$

This concludes the proof of the existence part.

Now the C^1-dependence of the function w on Q follows from the Implicit Function Theorem; see also [4], Proposition 8.7. In order to prove the second estimate in (3.2.17), let us consider the map $H : \mathbb{R}^3 \times H^1(\Omega_\epsilon) \times \mathbb{R} \times \mathbb{R} \longrightarrow H^1(\Omega_\epsilon) \times \mathbb{R}$ defined by

$$H(Q, w, \alpha, \epsilon) = \begin{pmatrix} I'_{\epsilon,N}\left(U_{Q,\epsilon} + w\right) - \alpha\frac{\partial U_{Q,\epsilon}}{\partial Q} \\ \left(w, \frac{\partial U_{Q,\epsilon}}{\partial Q}\right) \end{pmatrix}.$$

Then $w \in \left(T_{U_{Q,\epsilon}} Z_\epsilon\right)^\perp$ is a solution of $P_Q I'_{\epsilon,N}\left(U_{Q,\epsilon} + w\right) = 0$ if and only if $H(Q, w, \alpha, \epsilon) = 0$. Moreover, for $v \in H^1(\Omega_\epsilon)$ and $\beta \in \mathbb{R}$, there holds

$$\begin{aligned} \frac{\partial H}{\partial(w,\alpha)}(Q, w, \alpha, \epsilon)[v, \beta] &= \begin{pmatrix} I''_{\epsilon,N}\left(U_{Q,\epsilon} + w\right)[v] - \beta\frac{\partial U_{Q,\epsilon}}{\partial Q} \\ \left(v, \frac{\partial U_{Q,\epsilon}}{\partial Q}\right) \end{pmatrix} & (3.2.23)\\ &= \begin{pmatrix} I''_{\epsilon,N}\left(U_{Q,\epsilon}\right)[v] - \beta\frac{\partial U_{Q,\epsilon}}{\partial Q} \\ \left(v, \frac{\partial U_{Q,\epsilon}}{\partial Q}\right) \end{pmatrix} + O\left(\|w\| + \|w\|^{p-1}\right). \end{aligned}$$

To prove the last estimate it is sufficient to use the following inequality

$$|(a+b)^{p-1}-a^{p-1}| \leqslant \begin{cases} C(p)\,|b|^{p-1} & \text{for } p \leqslant 2, \\ C(p)\left(|b|+|b|^{p-1}\right) & \text{for } p > 2, \end{cases}$$

for $a, b \in \mathbb{R}$, $|a| \leqslant 1$, the Hölder's inequality and the Sobolev embedding. Using the invertibility of the operator $L_Q = P_Q \circ I''_{\epsilon,N}\left(U_{Q,\epsilon}\right) \circ P_Q$, it is easy to check that $\frac{\partial H}{\partial(w,\alpha)}(Q,0,0,\epsilon)$ is uniformly invertible in Q for ϵ small. Hence, by (3.2.22) and (3.2.23), also $\frac{\partial H}{\partial(w,\alpha)}(Q,w,\alpha,\epsilon)$ is uniformly invertible in Q for ϵ small. As a consequence, by the Implicit Function Theorem, the map $Q \mapsto \left(w_Q, \alpha_Q\right)$ is of class C^1. Now we are in position to provide the norm estimate of $\frac{\partial w(\epsilon,Q)}{\partial Q}$. Differentiating the equation

$$H\left(Q, w_Q, \alpha_Q, \epsilon\right) = 0$$

with respect to Q, we obtain

$$0 = \frac{\partial H}{\partial Q}(Q,w,\alpha,\epsilon) + \frac{\partial H}{\partial(w,\alpha)}(Q,w,\alpha,\epsilon)\frac{\partial\left(w_Q,\alpha_Q\right)}{\partial Q}.$$

Hence, by the uniform invertibility of $\frac{\partial H}{\partial(w,\alpha)}(Q,w,\alpha,\epsilon)$ it follows that

$$\begin{aligned}
\left\|\frac{\partial\left(w_Q,\alpha_Q\right)}{\partial Q}\right\| &\leqslant C\left\|\begin{pmatrix} I''_{\epsilon,N}\left(U_{Q,\epsilon}+w\right)\left[\frac{\partial U_{Q,\epsilon}}{\partial Q}\right]-\alpha\frac{\partial^2 U_{Q,\epsilon}}{\partial Q^2} \\ \left(w, \frac{\partial^2 U_{Q,\epsilon}}{\partial Q^2}\right) \end{pmatrix}\right\| \\
&\leqslant C\left(\left\|I''_{\epsilon,N}\left(U_{Q,\epsilon}+w\right)\left[\frac{\partial U_{Q,\epsilon}}{\partial Q}\right]\right\| + |\alpha| \cdot \left\|\frac{\partial^2 U_{Q,\epsilon}}{\partial Q^2}\right\| + \|w\| \cdot \left\|\frac{\partial^2 U_{Q,\epsilon}}{\partial Q^2}\right\|\right) \\
&\leqslant C\left(\left\|I''_{\epsilon,N}\left(U_{Q,\epsilon}+w\right)\left[\frac{\partial U_{Q,\epsilon}}{\partial Q}\right]\right\| + |\alpha| + \|w\| + \epsilon\right).
\end{aligned}$$

Note that α, similarly to w, satisfies $|\alpha| \leqslant C\epsilon$. By the estimate in (3.2.23) we obtain

$$\left\|I''_{\epsilon,N}\left(U_{Q,\epsilon}+w\right)\left[\frac{\partial U_{Q,\epsilon}}{\partial Q}\right]\right\| \leqslant \left\|I''_{\epsilon,N}\left(U_{Q,\epsilon}\right)\left[\frac{\partial U_{Q,\epsilon}}{\partial Q}\right]\right\| + C\left(\|w\| + \|w\|^{p-1}\right).$$

Using the fact that $I''\left(U_Q\right)\left[\frac{\partial U_Q}{\partial z_1}\right] = 0$ we obtain

$$\begin{aligned}
\left\|I''_{\epsilon,N}\left(U_{Q,\epsilon}+w\right)\left[\frac{\partial U_{Q,\epsilon}}{\partial Q}\right]\right\| \leqslant{} & \left\|I''_{\epsilon,N}\left(U_{Q,\epsilon}\right)\left[\frac{\partial U_Q}{\partial z_1}\right] - I''\left(U_Q\right)\left[\frac{\partial U_Q}{\partial z_1}\right]\right\| \\
& + C\epsilon + C\left(\|w\| + \|w\|^{p-1}\right).
\end{aligned}$$

For any $v \in H^1(K)$, one finds

$$|\left(I''_{\epsilon,N}\left(U_{Q,\epsilon}\right) - I''\left(U_Q\right)\right)\left[\frac{\partial U_Q}{\partial z_1}, v\right]| \leqslant p\int_{K\cap\Omega_\epsilon} |U_{Q,\epsilon} - U_Q|\frac{\partial U_Q}{\partial z_1} v + C\epsilon.$$

The last three formulas implies the estimate for $\frac{\partial w(\epsilon,Q)}{\partial Q}$. This concludes the proof. □

Now we can state the following result, which allows us to perform a finite-dimensional reduction of problem (3.0.2) on the manifold Z_ϵ.

Proposition 3.2.5. *The functional* $\Psi_\epsilon : Z_\epsilon \to \mathbb{R}$ *defined by* $\Psi_\epsilon(Q) = I_\epsilon\left(U_{Q,\epsilon} + w(\epsilon, Q)\right)$ *is of class* C^1 *in* Q *and satisfies*

$$\Psi'_\epsilon(Q) = 0 \qquad \Longrightarrow \qquad I'_{\epsilon,N}\left(U_{Q,\epsilon} + w(\epsilon, Q)\right) = 0.$$

Proof. This proposition can be proved using the arguments of Theorem 2.12 of [4]. From a geometric point of view, we consider the manifold

$$\tilde{Z}_\epsilon = \left\{U_{Q,\epsilon} + w(\epsilon, Q) : Q \in \Gamma\right\}.$$

Since (3.2.17) holds, we have that for ϵ small

$$T_{U_{Q,\epsilon}} Z_\epsilon \sim T_{U_{Q,\epsilon}+w(\epsilon,Q)} \tilde{Z}_\epsilon. \tag{3.2.24}$$

If $U_{Q,\epsilon} + w(\epsilon, Q)$ is a critical point of $I_{\epsilon,N}$ constrained on $\tilde{Z}_\epsilon$, then $I'_{\epsilon,N}\left(U_{Q,\epsilon} + w(\epsilon, Q)\right)$ is perpendicular to $T_{U_{Q,\epsilon}+w(\epsilon,Q)}\tilde{Z}_\epsilon$, and hence, from (3.2.24), is almost perpendicular to $T_{U_{Q,\epsilon}} Z_\epsilon$. Since, by construction of $\tilde{Z}_\epsilon$, it is $I'_{\epsilon,N}\left(U_{Q,\epsilon} + w(\epsilon, Q)\right) \in T_{U_{Q,\epsilon}} Z_\epsilon$, it must be $I'_{\epsilon,N}\left(U_{Q,\epsilon} + w(\epsilon, Q)\right) = 0$. This concludes the proof. □

3.2.1 *Proof of Theorem 3.0.2*

First of all we have

$$\begin{aligned}
\Psi_\epsilon(Q) &= I_{\epsilon,N}\left(U_{Q,\epsilon} + w(\epsilon, Q)\right) \\
&= I_{\epsilon,N}\left(U_{Q,\epsilon}\right) + I'_{\epsilon,N}\left(U_{Q,\epsilon}\right)[w(\epsilon, Q)] + O\left(\|w(\epsilon, Q)\|^2\right).
\end{aligned}$$

Now, using Lemma 3.2.1 and the estimate (3.2.17) we infer

$$\Psi_\epsilon(Q) = I_{\epsilon,N}\left(U_{Q,\epsilon}\right) + O\left(\epsilon^2\right).$$

Hence Lemma 3.2.3 yields

$$\Psi_\epsilon(Q) = C_0\,\alpha(Q) + O(\epsilon).$$

Therefore, if $Q \in \Gamma$ is a local strict maximum or minimum of the function α, the thesis follows from Proposition 3.2.5.

4 CONCENTRATION OF SOLUTIONS FOR A SINGULARLY PERTURBED MIXED PROBLEM IN NON-SMOOTH DOMAINS

INTRODUCTION

In this chapter we study the following singular perturbation problem with mixed Dirichlet and Neumann boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^n$ whose boundary $\partial\Omega$ is non smooth:

$$\begin{cases} -\epsilon^2 \Delta u + u = u^p & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial_N \Omega, & u = 0 \quad \text{on } \partial_D \Omega, \\ u > 0 & \text{in } \Omega. \end{cases} \tag{4.0.1}$$

Here $p \in \left(1, \frac{n+2}{n-2}\right)$ is subcritical, ν denotes the outer unit normal at $\partial\Omega$ and $\epsilon > 0$ is a small parameter. Moreover $\partial_N \Omega$, $\partial_D \Omega$ are two subsets of the boundary of Ω such that the union of their closures coincides with the whole $\partial\Omega$, and their intersection is an $(n-2)$-dimensional smooth singularity.

We are interested here in finding boundary spike-layers for the mixed problem (4.0.1). We call Γ the intersection of the closures of $\partial_N \Omega$ and $\partial_D \Omega$, and suppose that it is an $(n-2)$-dimensional smooth singularity. Then we can consider the function $\alpha : \Gamma \to \mathbb{R}$ which associates to every $Q \in \Gamma$ the opening angle at Q, $\alpha(Q)$. Moreover we denote by H the mean curvature of $\partial\Omega$ restricted to the closure of $\partial_N \Omega$, that is $H : \overline{\partial_N \Omega} \to \mathbb{R}$.

The main result of this chapter is the following:

Theorem 4.0.6. *Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be a bounded domain whose boundary $\partial\Omega$ has an $(n-2)$-dimensional smooth singularity, and $1 < p < \frac{n+2}{n-2}$ ($1 < p < +\infty$ if $n = 2$). Suppose that $\partial_N \Omega$, $\partial_D \Omega$ are disjoint open sets of $\partial\Omega$ such that the union of the closures is the whole boundary of Ω and such that their intersection Γ is the singularity. Suppose $Q \in \Gamma$ is such that $\alpha(Q) \neq 0$ and $H|_\Gamma$ is critical and non degenerate at Q, and that $\nabla H(Q) \neq 0$ points toward $\partial_D \Omega$. Then for $\epsilon > 0$ sufficiently small problem (4.0.1) admits a solution concentrating at Q.*

The general strategy for proving Theorem 4.0.6 relies on a finite-dimensional reduction, as outlined in Section 2.1. Namely, one finds first a manifold Z of approximate solutions to the given problem, which in our case are of the form (1.2.1), and solve the equation up to a vector parallel to the tangent plane of this manifold. To do this one can use the spectral properties of the linearization of (1.2.2), see Lemma 4.3.3. Then, see Theorem 2.1.6, one generates a new manifold $\tilde{Z}$ close to Z which represents a natural constraint for the Euler functional of (4.0.1), which is

$$\tilde{I}_\epsilon(u) = \frac{1}{2} \int_\Omega \left(\epsilon^2 |\nabla u|^2 + u^2 \right) dx - \frac{1}{p+1} \int_\Omega |u|^{p+1} dx, \quad u \in H^1_D(\Omega),$$

where $H^1_D(\Omega)$ is the space of functions $H^1(\Omega)$ which have zero trace on $\partial_D \Omega$. By *natural constraint* we mean a set for which constrained critical points of $\tilde{I}_\epsilon$ are true critical points.

Now, we want to have a good control of the functional $\tilde{I}_\epsilon |_{\tilde{Z}}$. Improving the accuracy of the functions in the original manifold Z, we make $\tilde{Z}$ closer to Z; in this way the main term in the constrained functional will be given by $\tilde{I}_\epsilon |_Z$, see Propositions 4.2.12, 4.2.14, 4.2.15. To find sufficiently good approximate solutions we start with those constructed in literature for the Neumann problem

$$\begin{cases} -\epsilon^2 \Delta u + u = u^p & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega. \end{cases} \tag{4.0.2}$$

(see Subsection 4.1.1) which reveal the role of the mean curvature. The problem is that these functions are non zero on $\partial_D \Omega$, and even if one use cut-off functions to annihilate them the corresponding error turns out to be too large. Following the line of [31] and [62], we will use the *projection operator* in $H^1(\Omega)$, which associates to every function in this space its closest element in $H^1_D(\Omega)$. To study the asymptotic behavior of this projection we will use the limit behavior of the solution U to (1.2.2):

$$\lim_{r \to +\infty} e^r r^{\frac{n-2}{2}} U(r) = c_{n,p}, \tag{4.0.3}$$

where $r = |x|$ and $c_{n,p}$ is a positive constant depending only on the dimension n and p, together with

$$\lim_{r \to +\infty} \frac{U'(r)}{U(r)} = -\lim_{r \to +\infty} \frac{U''(r)}{U(r)} = -1, \tag{4.0.4}$$

as it was done in some previous works, see for instance [46] and [82]. Moreover, we will work at a scale $d \simeq \epsilon |\log \epsilon|$, which is the order of the distance of the peak from Γ, see Remark 1.4.1 (b). At this scale both $\partial_N \Omega$ and $\partial_D \Omega$ look flat; so we can identify them with the hypersurfaces of equations $x_n = 0$ and $x_1 \tan \alpha + x_n = 0$, and their intersection with the set $\{x_1 = x_n = 0\}$. Note that $\alpha = \alpha(Q)$ is the angle between x_1 and x_n at a fixed point $Q \in \Gamma$. Then we can replace Ω with a suitable domain Σ_D, which in particular for $0 < \alpha \leqslant \pi$ is even with respect to the coordinate x_n, see the beginning of Subsections 4.2.1 and 4.2.2. Now, studying the projections in this domain, we will find functions which have zero x_n-derivative on $\{x_n = 0\} \setminus \partial \Sigma_D$, which mimics the Neumann boundary condition on $\partial_N \Omega$. After analyzing carefully the projection in Subsections 4.2.1, 4.2.2, we will be able to define a family of suitable approximate solutions to (4.0.1) which have sufficient accuracy for our analysis, estimated in Propositions 4.2.12, 4.2.14, 4.2.15.

We can finally apply the above mentioned perturbation method to reduce the problem to a finite-dimensional one, and study the functional constrained on $\tilde{Z}$. We obtain an expansion of the energy of the approximate solutions, which turns out to be

$$\begin{aligned} \tilde{I}_\epsilon (u_{\epsilon,Q}) &= \tilde{C}_0 \epsilon^n - \tilde{C}_1 \epsilon^{n+1} H(Q) + \epsilon^n e^{-2\frac{d_\epsilon}{\epsilon}(1+o(1))} \\ &\quad + \epsilon^n e^{-\frac{d_\epsilon}{\epsilon}\left(1 + \frac{\sqrt{2}\tan\alpha(Q)}{\sqrt{\tan^2\alpha(Q)+1}}\right)(1+o(1))} + o\left(\epsilon^{n+2}\right), \end{aligned}$$

in the case $0 < \alpha < \frac{\pi}{2}$, and

$$\tilde{I}_\epsilon (u_{\epsilon,Q}) = \tilde{C}_0 \epsilon^n - \tilde{C}_1 \epsilon^{n+1} H(Q) + \epsilon^n e^{-2\frac{d_\epsilon}{\epsilon}(1+o(1))} + o\left(\epsilon^{n+2}\right),$$

in the case $\frac{\pi}{2} \leqslant \alpha < 2\pi$. As for (1.2.6), we have that the first two terms come from the Neumann condition, while the others are related to the repulsive effect due to the Dirichlet condition. Let us notice that, in the first case, in the terms related to the Dirichlet condition appears the opening angle α, whereas in the second case it does not; this phenomenon comes from the fact that the distance of the point Q from the Dirichlet part $\partial_D\Omega$ depends on α only if $0<\alpha<\frac{\pi}{2}$.

Concerning the regularity of the solution, following the ideas in [34], it is possible to say that it is influenced by the presence of the angle. In fact, the solution is at least C^2 in the interior of the domain, far from the angle; whereas, near the angle, one can split the solution into a regular part and a singular one, whose regularity depends on the value of α. For more details about the regularity of solutions in non-smooth domains we refer the reader to the book [34].

The fact that the solution u is C^2 in the interior of the domain allows to say also that it is strictly positive, by using the strong Maximum Principle. In fact, we have that $u \geqslant 0$ in the domain. Moreover, if there exists a point x_0 in the interior of the domain such that $u(x_0)=0$, we can consider a ball centered at x_0 of small radius suct that it is contained in the domain; since in the ball u is C^2 we can conclude that u cannot be zero in x_0.

The plan of the chapter is the following. In Section 4.1 we collect some preliminary material: we recall some known results concerning the Neumann problem (4.0.2). In Section 4.2 we construct a model domain to deal with the interface, analyze the asymptotics of projections in H^1 and then construct approximate solution to (4.0.1). Finally in Section 4.3 we expand the functional on the natural constraint, prove the existence of critical points and deduce Theorem 4.0.6.

4.1 PRELIMINARIES

We want to find solutions to (4.0.1) with a specific asymptotic profile, so it is convenient to make the change of variables $x \mapsto \epsilon x$, and study (4.0.1) in the dilated domain

$$\Omega_\epsilon := \frac{1}{\epsilon}\Omega.$$

Then the problem becomes

$$\begin{cases} -\Delta u + u = u^p & \text{in } \Omega_\epsilon, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial_N\Omega_\epsilon, & u = 0 \quad \text{on } \partial_D\Omega_\epsilon, \\ u > 0 & \text{in } \Omega_\epsilon, \end{cases} \tag{4.1.1}$$

where $\partial_N\Omega_\epsilon$ and $\partial_D\Omega_\epsilon$ stand for the dilations of $\partial_N\Omega$ and $\partial_D\Omega$ respectively. Moreover we denote by Γ_ϵ the intersection of the closures of $\partial_N\Omega_\epsilon$ and $\partial_D\Omega_\epsilon$.

Solutions of (4.1.1) can be found as critical points of the Euler-Lagrange functional

$$I_\epsilon(u) = \frac{1}{2}\int_{\Omega_\epsilon}\left(|\nabla u|^2 + u^2\right)dx - \frac{1}{p+1}\int_{\Omega_\epsilon}|u|^{p+1}dx, \quad u \in H^1_D(\Omega_\epsilon).$$

Here $H^1_D(\Omega_\epsilon)$ denotes the space of functions in $H^1(\Omega_\epsilon)$ with zero trace on $\partial_D\Omega_\epsilon$.

4.1.1 *Approximate solutions for* (4.0.1) *with Neumann conditions*

In this subsection we recall some results from [4] and [31] concerning approximate solutions to the Neumann problem.

Recalling the new coordinate system that we introduced in Section 2.2, we assume that this coordinate system y is defined in $B_{\mu_0}(Q)$, with $\mu_0 > 0$ sufficiently small. Now, in this set of coordinates we choose a cut-off function χ_{μ_0} with the following properties

$$\begin{cases} \chi_{\mu_0}(x) = 1 & \text{in } B_{\frac{\mu_0}{4}}(Q), \\ \chi_{\mu_0}(x) = 0 & \text{in } \mathbb{R}^n \setminus B_{\frac{\mu_0}{2}}(Q), \\ |\nabla\chi_{\mu_0}| + |\nabla^2\chi_{\mu_0}| \leqslant C & \text{in } B_{\frac{\mu_0}{2}}(Q) \setminus B_{\frac{\mu_0}{4}}(Q), \end{cases}$$

and we define the approximate solution $\bar{u}_{\epsilon,Q}$ as

$$\bar{u}_{\epsilon,Q}(y) := \chi_{\mu_0}(\epsilon y)\left(U_Q(y) + \epsilon w_Q(y)\right), \tag{4.1.2}$$

where $U_Q(y) = U(y - Q)$ and w_Q is a suitable function obtained in Subsection 2.2 of [31] by a small modifications of Lemma 9.3 in [4], satisfying the following estimate

$$|w_Q(y)| + |\nabla w_Q(y)| + |\nabla^2 w_Q(y)| \leqslant C_\Omega\left(1 + |y|^K\right)e^{-|y|}, \tag{4.1.3}$$

where C_Ω and K are constants depending on Ω, H, n and p.

The next result collects estimates obtained following the same arguments of Lemmas 9.4, 9.7 and 9.8 in [4].

Proposition 4.1.1. *There exist $C, K > 0$ such that for ϵ small the following estimates hold*

$$\left|\frac{\partial \bar{u}_{\epsilon,Q}}{\partial \nu_g}\right|(y) \leqslant \begin{cases} C\epsilon^2\left(1 + |y|^K\right)e^{-|y|} & \text{for } |y| \leqslant \frac{\mu_0}{4\epsilon}, \\ Ce^{-\frac{1}{C\epsilon}} & \text{for } \frac{\mu_0}{4\epsilon} \leqslant |y| \leqslant \frac{\mu_0}{2\epsilon}; \end{cases}$$

$$|-\Delta_g\bar{u}_{\epsilon,Q} + \bar{u}_{\epsilon,Q} - \bar{u}^p_{\epsilon,Q}|(y) \leqslant \begin{cases} C\epsilon^2\left(1 + |y|^K\right)e^{-|y|} & \text{for } |y| \leqslant \frac{\mu_0}{4\epsilon}, \\ Ce^{-\frac{1}{C\epsilon}} & \text{for } \frac{\mu_0}{4\epsilon} \leqslant |y| \leqslant \frac{\mu_0}{2\epsilon}; \end{cases}$$

$$I_{\epsilon,N}\left(\bar{u}_{\epsilon,Q}\right) = \tilde{C}_0 - \tilde{C}_1\epsilon H(\epsilon Q) + o\left(\epsilon^2\right);$$

$$\frac{\partial}{\partial Q}I_{\epsilon,N}\left(\bar{u}_{\epsilon,Q}\right) = -\tilde{C}_1\epsilon^2 H'(\epsilon Q) + o\left(\epsilon^2\right),$$

where

$$\tilde{C}_0 = \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^n_+} U^{p+1}dy, \qquad \tilde{C}_1 = \left(\int_0^\infty r^n U_r^2 dr\right)\int_{S^n_+} y_n|y'|^2 d\sigma.$$

An immediate consequence of this proposition is that

$$\left\|I'_\epsilon\left(\bar{u}_{\epsilon,Q}\right)\right\| \leqslant C\epsilon^2 \qquad \text{for all } Q \in \partial_N\Omega_\epsilon \text{ such that } \operatorname{dist}(Q,\Gamma_\epsilon) \geqslant \frac{\mu_0}{\epsilon}, \tag{4.1.4}$$

where $C > 0$ is some fixed constant and μ_0 is as before.

To construct good approximate solutions to (4.1.1), we will start from a family of known functions which constitute good approximate solutions to (4.1.1) when we impose pure Neumann boundary conditions. Since we have to take into account the effect of the Dirichlet boundary conditions, we will modify these functions in a convenient way. Following the line of [31] and [62], we will use the *projection operator* onto $H^1_D(\Omega_\epsilon)$, which associates to every element in $H^1(\Omega_\epsilon)$ its closest point in $H^1_D(\Omega_\epsilon)$. Explicitly, this is constructed subtracting to any given $u \in H^1(\Omega_\epsilon)$ the solution to

$$\begin{cases} -\Delta v + v = 0 & \text{in } \Omega_\epsilon, \\ v = u & \text{on } \partial_D \Omega_\epsilon, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial_N \Omega_\epsilon. \end{cases} \tag{4.2.1}$$

This solution can be found variationally by looking at the following minimum problem

$$\inf_{v=u \text{ on } \partial_D\Omega_\epsilon} \left\{ \int_{\Omega_\epsilon} \left(|\nabla v|^2 + v^2 \right) dx \right\}.$$

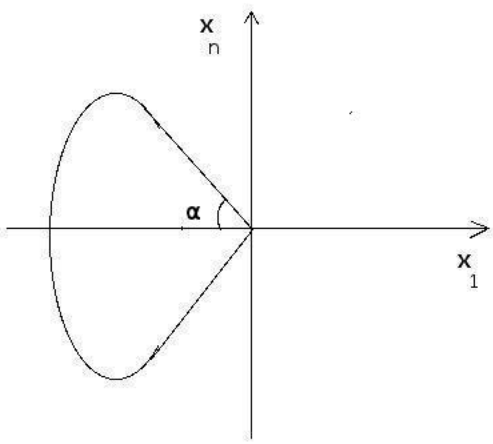
Instead of studying (4.2.1) directly, it is convenient to modify the domain in order that the region of the boundary near Γ_ϵ becomes flat. We fix $Q \in \Gamma_\epsilon$ and consider the opening angle of Γ_ϵ at Q, $\alpha = \alpha(Q)$. Since the construction of this new domain is different for $0 < \alpha \leqslant \pi$ and $\pi < \alpha < 2\pi$, we will study separately the two cases in the following two subsections.

4.2.1 *Case* $0 < \alpha \leqslant \pi$

For technical reasons we construct a domain Σ in the following way: we consider two hypersurfaces defined by the equations $x_1 \tan\alpha + x_n = 0$ and $x_1 \tan\alpha - x_n = 0$, which obviously intersect at $\{x_1 = x_n = 0\}$. Then we close the domain between the two hypersurfaces with $x_1 < 0$ if $0 < \alpha < \frac{\pi}{2}$ and with $x_1 > 0$ if $\frac{\pi}{2} \leqslant \alpha \leqslant \pi$ with a smooth surface, in such a way that the scaled domain

$$\Sigma_D = D\Sigma, \tag{4.2.2}$$

defined for a large number D, contains a sufficiently large cube. In Σ_D we denote by Γ_D the singularity, which lies on $\{x_1 = x_n = 0\}$. The following figure represents a section of the domain in the plane x_1, x_n.



The advantage of dealing with this set is that if we solve a Dirichlet problem in Σ_D with data even in x_n, then for suitable boundary conditions the solution in the upper part $\Sigma_D \cap \{x_n > 0\}$ will be qualitatively similar to that of (4.2.1).

Our next goal is to consider the following problem

$$\begin{cases} -\Delta\tilde{\varphi} + \tilde{\varphi} = 0 & \text{in } \Sigma_{dD}, \\ \tilde{\varphi} = U(\cdot - dQ_0) & \text{on } \partial\Sigma_{dD}, \end{cases} \tag{4.2.3}$$

where $Q_0 = (-1, 0, \cdots, 0)$. By a scaling of variables, this problem is equivalent to

$$\begin{cases} -\frac{1}{d^2}\Delta\varphi + \varphi = 0 & \text{in } \Sigma_D, \\ \varphi = U(d(\cdot - Q_0)) & \text{on } \partial\Sigma_D, \end{cases} \tag{4.2.4}$$

Asymptotic analysis of (4.2.4)

First of all we need to know if (4.2.4) is solvable. It follows from Lemma 3.1 in [31]; in fact, making a modification of some arguments in [34], they construct barrier functions for the operators Δ and $-\Delta + 1$ at all boundary points of the set Σ. This guarantees, via the classical Perron method, the existence of a solution for the problem (4.2.4).

If we consider the function $\phi = -\frac{1}{d}\log\varphi$, then ϕ satisfies

$$\begin{cases} \frac{1}{d}\Delta\phi - |\nabla\phi|^2 + 1 = 0 & \text{in } \Sigma_D, \\ \phi = -\frac{1}{d}\log(U(d(\cdot - Q_0))) & \text{on } \partial\Sigma_D. \end{cases} \tag{4.2.5}$$

Using the limit behavior of the function U given by (4.0.3), it is easy to show the following:

Lemma 4.2.1. *For any fixed constant $D > 0$ we have that*

$$-\frac{1}{d}\log(U(d(\cdot - Q_0))) \to |\cdot - Q_0| \qquad \text{uniformly on } \partial\Sigma_D \tag{4.2.6}$$

as $d \to +\infty$.

Since Lemma 4.2.1 states that the boundary datum is everywhere close to the function $|x - Q_0|$, it is useful to consider the following auxiliary problem

$$\begin{cases} \frac{1}{d}\Delta\phi - |\nabla\phi|^2 + 1 = 0 & \text{in } \Sigma_D, \\ \phi = |x - Q_0| & \text{on } \partial\Sigma_D. \end{cases} \tag{4.2.7}$$

Lemma 4.2.2. *Let $D > 1$ be a fixed constant. Then, when $d \to \infty$, problem (4.2.7) has a unique solution ϕ^d, which is everywhere positive, and which more precisely satisfies the estimates*

$$\frac{\tan\alpha}{\sqrt{\tan^2\alpha + 1}} < \phi^d(x) < C \qquad \text{in } \Sigma_D, \tag{4.2.8}$$

if $0 < \alpha < \frac{\pi}{2}$, and

$$1 < \phi^d(x) < C \qquad \text{in } \Sigma_D, \tag{4.2.9}$$

if $\frac{\pi}{2} \leqslant \alpha \leqslant \pi$, where C depends only on D and Σ.

Proof. Applying the transformation inverse to the one at the beginning of this subsection and using the existence of barrier functions for the operator $-\Delta+1$, as shown in [31], Lemma 3.1, we get existence. Uniqueness and positivity of ϕ^d follows from the maximum principle.

To prove the estimates (4.2.8) and (4.2.9), we can reason as in [31], Lemma 3.4, or in [62], Lemma 4.2. In the case $0<\alpha<\frac{\pi}{2}$, we have that $\phi^d_-(x)\equiv\frac{\tan\alpha}{\sqrt{\tan^2\alpha+1}}$ in Σ_D is a subsolution to (4.2.7), since $\operatorname{dist}(Q_0,\partial\Sigma_D)=\frac{\tan\alpha}{\sqrt{\tan^2\alpha+1}}$; whereas, in the case $\frac{\pi}{2}\leqslant\alpha\leqslant\pi$, we have that $\operatorname{dist}(Q_0,\partial\Sigma_D)=1$, and then the subsolution is given by $\phi^d_-(x)\equiv 1$. Moreover, in both the two cases, the function $\phi^d_+(x)=C+x_1$ is a supersolution for C sufficiently large. Then our claim follows. □

We next show some pointwise bounds on ϕ^d, which in particular imply a control on the gradient within some region in the boundary of Σ_D. We obtain gradient bounds only near smooth parts of the boundary, away from the singularity Γ_D.

Lemma 4.2.3. *Let $D>1$ be as in Lemma 4.2.2. Then, there exists a constant $C>0$ such that for any $\sigma>0$ sufficiently small there exist $\bar\delta>0$ and $d_\sigma>0$ so large that*

$$|\phi^d(x)-\phi^d(z_x)|\leqslant C|x-z_x|,\qquad z_x\in\partial\Sigma_D,\quad \operatorname{dist}(z_x,D\Gamma_D)\geqslant\sigma,\quad |x-z_x|\leqslant\bar\delta,\quad d\geqslant d_\sigma.$$

In the above formula z_x denotes the point in $\partial\Sigma_D$ closest to x.

Proof. Let us first consider the case $0<\alpha<\frac{\pi}{2}$. Let us fix $\sigma>0$ small and consider, for every $0<\delta<\bar\delta=\sigma\tan\alpha$, the points $x\in\Sigma_D$ of the form $z+\delta\nu(z)$, where $z\in\partial\Sigma_D$ and $\nu(z)$ is the inner unit normal at z. Note that there is no problem in the representation of x if $\operatorname{dist}(z,D\Gamma_D)\geqslant\sigma$; whereas if $\operatorname{dist}(z,D\Gamma_D)<\sigma$, we follow the inner normal direction given by $\nu(z)$ and stop at $x_n=0$ if we reach this hyperplane at a distance from the boundary smaller than $\bar\delta$. Let us call Λ_δ this set of points $x\in\Sigma_D$ at distance δ from the boundary. Note that the Λ_δ's are all disjoint as δ varies in $[0,\bar\delta]$. Now in Λ_δ we can define the functions

$$\begin{aligned}\phi_1(x)&=|z_1(x)-Q_0|+M\delta_1(x),\\ \phi_2(x)&=|z_2(x)-Q_0|+M\delta_2(x),\end{aligned}$$

where $z_1(x)$, $z_2(x)$ are the points in $\partial\Sigma_D$ closest to x with the n-th coordinate respectively positive and negative; $\delta_1(x)$, $\delta_2(x)$ give the distance of x from $z_1(x)$, $z_2(x)$. If we set

$$\hat\phi^d_+(x)=\min\{\phi_1(x),\phi_2(x)\},$$

we choose the constant M so large that $\hat\phi^d_+(x)>\phi^d(x)$ when $x\in\{z+\bar\delta\nu(z):z\in\partial\Sigma_D\}$. The existence of such constant M is guaranteed by Lemma 4.2.2.

Next we consider a smooth function $\rho\in\mathcal{C}^\infty_0(\mathbb{R}^n)$, such that $\operatorname{supp}\rho\subset B_1(0)$, and $\int_{\mathbb{R}^n}\rho(x)\,dx=1$. Moreover we define the function

$$\lambda(x)=-\frac{2}{\bar\delta}\delta^2(x)+2\delta(x),\qquad \text{for } x\in\{z+\delta\nu(z):z\in\partial\Sigma_D,\delta\in[0,\bar\delta]\}.$$

Then we construct a mollifiers

$$\rho_{\lambda(x)}(y)=\frac{1}{\lambda^n(x)}\rho\left(\frac{y}{\lambda(x)}\right),\tag{4.2.10}$$

in such a way that the support of each $\rho_{\lambda(x)}$ depends on the point x, and, in particular, it shrinks to a point when we are close to the boundary.

Finally we regularize $\hat{\phi}^d_+$ using the convolution with the mollifiers defined in (4.2.10). Then we obtain the following smooth function

$$\phi^d_+(x) = \left(\hat{\phi}^d_+ * \rho_{\lambda(\cdot)}\right)(x) = \int_{\mathbb{R}^n} \hat{\phi}^d_+(x-y)\,\rho_{\lambda(x)}(y)\,dy.$$

It is easy to see that, for $i = 1, \dots, n-1$,

$$\frac{\partial \phi_1}{\partial x_i} < o(1) - \frac{1}{C}M, \tag{4.2.11}$$

$$\frac{\partial \phi_2}{\partial x_i} < o(1) - \frac{1}{C}M. \tag{4.2.12}$$

Moreover, using (4.2.11) and (4.2.12), we have that

$$\begin{aligned}\frac{\partial \phi^d_+}{\partial x_i} &= \int_{\mathbb{R}^n} \frac{\partial \hat{\phi}^d_+}{\partial x_i}(x-y)\,\rho_{\lambda(x)}(y)\,dy + \int_{\mathbb{R}^n} \hat{\phi}^d_+(x-y)\,\frac{\partial \rho_{\lambda(x)}}{\partial \lambda}(y)\,\frac{\partial \lambda}{\partial x_i}(x)\,dy \\ &\leqslant o(1) - \frac{1}{C}M + \int_{\mathbb{R}^n} \hat{\phi}^d_+(x-y)\,\frac{\partial \rho_{\lambda(x)}}{\partial \lambda}(y)\,\frac{\partial \lambda}{\partial x_i}(x)\,dy. \end{aligned} \tag{4.2.13}$$

Now we need an estimate for the last term in (4.2.13), let us call it A. If we add and subtract $\hat{\phi}^d_+(x)$ in the integral, we obtain

$$\begin{aligned} A &= \int_{\mathbb{R}^n} \hat{\phi}^d_+(x)\,\frac{\partial \rho_{\lambda(x)}}{\partial \lambda}(y)\,\frac{\partial \lambda}{\partial x_i}(x)\,dy + \int_{\mathbb{R}^n} \left[\hat{\phi}^d_+(x-y) - \hat{\phi}^d_+(x)\right]\frac{\partial \rho_{\lambda(x)}}{\partial \lambda}(y)\,\frac{\partial \lambda}{\partial x_i}(x)\,dy \\ &= \frac{\partial \lambda}{\partial x_i}(x)\int_{\mathbb{R}^n}\left[\hat{\phi}^d_+(x-y) - \hat{\phi}^d_+(x)\right]\frac{\partial \rho_{\lambda(x)}}{\partial \lambda}(y)\,dy; \end{aligned}$$

in the last step we have used the fact that $\hat{\phi}^d_+(x)$ and $\frac{\partial \lambda}{\partial x_i}(x)$ do not depend on y, and the fact that $\int_{\mathbb{R}^n} \frac{\partial \rho_{\lambda(x)}}{\partial \lambda}(y)\,dy = \frac{\partial}{\partial \lambda}\int_{\mathbb{R}^n} \rho_{\lambda(x)}(y)\,dy = 0$, since $\int_{\mathbb{R}^n} \rho_{\lambda(x)}(y)\,dy = 1$, for every $\lambda > 0$. Now, from (4.2.10), a simple computation yields

$$\frac{\partial \rho_{\lambda(x)}}{\partial \lambda}(y) = -n\lambda^{-n-1}(x)\,\rho\left(\frac{y}{\lambda(x)}\right) - \lambda^{-n-2}(x)\,y\nabla\rho\left(\frac{y}{\lambda(x)}\right).$$

Then, using the fact that $\frac{\partial \lambda}{\partial x_i}(x) \simeq -Cx_i$, for some positive constant C, and making the change of variable $y = \lambda(x)z$, we have

$$A = C\lambda^{-1}(x)\,x_i \int_{\mathbb{R}^n}\left[\hat{\phi}^d_+(x - \lambda(x)z) - \hat{\phi}^d_+(x)\right]\cdot\left[\rho(z) + z\nabla\rho(z)\right]dz. \tag{4.2.14}$$

Since $\hat{\phi}^d_+$ is a Lipschitz function, from (4.2.14) we get that

$$A \leqslant Cx_i \int_{\mathbb{R}^n} |z|\cdot\left[\rho(z) + z\nabla\rho(z)\right]dz,$$

and then $A \leqslant o(1)$. It follows that, for M sufficiently large, the norm of $\nabla\phi^d_+$ can be arbitrarily big on its domain. By (4.2.8), if M is large then ϕ^d_+ is everywhere bigger than ϕ^d on $\Sigma_D \cap \left\{\mathrm{dist}(\cdot, \partial\Sigma_D) = \bar{\delta}\right\}$, so ϕ^d_+ is a supersolution of (4.2.7) in $\Sigma_D \cap \left\{\mathrm{dist}(\cdot, \partial\Sigma_D) < \bar{\delta}\right\}$.

On the other hand, we claim that the function $\phi^d_-(x) = |x - Q_0|$ is a subsolution of (4.2.7) in $\Sigma_D \cap \left\{\mathrm{dist}(\cdot, \partial\Sigma_D) < \bar{\delta}\right\}$. In fact, if we consider the set $\Sigma_D \setminus B_{\tilde{\delta}(d)}(Q_0)$, where $\tilde{\delta}(d)$ is a small positive number depending on d, we can see by easy computation that here ϕ^d_- satisfies

$$\frac{1}{d}\Delta\phi^d_- - |\nabla\phi^d_-|^2 + 1 = \frac{n-1}{d|x - Q_0|}.$$

Moreover, since ϕ^d is positive, we can choose $\tilde{\delta}(d)$ sufficiently small so that $\phi^d_- < \phi^d$. Hence we obtain that $\phi^d_- \leqslant \phi^d$ in the closure of $\Sigma_D \cap \{\operatorname{dist}(\cdot, \partial\Sigma_D) < \tilde{\delta}\}$.

Finally, the conclusion follows from the fact that ϕ^d_- and ϕ^d_+ coincide on the set

$$\{x \in \partial\Sigma_D : \operatorname{dist}(x, D\Gamma_D) \geqslant \sigma\}$$

and that we have uniform bounds on the gradient here, independently on d.

In the case $\frac{\pi}{2} \leqslant \alpha \leqslant \pi$, we can repeat essentially the same construction of the proof of Lemma 3.5 in [31] and obtain the same conclusion. □

Using the same arguments as in Lemma 3.6 in [31] we are able to extend the gradient estimate which follows from the previous lemma to a subset of the interior of the domain.

Lemma 4.2.4. *Let $D > 1$ be as in Lemma 4.2.2. Then, there exists a constant $C > 0$ such that for any $\sigma > 0$ sufficiently small there exists $d_\sigma > 0$ so large that*

$$|\nabla\phi^d(x)| \leqslant C \qquad \text{in } \{x \in \overline{\Sigma}_D : \operatorname{dist}(x, D\Gamma_D) \geqslant \sigma\}, \quad d \geqslant d_\sigma. \tag{4.2.15}$$

The next proposition is about the asymptotic behavior of the solutions of (4.2.7).

Lemma 4.2.5. *Let ϕ^d be the solution of (4.2.7), then we have that*

$$\phi^d(x) \to \overline{\phi}(x) := \inf_{z \in \partial\Sigma_D} (|x - z| + |z - Q_0|), \qquad \text{as } d \to \infty, \tag{4.2.16}$$

uniformly on the compact sets of $\overline{\Sigma}_D$.

Proof. We will show (4.2.16) in two steps:

1) we prove that the function on the right-hand side of (4.2.16) is the supremum of all the elements of

$$F = \{v \in W^{1,\infty}(\Sigma_D) : v(x) \leqslant |x - Q_0| \text{ on } \partial\Sigma_D, |\nabla v| \leqslant 1 \text{ a.e. in } \Sigma_D\};$$

2) we prove that for any sequence $d_k \to \infty$, there is a subsequence $d_{k_l} \to \infty$ such that $\phi^{d_{k_l}} \to \overline{\phi}$ uniformly on the compact sets of $\overline{\Sigma}_D$ as $d_{k_l} \to \infty$. Then it follows that $\phi^d \to \overline{\phi}$ uniformly on the compact sets of $\overline{\Sigma}_D$ as $d \to \infty$.

We first prove 1). To begin we show that $\overline{\phi} \in F$. If $x_1, x_2 \in \Sigma_D$ and $z_2 \in \partial\Sigma_D$ realizes the infimum for x_2, we have

$$|\overline{\phi}(x_1) - \overline{\phi}(x_2)| \leqslant ||x_1 - z_2| + |z_2 - Q_0| - |x_2 - z_2| - |z_2 - Q_0|| \leqslant |x_1 - x_2|.$$

Then, taking x_1, x_2 close, we get $\overline{\phi} \in W^{1,\infty}(\Sigma_D)$ and $|\nabla\overline{\phi}| \leqslant 1$ a. e. in Σ_D. Moreover, it is easy to see that $\overline{\phi}(x) = |x - Q_0|$ if $x \in \partial\Sigma_D$. We next show that $\overline{\phi}$ is the maximum element of F. We construct a δ neighborhood Σ_D^δ of Σ_D in this way: consider $Q_0 = (-1, 0, \cdots, 0)$ and, for every $z \in \partial\Sigma_D$, the line from Q_0 to z. If $\delta > 0$ is small enough, each point x in $\Sigma_D^\delta \setminus \Sigma_D$ is uniquely determined by the equation $x = z + \overline{\delta} r(z)$, where $z \in \partial\Sigma_D$ is the intersection point of the line from Q_0 to x with $\partial\Sigma_D$, $r(z)$ is the unit outer vector on the line, and $0 < \overline{\delta} < \frac{\delta}{\cos\theta(z)}$; here $\theta(z)$ is the angle between $r(z)$ and the unit outer normal at z, $\nu(z)$, in the plane generated by $r(z)$ and $\nu(z)$. Note that for the point on the boundary $z \in \{z_1 = z_n = 0\}$ we can consider $\nu(z)$ just taking the normal to the hypersurface defined by the equation $x_1 \tan\alpha + x_n = 0$ or to the one defined by the equation $x_1 \tan\alpha - x_n = 0$, and it is well defined since the angle $\theta(z)$ is the same for those points. In addition, the map $x \to (z, \overline{\delta})$ is continuous in $\Sigma_D^\delta \setminus \Sigma_D$.

Now, we can extend every $v \in F$ to a $\tilde{v} \in W^{1,\infty}(\Sigma_D^\delta)$, taking $v = \tilde{v}$ in Σ_D and $\tilde{v}(x) = v(z)$ for $x \in \Sigma_D^\delta \setminus \Sigma_D$. Moreover, if we consider the function

$$\tilde{K}(x) = \begin{cases} 1 & \text{in } \Sigma_D, \\ 1 + C\overline{\delta} & \text{in } \Sigma_D^\delta \setminus \Sigma_D, \end{cases}$$

for some large constant $C > 0$ independent of δ, we get $|\nabla \tilde{v}| \leqslant \tilde{K}$ a. e. in Σ_D^δ. Now, we regularize $\tilde{v}$ using the convolution with mollifiers, that is considering, for $\lambda > 0$ small enough, $v_\lambda := \tilde{v} * \rho_\lambda$, with $\rho_\lambda(x) = \lambda^{-n}\rho(x/\lambda)$, $\rho \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $\operatorname{supp}\rho \subset B_1(0)$, $\int_{\mathbb{R}^n} \rho(x)\,dx = 1$. Then we have

$$|\nabla v_\lambda| \leqslant |\nabla \tilde{v}| * \rho_\lambda \leqslant \tilde{K} * \rho_\lambda \leqslant 1 + C\lambda$$

on Σ_D and $v_\lambda \to v$ in $\mathcal{C}(\Sigma_D)$ as $\lambda \to 0$. Let now $x, y \in \Sigma_D$ and consider the function $\xi(t) = tx + (1-t)y$, for $t \in [0,1]$; then we can estimate

$$|v_\lambda(x) - v_\lambda(y)| \leqslant \int_0^1 |\nabla v_\lambda(\xi(t))| \cdot |\frac{d\xi}{dt}| dt \leqslant \int_0^1 |1 + C\lambda| \cdot |x-y| dt \leqslant (1 + C\lambda) \cdot |x-y|.$$

Letting $\lambda \to 0$, we obtain $|v(x) - v(y)| \leqslant |x-y|$. Hence $v(x) \leqslant v(y) + |x-y|$, and $v(x) \leqslant |y - Q_0| + |x-y|$ for all $y \in \partial\Sigma_D$. So $v \leqslant \overline{\phi}$.

We next prove 2). By gradient estimate and the Ascoli-Arzelà theorem we know that the ϕ^d's admit limit $\hat{\phi}$ in the whole closure of Σ_D. Moreover it is easy to see that $\hat{\phi}$ belong to the set F; hence $\hat{\phi} \leqslant \overline{\phi}$. We need then to prove only $\overline{\phi} \leqslant \hat{\phi}$. Let $v \in F$. Similarly to 1), we extend v to $\tilde{v}$ in Σ_D^δ and regularize $\tilde{v}$ to v_λ in such a way that we have $\|v - v_\lambda\|_{L^\infty(\Sigma_D)} \leqslant C\lambda$ and $|\nabla \tilde{v}| \leqslant \tilde{K}$. Hence as before we get $|\nabla v_\lambda| \leqslant 1 + C\lambda$ on Σ_D and $v_\lambda \to v$ in $\mathcal{C}(\Sigma_D)$ as $\lambda \to 0$. By simple computation we obtain that v_λ satisfies

$$\begin{cases} \frac{1}{d}\Delta v_\lambda - |\nabla v_\lambda|^2 + 1 + C\lambda + \frac{1}{d}A_\lambda \geqslant 0 & \text{in } \Sigma_D, \\ v_\lambda \leqslant |x - Q_0| + C\lambda & \text{on } \partial\Sigma_D, \end{cases}$$

where $A_\lambda \geqslant 0$. If we define

$$\tilde{v}_\lambda := \frac{v_\lambda}{\sqrt{1 + C\lambda + \frac{1}{d}A_\lambda}},$$

by comparison we deduce that

$$\tilde{v}_\lambda \leqslant \phi^{d\sqrt{1+C\lambda+\frac{1}{d}A_\lambda}} + C\lambda. \tag{4.2.17}$$

Choosing $d = d'_{k_l}$ in (4.2.17) such that

$$d_{k_l} = d'_{k_l}\sqrt{1 + C\lambda + \frac{1}{d'_{k_l}}A_\lambda},$$

we see that

$$\frac{v_\lambda}{\sqrt{1 + C\lambda}} \leqslant \hat{\phi} + C\lambda$$

as $d'_{k_l} \to \infty$. Then, letting $\lambda \to 0$, we obtain $v \leqslant \hat{\phi}$; in particular, $\overline{\phi} \leqslant \hat{\phi}$. Hence $\overline{\phi} = \hat{\phi}$. □

Next we analyze the asymptotic behavior of the solutions of (4.2.5). From now on in this subsection we study separately the two cases $0 < \alpha < \frac{\pi}{2}$ and $\frac{\pi}{2} \leqslant \alpha \leqslant \pi$. Let us consider the first case.

Proposition 4.2.6. *Suppose that $0 < \alpha < \frac{\pi}{2}$. Let D be a large fixed constant and Φ^d the solution of (4.2.5). Then we have*

$$\Phi^d(x) \to \min\{d_1(x), d_2(x)\}, \qquad \text{as } d \to \infty,$$

uniformly on the compact sets of $\overline{\Sigma}_D \cap \bar{B}_{\frac{D}{4}}(0)$, where

$$d_1(x) := \sqrt{\left(x_1 - \frac{\tan^2\alpha - 1}{\tan^2\alpha + 1}\right)^2 + |x''|^2 + \left(x_n - \frac{2\tan\alpha}{\tan^2\alpha + 1}\right)^2}, \tag{4.2.18}$$

$$d_2(x) := \sqrt{\left(x_1 - \frac{\tan^2\alpha - 1}{\tan^2\alpha + 1}\right)^2 + |x''|^2 + \left(x_n + \frac{2\tan\alpha}{\tan^2\alpha + 1}\right)^2}. \tag{4.2.19}$$

Remark 4.2.7. Note that d_1 and d_2 are the distance functions, respectively, from the point $Q_1 = \left(\frac{\tan^2\alpha-1}{\tan^2\alpha+1}, 0, \cdots, 0, \frac{2\tan\alpha}{\tan^2\alpha+1}\right)$, which is the symmetrical point to Q_0 with respect to the hypersurface defined by the equation $x_1 \tan\alpha + x_n = 0$, and from the point $Q_2 = \left(\frac{\tan^2\alpha-1}{\tan^2\alpha+1}, 0, \cdots, 0, -\frac{2\tan\alpha}{\tan^2\alpha+1}\right)$, which is the symmetrical point to Q_0 with respect to the hypersurface defined by the equation $x_1 \tan\alpha - x_n = 0$. So the function $\overline{\phi}(x)$ is even with respect to the coordinate x_n and a.e. differentiable. The problem is that it does not have zero x_n-derivative on $\{x_n = 0\}$.

Proof. If ϕ^d is a solution of (4.2.7), it is easy to see that

$$\phi^d + \sup_{x \in \partial\Sigma_D} ||x - Q_0| + \frac{1}{d}\log(U(d(x - Q_0)))|$$

is a supersolution of (4.2.5) and

$$\phi^d - \sup_{x \in \partial\Sigma_D} ||x - Q_0| + \frac{1}{d}\log(U(d(x - Q_0)))|$$

is a subsolution. Then Φ^d must lie in between these two functions. Hence, by Lemma 4.2.1, it is sufficient to prove the analogous statement for ϕ^d. The proof of the latter fact is a consequence of Lemma 4.2.5 and the following Lemma 4.2.8. ☐

Lemma 4.2.8. *Suppose that $0 < \alpha < \frac{\pi}{2}$. If $\overline{\phi}(x)$ is as in (4.2.16), then*

$$\overline{\phi}(x) = \min\{d_1(x), d_2(x)\}, \qquad x \in \bar{B}_{\frac{D}{4}}(0),$$

where d_1 and d_2 are as in (4.2.18) and (4.2.19).

Proof. Consider a point $x = (x_1, \cdots, x_n)$ with $x_n \geqslant 0$.

By construction of Σ_D, the point $z \in \partial\Sigma_D$ which realizes the infimum will necessarily belong to the set $\{\{x_1 \tan\alpha + x_n = 0\} \cap \{x_1 < 0\}\}$. This implies that

$$\overline{\phi}(x) = \inf_{z \in \{\{x_1 \tan\alpha + x_n = 0\} \cap \{x_1 < 0\}\}} (|x - z| + |z - Q_0|).$$

Now we can reason as follows: given x, the level sets of the function $z \to |x-z|+|z-Q_0|$ are the axially symmetric ellipsoids with focal points x and Q_0. The smaller is the ellipsoid, the smaller is the value of this function; so we are reduced to find the smallest ellipsoid which intersects $\{\{x_1 \tan\alpha + x_n = 0\} \cap \{x_1 < 0\}\}$. We note that if we fix x_1, x_n and vary only x'', the corresponding infimum z has the same z_1, z_n and different z''; so we can determine z_1, z_n in the simplest case $x'' = (0, \cdots, 0)$, and obviously $z'' = (0, \cdots, 0)$. Then we are reduced to consider the minimum problem

$$\min_{(z_1,z_n)\in\{\{x_1 \tan\alpha+x_n=0\}\cap\{x_1<0\}\}} \left(\sqrt{(x_1-z_1)^2+(x_n+\tan\alpha z_1)^2}+\sqrt{(z_1+1)^2+\tan^2\alpha z_1^2}\right).$$

Deriving with respect to the variable z_1 we obtain that at a minimum point

$$\frac{-(x_1-z_1)+\tan\alpha\,(x_n+\tan\alpha z_1)}{\sqrt{(x_1-z_1)^2+(x_n+\tan\alpha z_1)^2}}+\frac{(z_1+1)+\tan^2\alpha z_1}{\sqrt{(z_1+1)^2+\tan^2\alpha z_1^2}}=0,$$

which implies

$$z_1 = \frac{-2\tan\alpha x_1 + \left(\tan^2\alpha - 1\right)x_n}{\left(\tan^2\alpha+1\right)\left(\tan\alpha x_1 + x_n - \tan\alpha\right)}, \tag{4.2.20}$$

$$z_n = \frac{2\tan^2\alpha x_1 - \tan\alpha\left(\tan^2\alpha - 1\right)x_n}{\left(\tan^2\alpha+1\right)\left(\tan\alpha x_1 + x_n - \tan\alpha\right)}. \tag{4.2.21}$$

Now assume that $x'' \neq (0, \cdots, 0)$ and x_1, x_n are as before. By the previous observation we know that the coordinates z_1, z_n of the corresponding infimum are given by (4.2.20) and (4.2.21). So we have to determine only z''. To do this let us consider the minimum problem

$$\min_{z''\in\mathbb{R}^{n-2}} \left(\sqrt{(x_1-z_1)^2+|x''-z''|^2+(x_n+\tan\alpha z_1)^2}+\sqrt{(z_1+1)^2+|z''|^2+\tan^2\alpha z_1^2}\right). \tag{4.2.22}$$

Again by differentiation we obtain that a minimum point must satisfy

$$\frac{z''-x''}{\sqrt{(x_1-z_1)^2+|x''-z''|^2+(x_n+\tan\alpha z_1)^2}}+\frac{z''}{\sqrt{(z_1+1)^2+|z''|^2+\tan^2\alpha z_1^2}}=0,$$

which gives

$$z'' = x''\frac{\sqrt{(z_1+1)^2+\tan^2\alpha z_1^2}}{\sqrt{(x_1-z_1)^2+(x_n+\tan\alpha z_1)^2}+\sqrt{(z_1+1)^2+\tan^2\alpha z_1^2}}. \tag{4.2.23}$$

If we plug (4.2.20), (4.2.21) and (4.2.23) into (4.2.22), we obtain that $\overline{\phi}(x) = d_1(x)$. Reasoning in the same way for points with $x_n < 0$, we have $\overline{\phi}(x) = d_2(x)$. Then we get the conclusion. □

Remark 4.2.9. Note that $\overline{\phi}$ is a viscosity solution of the Hamilton-Jacobi equation $|\nabla\phi|^2 = 1$ in Σ_D. In fact, what we have to show is that

i) $|p|^2 \leqslant 1$, for every $x \in \Sigma_D$ and every $p \in D^+\overline{\phi}(x)$,

ii) $|p|^2 \geqslant 1$, for every $x \in \Sigma_D$ and every $p \in D^-\overline{\phi}(x)$, where $D^+\overline{\phi}(x)$ and $D^-\overline{\phi}(x)$ are respectively the superdifferential and the subdifferential of $\overline{\phi}$ at x. Now we can use the description of $D^+\overline{\phi}(x)$ and $D^-\overline{\phi}(x)$ given in Theorem 3.4.4 in [15]: let $\Omega \subset \mathbb{R}^n$ be open and $S \subset \mathbb{R}^m$ be compact; let $F = F(s,x)$ be continuous in $S \times \Omega$ together with its partial derivative $D_x F$, and let us define $u(x) = \min_{s \in S} F(s,x)$; given $x \in \Omega$, let us set

$$M(x) = \{s \in S : u(x) = F(s,x)\}, \qquad Y(x) = \{D_x F(s,x) : s \in M(x)\}.$$

Then, for any $x \in \Omega$,

$$D^+ u(x) = \mathrm{co}\,(Y(x)), \tag{4.2.24}$$

and

$$D^- u(x) = \begin{cases} \{p\} & \text{if } Y(x) = p, \\ \emptyset & \text{if } Y(x) \text{ is not a singleton.} \end{cases} \tag{4.2.25}$$

Now we can take $\Omega = \Sigma_D$, $S = \{Q_1, Q_2\}$ and $\overline{\phi}(x) = \min_{i \in \{1,2\}}\{d_i(x)\}$; so

$$M(x) = \left\{Q_i : \overline{\phi}(x) = d_i(x)\right\}, \qquad Y(x) = \{D_x d_i(x) : Q_i \in M(x)\}.$$

Then, using (4.2.24) and (4.2.25), it is easy to see that, if we take $x \in \Sigma_D$ with $x_n > 0$, then $D^+\overline{\phi}(x) = D^-\overline{\phi}(x) = \{D_x d_1(x)\}$; in the same way, if $x_n < 0$, then $D^+\overline{\phi}(x) = D^-\overline{\phi}(x) = \{D_x d_2(x)\}$. So in these two cases properties i), ii) are trivially verified. In the case $x_n = 0$, we have that $\overline{\phi}(x) = d_1(x) = d_2(x)$; then $M(x) = \{Q_1, Q_2\}$ and $Y(x) = \{D_x d_1(x), D_x d_2(x)\}$. Hence, using again (4.2.24), (4.2.25), we obtain $D^+\overline{\phi}(x) = \mathrm{co}\left\{\frac{x-Q_1}{d_1(x)}, \frac{x-Q_2}{d_2(x)}\right\} = \frac{x-\mathrm{co}\{Q_1,Q_2\}}{\overline{\phi}(x)}$ and $D^-\overline{\phi}(x) = \emptyset$. Then we have only to prove property i), since ii) is again trivially verified. To show i) it is sufficient to observe that every $p \in D^+\overline{\phi}(x)$ is of the form $p = \frac{x-Q}{\overline{\phi}(x)}$, where Q belongs to the line joining Q_1 to Q_2, and that $|x - Q| \leqslant \overline{\phi}(x)$.

Let us consider now the case $\frac{\pi}{2} \leqslant \alpha \leqslant \pi$. We have the analogous of the Proposition 4.2.6.

Proposition 4.2.10. *Suppose that $\frac{\pi}{2} \leqslant \alpha \leqslant \pi$. Let D be a large fixed constant and Φ^d the solution of* (4.2.5)*. Then we have*

$$\Phi^d(x) \to \bar{\Phi}(x), \qquad \text{as } d \to \infty, \tag{4.2.26}$$

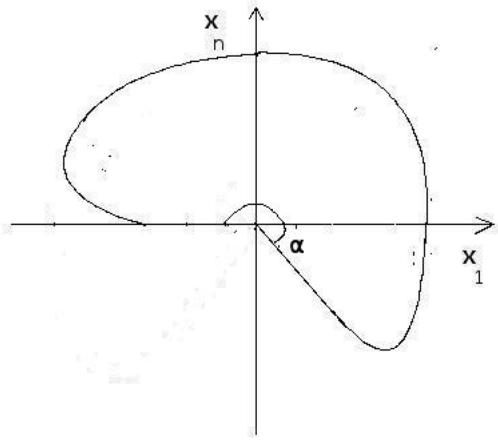
uniformly on the compact sets of $\overline{\Sigma}_D \cap \bar{B}_{\frac{D}{4}}(0)$, where

$$\bar{\Phi}(x) = \begin{cases} \min\{d_1(x), d_2(x)\}, & \text{if } \tan\alpha \leqslant \frac{x_1 - \sqrt{x_1^2 + x_n^2}}{x_n}, \\ \sqrt{\left(1 + \sqrt{x_1^2 + x_n^2}\right)^2 + |x''|^2}, & \text{if } \tan\alpha \geqslant \frac{x_1 - \sqrt{x_1^2 + x_n^2}}{x_n}. \end{cases} \tag{4.2.27}$$

Proof. We can reason as in the proof of Proposition 4.2.6, obtaining that it is sufficient to show the convergence in (4.2.26) for the function ϕ^d. To prove the latter assertion we have to use Lemma 4.2.5, together with the fact that in the case $\frac{\pi}{2} \leqslant \alpha \leqslant \pi$ the function $\overline{\phi}$ defined in (4.2.16) is equal to that one defined in (4.2.27). We can obtain this expression by mixing the arguments used in the proof of Lemma 4.2.8 and those used in Lemma 3.9 in [31]. □

4.2.2 *Case $\pi < \alpha < 2\pi$*

In this case we construct the domain Σ in the following way: we consider the set $\{x_n = 0\} \cap \{x_1 \leqslant 0\}$ and the hypersurface defined by the equation $x_1 \tan\alpha + x_n = 0$ with $x_n \leqslant 0$. Then we close the domain with a smooth surface; the following figure represents a section of the domain in the plane x_1, x_n.



We define the scaled domain Σ_D as in (4.2.2) and denote by Γ_D the singularity, which lies on $\{x_1 = x_n = 0\}$. As in the previous case, the solution of a Dirichlet problem in Σ_D will be qualitatively similar to that of (4.2.1).

We have to study the asymptotic behavior of the solution of the problem

$$\begin{cases} -\frac{1}{d^2}\Delta\varphi + \varphi = 0 & \text{in } \Sigma_D, \\ \varphi = U\left(d\left(\cdot - Q_0\right)\right) & \text{on } \partial\Sigma_D, \end{cases}$$

To do this we consider the function $\phi = -\frac{1}{d}\log\varphi$, which satisfies

$$\begin{cases} \frac{1}{d}\Delta\phi - |\nabla\phi|^2 + 1 = 0 & \text{in } \Sigma_D, \\ \phi = -\frac{1}{d}\log\left(U\left(d\left(\cdot - Q_0\right)\right)\right) & \text{on } \partial\Sigma_D. \end{cases} \tag{4.2.28}$$

Since the asymptotic analysis is very similar to that one made in Subsection 4.2.1 for $0 < \alpha \leqslant \pi$ we will not repeat the computations. What we obtain is the following result:

Proposition 4.2.11. *Suppose that $\pi < \alpha < 2\pi$. Let D be a large fixed constant and Φ^d the solution of (4.2.28). Then we have*

$$\Phi^d(x) \to \operatorname{dist}(x, Q_0) = \sqrt{(x_1+1)^2 + |x'|^2}, \qquad \text{as } d \to \infty,$$

uniformly on the compact sets of $\overline{\Sigma}_D \cap \bar{B}_{\frac{D}{4}}(0)$.

4.2.3 *Definition of the approximate solutions*

In order to apply the theory in Section 2.1, in this subsection we construct a manifold of approximate solutions to (4.1.1). Since the limit function of the solutions of (4.2.5) is not the same for different angles α, as we have seen in Subsections 4.2.1 and 4.2.2, we will distinguish the cases. We will give the precise construction only for $0 < \alpha < \frac{\pi}{2}$; in fact in this case the computations are quite different from the flat case $\alpha = \pi$. In the other cases the estimates for the approximate solutions are the same (for $\frac{\pi}{2} \leqslant \alpha \leqslant \pi$) or very similar (for $\pi < \alpha < 2\pi$) to that ones obtained in [31], Subsection 3.2, and then we will omit the proofs.

Case $0 < \alpha < \frac{\pi}{2}$

Since the function $\bar{u}_{\epsilon,Q}$ defined in Subsection 4.1.1 is an approximate solution of (4.1.1) with pure Neumann boundary conditions, we need to modify it in the following way. If Φ^d the solution of (4.2.5), the function

$$\Xi_d(y) = e^{-d\Phi^d\left(\frac{y}{d}+Q_0\right)} \tag{4.2.29}$$

solves the problem

$$\begin{cases} -\Delta \Xi_d + \Xi_d = 0 & \text{in } d\left(\Sigma_D - Q_0\right), \\ \Xi_d = U(\cdot) & \text{on } d\left(\partial\Sigma_D - Q_0\right). \end{cases} \tag{4.2.30}$$

We can obtain a solution to (4.2.30) looking at the minimum problem

$$\inf_{v = U \text{ on } d(\partial\Sigma_D - Q_0)} \left\{ \int_{d(\Sigma_D - Q_0)} \left(|\nabla v|^2 + v^2\right) dy \right\}. \tag{4.2.31}$$

From (4.2.31) we can derive norm estimate on Ξ_d. In fact, we can take a cut-off function $\chi_1 : d\left(\overline{\Sigma}_D - Q_0\right) \to \mathbb{R}$ such that

$$\begin{cases} \chi_1(y) = 1 & \text{for } \operatorname{dist}\left(y, d\left(\partial\Sigma_D - Q_0\right)\right) \leqslant \frac{1}{2}, \\ \chi_1(y) = 0 & \text{for } y \in d\left(\Sigma_D - Q_0\right),\ \operatorname{dist}\left(y, d\left(\partial\Sigma_D - Q_0\right)\right) \geqslant 1, \\ |\nabla \chi_1(y)| \leqslant 4 & \text{for all } y, \end{cases}$$

and then consider the function $\bar{v}(y) = \chi_1(y)\, U(y)$. It is easy to see that $\|\bar{v}\|_{H^1(d(\Sigma_D - Q_0))} \leqslant e^{-d(1+o(1))}$, so by (4.2.31) we find that

$$\|\Xi_d\|_{H^1(d(\Sigma_D - Q_0))} \leqslant \|\bar{v}\|_{H^1(d(\Sigma_D - Q_0))} \leqslant e^{-d(1+o(1))}. \tag{4.2.32}$$

We can also obtain pointwise estimates on Ξ_d. In fact, from Proposition 4.2.6 we obtain that, as $d \to +\infty$,

$$\Xi_d(y) = \exp\left[-\min\left\{\sqrt{\left(y_1 - d - \frac{d\left(\tan^2\alpha - 1\right)}{\tan^2\alpha + 1}\right)^2 + |y''|^2 + \left(y_n \mp \frac{2d\tan\alpha}{\tan^2\alpha + 1}\right)^2}\right\}\right] \cdot e^{o(d)}, \tag{4.2.33}$$

for $y \in d\left(V - Q_0\right)$, where V is any set compactly contained in $\overline{\Sigma}_D$. Finally, we have pointwise estimates for the gradient of Ξ_d. Indeed, using the uniform convergence in (4.2.6) and reasoning as in the proof of Lemmas 4.2.3 and 4.2.4, we obtain that (4.2.15) holds true also for Φ_d. Then we can apply the arguments in [46] (see in particular Proposition 1.4, Lemma 1.5

and Lemma B.1) to conclude that $\nabla\Phi_d \to \nabla\overline{\phi}$ uniformly as $d \to +\infty$ in any set compactly contained in $\overline{\Sigma}_D$ on which $\nabla\overline{\phi}$ is defined. This convergence implies that, as $d \to +\infty$,

$$\begin{aligned}
&\nabla\Xi_d(y) = \\
&\quad -\exp\left[-\min\left\{\sqrt{\left(y_1 - d - \frac{d\left(\tan^2\alpha - 1\right)}{\tan^2\alpha + 1}\right)^2 + |y''|^2 + \left(y_n \mp \frac{2d\tan\alpha}{\tan^2\alpha + 1}\right)^2}\right\}\right] \\
&\quad \cdot e^{o(d)} \cdot \left(\nabla\overline{\phi}\left(\frac{y}{d} + Q_0\right) + o(1)\right),
\end{aligned} \tag{4.2.34}$$

for $y \in d\,(V - Q_0)$, where V is as before.

Now, we want to obtain similar bounds and estimates for $\frac{\partial\Xi_d}{\partial d}$ and its gradient. Using the definition of $\Xi_d(y) = \varphi\left(\frac{y}{d} + Q_0\right)$ and the fact that also φ depends on d, we have that

$$\frac{\partial\Xi_d}{\partial d}(y) = \frac{\partial\varphi}{\partial d}\left(\frac{y}{d} + Q_0\right) - \frac{y}{d^2} \cdot \nabla\varphi\left(\frac{y}{d} + Q_0\right). \tag{4.2.35}$$

Since φ is the solution of (4.2.4), we can differentiate (4.2.4) obtaining

$$\begin{cases} -\frac{1}{d^2}\Delta\frac{\partial\varphi}{\partial d} + \frac{\partial\varphi}{\partial d} = -\frac{2}{d^3}\Delta\varphi = -\frac{2}{d}\varphi & \text{in } \Sigma_D, \\ \frac{\partial\varphi}{\partial d}(x) = \nabla U\left(d\,(x - Q_0)\right) \cdot (x - Q_0) & \text{on } \partial\Sigma_D, \end{cases} \tag{4.2.36}$$

Because of the asymptotic behavior of U at infinity, there exists a positive constant C_D such that for d large we have

$$\frac{1}{C_D}U\left(d\,(x - Q_0)\right) \leqslant -\nabla U\left(d\,(x - Q_0)\right) \cdot (x - Q_0) \leqslant C_D U\left(d\,(x - Q_0)\right). \tag{4.2.37}$$

Hence, from (4.2.4), (4.2.37), the fact that $\varphi > 0$ and the maximum principle we obtain that $\sigma := -\frac{\partial\varphi}{\partial d} \geqslant \frac{1}{C_D}\varphi$ in $\widehat{\Gamma}_D$. Moreover, as for (4.2.5) we can check that the function $\Upsilon^d := -\frac{1}{d}\log\sigma$ satisfies

$$\begin{cases} \frac{1}{d}\Delta\Upsilon^d + |\nabla\Upsilon^d|^2 + 1 - \frac{\varphi}{d\sigma} = 0 & \text{in } \Sigma_D, \\ \Upsilon^d(x) = -\frac{1}{d}\log\left(-\nabla U\left(d\,(x - Q_0)\right) \cdot (x - Q_0)\right) & \text{on } \partial\Sigma_D, \end{cases} \tag{4.2.38}$$

Since $\frac{\varphi}{\sigma}$ stays bounded, $\frac{\varphi}{d\sigma}$ tends to zero as $d \to +\infty$. Moreover, using again the asymptotic behavior of U at infinity, we can say that the boundary datum in (4.2.38) converges in every smooth sense (where $\partial\Sigma_D$ is regular) to $|x - Q_0|$ as $d \to +\infty$. As a consequence, the previous analysis adapts to Υ^d and allows to conclude that still

$$\Upsilon^d \to \overline{\phi} \quad \text{and} \quad \nabla\Upsilon^d \to \nabla\overline{\phi} \tag{4.2.39}$$

uniformly as $d \to +\infty$ in any set compactly contained in $\overline{\Sigma}_D$ on which $\nabla\overline{\phi}$ is defined.

From (4.2.36), reasoning as for (4.2.32), we have that

$$\left\|\frac{\partial\varphi}{\partial d}\left(\frac{\cdot}{d} + Q_0\right)\right\|_{H^1(d(\Sigma_D - Q_0))} \leqslant e^{-d(1+o(1))}. \tag{4.2.40}$$

On the other hand, from (4.2.30) one finds that the function $\varpi := \frac{y}{d^2} \cdot \nabla\varphi\left(\frac{y}{d} + Q_0\right) = \frac{y}{d} \cdot \nabla\Xi_d(y)$ satisfies

$$-\Delta\varpi + \varpi = -\frac{2}{d}\Xi_d \qquad \text{in } d\,(\Sigma_D - Q_0).$$

To control the boundary value of ϖ we divide $\partial d\left(\Sigma_D - Q_0\right)$ into its intersection with $\{y_n = 0\}$ and its complement. In the first region we have simply that $\varpi = \frac{y}{d} \cdot \nabla U\left(y\right)$. In the second instead the estimates in (4.2.33) and (4.2.34) hold true, which shows that the L^2 norm of the trace of ϖ on $\partial d\left(\Sigma_D - Q_0\right)$ is of order $e^{-d(1+o(1))} + e^{-d\left[1+\frac{2\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right](1+o(1))}$. This fact and the latter formula imply that

$$\|\varpi\|_{H^1(d(\Sigma_D - Q_0))} \leqslant e^{-d(1+o(1))} + e^{-d\left[1+\frac{2\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right](1+o(1))}. \tag{4.2.41}$$

Then, from (4.2.40) and (4.2.41), we conclude that

$$\left\|\frac{\partial \Xi_d}{\partial d}\right\|_{H^1(d(\Sigma_D - Q_0))} \leqslant e^{-d(1+o(1))} + e^{-d\left[1+\frac{2\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right](1+o(1))}. \tag{4.2.42}$$

Now, using the fact that $\varphi \leqslant C_D|\frac{\partial\varphi}{\partial d}|$ and (4.2.39), together with the Harnack inequality (which implies $|\nabla\varphi| \leqslant Cd\varphi$ in $d\left(V - Q_0\right)$) one also finds

$$\begin{aligned}\frac{\partial \Xi_d}{\partial d}\left(y\right) \quad &= \quad -\exp\left[-\min\left\{\sqrt{\left(y_1 - \frac{2d\tan^2\alpha}{\tan^2\alpha+1}\right)^2 + |y''|^2 + \left(y_n \mp \frac{2d\tan\alpha}{\tan^2\alpha+1}\right)^2}\right\}\right] \\ &\qquad \cdot e^{o(d)} \cdot \left(1 + o\left(\frac{|y|}{d}\right)\right),\end{aligned} \tag{4.2.43}$$

and

$$\begin{aligned}|\nabla\frac{\partial \Xi_d}{\partial d}\left(y\right)| \quad &\leqslant \quad \exp\left[-\min\left\{\sqrt{\left(y_1 - \frac{2d\tan^2\alpha}{\tan^2\alpha+1}\right)^2 + |y''|^2 + \left(y_n \mp \frac{2d\tan\alpha}{\tan^2\alpha+1}\right)^2}\right\}\right] \\ &\qquad \cdot e^{o(d)},\end{aligned} \tag{4.2.44}$$

for $d\left(V - Q_0\right)$ and $d \to +\infty$.

After these preliminaries, we are now in position to introduce our approximate solutions. Let us define two smooth non negative cut-off functions $\chi_D : \mathbb{R}^n \to \mathbb{R}$, $\chi_0 : \mathbb{R} \to \mathbb{R}$ satisfying respectively

$$\begin{cases} \chi_D\left(y\right) = 1 & \text{for } |y| \leqslant \frac{dD}{16}, \\ \chi_D\left(y\right) = 0 & \text{for } |y| \geqslant \frac{dD}{8}, \\ |\nabla\chi_D| \leqslant \frac{32}{dD} & \text{on } \mathbb{R}^n, \end{cases} \tag{4.2.45}$$

and

$$\begin{cases} \chi_0\left(y\right) = 1 & \text{for } y \leqslant 0, \\ \chi_0\left(y\right) = 0 & \text{for } y \geqslant 1, \\ \chi_0 \text{ is non increasing} & \text{on } \mathbb{R}. \end{cases} \tag{4.2.46}$$

Now, using the new coordinates y introduced at the end of Section 2.2, we define

$$u_{\epsilon,Q}\left(y\right) := \chi_{\mu_0}\left(\epsilon y\right)\left[\left(U_Q\left(y\right) - \Xi_d\left(y\right)\right)\chi_D\left(y\right) + \epsilon w_Q\left(y\right)\chi_0\left(y_1 - d\right)\right]. \tag{4.2.47}$$

Following the line of [31] we prove that the $u_{\epsilon,Q}$'s are good approximate solutions to (4.1.1) for suitable conditions of Q.

Proposition 4.2.12. *Let μ_0 be the constant appearing in Subsection 4.1.1. Then there exists another constant C_Ω, independent of ϵ, such that, for $C_\Omega \leqslant d \leqslant \frac{1}{\epsilon C_\Omega}$ and for $Dd < \frac{\mu_0}{\epsilon C_\Omega}$, the functions $u_{\epsilon,Q}$ satisfy*

$$\begin{aligned}\left\| I'_\epsilon\left(u_{\epsilon,Q}\right)\right\| &\leqslant C\Big(\epsilon^2+\epsilon e^{-d(1+o(1))}+e^{-d\left[\frac{1}{2}\sqrt{\frac{D\tan\alpha(\tan\alpha+1)}{\tan^2\alpha+1}}+\frac{2\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right](1+o(1))} \\ &\qquad +e^{-\frac{d(p+1)}{2}(1+o(1))}+e^{-d\left(\frac{p}{2}+\frac{\sqrt{2}\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right)(1+o(1))}\Big), \end{aligned} \tag{4.2.48}$$

for a fixed $C>0$ and for ϵ sufficiently small.

Proof. Using the coordinates y, we can split $u_{\epsilon,Q}(y)=\bar{u}_{\epsilon,Q}(y)+\check{u}_{\epsilon,Q}(y)$, where $\bar{u}_{\epsilon,Q}$ is defined in (4.1.2) and

$$\check{u}_{\epsilon,Q}(y)=\chi_{\mu_0}(\epsilon y)\left[(\chi_D(y)-1)U_Q(y)-\chi_D(y)\Xi_d(y)+\epsilon(\chi_0(y_1-d)-1)w_Q(y)\right]. \tag{4.2.49}$$

Then, if we test the gradient of I_ϵ at $u_{\epsilon,Q}$ on any function $v\in H^1_D(\Omega_\epsilon)$, we obtain

$$\begin{aligned} I'_\epsilon\left(u_{\epsilon,Q}\right)[v] &= \int_{\Omega_\epsilon}\left(\nabla_g u_{\epsilon,Q}\nabla_g v+u_{\epsilon,Q}v\right)dy-\int_{\Omega_\epsilon}u^p_{\epsilon,Q}v\,dy \\ &= \int_{\Omega_\epsilon}\left(\nabla_g \bar{u}_{\epsilon,Q}\nabla_g v+\bar{u}_{\epsilon,Q}v\right)dy-\int_{\Omega_\epsilon}\bar{u}^p_{\epsilon,Q}v\,dy \\ &\quad +\int_{\Omega_\epsilon}\left(\nabla_g \check{u}_{\epsilon,Q}\nabla_g v+\check{u}_{\epsilon,Q}v\right)dy-\int_{\Omega_\epsilon}\left(\bar{u}^p_{\epsilon,Q}-u^p_{\epsilon,Q}\right)v\,dy \\ &= I'_\epsilon\left(\bar{u}_{\epsilon,Q}\right)[v]+A_1+A_2, \end{aligned} \tag{4.2.50}$$

where

$$A_1=\int_{\Omega_\epsilon}\left(\nabla_g\check{u}_{\epsilon,Q}\nabla_g v+\check{u}_{\epsilon,Q}v\right)dy; \qquad A_2=\int_{\Omega_\epsilon}\left(\bar{u}^p_{\epsilon,Q}-u^p_{\epsilon,Q}\right)v\,dy.$$

By Proposition 4.1.1 and in particular by (4.1.4) we have that $I'_\epsilon\left(\bar{u}_{\epsilon,Q}\right)[v]$ is of order at most ϵ^2. Hence we only need to estimate A_1 and A_2 in the last line of (4.2.50).

To estimate A_1 we divide further $\check{u}_{\epsilon,Q}=\check{u}_{\epsilon,Q,1}+\check{u}_{\epsilon,Q,2}+\check{u}_{\epsilon,Q,3}$, where

$$\check{u}_{\epsilon,Q,1}(y)=\chi_{\mu_0}(\epsilon y)(\chi_D(y)-1)U_Q(y); \qquad \check{u}_{\epsilon,Q,2}(y)=\chi_{\mu_0}(\epsilon y)\chi_D(y)\Xi_d(y);$$
$$\check{u}_{\epsilon,Q,3}(y)=\chi_{\mu_0}(\epsilon y)\epsilon(\chi_0(y_1-d)-1)w_Q(y).$$

Then we write $A_1=A_{1,1}+A_{1,2}+A_{1,3}$, with

$$A_{1,i}=\int_{\Omega_\epsilon}\left(\nabla_g\check{u}_{\epsilon,Q,i}\nabla_g v+\check{u}_{\epsilon,Q,i}v\right)dy, \qquad i=1,2,3.$$

Since $\chi_D(y)$ is identically equal to 1 for $|y|\leqslant\frac{dD}{16}$ and since $\chi_0(y_1-d)-1=0$ for $y_1\leqslant d$, from (4.0.3) and (4.1.3) we get

$$|A_{1,1}|\leqslant e^{-\frac{dD}{16}(1+o(1))}\|v\|_{H^1_D(\Omega_\epsilon)}; \qquad |A_{1,3}|\leqslant C\epsilon\left(1+|d|^K\right)e^{-d}\|v\|_{H^1_D(\Omega_\epsilon)}. \tag{4.2.51}$$

To control $A_{1,2}$ we write that

$$\begin{aligned} A_{1,2} &= \int_{\Omega_\epsilon}\left(\nabla_g\check{u}_{\epsilon,Q,2}\nabla_g v+\check{u}_{\epsilon,Q,2}v\right)dy=\int_{\Omega_\epsilon}\left(g^{ij}\partial_i\check{u}_{\epsilon,Q,2}\partial_j v+\check{u}_{\epsilon,Q,2}v\right)dy \\ &= \int_{\Omega_\epsilon}\left(\nabla\check{u}_{\epsilon,Q,2}\nabla v+\check{u}_{\epsilon,Q,2}v\right)dy+\int_{\Omega_\epsilon}\left(g^{ij}-\delta^{ij}\right)\partial_i\check{u}_{\epsilon,Q,2}\partial_j v\,dy. \end{aligned}$$

From the condition (c) in Subsection 2.2 we have that $|g^{ij} - \delta^{ij}| \leqslant C\epsilon|y|$; then

$$\left|A_{1,2} - \int_{\Omega_\epsilon} \left(\nabla \check{u}_{\epsilon,Q,2} \nabla v + \check{u}_{\epsilon,Q,2} v\right) dy\right| \leqslant C\epsilon \left(\int_{\Omega_\epsilon} |y|^2 |\nabla \check{u}_{\epsilon,Q,2}|^2 dy\right)^{\frac{1}{2}} \|v\|_{H^1_D(\Omega_\epsilon)}.$$

Since the support of $\check{u}_{\epsilon,Q,2}$ is contained in the set $\left\{|y| \leqslant \frac{dD}{8}\right\}$, we obtain from the last formula and (4.2.32) that

$$\left|A_{1,2} - \int_{\Omega_\epsilon} \left(\nabla \check{u}_{\epsilon,Q,2} \nabla v + \check{u}_{\epsilon,Q,2} v\right) dy\right| \leqslant C\epsilon d D e^{-d(1+o(1))} \|v\|_{H^1_D(\Omega_\epsilon)}.$$

Now, since Ξ_d satisfies (4.2.30), we have

$$\begin{aligned}\int_{\Omega_\epsilon} \left(\nabla \check{u}_{\epsilon,Q,2} \nabla v + \check{u}_{\epsilon,Q,2} v\right) dy &= \int_{\Omega_\epsilon} \Big(\nabla \left(\Xi_d(y)\left(\chi_{\mu_0}(\epsilon y)\chi_D(y) - 1\right)\right) \nabla v \\ &\quad + \Xi_d(y)\left(\chi_{\mu_0}(\epsilon y)\chi_D(y) - 1\right) v\Big) dy. \qquad (4.2.52)\end{aligned}$$

Since also $Dd < \frac{1}{C_\Omega}\frac{\mu_0}{\epsilon}$, the function $\chi_{\mu_0}(\epsilon y)\chi_D(y) - 1$ is identically zero in the set $\left\{|y| \leqslant \frac{dD}{16}\right\}$ if C_Ω is sufficiently large. Then, using (4.2.33), (4.2.34) and the Hölder inequality, we find that (also for D large)

$$\begin{aligned}&\left|\left|\int_{\Omega_\epsilon} \left(\nabla \left(\Xi_d(y)\left(\chi_{\mu_0}(\epsilon y)\chi_D(y) - 1\right)\right) \nabla v + \Xi_d(y)\left(\chi_{\mu_0}(\epsilon y)\chi_D(y) - 1\right) v\right) dy\right| \right.\\ &\qquad \leqslant e^{-\left[\frac{dD}{16} + \frac{d}{2}\sqrt{\frac{D\tan\alpha(\tan\alpha+1)}{\tan^2\alpha+1}} + \frac{2d\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right](1+o(1))} \|v\|_{H^1_D(\Omega_\epsilon)}. \qquad (4.2.53)\end{aligned}$$

The last three formulas imply

$$|A_{1,2}| \leqslant C\left(\epsilon d D e^{-d(1+o(1))} + e^{-\left[\frac{dD}{16} + \frac{d}{2}\sqrt{\frac{D\tan\alpha(\tan\alpha+1)}{\tan^2\alpha+1}} + \frac{2d\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right](1+o(1))}\right) \|v\|_{H^1_D(\Omega_\epsilon)}.$$

From (4.2.51) and the latter formula it follows that

$$\begin{aligned}|A_1| &\leqslant C\Big[\epsilon d D e^{-d(1+o(1))} + e^{-\left[\frac{dD}{16} + \frac{d}{2}\sqrt{\frac{D\tan\alpha(\tan\alpha+1)}{\tan^2\alpha+1}} + \frac{2d\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right](1+o(1))} \\ &\quad + \epsilon\left(1 + |d|^K\right) e^{-d}\Big] \cdot \|v\|_{H^1_D(\Omega_\epsilon)}. \qquad (4.2.54)\end{aligned}$$

It remains to estimate A_2. First of all, let us recall that the following inequality holds:

$$|\bar{u}^p_{\epsilon,Q} - u^p_{\epsilon,Q}| \leqslant \begin{cases} C|\bar{u}_{\epsilon,Q}|^{p-1}|\check{u}_{\epsilon,Q}| & \text{for } \check{u}_{\epsilon,Q} \in \left(0, \frac{1}{2}\bar{u}_{\epsilon,Q}\right), \\ C|\bar{u}_{\epsilon,Q}|^{p-1}|\check{u}_{\epsilon,Q}| + C|\check{u}_{\epsilon,Q}|^p & \text{otherwise,} \end{cases} \qquad (4.2.55)$$

for a fixed constant C depending only on p. Moreover, using (4.0.3) and (4.1.3), we can say that there exists a small constant $c_{K,n}$ such that

$$\bar{u}_{\epsilon,Q}(y) \geqslant \frac{7}{8}\frac{e^{-|y|}}{1 + |y|^{\frac{n-1}{2}}}; \qquad \text{for } |y| \leqslant \frac{1}{\epsilon^{c_{K,n}}}.$$

We divide next Ω_ϵ into the two regions

$$B_1 = \left\{|y| < \min\left\{\frac{d}{2}, \frac{1}{\epsilon^{c_{K,n}}}\right\}\right\}; \qquad B_2 = \Omega_\epsilon \setminus B_1.$$

For $y \in B_1$ we have that $\chi_{\mu_0}(\epsilon y) \equiv 1$, $\chi_D(y) \equiv 1$, $\chi_0(y_1 - d) \equiv 1$, and hence $\check{u}_{\epsilon,Q}(y) \equiv -\Xi_d(y)$. By (4.2.33) we have also that $|\check{u}_{\epsilon,Q}(y)| = |\Xi_d(y)| \leqslant e^{-\frac{d}{2} - \frac{\sqrt{2} d \tan\alpha}{\sqrt{\tan^2\alpha+1}} + o(d)} < \frac{1}{2}\bar{u}_{\epsilon,Q}$ for $y \in B_1$. This fact, (4.2.55) and the Hölder inequality yield

$$\int_{B_1} |\bar{u}^p_{\epsilon,Q} - u^p_{\epsilon,Q}||v| dy \leqslant C \int_{B_1} |\bar{u}_{\epsilon,Q}|^{p-1} |\check{u}_{\epsilon,Q}||v| dy \leqslant C e^{-\frac{d}{2} - \frac{\sqrt{2} d \tan\alpha}{\sqrt{\tan^2\alpha+1}} + o(d)} \|v\|_{H^1_D(\Omega_\epsilon)}.$$

On the other hand , in B_2 we have that $|\bar{u}^p_{\epsilon,Q}| < C\left(e^{-\frac{d}{2}+o(d)} + e^{-\frac{1+o(1)}{\epsilon^c K,n}}\right)$ and that $|\check{u}_{\epsilon,Q}| \leqslant e^{-d+o(d)}$; therefore (4.2.55) and the Hölder inequality imply again

$$\begin{aligned} \int_{B_2} |\bar{u}^p_{\epsilon,Q} - u^p_{\epsilon,Q}||v| dy \quad &\leqslant \quad C\left[\left(e^{-\frac{(p-1)d}{2}+o(d)} + e^{-\frac{p-1+o(1)}{\epsilon^c K,n}}\right) e^{-d+o(d)} + e^{-pd+o(d)}\right] \\ &\quad \cdot \|v\|_{H^1_D(\Omega_\epsilon)}. \end{aligned}$$

The last two formulas provide

$$\begin{aligned} |A_2| \quad \leqslant \quad C &\left[e^{-\frac{dp}{2} - \frac{\sqrt{2} d \tan\alpha}{\sqrt{\tan^2\alpha+1}} + o(d)} + e^{-pd+o(d)} + \left(e^{-\frac{(p-1)d}{2}+o(d)} + e^{-\frac{p-1+o(1)}{\epsilon^c K,n}}\right) e^{-d+o(d)}\right] \\ &\cdot \|v\|_{H^1_D(\Omega_\epsilon)}. \end{aligned} \tag{4.2.56}$$

Finally, we obtain the conclusion from (4.1.4), (4.2.50), (4.2.54) and (4.2.56). □

We have next another estimate for the functional I_ϵ, which allows to say that the condition $ii)'$ in Section 2.1 holds true for I_ϵ and the manifold of the $u_{\epsilon,Q}$'s.

Proposition 4.2.13. *Let μ_0 be the constant appearing in Subsection 4.1.1. Then there exists another constant C_Ω, independent of ϵ, such that, for $C_\Omega \leqslant d \leqslant \frac{1}{\epsilon C_\Omega}$ and for $Dd < \frac{\mu_0}{\epsilon C_\Omega}$, the functions $u_{\epsilon,Q}$ satisfy*

$$\begin{aligned} \left\|I''_\epsilon(u_{\epsilon,Q})[q]\right\| \quad \leqslant \quad C\Big(\epsilon^2 + \epsilon e^{-d(1+o(1))} + e^{-d\left(\frac{1}{2}\sqrt{\frac{D\tan\alpha(\tan\alpha+1)}{\tan^2\alpha+1}} + \frac{2\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right)(1+o(1))} \\ + e^{-\frac{d(p+1)}{2}(1+o(1))} + e^{-d\left(\frac{p}{2} + \frac{\sqrt{2}\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right)(1+o(1))}\Big) \|q\|, \end{aligned} \tag{4.2.57}$$

for some fixed $C > 0$ and for ϵ sufficiently small. In the above formula q represents a vector in $H^1_D(\Omega_\epsilon)$ which is tangent to the manifold of the $u_{\epsilon,Q}$'s (when Q varies).

Proof. Since the arguments are quite similar to those in the proof of Proposition 4.2.12, we will be rather quick. Using the fact that $\det(g^{ij}) = 1$ and the first line in (4.2.50), for any given test function $v \in H^1_D(\Omega_\epsilon)$ we can write that

$$I'_\epsilon(u_{\epsilon,Q})[v] = \sum_{i,j} \int_{\mathbb{R}^n_+} \left(g^{ij}\partial_i u_{\epsilon,Q}\partial_j v + u_{\epsilon,Q} v\right) dy - \int_{\mathbb{R}^n_+} u^p_{\epsilon,Q} v dy.$$

We want to differentiate next with respect to the parameter Q, taking first a variation q_T of the point Q for which d stays fixed, namely we take the tangential derivative to the level set of the distance d to the interface. Let us notice that in the above formula the dependence on

Q is in the metric coefficients g^{ij} and in the function w_Q appearing in the expression of $u_{\epsilon,Q}$ (see (4.2.47)). Therefore we obtain

$$\begin{aligned}\frac{\partial}{\partial Q_T} I'_\epsilon\left(u_{\epsilon,Q}\right)[v] &= I''_\epsilon\left(u_{\epsilon,Q}\right)\left[\frac{\partial u_{\epsilon,Q}}{\partial Q_T}, v\right] \\ &= \sum_{i,j}\int_{\mathbb{R}^n_+}\frac{\partial g^{ij}}{\partial Q_T}\partial_i u_{\epsilon,Q_T}\partial_j v dy + \sum_{i,j}\int_{\mathbb{R}^n_+}\left(g^{ij}\partial_i\frac{\partial u_{\epsilon,Q}}{\partial Q_T}\partial_j v + \frac{\partial u_{\epsilon,Q}}{\partial Q_T}v\right)dy \\ &\quad -p\int_{\mathbb{R}^n_+} u^{p-1}_{\epsilon,Q}\frac{\partial u_{\epsilon,Q}}{\partial Q_T} v dy. \qquad (4.2.58)\end{aligned}$$

From Remark 2.2.1 (ii) we have that $\frac{\partial g^{ij}}{\partial Q_T}$ is of order $\epsilon^2|y|$. Moreover, computing the expression of $\frac{\partial u_{\epsilon,Q}}{\partial Q_T}$ we obtain $\frac{\partial u_{\epsilon,Q}}{\partial Q_T} = \epsilon\chi_{\mu_0}(\epsilon y)\chi_0(y_1 - d)\frac{\partial w_Q}{\partial Q_T} = o\left(\epsilon^2\left(1+|y|^K\right)e^{-|y|}\right)$, see Subsection 2.2 in [31]. Reasoning as in the proof of Proposition 4.2.12 we then have

$$\left\|\frac{\partial}{\partial Q_T} I'_\epsilon\left(u_{\epsilon,Q}\right)[v]\right\| \leqslant C\epsilon^2\|v\|_{H^1_D(\Omega_\epsilon)} \qquad \text{for every } v \in H^1_D(\Omega_\epsilon). \qquad (4.2.59)$$

On the other hand, when we take a variation q_d of Q along the gradient of d, similarly to (4.2.58) we get

$$\begin{aligned}\frac{\partial}{\partial Q_d} I'_\epsilon\left(u_{\epsilon,Q}\right)[v] &= I''_\epsilon\left(u_{\epsilon,Q}\right)\left[\frac{\partial u_{\epsilon,Q}}{\partial Q_d}, v\right] \\ &= \sum_{i,j}\int_{\mathbb{R}^n_+}\frac{\partial g^{ij}}{\partial Q_d}\partial_i u_{\epsilon,Q_d}\partial_j v dy + \sum_{i,j}\int_{\mathbb{R}^n_+}\left(g^{ij}\partial_i\frac{\partial u_{\epsilon,Q}}{\partial Q_d}\partial_j v + \frac{\partial u_{\epsilon,Q}}{\partial Q_d}v\right)dy \\ &\quad -p\int_{\mathbb{R}^n_+} u^{p-1}_{\epsilon,Q}\frac{\partial u_{\epsilon,Q}}{\partial Q_d} v dy. \qquad (4.2.60)\end{aligned}$$

Concerning the derivatives of g^{ij} with respect to Q_d we can argue exactly as for Q_T, to find

$$\left|\sum_{i,j}\int_{\mathbb{R}^n_+}\frac{\partial g^{ij}}{\partial Q_d}\partial_i u_{\epsilon,Q_d}\partial_j v dy\right| \leqslant C\epsilon^2\|v\|_{H^1_D(\Omega_\epsilon)}.$$

Now, computing the derivative of $u_{\epsilon,Q}$ with respect to Q_d is more complicated than the previous case, because $\frac{\partial u_{\epsilon,Q}}{\partial Q_d}$ has a more involved expression. If we assume that the cut-off function $\chi_D(y)$ is defined as $\bar{\chi}_D\left(\frac{y}{d}\right)$ for some fixed $\bar{\chi}_D$, we obtain

$$\begin{aligned}\frac{\partial u_{\epsilon,Q}}{\partial Q_d} &= -\chi_{\mu_0}\chi_D\frac{\partial \Xi_d}{\partial d} + \frac{1}{d^2}\chi_{\mu_0}\left(\Xi_d - U_Q\right)y\cdot\nabla\bar{\chi}_D\left(\frac{y}{d}\right) + \epsilon\chi_{\mu_0}w_Q\frac{\partial\chi_0(y_1-d)}{\partial Q_d} \\ &\quad +\epsilon\chi_{\mu_0}\chi_0(y_1-d)\frac{\partial w_Q}{\partial Q_d}. \qquad (4.2.61)\end{aligned}$$

It is easy to see that the last two terms in the right hand side give a contribution to (4.2.60) of order at most $\epsilon e^{d(1+o(1))}\|v\|_{H^1_D(\Omega_\epsilon)}$ and $\epsilon^2 e^{d(1+o(1))}\|v\|_{H^1_D(\Omega_\epsilon)}$ respectively. Concerning the second one, we can use the fact that the support of $\nabla\chi_D$ is contained in the set $\left\{|y| \geqslant \frac{dD}{16}\right\}$, together with (4.2.33), (4.2.34) to see that the contribution of this term is at most of order

$$\left(e^{-\left(\frac{dD}{16}+\frac{d}{2}\sqrt{\frac{D\tan\alpha(\tan\alpha+1)}{\tan^2\alpha+1}}+\frac{2d\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right)(1+o(1))} + e^{-\frac{dD}{16}(1+o(1))}\right)\|v\|_{H^1_D(\Omega_\epsilon)}.$$

We can then focus on the first term in the right hand side of (4.2.61), and consider the quantity

$$-\sum_{i,j}\int_{\mathbb{R}^n_+}\left(g^{ij}\partial_i\left(\chi_{\mu_0}\chi_D\frac{\partial\Xi_d}{\partial d}\right)\partial_j v+\chi_{\mu_0}\chi_D\frac{\partial\Xi_d}{\partial d}v\right)dy+p\int_{\mathbb{R}^n_+}u_{\epsilon,Q}^{p-1}\chi_{\mu_0}\chi_D\frac{\partial\Xi_d}{\partial d}v dy. \tag{4.2.62}$$

Now, using condition (c) at the end of Section 2.2 and (4.2.42), if we substitute the coefficients g^{ij} with the Kronecker symbols we find a difference of order

$$\epsilon\left(e^{-d(1+o(1))}+e^{-d\left(1+\frac{2\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right)(1+o(1))}\right).$$

Next, since Ξ_d satisfies $-\Delta\Xi_d+\Xi_d=0$, when we differentiate with respect to d we get the same equation for $\frac{\partial\Xi_d}{\partial d}$, so reasoning as for (4.2.52), (4.2.53), together with (4.2.43), (4.2.44), we find

$$\left|\int_{\mathbb{R}^n_+}\left(\nabla\left(\chi_{\mu_0}\chi_D\frac{\partial\Xi_d}{\partial d}\right)\cdot\nabla v+\chi_{\mu_0}\chi_D\frac{\partial\Xi_d}{\partial d}v\right)dy\right|\leqslant$$
$$Ce^{-\left(\frac{dD}{16}+\frac{d}{2}\sqrt{\frac{D\tan\alpha(\tan\alpha+1)}{\tan^2\alpha+1}}+\frac{2d\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right)(1+o(1))}\cdot\|v\|_{H^1_D(\Omega_\epsilon)}.$$

It remains to estimate the last term in (4.2.62). Using (4.2.42), (4.2.43) and the exponential decay of $u_{\epsilon,Q}$ and reasoning with argument similar to those for (4.2.56), we find that it is of order

$$e^{-d(1+o(1))}\left(e^{-d\left(\frac{p-2}{2}+\frac{\sqrt{2}\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right)}+e^{-\frac{d(p-1)}{2}}+o\left(\epsilon^2\right)\right)\|v\|_{H^1_D(\Omega_\epsilon)}.$$

All the above comments yield that

$$\left\|\frac{\partial}{\partial Q_d}I'_\epsilon\left(u_{\epsilon,Q}\right)[v]\right\|\leqslant$$
$$C\left(\epsilon^2+\epsilon e^{-d(1+o(1))}+e^{-d\left[\frac{1}{2}\sqrt{\frac{D\tan\alpha(\tan\alpha+1)}{\tan^2\alpha+1}}+\frac{2\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right](1+o(1))}\right)\|v\|_{H^1_D(\Omega_\epsilon)}$$
$$+C\left(e^{-\frac{d(p+1)}{2}(1+o(1))}+e^{-d\left(\frac{p}{2}+\frac{\sqrt{2}\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right)(1+o(1))}\right)\|v\|_{H^1_D(\Omega_\epsilon)}. \tag{4.2.63}$$

From (4.2.59) and (4.2.63) we finally obtain the desired conclusion. □

Case $\frac{\pi}{2}\leqslant\alpha\leqslant\pi$

In this subsection we introduce the manifold of approximate solutions in the case $\frac{\pi}{2}\leqslant\alpha\leqslant\pi$. Since the construction is substantially the same as in the previous subsection, we will be rather sketchy.

Let us consider the solution of (4.2.5), Φ^d, and the function Ξ_d defined in (4.2.29). Reasoning as at the beginning of the Subsection 4.2.3, we derive norm estimate for Ξ_d:

$$\|\Xi_d\|_{H^1(d(\Sigma_D-Q_0))}\leqslant e^{-d(1+o(1))}.$$

Moreover, from Proposition 4.2.10 we also obtain pointwise estimates for Ξ_d and its gradient.

Now, using the cut-off functions (4.2.45), (4.2.46), we define, in the new coordinates y introduced in Subsection 4.1.1, the functions

$$u_{\epsilon,Q}(y) := \chi_{\mu_0}(\epsilon y)\left[\left(U_Q(y) - \Xi_d(y)\right)\chi_D(y) + \epsilon w_Q(y)\chi_0(y_1 - d)\right].$$

Following the line of the Subsection 4.2.3 we prove that the $u_{\epsilon,Q}$'s are good approximate solutions to (4.1.1) for suitable conditions of Q. Since the computations in the following proposition are the same as in Proposition 3.12 and Proposition 3.13 in [31] we will omit the proof.

Proposition 4.2.14. *Let μ_0 be the constant appearing in Subsection 4.1.1. Then there exists another constant C_Ω, independent of ϵ, such that, for $C_\Omega \leqslant d \leqslant \frac{1}{\epsilon C_\Omega}$ and for $Dd < \frac{\mu_0}{\epsilon C_\Omega}$, the functions $u_{\epsilon,Q}$ satisfy*

$$\left\|I'_\epsilon\left(u_{\epsilon,Q}\right)\right\| \leqslant C\left(\epsilon^2 + \epsilon e^{-d(1+o(1))} + e^{-\frac{d(p+1)}{2}(1+o(1))} + e^{-\frac{3}{2}d(1+o(1))}\right), \tag{4.2.64}$$

and

$$\begin{aligned}\left\|I''_\epsilon\left(u_{\epsilon,Q}\right)[q]\right\| &\leqslant \\ &C\left(\epsilon^2 + \epsilon\exp^{-d(1+o(1))} + e^{-\frac{d(p+1)}{2}(1+o(1))} + e^{-\frac{3}{2}d(1+o(1))}\right)\|q\|,\end{aligned} \tag{4.2.65}$$

for some fixed $C > 0$ and for ϵ sufficiently small. In (4.2.65) q represents a vector in $H^1_D(\Omega_\epsilon)$ which is tangent to the manifold of the $u_{\epsilon,Q}$'s (when Q varies).

Case $\pi < \alpha < 2\pi$

In this subsection we introduce the manifold of approximate solutions in the case $\pi < \alpha < 2\pi$. Also in this case we will be very quick, since the construction is the same as in the previous subsections.

Let us consider the solution of (4.2.5), Φ^d, and the function Ξ_d defined in (4.2.29). Reasoning as at the beginning of the Subsection 4.2.3, we derive norm estimate for Ξ_d:

$$\|\Xi_d\|_{H^1(d(\Sigma_D - Q_0))} \leqslant e^{-d(1+o(1))}.$$

Moreover, from Proposition 4.2.11 we also obtain pointwise estimates for Ξ_d and its gradient.

Now, using the cut-off functions (4.2.45), (4.2.46), we define, in the new coordinates y introduced in Subsection 4.1.1, the functions

$$u_{\epsilon,Q}(y) := \chi_{\mu_0}(\epsilon y)\left[\left(U_Q(y) - \Xi_d(y)\right)\chi_D(y) + \epsilon w_Q(y)\chi_0(y_1 - d)\right].$$

Following the line of the Subsection 4.2.3 we obtain that the $u_{\epsilon,Q}$'s are good approximate solutions to (4.1.1) for suitable conditions of Q. Since the computations in the following proposition are very similar to those in Proposition 3.12 and Proposition 3.13 in [31] we will omit the proof.

Proposition 4.2.15. *Let μ_0 be the constant appearing in Subsection 4.1.1. Then there exists another constant C_Ω, independent of ϵ, such that, for $C_\Omega \leqslant d \leqslant \frac{1}{\epsilon C_\Omega}$ and for $Dd < \frac{\mu_0}{\epsilon C_\Omega}$, the functions $u_{\epsilon,Q}$ satisfy*

$$\left\|I'_\epsilon\left(u_{\epsilon,Q}\right)\right\| \leqslant C\left(\epsilon^2 + \epsilon e^{-d(1+o(1))} + e^{-\frac{d(p+1)}{2}(1+o(1))} + e^{-\frac{d}{2}(1+o(1))}\right), \tag{4.2.66}$$

and

$$\left\|I_\epsilon''\left(u_{\epsilon,Q}\right)[q]\right\| \leqslant C\left(\epsilon^2+\epsilon\exp^{-d(1+o(1))}+e^{-\frac{d(p+1)}{2}(1+o(1))}+e^{-\frac{d}{2}(1+o(1))}\right)\|q\|, \quad (4.2.67)$$

for some fixed $C>0$ and for ϵ sufficiently small. In (4.2.67) q represents a vector in $H_D^1(\Omega_\epsilon)$ which is tangent to the manifold of the $u_{\epsilon,Q}$'s (when Q varies).

4.3 PROOF OF THEOREM 4.0.6

To prove our main Theorem we need to derive an expansion in terms of Q and ϵ of the energy of approximate solutions. Then we can apply the abstract theory in Section 2.1 to obtain the existence result.

In the case $\frac{\pi}{2}\leqslant\alpha\leqslant\pi$ the energy expansions for the approximate solutions $u_{\epsilon,Q}$ are the same as in the case $\alpha=\pi$, see Proposition 4.1 and Proposition 4.2 in [31]. Then also the definition of the critical manifold and the study of the reduced functional are the same. Therefore for the proof of Theorem 4.0.6 in the case $\frac{\pi}{2}\leqslant\alpha\leqslant\pi$ we refer the reader to Section 4 in [31].

In the case $\pi<\alpha<2\pi$, even if the approximate solutions are different from the previous case, the energy expansions turn out to be the same. Then also in this case we omit the proof of Theorem 4.0.6 and refer the reader to Section 4 in [31].

In the case $0<\alpha<\frac{\pi}{2}$ the energy expansions are quite different, so we will give the proof in the details.

From now on we will assume $0<\alpha<\frac{\pi}{2}$.

4.3.1 *Energy expansions for the approximate solutions $u_{\epsilon,Q}$*

Here we expand $I_\epsilon\left(u_{\epsilon,Q}\right)$ in terms of Q and ϵ, where $u_{\epsilon,Q}$ is the function defined in (4.2.47).

Proposition 4.3.1. *For $\epsilon\to0$ and $d=d(Q)\to+\infty$, the following expansion holds*

$$I_\epsilon\left(u_{\epsilon,Q}\right)=\tilde{C}_0-\tilde{C}_1\epsilon H(\epsilon Q)+e^{-2d(1+o(1))}+e^{\left(-d-\frac{d\sqrt{2}\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right)(1+o(1))}+o\left(\epsilon^2\right), \quad (4.3.1)$$

where $\tilde{C}_0$ and $\tilde{C}_1$ are the constants in Proposition 4.1.1.

Proof. As in the proof of Proposition 4.2.12, let us write $u_{\epsilon,Q}(y)=\bar{u}_{\epsilon,Q}(y)+\check{u}_{\epsilon,Q}(y)$, see (4.1.2) and (4.2.49). Then, using the coordinates y introduced in Subsection 4.1.1, we find that

$$I_\epsilon\left(u_{\epsilon,Q}\right)=I_\epsilon\left(\bar{u}_{\epsilon,Q}\right)+\int_{\Omega_\epsilon}\left(\nabla_g\bar{u}_{\epsilon,Q}\nabla_g\check{u}_{\epsilon,Q}+\bar{u}_{\epsilon,Q}\check{u}_{\epsilon,Q}\right)dy$$
$$+\frac{1}{2}\int_{\Omega_\epsilon}\left(|\nabla_g\check{u}_{\epsilon,Q}|^2+\check{u}_{\epsilon,Q}^2\right)dy+\frac{1}{p+1}\int_{\Omega_\epsilon}\left(|\bar{u}_{\epsilon,Q}|^{p+1}-|u_{\epsilon,Q}|^{p+1}\right)dy \quad (4.3.2)$$

Using condition (c) at the end of Section 2.2 we have that

$$\left|\int_{\Omega_\epsilon}\left(\nabla_g\bar{u}_{\epsilon,Q}\nabla_g\check{u}_{\epsilon,Q}+\bar{u}_{\epsilon,Q}\check{u}_{\epsilon,Q}\right)dy-\int_{\mathbb{R}^n_+}\left(\nabla\bar{u}_{\epsilon,Q}\nabla\check{u}_{\epsilon,Q}+\bar{u}_{\epsilon,Q}\check{u}_{\epsilon,Q}\right)dy\right|$$
$$\leqslant C\epsilon\int_{\mathbb{R}^n_+}|y|\cdot|\nabla\bar{u}_{\epsilon,Q}|\cdot|\nabla\check{u}_{\epsilon,Q}|dy; \quad (4.3.3)$$

$$\left|\int_{\Omega_\epsilon}\left(|\nabla_g \check{u}_{\epsilon,Q}|^2+\check{u}_{\epsilon,Q}^2\right)dy-\int_{\mathbb{R}^n_+}\left(|\nabla \check{u}_{\epsilon,Q}|^2+\check{u}_{\epsilon,Q}^2\right)dy\right|\leqslant C\epsilon\int_{\mathbb{R}^n_+}|y||\nabla \check{u}_{\epsilon,Q}|^2dy. \tag{4.3.4}$$

Concerning (4.3.3), we can divide the domain of integration into $B_{\frac{d}{2}}(0)$ and its complement and use (4.0.3), (4.1.3), (4.2.32), (4.2.33), (4.2.34) to find

$$C\epsilon\int_{\mathbb{R}^n_+}|y|\cdot|\nabla\bar{u}_{\epsilon,Q}|\cdot|\nabla\check{u}_{\epsilon,Q}|dy\leqslant C\epsilon\left(e^{-\frac{3}{2}d(1+o(1))}+e^{-d\left(1+\frac{\sqrt{2}\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right)(1+o(1))}\right).$$

For (4.3.4), the same estimates yield

$$C\epsilon\int_{\mathbb{R}^n_+}|y|\cdot|\nabla\check{u}_{\epsilon,Q}|^2dy\leqslant C\epsilon\left(e^{-2d(1+o(1))}+e^{-d\left(1+\frac{\sqrt{2}\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right)(1+o(1))}\right).$$

The last two formulas, (4.3.2), (4.3.3), (4.3.4) imply

$$\begin{aligned}I_\epsilon\left(u_{\epsilon,Q}\right) &= I_\epsilon\left(\bar{u}_{\epsilon,Q}\right)+\int_{\mathbb{R}^n_+}\left(\nabla\bar{u}_{\epsilon,Q}\nabla\check{u}_{\epsilon,Q}+\bar{u}_{\epsilon,Q}\check{u}_{\epsilon,Q}\right)dy\\ &\quad+\frac{1}{2}\int_{\mathbb{R}^n_+}\left(|\nabla\check{u}_{\epsilon,Q}|^2+\check{u}_{\epsilon,Q}^2\right)dy\\ &\quad+\frac{1}{p+1}\int_{\Omega_\epsilon}\left(|\bar{u}_{\epsilon,Q}|^{p+1}-|u_{\epsilon,Q}|^{p+1}\right)dy\\ &\quad+o\left(\epsilon\left(e^{-\frac{3}{2}d(1+o(1))}+e^{-d-\frac{\sqrt{2}d\tan\alpha}{\sqrt{\tan^2\alpha+1}}}\right)\right).\end{aligned}\tag{4.3.5}$$

Using the same notation as in the proof of Proposition 4.2.12, we write $\check{u}_{\epsilon,Q}=\check{u}_{\epsilon,Q,1}+\check{u}_{\epsilon,Q,2}+\check{u}_{\epsilon,Q,3}$. Formulas (4.0.3) and (4.1.3)) imply

$$\left|\int_{\mathbb{R}^n_+}\left(\nabla\bar{u}_{\epsilon,Q}\nabla\check{u}_{\epsilon,Q,1}+\bar{u}_{\epsilon,Q}\check{u}_{\epsilon,Q,1}\right)dy\right|\leqslant Ce^{-\frac{dD}{16}(1+o(1))};$$

$$\left|\int_{\mathbb{R}^n_+}\left(\nabla\bar{u}_{\epsilon,Q}\nabla\check{u}_{\epsilon,Q,3}+\bar{u}_{\epsilon,Q}\check{u}_{\epsilon,Q,3}\right)dy\right|\leqslant C\epsilon e^{-2d(1+o(1))},$$

from which we deduce that

$$\begin{aligned}\int_{\mathbb{R}^n_+}\left(\nabla\bar{u}_{\epsilon,Q}\nabla\check{u}_{\epsilon,Q}+\bar{u}_{\epsilon,Q}\check{u}_{\epsilon,Q}\right)dy &= \int_{\mathbb{R}^n_+}\left(\nabla\bar{u}_{\epsilon,Q}\nabla\check{u}_{\epsilon,Q,2}+\bar{u}_{\epsilon,Q}\check{u}_{\epsilon,Q,2}\right)dy\\ &\quad+o\left(e^{-\frac{dD}{16}(1+o(1))}+\epsilon e^{-2d(1+o(1))}\right).\end{aligned}$$

Similar estimates also yield

$$\begin{aligned}\int_{\mathbb{R}^n_+}\left(|\nabla\check{u}_{\epsilon,Q}|^2+\check{u}_{\epsilon,Q}^2\right)dy &= \int_{\mathbb{R}^n_+}\left(|\nabla\check{u}_{\epsilon,Q,2}|^2+\check{u}_{\epsilon,Q,2}^2\right)dy\\ &\quad+o\left(e^{-\frac{dD}{16}-d(1+o(1))}+\epsilon e^{-2d(1+o(1))}\right).\end{aligned}$$

From a straightforward computation one finds that for any function v

$$\begin{aligned}&\nabla\check{u}_{\epsilon,Q,2}\nabla v+\check{u}_{\epsilon,Q}^2 v=\\ &\qquad\nabla\Xi_d\cdot\nabla\left(\chi_{\mu_0}(\epsilon\cdot)\chi_D v\right)+\Xi_d\chi_{\mu_0}(\epsilon\cdot)\chi_D v+\nabla\left(\chi_{\mu_0}(\epsilon\cdot)\chi_D\right)\left(\Xi_d\nabla v-v\nabla\Xi_d\right).\end{aligned}$$

Applying this relation for $v = \bar{u}_{\epsilon,Q}$ and $v = \check{u}_{\epsilon,Q,2}$ respectively, and using (4.0.3), (4.1.3), (4.2.32), (4.2.33) and (4.2.34) we find that

$$\begin{aligned}
&\int_{\mathbb{R}^n_+} \left(\nabla \bar{u}_{\epsilon,Q} \nabla \check{u}_{\epsilon,Q,2} + \bar{u}_{\epsilon,Q} \check{u}_{\epsilon,Q,2}\right) dy = \\
&\quad \int_{\mathbb{R}^n_+} \left(\nabla \left(\chi_{\mu_0}(\epsilon \cdot) \chi_D \bar{u}_{\epsilon,Q}\right) \nabla \Xi_d + \chi_{\mu_0}(\epsilon \cdot) \chi_D \bar{u}_{\epsilon,Q} \Xi_d\right) dy \\
&\quad + o\Big(e^{-\left(d + \frac{\sqrt{2} d \tan\alpha}{\sqrt{\tan^2\alpha + 1}}\right)(1+o(1))} + e^{-\frac{3}{2} d (1+o(1))} \\
&\qquad + e^{-\left(\frac{dD}{16} + \frac{d}{2}\sqrt{\frac{D \tan\alpha (\tan\alpha + 1)}{\tan^2\alpha + 1}} + \frac{2 d \tan\alpha}{\sqrt{\tan^2\alpha + 1}}\right)(1+o(1))}\Big);
\end{aligned}$$

$$\int_{\mathbb{R}^n_+} \left(|\nabla \check{u}_{\epsilon,Q,2}|^2 + \check{u}^2_{\epsilon,Q,2}\right) dy = \int_{\mathbb{R}^n_+} \left(|\nabla \left(\chi_{\mu_0}(\epsilon \cdot) \chi_D \Xi_d\right)|^2 + \left(\chi_{\mu_0}(\epsilon \cdot) \chi_D \Xi_d\right)^2\right) dy.$$

Using now the fact that, by our construction, the function

$$\chi_{\mu_0}(\epsilon \cdot) \chi_D u_{\epsilon,Q} = \chi_{\mu_0}(\epsilon \cdot) \chi_D \left(\bar{u}_{\epsilon,Q} + \check{u}_{\epsilon,Q}\right)$$

vanishes on $d\left(\partial \Sigma_D - Q_0\right)$, from (4.2.30) we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n_+} \left(\nabla \left(\chi_{\mu_0}(\epsilon \cdot) \chi_D \bar{u}_{\epsilon,Q}\right) \nabla \Xi_d + \chi_{\mu_0}(\epsilon \cdot) \chi_D \bar{u}_{\epsilon,Q} \Xi_d\right) dy \\
&+ \frac{1}{2} \int_{\mathbb{R}^n_+} \left(|\nabla \left(\chi_{\mu_0}(\epsilon \cdot) \chi_D \Xi_d\right)|^2 + \left(\chi_{\mu_0}(\epsilon \cdot) \chi_D \Xi_d\right)^2\right) dy \\
&\quad = \frac{1}{2} \int_{\mathbb{R}^n_+} \left(\nabla \left(\chi_{\mu_0}(\epsilon \cdot) \chi_D \bar{u}_{\epsilon,Q}\right) \nabla \Xi_d + \chi_{\mu_0}(\epsilon \cdot) \chi_D \bar{u}_{\epsilon,Q} \Xi_d\right) dy.
\end{aligned}$$

From (4.3.5) and the last eight formulas we find

$$\begin{aligned}
I_\epsilon\left(u_{\epsilon,Q}\right) &= I_\epsilon\left(\bar{u}_{\epsilon,Q}\right) + \frac{1}{2} \int_{\mathbb{R}^n_+} \left(\nabla \bar{u}_{\epsilon,Q} \nabla \check{u}_{\epsilon,Q} + \bar{u}_{\epsilon,Q} \check{u}_{\epsilon,Q}\right) dy \\
&\quad + \frac{1}{p+1} \int_{\Omega_\epsilon} \left(|\bar{u}_{\epsilon,Q}|^{p+1} - |u_{\epsilon,Q}|^{p+1}\right) dy \\
&\quad + o\Bigg(e^{-\frac{dD}{16}(1+o(1))} + e^{-d - \frac{\sqrt{2} d \tan\alpha}{\sqrt{\tan^2\alpha + 1}}(1+o(1))} \\
&\qquad + \epsilon\left(e^{-\frac{3}{2} d (1+o(1))} + e^{-d - \frac{\sqrt{2} d \tan\alpha}{\sqrt{\tan^2\alpha + 1}}(1+o(1))}\right)\Bigg).
\end{aligned}$$

From (4.0.3), (4.1.3), (4.1.4) and (4.2.32) we have that

$$\begin{aligned}
\int_{\mathbb{R}^n_+} \left(\nabla \bar{u}_{\epsilon,Q} \nabla \check{u}_{\epsilon,Q} + \bar{u}_{\epsilon,Q} \check{u}_{\epsilon,Q}\right) dy &= I'_\epsilon\left(\bar{u}_{\epsilon,Q}\right)\left[\check{u}_{\epsilon,Q}\right] + \int_{\Omega_\epsilon} |\bar{u}_{\epsilon,Q}|^p \check{u}_{\epsilon,Q} \, dy \\
&\leqslant C\epsilon^2 e^{-d(1+o(1))} + \int_{\Omega_\epsilon} |\bar{u}_{\epsilon,Q}|^p \check{u}_{\epsilon,Q} \, dy,
\end{aligned}$$

and then

$$\begin{aligned}
I_\epsilon\left(u_{\epsilon,Q}\right) &= I_\epsilon\left(\bar{u}_{\epsilon,Q}\right) + \frac{1}{2}\int_{\Omega_\epsilon} |\bar{u}_{\epsilon,Q}|^p \check{u}_{\epsilon,Q}\,dy + \frac{1}{p+1}\int_{\Omega_\epsilon}\left(|\bar{u}_{\epsilon,Q}^{p+1} - |u_{\epsilon,Q}|^{p+1}\right)dy \\
&\quad + o\Bigg(e^{-\frac{dD}{16}(1+o(1))} + e^{-d-\frac{\sqrt{2}d\tan\alpha}{\sqrt{\tan^2\alpha+1}}(1+o(1))} \\
&\qquad + \epsilon\left(e^{-\frac{3}{2}d(1+o(1))} + e^{-d-\frac{\sqrt{2}d\tan\alpha}{\sqrt{\tan^2\alpha+1}}(1+o(1))}\right)\Bigg) \\
&\quad + o\left(\epsilon^2 e^{-d(1+o(1))}\right). \qquad (4.3.6)
\end{aligned}$$

Using a Taylor expansion we can write that

$$\begin{aligned}
&|\bar{u}_{\epsilon,Q}|^{p+1} - |u_{\epsilon,Q}|^{p+1} \\
&= \begin{cases} -(p+1)|\bar{u}_{\epsilon,Q}|^p|\check{u}_{\epsilon,Q}| + o\left(|\bar{u}_{\epsilon,Q}|^{p-1}\check{u}_{\epsilon,Q}^2\right) & \text{for } \check{u}_{\epsilon,Q}\in\left(0,\frac{1}{2}\bar{u}_{\epsilon,Q}\right), \\ o\left(|\bar{u}_{\epsilon,Q}|^p|\check{u}_{\epsilon,Q}| + |\check{u}_{\epsilon,Q}|^{p+1}\right) & \text{otherwise,}\end{cases} \qquad (4.3.7)
\end{aligned}$$

As for the estimate of A_2 in (4.2.56), we divide the domain into the two regions B_1, B_2, and deduce that

$$\begin{aligned}
\frac{1}{p+1}\int_{\Omega_\epsilon}\left(|\bar{u}_{\epsilon,Q}|^{p+1} - |u_{\epsilon,Q}|^{p+1}\right)dy &= -\int_{\Omega_\epsilon}|\bar{u}_{\epsilon,Q}|^p\check{u}_{\epsilon,Q}\,dy \\
+o\left(e^{-\frac{d(p+1)}{2}-\frac{2d\sqrt{2}\tan\alpha}{\sqrt{\tan^2\alpha+1}}(1+o(1))} + e^{-\frac{d(p+2)}{2}(1+o(1))} + e^{-d(1+o(1))}e^{-\frac{1}{\epsilon^{c_{K,n}}}}\right).
\end{aligned}$$

Therefore using (4.3.6) the energy becomes

$$\begin{aligned}
I_\epsilon\left(u_{\epsilon,Q}\right) &= I_\epsilon\left(\bar{u}_{\epsilon,Q}\right) - \frac{1}{2}\int_{\Omega_\epsilon}|\bar{u}_{\epsilon,Q}|^p\check{u}_{\epsilon,Q}\,dy \\
&+o\left(e^{-d-\frac{\sqrt{2}d\tan\alpha}{\sqrt{\tan^2\alpha+1}}(1+o(1))} + e^{-\frac{d(p+2)}{2}(1+o(1))}\right) \\
&+o\left(\epsilon\left(\exp^{-\frac{3}{2}(1+o(1))} + e^{-d-\frac{\sqrt{2}d|\tan\alpha|}{\sqrt{\tan^2\alpha+1}}(1+o(1))}\right) + \epsilon^2 e^{-d(1+o(1))}\right).
\end{aligned}$$

From (4.2.33), the expression of $\check{u}_{\epsilon,Q}$ and the estimates in the same spirit as above one finds that

$$\int_{\Omega_\epsilon}|\bar{u}_{\epsilon,Q}|^p\check{u}_{\epsilon,Q}\,dy = -\left(e^{-2d(1+o(1))} + e^{-d-\frac{\sqrt{2}d\tan\alpha}{\sqrt{\tan^2\alpha+1}}(1+o(1))}\right),$$

and hence from Proposition 4.1.1 we finally find

$$\begin{aligned}
I_\epsilon\left(u_{\epsilon,Q}\right) &= \tilde{C}_0 - \tilde{C}_1\epsilon H(\epsilon Q) + O\left(\epsilon^2\right) + e^{-2d(1+o(1))} + e^{-d-\frac{\sqrt{2}d\tan\alpha}{\sqrt{\tan^2\alpha+1}}(1+o(1))} \\
&+o\left(e^{-d-\frac{\sqrt{2}d\tan\alpha}{\sqrt{\tan^2\alpha+1}}(1+o(1))} + e^{-\frac{d(p+2)}{2}(1+o(1))}\right) \\
&+o\left(\epsilon\left(e^{-\frac{3}{2}(1+o(1))} + e^{-d-\frac{\sqrt{2}d\tan\alpha}{\sqrt{\tan^2\alpha+1}}(1+o(1))}\right) + \epsilon^2 e^{-d(1+o(1))}\right). \qquad (4.3.8)
\end{aligned}$$

The conclusion follows from the Schwartz inequality. □

We give also a related result about the computation of the derivative of the energy with respect to Q. Again, we will be rather sketchy in the proof since the arguments are quite similar to the previous ones.

Proposition 4.3.2. *For $\epsilon \to 0$ and $d = d(Q) \to +\infty$, the following expansions hold*

$$\frac{\partial}{\partial Q_T} I_\epsilon \left(u_{\epsilon,Q}\right) = -\tilde{C}_1 \epsilon^2 \nabla_T H(\epsilon Q) + o\left(\epsilon^2\right); \tag{4.3.9}$$

$$\frac{\partial}{\partial Q_d} I_\epsilon \left(u_{\epsilon,Q}\right) = -\tilde{C}_1 \epsilon^2 \nabla_d H(\epsilon Q) - e^{\left(-d - \frac{d\sqrt{2}\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right)(1+o(1))} + o\left(\epsilon^2\right), \tag{4.3.10}$$

where $\tilde{C}_0$ and $\tilde{C}_1$ are the constants in Proposition 4.1.1.

Proof. After some elementary calculations, recalling the definition of $\bar{u}_{\epsilon,Q}$ in (4.1.2), we can write

$$\begin{aligned} I'_\epsilon\left(u_{\epsilon,Q}\right)\left[\frac{\partial u_{\epsilon,Q}}{\partial Q}\right] &= \frac{\partial}{\partial Q} I_\epsilon\left(\bar{u}_{\epsilon,Q}\right) + \int_{\Omega_\epsilon}\left(\nabla_g \bar{u}_{\epsilon,Q} \nabla_g \frac{\partial \check{u}_{\epsilon,Q}}{\partial Q} + \bar{u}_{\epsilon,Q}\frac{\partial \check{u}_{\epsilon,Q}}{\partial Q}\right) dy \\ &\quad - \int_{\Omega_\epsilon} \bar{u}^p_{\epsilon,Q} \frac{\partial \check{u}_{\epsilon,Q}}{\partial Q} dy \\ &\quad + \int_{\Omega_\epsilon}\left(\nabla_g \check{u}_{\epsilon,Q}\nabla_g \frac{\partial u_{\epsilon,Q}}{\partial Q} + \check{u}_{\epsilon,Q}\frac{\partial u_{\epsilon,Q}}{\partial Q}\right) dy \\ &\quad + \int_{\Omega_\epsilon}\left(\bar{u}^p_{\epsilon,Q} - u^p_{\epsilon,Q}\right)\frac{\partial u_{\epsilon,Q}}{\partial Q} dy, \end{aligned} \tag{4.3.11}$$

where $\check{u}_{\epsilon,Q} = u_{\epsilon,Q} - \bar{u}_{\epsilon,Q}$ was defined in (4.2.49). The first term on the right hand side is estimated in Proposition 4.1.1. The next two, integrating by parts and using Proposition 4.1.1, can be estimated in terms of a quantity like

$$C\epsilon^2 \int_{\Omega_\epsilon}\left(1 + |y|^K\right)\left|\frac{\partial \check{u}_{\epsilon,Q}}{\partial Q}\right| dy.$$

From the same arguments as in the proof of Proposition 4.2.13 one deduces that the latter integral is of order $\epsilon^2\left(e^{-2d(1+o(1))} + e^{-d - \frac{\sqrt{2}d\tan\alpha}{\sqrt{\tan^2\alpha+1}}(1+o(1))}\right)$. To control the first integral in the last line of (4.3.11) we can reason as for the estimate of $A_{1,2}$ in the proof of Proposition 4.2.12 to see that this is of order $e^{-d(1+o(1))}\left(\epsilon + e^{-d(1+o(1))}\right)\left\|\frac{\partial u_{\epsilon,Q}}{\partial Q}\right\|_{H^1_D(\Omega_\epsilon)}$. From the proof of Proposition 4.2.13 one can deduce that $\left\|\frac{\partial u_{\epsilon,Q}}{\partial Q}\right\|_{H^1_D(\Omega_\epsilon)} \leqslant C\left(\epsilon^2 + e^{-d(1+o(1))}\right)$, and hence the integral under interest is controlled by $o\left(\epsilon^2\right) + e^{-3d(1+o(1))}$.

Finally, the last term in (4.3.11) can be estimated using a Taylor expansion as for the term A_2 in the proof of Proposition 4.2.12, and up to higher order is given by

$$p\int_{\mathbb{R}^n_+} U_Q^{p-1}(y)\, \check{u}_{\epsilon,Q} \nabla U_Q(y) \cdot q\, dy,$$

where q stands either for the variation of Q in the coordinates y. If q preserves d, the latter integral gives a negligible contribution, and we find (4.3.9). If instead q is directed toward the gradient of d the above estimates (and in particular (4.2.33)) allow to deduce (4.3.10). □

4.3.2 *Finite-dimensional reduction and study of the constrained functional*

We apply now the abstract setting described in Section 2.1. In fact, the following two Lemmas hold.

Lemma 4.3.3. *If C_Ω is as in the previous section and if we choose*

$$Z_\epsilon = \left\{ u_{\epsilon,Q} : C_\Omega < d < \frac{1}{\epsilon C_\Omega} \right\},$$

then the properties i)′, iii)′ *and* iv)′ *in Section 2.1 hold true, with $\gamma = \min\{1, p-1\}$.*

Proof. It is immediate to prove that i)′ and iii)′ hold; in particular, the value of γ comes from the standard properties of Nemitski operators. Property iv)′ can be easily deduced from the fact that the kernel of the linearization of (1.2.2) in the half space is spanned by $\frac{\partial U}{\partial x_1}, \dots, \frac{\partial U}{\partial x_{n-1}}$, as proved in [63], and from some localization arguments which can be found in Subsections 4.2, 9.2 and 9.3 of [4]. □

Lemma 4.3.4. *For any small positive constant δ, if we take*

$$Z_\epsilon = \left\{ u_{\epsilon,Q} : (2-\delta)\,|\log \epsilon| < d < \frac{1}{\epsilon C_\Omega} \right\},$$

then also property ii)′ *in Section 2.1 holds true, with*

$$f(\epsilon) = \epsilon^{\min\left\{3-\delta, \frac{p+1}{2}(2-\delta), (2-\delta)\left(\frac{1}{2}\sqrt{\frac{D\tan\alpha(\tan\alpha+1)}{\tan^2\alpha+1}} + \frac{2\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right), (2-\delta)\left(\frac{p}{2} + \frac{\sqrt{2}\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right)\right\}}.$$

Proof. This lemma simply follows from Propositions 4.2.12 and 4.2.13. □

As a corollary of the above two lemmas we can apply Proposition 2.1.5 and Theorem 2.1.6, so we expand next the reduced functional and its gradient on the natural constraint $\tilde{Z}_\epsilon$.

Proposition 4.3.5. *With the choice of $\tilde{Z}_\epsilon$ in Lemma 4.3.4, if w_ϵ is given by Proposition 2.1.5, then we have*

$$\begin{aligned} \mathbf{I}_\epsilon\left(u_{\epsilon,Q}\right) &:= I_\epsilon\left(u_{\epsilon,Q} + w_\epsilon\left(u_{\epsilon,Q}\right)\right) \\ &= \tilde{C}_0 - \tilde{C}_1 \epsilon H(\epsilon Q) + e^{-2d(1+o(1))} \\ &\quad + e^{-d\left(1+\frac{\sqrt{2}\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right)(1+o(1))} + o\left(\epsilon^2\right); \end{aligned} \tag{4.3.12}$$

$$\frac{\partial}{\partial Q_T}\mathbf{I}_\epsilon\left(u_{\epsilon,Q}\right) = -\tilde{C}_1 \epsilon^2 \nabla_T H(\epsilon Q) + o\left(\epsilon^2\right); \tag{4.3.13}$$

$$\frac{\partial}{\partial Q_d}\mathbf{I}_\epsilon\left(u_{\epsilon,Q}\right) = -\tilde{C}_1 \epsilon^2 \nabla_d H(\epsilon Q) + e^{-d\left(1+\frac{\sqrt{2}\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right)(1+o(1))} + o\left(\epsilon^2\right), \tag{4.3.14}$$

as $\epsilon \to 0$, where $\tilde{C}_0$ and $\tilde{C}_1$ are as in Proposition 4.3.1 and where Q_T, Q_d are as in the proof of Proposition 4.2.13.

Proof. By Propositions 2.1.5 and 4.2.12 we have that

$$\begin{aligned}\left\|w_\epsilon\left(u_{\epsilon,Q}\right)\right\| &\leqslant C_1\left\|I'_\epsilon\left(u_{\epsilon,Q}\right)\right\| \\ &\leqslant C\left(\epsilon^2+\epsilon e^{-d(1+o(1))}\right) \\ &\quad +C\left(e^{-d\left(\frac{1}{2}\sqrt{\frac{D\tan\alpha(\tan\alpha+1)}{\tan^2\alpha+1}}+\frac{2\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right)(1+o(1))}\right. \\ &\quad\left. +e^{-d\left(\frac{p}{2}+\frac{\sqrt{2}\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right)(1+o(1))}+e^{-\frac{d(p+1)}{2}(1+o(1))}\right).\end{aligned}$$

From the regularity of I_ϵ and Proposition 4.3.1 we then have

$$\begin{aligned}&I_\epsilon\left(u_{\epsilon,Q}+w_\epsilon\left(u_{\epsilon,Q}\right)\right) \\ &\quad = I_\epsilon\left(u_{\epsilon,Q}\right)+I'_\epsilon\left(u_{\epsilon,Q}\right)\left[w_\epsilon\left(u_{\epsilon,Q}\right)\right]+o\left(\left\|w_\epsilon\left(u_{\epsilon,Q}\right)\right\|^2\right) \\ &\quad = \tilde{C}_{-1}-\tilde{C}_1\epsilon H(\epsilon Q)+e^{-2d(1+o(1))}+e^{-d\left(1+\frac{\sqrt{2}\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right)(1+o(1))} \\ &\qquad +o\left(\epsilon^2\right)+o\left(\epsilon^{6-2\delta}+\epsilon^{(p+1)(2-\delta)}\right. \\ &\qquad\quad \left.+\epsilon^{(2-\delta)\left(\sqrt{\frac{D\tan\alpha(\tan\alpha+1)}{\tan^2\alpha+1}}+\frac{4\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right)}+\epsilon^{(2-\delta)\left(p+\frac{2\sqrt{2}\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right)}\right).\end{aligned}$$

This immediately gives (4.3.12), since $p>1$ and δ is small.

The remaining two estimates are also rather immediate for $p\geqslant 2:$ in fact in this case property $iii)'$ in Section 2.1 holds true for $\gamma=1$, so we also have $\left\|\partial_Q w_\epsilon\right\|\leqslant Cf(\epsilon)$ by the last statement in Proposition 2.1.5. This, together with the Lipschitzianity of I'_ϵ implies that

$$\begin{aligned}\frac{\partial}{\partial Q}\mathbf{I}_\epsilon\left(u_{\epsilon,Q}\right) &= I'_\epsilon\left(u_{\epsilon,Q}+w_\epsilon\right)\left[\partial_Q u_{\epsilon,Q}+\partial_Q w_\epsilon\right] \\ &= \frac{\partial}{\partial Q}I_\epsilon\left(u_{\epsilon,Q}\right)+I''_\epsilon\left(u_{\epsilon,Q}\right)\left[w_\epsilon,\partial_Q u_{\epsilon,Q}\right]+I''_\epsilon\left(u_{\epsilon,Q}\right)\left[w_\epsilon,\partial_Q w_\epsilon\right] \\ &\quad +\left\|w_\epsilon\right\|^{\gamma+1}\left(\left\|\partial_Q u_{\epsilon,Q}\right\|+\left\|\partial_Q w_\epsilon\right\|\right) \\ &= \frac{\partial}{\partial Q}I_\epsilon\left(u_{\epsilon,Q}\right)+o\left(f(\epsilon)^2\right) \\ &= \frac{\partial}{\partial Q}I_\epsilon\left(u_{\epsilon,Q}\right)+o\left(\epsilon^{6-2\delta}+\epsilon^{(p+1)(2-\delta)}\right. \\ &\quad \left.+\epsilon^{(2-\delta)\left(\sqrt{\frac{D\tan\alpha(\tan\alpha+1)}{\tan^2\alpha+1}}+\frac{4\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right)}+\epsilon^{(2-\delta)\left(p+\frac{2\sqrt{2}\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right)}\right),\end{aligned} \tag{4.3.15}$$

since $\gamma=1$. The last two estimates then follow from Proposition 4.3.2.

For the case $1<p<2$, we reason as in the proof of Proposition 4.5 in [31] to obtain the estimates. This concludes the proof. □

4.3.3 *Proof of Theorem* 4.0.6

We use degree theory and the previous expansions. First of all, since Q is non degenerate for $H|_\Gamma$, we can find a small neighborhood V of Q in Γ such that $\nabla H|_\Gamma\neq 0$ on ∂V and such that in some set of coordinates

$$\deg\left(\nabla H|_\Gamma,V,0\right)\neq 0.$$

Then, if δ is as in Lemma 4.3.4, we choose $0 < \beta < \frac{\delta}{2}$, and consider the set

$$Y = \{(d, Q) : d \in ((2-\beta)|\log \epsilon|, (2+\beta)|\log \epsilon|), \epsilon Q \in V\}.$$

Since $\nabla H|_\Gamma(Q)$ corresponds to $\nabla_T H(\epsilon Q)$ in the scaled domain Ω_ϵ, by using (4.3.13) and our choice of V we know that, as $\epsilon \to 0$

$$\nabla_{Q_T} \mathbf{I}_\epsilon(u_{\epsilon,Q}) = -\tilde{C}_1 \epsilon^2 \nabla_T H(\epsilon Q) + o\left(\epsilon^2\right) \neq 0 \qquad \text{on } \frac{1}{\epsilon}\partial V. \tag{4.3.16}$$

On the other hand, by (4.3.14) we also have

$$\nabla_{Q_d} \mathbf{I}_\epsilon(u_{\epsilon,Q}) = -\epsilon^{(2-\beta)\left(1+\frac{\sqrt{2}\tan\alpha}{\sqrt{\tan^2\alpha+1}}\right)} \qquad \text{for } d = (2-\beta)|\log \epsilon|, \tag{4.3.17}$$

and

$$\nabla_{Q_d} \mathbf{I}_\epsilon(u_{\epsilon,Q}) = -\tilde{C}_1 \epsilon^2 \nabla_d H(\epsilon Q) + o\left(\epsilon^2\right), \qquad \text{for } d = (2+\beta)|\log \epsilon|. \tag{4.3.18}$$

Since we are assuming that the gradient of H points toward $\partial_D \Omega$ near the interface Γ, $\nabla_d H(\epsilon Q)$ is negative and therefore the two d-derivatives in the last two formulas have opposite signs. It follows from the product formula for the degree and (4.3.16)-(4.3.18) that

$$\deg(\nabla \mathbf{I}_\epsilon, Y, 0) = -\deg(\nabla H|_\Gamma, V, 0) \neq 0,$$

which proves the existence of a critical point for $\mathbf{I}_\epsilon$ in Y. Since we can choose V and β arbitrarily small, the solution has the asymptotic behavior required by the theorem, and more precisely by Remark 1.4.1 (b): the uniqueness of the global maximum follows from the asymptotics of the solution and standard elliptic regularity estimates.

Remark 4.3.6. To prove also the assertion in Remark 1.4.1 (a), using (4.3.12) in the case of local maximum it is easy to construct an open set of Z_ϵ where the maximum of $\mathbf{I}_\epsilon$ at the interior is strictly larger than the maximum at the boundary. On the other hand, when we have a local minimum, one can construct a mountain-pass path connecting the two points parametrized by $\left(\frac{1}{\epsilon}Q, (2-\beta)|\log \epsilon|\right)$ and $\left(\frac{1}{\epsilon}Q, (2+\beta)|\log \epsilon|\right)$. Using a suitably truncated pseudo-gradient flow, one can prove that the evolution of the path remains inside $\frac{1}{\epsilon}V \times ((2-\beta)|\log \epsilon|, (2+\beta)|\log \epsilon|)$, and still find a critical point of $\mathbf{I}_\epsilon$.

Part III

EXISTENCE PROBLEMS INVOLVING THE FRACTIONAL LAPLACIAN

5 EXISTENCE AND SYMMETRY RESULTS FOR A SCHRÖDINGER TYPE PROBLEM INVOLVING THE FRACTIONAL LAPLACIAN

INTRODUCTION

In this chapter we will deal with the following problem

$$\begin{cases} (-\Delta)^s u + u = |u|^{p-1} u \ \text{ in } \mathbb{R}^n, \\ u \in H^s(\mathbb{R}^n), \ u \not\equiv 0, \end{cases} \tag{5.0.1}$$

where $H^s(\mathbb{R}^n)$ denotes the fractional Sobolev space; we immediately refer to Section 5.1.2 for the definitions of the space $H^s(\mathbb{R}^n)$ and of variational solutions to (5.0.1).

Precisely, we are interested in existence and symmetry properties of the variational solutions u to (5.0.1), as stated in the following

Theorem 5.0.7. *Let $s \in (0,1)$ and $p \in (1, (n+2s)/(n-2s))$, with $n > 2s$. There exists a solution $u \in H^s(\mathbb{R}^n)$ to problem (5.0.1) which is positive and spherically symmetric.*

Note that the upper bound on the exponent p is exactly $2^*_s + 1$, where $2^*_s = 2n/(n-2s)$ is the critical Sobolev exponent of the embedding $H^s \hookrightarrow L^p$.

The proof of Theorem 5.0.7 extends part of that of Theorem 2 in [9]; in particular, we will apply the variational approach by the constrained method mentioned above, for the energy functional related to (5.0.1), that is

$$\mathcal{E}(u) := \frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dx \, dy + \int_{\mathbb{R}^n} \left(\frac{1}{2} |u(x)|^2 - \frac{1}{p+1} |u(x)|^{p+1} \right) dx. \tag{5.0.2}$$

It is worth mentioning that the results in Theorem 5.0.7 for $n = 1$ have been obtained in [83], where modulation stability of ground states solitary wave solutions of nonlinear Schrödinger equations has been studied, via an *unconstrained* variational approach within the "concentration-compactness" framework of P. L. Lions ([48, 49]). Also, in the more recent papers [50] and [51], an alternative approach has been presented, which permits to handle a very general context, also including the equations we are dealing with (see, in addition, [53], where the decay of solutions is analyzed in the case $s = 1/2$).

Here, we will present a very simple proof, whose general strategy will follow the original argument in [9]. The method used here (and in [9]) relies on the selection of a specific minimizing sequence composed of radial functions: though this idea is now classical, we thought it was interesting to point out that this argument also works in the case of the fractional Laplacian. Clearly, we need to operate various technical modifications due to the non-locality of the fractional Laplacian operator (and of the correspondent norm $H^s(\mathbb{R}^n)$). Moreover, we will need some energy estimates and preliminary results, also including the analogue of the classical Polya-Szegö inequality, as given in forthcoming Section 5.1.3.

As for the precise decay of the solution found, a precise bound may be obtained via the construction of exact barriers (see Lemma 3.1 in [71] and, also, Lemma 8 in [64]). Also, it could be taken into account to extend all the results above in order to investigate a problem of type (5.0.1) by substituting the nonlinearity with an odd continuous function satisfying standard growth assumptions, in the same spirit of [9].[1]

1 After completing this project, we have heard of an interesting work, where related results have been presented by using different techniques (see [28]).

The chapter is organized as follows. In Section 5.1 below, we fix notation and we state and prove some preliminary results. Section 5.2 is devoted to the proof of Theorem 5.0.7.

5.1 PRELIMINARY RESULTS

In this section, we state and prove a few preliminary results that we will need in the rest of the paper. First, we will recall some definitions involving the fractional Laplacian operator and we give the definition of the solutions to the problem we are dealing with.

5.1.1 *Notation*

We consider the Schwartz space $\mathscr{S}$ of rapidly decaying C^∞ functions in $\mathbb{R}^n$, with the corresponding topology generated by the seminorms

$$p_k(\varphi) = \sup_{x\in\mathbb{R}^n} (1+|x|)^k \sum_{|\alpha|\leqslant k} |D^\alpha \varphi(x)|, \quad k = 0, 1, 2, \dots,$$

where $\varphi \in \mathscr{S}(\mathbb{R}^n)$. Let $\mathscr{S}'(\mathbb{R}^n)$ be the set of all tempered distributions, that is the topological dual of $\mathscr{S}(\mathbb{R}^n)$. As usual, for any $\varphi \in \mathscr{S}(\mathbb{R}^n)$, we denote by

$$\mathscr{F}\varphi(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} \varphi(x)\, dx$$

the Fourier transform of φ and we recall that one can extend $\mathscr{F}$ from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}'(\mathbb{R}^n)$.

For any $s \in (0,1)$, the fractional Sobolev space $H^s(\mathbb{R}^n)$ is defined by

$$H^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) \;:\; \frac{|u(x)-u(y)|}{|x-y|^{\frac{n}{2}+s}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n) \right\}, \tag{5.1.1}$$

endowed with the natural norm

$$\|u\|_{H^s(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |u|^2\, dx + \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dx\, dy \right)^{\frac{1}{2}},$$

where the term

$$[u]_{H^s(\mathbb{R}^n)} = \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^n)} := \left(\iint_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dx\, dy \right)^{\frac{1}{2}} \tag{5.1.2}$$

is the so-called *Gagliardo semi-norm* of u.

5.1.2 *A few basic results on the fractional Laplacian and setting of the problem*

In the following, we make use of equivalent definitions of the fractional Laplacian and the Gagliardo semi-norm via the Fourier transform. Indeed, the fractional Laplacian $(-\Delta)^s$ can be seen as a pseudo-differential operator of symbol $|\xi|^s$, as stated in the following

Proposition 5.1.1. (see, e.g., *[22, Proposition 3.3]* or *[78, Section 3]*). *Let $s \in (0,1)$ and let $(-\Delta)^s : \mathscr{S} \to L^2(\mathbb{R}^n)$ be the fractional operator defined by* (1.5.2)*. Then, for any $u \in \mathscr{S}$,*

$$(-\Delta)^s u = \mathscr{F}^{-1}(|\xi|^{2s}(\mathscr{F}u)) \quad \forall \xi \in \mathbb{R}^n,$$

up to a multiplicative constant.

Analogously, one can see that the fractional Sobolev space $H^s(\mathbb{R}^n)$, given by (5.1.1), can be defined via the Fourier transform as follows

$$H^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) \ : \ \int_{\mathbb{R}^n} (1 + |\xi|^{2s}) |\mathscr{F} u(\xi)|^2 \, d\xi < +\infty \right\}. \tag{5.1.3}$$

This is a natural consequence of the equivalence stated in the following proposition, whose proof relies on the Plancherel formula.

Proposition 5.1.2. (see, e.g., *[22, Proposition 3.4]). Let $s \in (0,1)$. For any $u \in H^s(\mathbb{R}^n)$*

$$[u]^2_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\xi|^{2s} |\mathscr{F} u(\xi)|^2 \, d\xi, \tag{5.1.4}$$

up to a multiplicative constant.

Finally, we recall the definition of variational solutions $u \in H^s(\mathbb{R}^n)$ to

$$(-\Delta)^s u + u = |u|^{p-1} u \quad \text{in } \mathbb{R}^n, \quad u \not\equiv 0, \tag{5.1.5}$$

where $p > 1$.

For any $s \in (0,1)$, a measurable function $u : \mathbb{R}^n \to \mathbb{R}$ is a variational solution to (5.1.5) if

$$\begin{aligned} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\big(u(x) - u(y)\big)\big(\varphi(x) - \varphi(y)\big)}{|x-y|^{n+2s}} \, dx \, dy + \int_{\mathbb{R}^n} u(x) \varphi(x) \, dx \\ = \int_{\mathbb{R}^n} |u(x)|^{p-1} u(x) \varphi(x) \, dx, \end{aligned} \tag{5.1.6}$$

for any function $\varphi \in C^1_0(\mathbb{R}^n)$.

As stated in the Introduction, a natural method to solve (5.1.5) is to look for critical points of the related energy functional $\mathcal{E}$ on the space $H^s(\mathbb{R}^n)$ defined in (5.0.2), that is

$$\mathcal{E}(u) := \frac{1}{2} [u]^2_{H^s(\mathbb{R}^n)} - \int_{\mathbb{R}^n} G(u) \, dx, \tag{5.1.7}$$

where $[u]_{H^s}$ is defined by 5.1.2 and we denoted by G the function

$$G(u) := \frac{1}{p+1} |u|^{p+1} - \frac{1}{2} |u|^2. \tag{5.1.8}$$

Therefore, from now on we will focus on the following variational problem

$$\min \left\{ [u]^2_{H^s(\mathbb{R}^n)} : u \in H^s(\mathbb{R}^n), \int_{\mathbb{R}^n} G(u) \, dx = 1 \right\}. \tag{5.1.9}$$

5.1.3 *Tools*

For any measurable function u consider the corresponding symmetric radial decreasing rearrangement u^*, whose classical definition and basic properties can be found, for instance, in [43, Chapter 2]. As in the classic case (i.e., the Polya-Szegö inequality [68]), also in the fractional framework the energy of u^* decreases with respect to that of u. Again, by using the Fourier characterization of $[u]_{H^s(\mathbb{R}^n)}$ given by Proposition (5.1.2), one can plainly apply the symmetrization lemma by Beckner ([8]; see also [2]) to obtain the following

Lemma 5.1.3. (see, e.g., *[65, Theorem 1.1]*). *Let $s \in (0,1)$. For any $u \in H^s(\mathbb{R}^n)$, the following inequality holds*

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u^*(x)-u^*(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \leqslant \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy, \tag{5.1.10}$$

where u^* denotes the symmetric radial decreasing rearrangement of u.

Next we recall two results which we will use in the proof of Theorem 5.0.7 (see, in particular, Step 2 there). The first one is the following *radial lemma*.

Lemma 5.1.4. *Let $u \in L^2(\mathbb{R}^n)$ be a nonnegative radial decreasing function. Then*

$$|u(x)| \leqslant \left(\frac{n}{\omega_{n-1}}\right)^{1/2} |x|^{-n/2}\|u\|_{L^2(\mathbb{R}^n)}, \quad \forall x \neq 0,$$

where ω_{n-1} is the Lebesgue measure of the unit sphere in $\mathbb{R}^n$.

Proof. Setting $r = |x|$, we have that, for every $r > 0$,

$$\|u\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |u(x)|^2\,dx \geqslant \omega_{n-1}\int_0^R |u(r)|^2 r^{n-1}\,dr \geqslant \omega_{n-1}|u(R)|^2\frac{R^n}{n},$$

where in the last inequality we used the fact that u is decreasing. □

The second result is a *compactness lemma* due to Strauss [75] (see also [9, Theorem A.I] for a simple proof).

Lemma 5.1.5. *Let $P, Q : \mathbb{R} \to \mathbb{R}$ be two continuous functions satisfying*

$$\frac{P(t)}{Q(t)} \to 0, \quad \text{as } |t| \to +\infty. \tag{5.1.11}$$

Let $u_k : \mathbb{R}^n \to \mathbb{R}$ be a sequence of measurable functions such that

$$\sup_k \int_{\mathbb{R}^n} |Q(u_k(x))|\,dx < +\infty, \tag{5.1.12}$$

and

$$P(u_k(x)) \to v(x) \quad \text{a.e. in } \mathbb{R}^n \quad \text{as } k \to +\infty. \tag{5.1.13}$$

Then, for every bounded Borel set B, we have

$$\int_B |P(u_k(x)) - v(x)|\,dx \to 0 \quad \text{as } k \to +\infty. \tag{5.1.14}$$

If we further assume that

$$\frac{P(t)}{Q(t)} \to 0 \quad \text{as } t \to 0, \tag{5.1.15}$$

and

$$u_k(x) \to 0 \quad \text{as } |x| \to +\infty, \quad \text{uniformly with respect to } k, \tag{5.1.16}$$

then $P(u_k)$ converges to v in $L^1(\mathbb{R}^n)$ as $k \to +\infty$.

We conclude this section with the following Lemma 5.1.6, in which we state and prove some H^s estimates, which, in turn, imply that there exists a nontrivial competitor for the variational problem (5.1.9), as described in the subsequent Remark 5.1.8.

Lemma 5.1.6. *Let ζ, $R > 0$. For any $t \geqslant 0$ let*

$$v_R(t) := \begin{cases} \zeta & \text{if } t \in [0, R], \\ \zeta\,(R+1-t) & \text{if } t \in (R, R+1), \\ 0 & \text{if } t \in [R+1, +\infty). \end{cases}$$

For any $x \in \mathbb{R}^n$, let $w_R(x) := v_R(|x|)$.

Then, $w_R \in H^s(\mathbb{R}^n)$ for any $s \in (0,1)$ and there exists $C(n,s,R) > 0$ such that $\|w_R\|_{H^s(\mathbb{R}^n)} \leqslant C(n,s,R)\,\zeta$.

Proof. We take $\zeta := 1$ (the general case follows by multiplication by ζ). Notice that w_R is uniformly Lipschitz and vanishes outside B_{R+1}. In particular $w_R \in H^1(B_{R+1})$. Also, if $x \in B_{R+1} \setminus B_R$ and $y \in B_{R+2} \setminus B_{R+1}$, we have

$$|w_R(x) - w_R(y)| = R + 1 - |x| \leqslant |y| - |x| \leqslant |x - y|,$$

therefore

$$\begin{aligned} &\iint_{B_{R+1}\times(\mathbb{R}^n\setminus B_{R+1})} \frac{|w_R(x)-w_R(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ \leqslant &\iint_{(B_{R+1}\setminus B_R)\times(B_{R+2}\setminus B_{R+1})} \frac{|w_R(x)-w_R(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + C_1(n,s,R) \leqslant C_2(n,s,R). \end{aligned}$$

Hence, by Proposition 2.2 in [22],

$$\begin{aligned} \|w_R\|_{H^s(\mathbb{R}^n)} &\leqslant C\left(\iint_{B_{R+1}\times(\mathbb{R}^n\setminus B_{R+1})} \frac{|w_R(x)-w_R(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \|w_R\|_{H^s(B_{R+1})}\right) \\ &\leqslant C_3(n,s,R)\left(1 + \|w_R\|_{H^1(B_{R+1})}\right) \leqslant C_4(n,s,R), \end{aligned}$$

which proves the desired result. □

Remark 5.1.7. Here is another proof of Lemma 5.1.6 based on an interpolation inequality: given $u \in H^1(\mathbb{R}^n)$, by Proposition 5.1.2, using the Hölder inequality with exponents $1/s$ and $1/(1-s)$, we have

$$\begin{aligned} [u]_{H^s(\mathbb{R}^n)} &= \sqrt{\int_{\mathbb{R}^n} |\xi|^{2s} |\mathscr{F}u(\xi)|^{2s} |\mathscr{F}u(\xi)|^{2(1-s)}\,d\xi} \\ &\leqslant \left(\int_{\mathbb{R}^n} |\xi|^2 |\mathscr{F}u(\xi)|^2\,d\xi\right)^{s/2} \left(\int_{\mathbb{R}^n} |\mathscr{F}u(\xi)|^2\,d\xi\right)^{(1-s)/2} \\ &= [u]^s_{H^1(\mathbb{R}^n)}\,\|u\|^{1-s}_{L^2(\mathbb{R}^n)}, \end{aligned}$$

which clearly implies Lemma 5.1.6 by choosing $u := w_R$.

Remark 5.1.8. By Lemma 5.1.6, the set in the minimum problem (5.1.9) is not empty. Indeed, if $w_R \in H^s(\mathbb{R}^n)$ is defined as in Lemma 5.1.6, we have that

$$\begin{aligned}\int_{\mathbb{R}^n} G\left(w_R(x)\right)\, dx &= \int_{B_{R+1}} G\left(w_R(x)\right)\, dx \\ &= \int_{B_R} G\left(w_R(x)\right)\, dx + \int_{B_{R+1}\setminus B_R} G\left(w_R(x)\right)\, dx \\ &\geqslant G\left(\zeta\right)|B_R| - |B_{R+1}\setminus B_R| \left(\max_{t\in[0,\zeta]} |G(t)|\right),\end{aligned}$$

where $|\cdot|$ denotes the Lebesgue measure. This implies that there exist two positive constants C_1 and C_2 such that

$$\int_{\mathbb{R}^n} G\left(w_R(x)\right)\, dx \geqslant C_1 R^n - C_2 R^{n-1},$$

and so we can choose $R > 0$ large enough such that $\int_{\mathbb{R}^n} G\left(w_R(x)\right)\, dx > 0$.

Now we make the scale change $w_{R,\sigma}(x) = w_R\left(x/\sigma\right)$, and a suitable choice of $\sigma > 0$, so that

$$\int_{\mathbb{R}^n} G\left(w_{R,\sigma}(x)\right)\, dx = \sigma^n \int_{\mathbb{R}^n} G\left(w_R(x)\right)\, dx = 1.$$

5.2 PROOF OF THEOREM 5.0.7

In the same spirit of the proof of Theorem 2 in [9], we divide that of Theorem 5.0.7 in a few steps. For the reader's convenience, we will give full details of the proof, by taking into account the preliminary results in Section 5.1.3 together with the modifications due to the presence of the fractional Sobolev spaces.

Proof.
Step 1 - A minimizing sequence u_k. Consider a sequence $\{u_k\} \subseteq H^s(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} G(u_k)\, dx = 1$$

and

$$\lim_{k\to+\infty} [u_k]^2_{H^s(\mathbb{R}^n)} = \inf\left\{[u]^2_{H^s(\mathbb{R}^n)} : u \in H^s(\mathbb{R}^n), \int_{\mathbb{R}^n} G(u)\, dx = 1\right\} \geqslant 0. \tag{5.2.1}$$

By triangle inequality,

$$\big||u_k(x)| - |u_k(y)|\big| \leqslant |u_k(x) - u_k(y)|$$

thus the Gagliardo semi-norm of $|u_k|$ is not bigger than the one of u_k.

So, without loss of generality, we may suppose that u_k is nonnegative. Let u_k^* denote the symmetric radial decreasing rearrangement of u_k. Then

$$\int_{\mathbb{R}^n} G(u_k^*)\, dx = \int_{\mathbb{R}^n} G(u_k)\, dx = 1$$

and so, in view of Lemma 5.1.3, we have that $\{u_k^*\}$ is also a minimizing sequence.

These observations imply that we can select a sequence $\{u_k\}$ in such a way that, for every $k \in \mathbb{N}$, u_k is nonnegative, spherically symmetric and decreasing in $r = |x|$.

Step 2 - A priori estimates for u_k. We want to obtain bounds uniform in k on $\|u_k\|_{L^q(\mathbb{R}^n)}$, for every $2 \leqslant q \leqslant 2n/(n-2s)$, and on $\|u_k\|_{H^s(\mathbb{R}^n)}$.

We begin with $\|u_k\|_{H^s(\mathbb{R}^n)}$. Clearly, by (5.2.1), $[u_k]^2_{H^s(\mathbb{R}^n)} \leqslant C$ for some positive constant C (recall also Remark 5.1.8). Therefore, it remains to prove that $\|u_k\|_{L^2(\mathbb{R}^n)}$ is bounded. To do this, we set

$$g_1(t) := |t|^{p-1}t, \qquad g_2(t) := t, \qquad G_1(t) := \frac{1}{p+1}|t|^{p+1} \qquad \text{and} \qquad G_2(t) := \frac{1}{2}|t|^2.$$

Then

$$g(t) = g_1(t) - g_2(t),$$

and so

$$G(z) = \int_0^z g(t)dt = \int_0^z g_1(t)dt - \int_0^z g_2(t)dt = G_1(z) - G_2(z), \quad \forall z \geqslant 0. \tag{5.2.2}$$

Since $1 < p < (n+2s)/(n-2s)$, we have that for every $\epsilon > 0$ there exists a positive constant C_ϵ such that $g_1(t) \leqslant C_\epsilon |t|^{\frac{n+2s}{n-2s}} + \epsilon g_2(t)$ for any $t \geqslant 0$. This implies that $G_1(z) \leqslant C_\epsilon |z|^{\frac{2n}{n-2s}} + \epsilon G_2(z)$ for any $z \geqslant 0$. Choosing $\epsilon = 1/2$, we get

$$G_1(z) \leqslant C|z|^{\frac{2n}{n-2s}} + \frac{1}{2}G_2(z). \tag{5.2.3}$$

Now, the condition $\int_{\mathbb{R}^n} G(u_k)\,dx = 1$ can be written in the following form

$$\int_{\mathbb{R}^n} G_1(u_k)\,dx = \int_{\mathbb{R}^n} G_2(u_k)\,dx + 1. \tag{5.2.4}$$

Putting together (5.2.3) and (5.2.4), we obtain

$$\frac{1}{2}\int_{\mathbb{R}^n} G_2(u_k)\,dx + 1 \leqslant C\int_{\mathbb{R}^n} |u_k|^{\frac{2n}{n-2s}}\,dx. \tag{5.2.5}$$

Now we use the fractional Sobolev embedding theorem (see, e.g., [22, Theorem 6.5]) to say that

$$\|u_n\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} \leqslant C[u_k]_{H^s(\mathbb{R}^n)},$$

where the constant C does not depend on k. Thus, since u_k is a minimizing sequence, the boundedness of $[u_k]^2_{H^s(\mathbb{R}^n)}$ yields that of $\|u_k\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)}$. By the definition of G_2, the inequality in (5.2.5) implies that

$$\frac{1}{2}\int_{\mathbb{R}^n} u_k^2\,dx = \int_{\mathbb{R}^n} G_2(u_k)\,dx \leqslant C,$$

and thus we bound $\|u_k\|^2_{L^2(\mathbb{R}^n)}$ (and so $\|u_k\|^2_{H^s(\mathbb{R}^n)}$) uniformly in k.

Finally, by the bounds on $\|u_k\|_{L^2(\mathbb{R}^n)}$ and $\|u_k\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)}$, using the Hölder inequality, we obtain that $\|u_k\|_{L^q(\mathbb{R}^n)} \leqslant C$ for every $2 \leqslant q \leqslant 2n/(n-2s)$.

Step 3 - Passage to the limit and conclusion of the proof. Since $u_k \in L^2(\mathbb{R}^n)$ is a sequence of nonnegative radial decreasing functions, we can apply Lemma 5.1.4 to get

$$|u_k(x)| \leqslant \left(\frac{n}{\omega_{n-1}}\right)^{\frac{1}{2}} |x|^{-n/2}\|u_k\|_{L^2(\mathbb{R}^n)}. \tag{5.2.6}$$

From the previous step we have that u_k is uniformly bounded in $L^2(\mathbb{R}^n)$; then $|u_k(x)| \leqslant C|x|^{-n/2}$, with C independent of k. This implies that $u_k(x) \to 0$ as $|x| \to +\infty$ uniformly with respect to k. Now, since u_k is bounded in $H^s(\mathbb{R}^n)$, we can extract a subsequence of u_k, again denoted by u_k, such that u_k converges weakly in $H^s(\mathbb{R}^n)$ and almost everywhere in $\mathbb{R}^n$ to a function $\overline{u}$. Moreover, by construction, $\overline{u} \in H^s(\mathbb{R}^n)$ is spherically symmetric and decreasing in r.

Now, in order to apply Lemma 5.1.5 (with $P := G_1$), consider the polynomial function Q defined by

$$Q(t) := t^2 + |t|^{\frac{2n}{n-2s}}.$$

Since the sequence u_k is uniformly bounded in $L^2(\mathbb{R}^n)$ and in $L^{\frac{2n}{n-2s}}(\mathbb{R}^n)$, we have that Q satisfies

$$\int_{\mathbb{R}^n} |Q(u_k(x))|\, dx = \int_{\mathbb{R}^n} \left(u_k^2(x) + |u_k(x)|^{\frac{2n}{n-2s}}\right) dx \leqslant C, \quad \text{for every } k \in \mathbb{N}.$$

Moreover, if G_1 is defined as in the previous step, by the fact that $p \in \left(1, \frac{n+2s}{n-2s}\right)$ we derive

$$\frac{G_1(t)}{Q(t)} \to 0, \quad \text{as } t \to +\infty \text{ and } t \to 0.$$

Since u_k converges almost everywhere in $\mathbb{R}^n$ to $\overline{u}$, we have that also $G_1(u_k)$ converges $G_1(\overline{u})$. Finally, $u_k(x) \to 0$ as $|x| \to +\infty$ uniformly with respect to n. Therefore Lemma 5.1.5 holds, getting

$$\int_{\mathbb{R}^n} G_1(u_k(x))\, dx \to \int_{\mathbb{R}^n} G_1(\overline{u}(x))\, dx \quad \text{as } k \to +\infty.$$

Thus, using Fatou's Lemma in (5.2.4), we obtain that

$$\int_{\mathbb{R}^n} G_1(\overline{u}(x))\, dx \geqslant \int_{\mathbb{R}^n} G_2(\overline{u}(x))\, dx + 1, \tag{5.2.7}$$

that is

$$\int_{\mathbb{R}^n} G(\overline{u}(x))\, dx \geqslant 1.$$

On the other hand, using again Fatou's Lemma, we have that

$$\begin{aligned} [\overline{u}]^2_{H^s(\mathbb{R}^n)} &\leqslant \lim_{k \to +\infty} [u_k]^2_{H^s(\mathbb{R}^n)} \\ &= \inf \left\{ [u]^2_{H^s(\mathbb{R}^n)} : u \in H^s(\mathbb{R}^n), \int_{\mathbb{R}^n} G(u)\, dx = 1 \right\}. \end{aligned} \tag{5.2.8}$$

Now, suppose by contradiction that $\int_{\mathbb{R}^n} G(\overline{u}(x))\, dx > 1$. Then, by the scale change $\overline{u}_\sigma(x) = \overline{u}(x/\sigma)$, we have

$$\int_{\mathbb{R}^n} G(\overline{u}_\sigma(x))\, dx = \sigma^n \int_{\mathbb{R}^n} G(\overline{u}(x))\, dx = 1 \tag{5.2.9}$$

for some

$$\sigma \in (0, 1). \tag{5.2.10}$$

Moreover, we have

$$\begin{aligned} [\overline{u}_\sigma]^2_{H^s(\mathbb{R}^n)} &= \sigma^{n-2s} [\overline{u}]^2_{H^s(\mathbb{R}^n)} \\ &\leqslant \sigma^{n-2s} \inf \left\{ [u]^2_{H^s(\mathbb{R}^n)} : u \in H^s(\mathbb{R}^n), \int_{\mathbb{R}^n} G(u)\, dx = 1 \right\}, \end{aligned} \tag{5.2.11}$$

due to (5.2.8), and

$$\inf\left\{[u]^2_{H^s(\mathbb{R}^n)} : u \in H^s(\mathbb{R}^n), \int_{\mathbb{R}^n} G(u)\,dx = 1\right\} \leqslant [\overline{u}_\sigma]^2_{H^s(\mathbb{R}^n)},$$

thanks to (5.2.9). Combining the last two inequalities and recalling (5.2.10), we get

$$\inf\left\{[u]^2_{H^s(\mathbb{R}^n)} : u \in H^s(\mathbb{R}^n), \int_{\mathbb{R}^n} G(u)\,dx = 1\right\} = 0,$$

hence also $[\overline{u}]^2_{H^s(\mathbb{R}^n)} = 0$. Then $\overline{u} \equiv 0$, which is in contradiction with (5.2.7). Therefore, $\int_{\mathbb{R}^n} G\left(\overline{u}(x)\right)\,dx = 1$ and $[\overline{u}]_{H^s(\mathbb{R}^n)} = \inf\left\{[u]_{H^s(\mathbb{R}^n)} : u \in H^s(\mathbb{R}^n), \int G(u)\,dx = 1\right\}$; that is, $\overline{u}$ solves the minimization problem (5.1.9). $\square$

Part IV

FURTHER PROJECTS AND PERSPECTIVES

Concentration phenomena of mountain pass solutions. An interesting topic of research is to detect whether the mountain pass solutions of a mixed Dirichlet and Neumann problem in non-smooth domains concentrate either at interface points or at the interior points of the Neumann part. In particular, we would like to study the boundary concentration phenomena of the equation

$$\begin{cases} -\epsilon^2 \Delta u + u = u^p & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial_N \Omega, & u = 0 \quad \text{on } \partial_D \Omega, \\ u > 0 & \text{in } \Omega, \end{cases}$$

where Ω is a Lipschitz (but not smooth) set, $p \in \left(1, \frac{n+2}{n-2}\right)$ and $\partial_N \Omega$, $\partial_D \Omega$ are two subsets of the boundary of Ω such that the union of their closures coincides with the whole $\partial\Omega$, and their intersection is an $(n-2)$-dimensional smooth singularity.

The main difficulty with respect to the smooth case (dealt with in [32]) lies in the use of the moving plane method, which can be obstructed by the presence of the angle. To deal with this difficulty, we plan to obtain a careful asymptotic expansion of the solution near the angle, which should allow us to use the moving plane, at least for some range of the opening of the angle.

Also, it would be interesting to exclude concentration on the Neumann part by adapting the technique of [32].

Concentration phenomena for fractional operators. We would like to investigate also the concentration phenomena of elliptic equations driven by the fractional Laplacian. For instance, a natural question is whether the technique developed in this thesis may be adapted to the construction of solutions concentrating either in the interior or along the boundary of the domain for an equation of the type

$$\epsilon^{2s}(-\Delta)^s u + u = u^p, \qquad \text{for } s \in (0, 1).$$

With regard to this, a first step was performed in [26] where we constructed solutions in the whole of the space which might be used as the leading order of a perturbation argument. This project is very challenging and to complete it one needs to overcome several conceptual complications that arise in the fractional setting.

First of all, to apply the perturbation argument, some type of non-degeneracy condition is required. Checking this condition will be much harder than in the classical case, since it is usually very difficult to compute explicitly fractional derivatives and singular integrals. A partial answer in this direction is given by a very recent preprint [20], where a non-degeneracy condition was checked in a particular case.

It may be possible that a dramatic change of perspective is needed to attack the non-degeneracy condition in the fractional case, for instance by reducing to check such condition in a somehow generic sense or only for some specific choice of operators.

Also, we recall that the decay of the solutions in the fractional case is only polynomial, and not exponential as it happens in the classical case and this feature is, of course, the source of major complications. As a matter of fact, a slow decay of the solutions in the entire space may reflect an additional difficulty in localizing possible interior concentration points.

Moreover, if dealing with Neumann or mixed boundary conditions, one needs to understand how such prescription translates into the fractional setting. The main difficulty is that solutions are, in general, uniformly continuous but not C^1 up to the boundary, hence the normal derivative is not defined in this case. One possibility could be to replace the normal derivative at a point $x_0 \in \partial\Omega$ with a fractional derivative of order s of the type

$$\lim_{\Omega \ni x \to x_0} \frac{u(x) - u(x_0)}{\big(\mathrm{dist}(x, \partial\Omega)\big)^s}.$$

Another possibility could be to test the associated Euler-Lagrange functional on the functions with finite H^s-norm.

Fractional perimeters. With respect to asymptotic features of fractional problems, we think that a nice topic of research is also given by the s-perimeters. Namely the following functional is studied in [13]: given a fixed bounded domain Ω and a measurable set $E \subseteq \mathbb{R}^n$, let

$$\mathrm{Per}_s(E;\Omega) := \mathcal{J}_s(E \cap \Omega, \mathbb{R}^n \setminus E) + \mathcal{J}_s(\Omega \setminus E, E \setminus \Omega),$$

with

$$\mathcal{J}_s(A,B) := \left(\frac{1}{2} - s\right) s \iint_{A \times B} \frac{dx\, dy}{|x-y|^{n+2s}}$$

for any disjoint, measurable sets A, B and for a fixed $s \in (0, 1/2)$. The reason for which the above functional may be seen as a fractional perimeter is that, as $s \nearrow 1/2$ the above functional approaches the classical perimeter of E in Ω, up to a normalizing dimensional constant and in a sense made precise in [14] and [6].

An interesting topic of research is, we believe, the asymptotics of this fractional perimeter as $s \searrow 0$. A first result in this direction was given by the recent preprint [25], where the limit as $s \searrow 0$ of the fractional perimeter is related to a convex combination of the Lebesgue measures of $\Omega \cap E$ and $\Omega \setminus E$, the interpolation parameter being given by an averaged volume of $E \setminus \Omega$ (provided that the limit exists, some counterexamples are also constructed in [25]). This topic of research probably deserves some further investigation, in terms of geometric and functional convergence. For instance, it would be desirable to decide whether some uniform limit on the behavior of the set may be obtained or whether the minimality conditions make sense in the limit. To answer these type of questions some new idea is needed since the constants of the geometric and functional estimates in [13], [14] and [6] may blow-up as $s \searrow 0$. Also, it would be desirable to build perturbations argument that bifurcate from $s = 0$ towards a small $s > 0$. This is again a challenging problem, since the case $s = 0$ does not seem to have any regularity theory.

Free boundary problems. Another interesting topic of research is trying to better understand the free boundary problems for the fractional Laplacian, for instance when we are in presence of two phases driven by different powers of the Laplacian. Some technical difficulties here are that the fractional Dirichlet energy is not additive (different from the classical case) and that the set in which the phase change may charge the energy too (because it may interact with both the phases due to the non-local energy effect), so we expect that the regularity theory in this case is considerably more subtle than in the local framework.

Elliptic systems. Other research projects concern the systems of the semilinear elliptic partial differential equations arising in biology in order to study the cohexistence and segregation of different species, such as

$$\begin{cases} \Delta u = uv^2, \\ \Delta v = vu^2, \\ u, v > 0. \end{cases}$$

Following the work in [11] we know that the positive solutions of these systems of equations in low dimension possess several geometric properties and enjoy additional symmetry features. It would be interesting to generalize these results to other types of nonlinearities and operators. For instance, one should be able to include the singular or degenerate cases driven by the p-Laplacian operator, the case of fractional diffusion and the one of stratified hambient space. To consider these cases, it will be useful to follow the geometric technique of [27], as extended in [69] and [70].

[1] R. A. Adams: *Sobolev Spaces*. Academic Press, New York (1975).

[2] F. J. Almgren, E. H. Lieb: Symmetric decreasing rearrangement is sometimes continuous. *J. Amer. Math. Soc.* 2 (1989), 683–773.

[3] A. Ambrosetti, M. Badiale, S. Cingolani: Semiclassical states of nonlinear Schrödinger equations. *Arch. Rat. Mech. Anal.* 140 (1997), 285–300.

[4] A. Ambrosetti, A. Malchiodi: *Perturbation Methods and Semilinear Elliptic Problems on* $\mathbb{R}^n$. Birkhäuser, Progr. in Math. 240 (2005).

[5] A. Ambrosetti, A. Malchiodi, S. Secchi: Multiplicity results for some nonlinear singularly perturbed elliptic problems on $\mathbb{R}^n$. *Arch. Rat. Mech. Anal.* 159 (2001), no. 3, 253–271.

[6] L. Ambrosio, G. De Philippis and L. Martinazzi: Γ-convergence of nonlocal perimeter functionals. *Manuscripta Math* 134 (2011), 377–403.

[7] B. Barrios, E. Colorado, A. De Pablo and U. Sánchez: On some Critical Problems for the Fractional Laplacian Operator. *J. Differential Equations*, **252**, n. 11, 6133–6162 (2012)

[8] W. Beckner: Sobolev inequalities, the Poisson semigroup, and analysis on the sphere on S^N. *Proc. Natl. Acad. Sci.* 89 (1992), 4816–4819.

[9] H. Berestycki, P.-L. Lions: Nonlinear scalar field equations, I. Existence of a ground state. *Arch. Rational Mech. Anal.* 82 (1983), no. 4, 313–345.

[10] H. Berestycki, P.-L. Lions: Nonlinear scalar field equations, II. Existence of infinitely many solutions. *Arch. Rational Mech. Anal.* 82 (1983), no. 4, 347–375.

[11] H. Berestycki, S. Terracini, K. Wang, J. Wei: Existence and stability of entire solutions of an elliptic system modeling phase separation. *Preprint* (2012).

[12] C. Brändle, E. Colorado, A. de Pablo: A concave-convex elliptic problem involving the fractional Laplacian. To appear in *Proc. Roy. Soc. Edinburgh Sect. A* (2012), `http://arxiv.org/abs/1006.4510v2`

[13] L. Caffarelli, J.-M. Roquejoffre and O. Savin: Non-local minimal surfaces. *Comm. Pure Appl. Math.* 63 (2010), 1111–1144.

[14] L. Caffarelli, E. Valdinoci: Uniform estimates and limiting arguments for nonlocal minimal surfaces. *Calc. Var. Partial Differential Equations* 41 (2011), no. 1-2, 203–240.

[15] P. Cannarsa, C. Sinestrari: *Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control*. Progress in Nonlinear Differential Equations and their Applications 58, Birkhäuser, Boston (2004).

[16] R.G. Casten, C.J. Holland: Instability results for reaction diffusion equations with Neumann boundary conditions. *J. Diff. Eq.* 27 (1978), no. 2, 266–273.

[17] I. Chavel: *Eigenvalues in Riemannian geometry*. Academic Press, New York (1984).

[18] E. N. Dancer, J. Wei: On the effect of domain topology in a singular perturbation problem. *Topol. Methods Nonlinear Anal.* 11 (1998), no. 2, 227–248.

[19] E. N. Dancer, S. Yan: Multipeak solutions for a singularly perturbed Neumann problem. *Pacific J. Math.* 189 (1999), no. 2, 241–262.

[20] J. Dávila, M. Del Pino, Y. Sire: Non degeneracy of the bubble in the critical case for non local equations. *Preprint* (2012).

[21] M. Del Pino, P. Felmer, J. Wei: On the role of the mean curvature in some singularly perturbed Neumann problems. *S.I.A.M. J. Math. Anal.* 31 (1999), 63–79.

[22] E. Di Nezza, G. Palatucci, E. Valdinoci: Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. math.*, 136 (2012), no. 5, 521–573.

[23] S. Dipierro: Concentration of solutions for a singularly perturbed Neumann problem in non-smooth domains. *Ann. Inst. H. Poincaré (C) Anal. Non Linéaire* 28 (2011), no. 1, 107–126.

[24] S. Dipierro: Concentration of solutions for a singularly perturbed mixed problem in non-smooth domains. *J. Differential Equations* 254 (2013), no.1, 30–66.

[25] S. Dipierro, A. Figalli, G. Palatucci, E. Valdinoci: Asymptotics of the s-perimeter as $s \searrow 0$. *Discrete Contin. Dyn. Syst.*, `http://arxiv.org/abs/1204.0750`

[26] S. Dipierro, G. Palatucci, E. Valdinoci: Existence and symmetry results for a Schrödinger type problem involving the fractional Laplacian. *Le Matematiche* (2012), `http://arxiv.org/abs/1202.0576`http://arxiv.org/abs/1202.0576

[27] A. Farina, B. Sciunzi, E. Valdinoci: Bernstein and De Giorgi type problems: new results via a geometric approach. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 7, no. 4, 741–791 (2008).

[28] P. Felmer, A. Quaas, J. Tan: Positive solutions of nonlinear Schrödinger equation with the fractional Laplacian. *Proc. Roy. Soc. Edinburgh Sect A.*

[29] A. Floer, A. Weinstein: Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential. *J. Funct. Anal.* 69 (1986), 397–408.

[30] R. L. Frank, E. Lenzmann: Uniqueness and nondegeneracy of ground states for $(-\Delta)^s Q + Q - Q^{\alpha-1} = 0$ in $\mathbb{R}$. *Acta Math.*, `http://arxiv.org/abs/1009.4042v1`

[31] J. Garcia Azorero, A. Malchiodi, L. Montoro, I. Peral: Concentration of solutions for some singularly perturbed mixed problems. Part I: existence results. *Archive Rat. Mech. Anal.* 196 (2010), no. 3, 907–950.

[32] J. Garcia Azorero, A. Malchiodi, L. Montoro, I. Peral: Concentration of solutions for some singularly perturbed mixed problems. Part II: asymptotic of minimal energy solutions. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 27 (2010), 37–56.

[33] A. Gierer, H. Meinhardt: A theory of biological pattern formation. *Kybernetik (Berlin)* 12 (1972), 30–39.

[34] P. Grisvard: *Elliptic problems in nonsmooth domains*. Pitman, London (1985).

[35] H. Groemer: *Geometric applications of Fourier series and spherical harmonics*. Encyclopedia of Mathematics and its Application 61, Cambridge University Press, Cambridge (1996).

[36] M. Grossi: Some results on a class of nonlinear Schrödinger equations. *Math. Z.* 235 (2000), no. 4, 687–705.

[37] M. Grossi, A. Pistoia, J. Wei: Existence of multipeak solutions for a semilinear Neumann problem via non smooth critical point theory. *Calc. Var. Partial Differential Equations* 11 (2000), no. 2, 143–175.

[38] C. Gui: Multipeak solutions for a semilinear Neumann problem. *Duke Math. J.* 84 (1996), no. 3, 739–769.

[39] C. Gui, J. Wei: Multiple interior peak solutions for some singularly perturbed Neumann problems. *J. Differential Equations* 158 (1999), no. 1, 1–27.

[40] C. Gui, J. Wei: On multiple mixed interior and boundary peak solutions for some singularly perturbed Neumann problems. *Canad. J. Math.* 52 (2000), no. 3, 522–538.

[41] C. Gui, J. Wei, M. Winter: Multiple boundary peak solutions for some singularly perturbed Neumann problems. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 17 (2000), no. 1, 47–82.

[42] E. W. Hobson: *The theory of Spherical and Ellipsoidal Harmonics*. Chelsea Pub. Co. (1955).

[43] B. Kawohl: *Rearrangements and convexity of level sets in PDE*. Springer- Verlag, Berlin, (1985).

[44] M.K. Kwong: Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^n$. *Arch. Rat. Mech. Anal.* 105 (1989), 243–266.

[45] Y. Y. Li: On a singularly perturbed equation with Neumann boundary conditions. *Comm. Partial Differential Equations* 23 (1998), no. 3-4, 487–545.

[46] Y.Y. Li, L. Nirenberg: The Dirichlet problem for singularly perturbed elliptic equation. *Comm. Pure Appl. Math.* 51 (1998), 1445–1490.

[47] C.S. Lin, W.M. Ni, I. Takagi: Large amplitude stationary solutions to a chemotaxis systems. *J. Differential Equations* 72 (1988), 1–27.

[48] P.-L. Lions: The concentration compactness principle in the calculus of variations: the locally compact case. Parts 1. *Ann. I.H.P. Anal. Nonlin.* 1 (1984), no. 2, 109–145.

[49] P.-L. Lions: The concentration compactness principle in the calculus of variations: the locally compact case. Parts 2. *Ann. I.H.P. Anal. Nonlin.* 1 (1985), no. 4, 223–283.

[50] O. Lopes: Nonlocal variational problems arising in long wave propagation. *ESAIM: Control, Optimization and Calculus of Variations* 5 (2000), 501–528.

[51] O. Lopes, M. Maris: Symmetry of minimizers for some nonlocal variational problems. *J. Funct. Anal.* 254 (2008), 535–592.

[52] A. Malchiodi: *Concentration of solutions for some singularly perturbed Neumann problems*. Geometric analysis and PDEs, 63–115, Lecture Notes in Math., 1977, Springer, Dordrecht (2009).

[53] M. Maris: On the existence, regularity and decay of solitary waves to a generalized Benjamin-Ono equation. *Nonlinear Analysis* 51 (2002), 1073–1085.

[54] H. Matano: Asymptotic behavior and stability of solutions of semilinear diffusion equations. *Publ. Res. Inst. Math. Sci.* 15 (1979), 401–454.

[55] C. Müller: *Analysis of spherical symmetries in euclidean spaces*. Applied Mathematical Sciences 129, Springer-Verlag, New York (1998).

[56] C. Müller: *Spherical Harmonics*. Lecture Notes in Math. 17, Springer-Verlag, Berlin, Heidelberg, New York (1966).

[57] W.M. Ni: Diffusion, cross-diffusion, and their spike-layer steady states. *Notices Amer. Math. Soc.* 45 (1998), no. 1, 9–18.

[58] W. M. Ni, X. B. Pan, I. Takagi: Singular behavior of least-energy solutions of a semilinear Neumann problem involving critical Sobolev exponents. *Duke Math. J.* 67 (1992), no. 1, 1–20.

[59] W.M. Ni, I. Takagi: On the shape of least-energy solution to a semilinear Neumann problem. *Comm. Pure Appl. Math.* 41 (1991), 819–851.

[60] W. M. Ni, I. Takagi: Locating the peaks of least-energy solutions to a semilinear Neumann problem. *Duke Math. J.* 70 (1993), 247–281.

[61] W.M. Ni, I. Takagi, E. Yanagida: Stability of least energy patterns of the shadow system for an activator-inhibitor model. Recent topics in mathematics moving toward science and engineering. *Japan J. Indust. Appl. Math.* 18 (2001), no. 2, 259–272.

[62] W.M. Ni, J. Wei: On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems. *Comm. Pure Appl. Math.* 48 (1995), no. 7, 731–768.

[63] Y.G. Oh: On positive Multi-lump bound states of nonlinear Schrödinger equations under multiple well potentials. *Comm. Math. Phys.* 131 (1990), 223–253.

[64] G. Palatucci, O. Savin, E. Valdinoci: Local and global minimizers for a variational energy involving a fractional norm. *Ann. Mat. Pura App.*, DOI: 10.1007/s10231-011-0243-9, `http://dx.doi.org/10.1007/s10231-011-0243-9`

[65] Y.J. Park: Fractional Polya-Szegö inequality. *J. Chungcheong Math. Soc.* 24 (2011), no. 2, 267–271.

[66] A. Pisante, G. Palatucci: Sobolev embeddings and concentration-compactness alternative for fractional Sobolev spaces. *Preprint* (2011), `http://mipa.unimes.fr/preprints.html`

[67] S. Pohozaev: Eigenfunctions of the Equations $\Delta u = \lambda f(u)$. *Soviet Math. Dkl.* 6 (1965), 1408–1411.

[68] G. Polya, G. Szegö: Inequalities for the capacity of a condenser. *Amer. J. Math.* 67 (1945), 1–32.

[69] Y. Sire, E. Valdinoci: Fractional Laplacian phase transitions and boundary reactions: a geometric inequality and a symmetry result. *J. Funct. Anal.* 256 (2009), no. 6, 1842–1864.

[70] O. Savin, E. Valdinoci: Elliptic PDEs with fibered nonlinearities. *J. Geom. Anal.* 19 (2009), no. 2, 420–432.

[71] O. Savin, E. Valdinoci: Density estimates for a variational model driven by the Gagliardo norm. Preprint (2011), http://arxiv.org/abs/1007.2114v3

[72] R. Servadei, E. Valdinoci: Variational methods for non-local operators of elliptic type. *Preprint* (2011), http://www.ma.utexas.edu/mp_arc-bin/mpa?yn=11-131

[73] R. Servadei, E. Valdinoci: Mountain Pass solutions for non-local elliptic operators. *J. Math. Anal. Appl.* 389 (2012), 887–898.

[74] J. Shi: Semilinear Neumann boundary value problems on a rectangle. *Trans. Amer. Math. Soc.* 354 (2002), 3117–3154.

[75] W. A. Strauss: Existence of solitary waves in higher dimensions. *Comm. Math. Phys.* 55 (1977), 149–162.

[76] J. Tan: The Brezis-Nirenberg type problem involving the square root of the Laplacian. *Calc. Var. Partial Differential Equations* 42 (2012), no. 1, 21–41.

[77] A. M. Turing: The chemical basis of morphogenesis. *Phil. Trans. Royal Soc. London, Series B, Biological Sciences* 237 (1952), 37–72.

[78] E. Valdinoci: From the lung jump random walk to the fractional Laplacian. *Bol. Soc. Esp. Mat. Apl.* S$\vec{e}$MA 49 (2009), 33–44.

[79] Z. Q. Wang: On the existence of multiple, single-peaked solutions for a semilinear Neumann problem. *Arch. Rational Mech. Anal.* 120 (1992), no. 4, 375–399.

[80] J. Wei: On the boundary spike layer solutions of a singularly perturbed semilinear Neumann problem. *J. Differential Equations* 134 (1997), no. 1, 104–133.

[81] J. Wei: On the construction of single-peaked solutions to a singularly perturbed semilinear Dirichlet problem. *J. Differential Equations* 129 (1996), no. 2, 315–333.

[82] J. Wei: On the effect of domain geometry in singular perturbation problems. *Differential Integral Equations* 13 (2000), no. 1-3, 15–45.

[83] M. Weinstein: Existence and dynamic stability of solitary wave solutions of equations arising in long wave propagation. *Comm. Partial Differential Equation* 12 (1987), 1133–1173.

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