Electrically and Magnetically Charged Solitons in Gauge Field Theory

Yisong Yang

Polytechnic Institute of New York University

Talk at the conference “Differential and Topological Problems in Modern Theoretical Physics”, SISSA, Trieste, Italy, April 26 – 30, 2010
Outline

1. From Maxwell equations to monopoles and dyons
2. Monopoles and dyons in gauge field theory
3. The Julia–Zee theorem
4. The Chern–Simons vortices
5. Self-dual Chern–Simons equations
6. Summary
Magnetism, monopoles, and P. Curie

P. Curie – Scientific giant in magnetism

Curie’s law (for magnetization of paramagnetic materials):

\[ M = \frac{C}{T} B \]

Curie constant, Curie point.

Magnetic monopoles?


Scholar.google citation count: 4

P. Dirac: Born on August 8, 1902
Mathematics and formulation of monopoles by Dirac

New mathematics with the birth of Quantum Mechanics is needed


Scholar.google citation count: 1396

Quoted from the first paragraph of the Dirac paper:

“The steady progress of physics requires for its theoretical formulation a mathematics that gets continually more advanced. This is only natural and to be expected.”

Dirac’s toy: The Maxwell equations
Maxwell equations

Variational structure

Spacetime coordinates $t = x^0, (x, y, z) = (x^i), x^\mu = (x^0, x^i)$

Gauge field $A_\mu, \mu = 0, 1, 2, 3$

Electromagnetic field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

4-Current density $J^\mu = (\rho, J)$

Charge density $J^0 = \rho$

Current density $J^k = J$

Action density (the Lagrangian density)

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu$$

Maxwell equations

$$\partial_\nu F^{\mu\nu} = - J^\mu$$
Static situation

$$\partial_0 = 0$$

$$A_0$$ – electric potential

$$A_i = A$$ – magnetic potential

$$E = \nabla A_0$$ – electric field

$$B = \nabla \times A$$ – magnetic field

Maxwell equations

$$\nabla \cdot E = \rho, \quad \nabla \times B = J, \quad \nabla \cdot B = 0, \quad \nabla \times E = 0$$

or

$$\Delta A_0 = \rho$$

$$\nabla \times \nabla \times A = J$$
The Maxwell equations are the Euler–Lagrange equations of the action functional

\[ I(A_0, A) = \int_{\mathbb{R}^3} \left\{ -|\nabla A_0|^2 + |\nabla \times A|^2 - A_0 \rho + A \cdot J \right\} \]

which is indefinite.

**Finite-energy condition**

\[ E = \int_{\mathbb{R}^3} \{E^2 + B^2\} \]

\[ = \int_{\mathbb{R}^3} \{|\nabla A_0|^2 + |\nabla \times A|^2\} < \infty \]
The point charge problems

**Point electric charge**

With

\[ \rho = 4\pi e \delta(x), \quad J = 0 \]

we can solve the Maxwell equations to get

\[ \mathbf{E} = \nabla A_0 = \frac{e}{|x|^3} x \quad \text{(Coulomb's law)} \]

\[ \mathbf{B} = 0 \]

Infinite energy

\[ \int_{\mathbb{R}^3} \mathbf{E}^2 = \int_0^\infty \frac{4\pi e^2}{r^2} \, dr = \infty \]
Point magnetic charge (a “monopole”)  
As before, we solve  
\[ \nabla \cdot \mathbf{B} = 4\pi m \delta(x) \]

to obtain  
\[ \mathbf{B} = \frac{m}{|x|^3} x \]  
(“a magnetic Coulomb’s law”)  

We arrive at infinite energy problem again.  
The real problem is more severe than this – inconsistency with the existence of a magnetic potential \( \mathbf{A} \):  
\[ \mathbf{B} = \nabla \times \mathbf{A} \]

which requires \( \nabla \cdot \mathbf{B} = 0 \) in all \( \mathbb{R}^3 \) ("magnetic field is solenoidal").
A deeper insight is needed to resolve the puzzle.

**Topological structure**

We hope to see what is needed to represent the magnetic field $\mathbf{B}$ of a monopole smoothly over the punctured space

$$M = \mathbb{R}^3 \setminus \{0\}$$

by a vector potential field $\mathbf{A}$.

In fact, if we insist to achieve a smooth relation $\mathbf{B} = \nabla \times \mathbf{A}$ in $M$, we run into trouble with the Stokes theorem:

$$4\pi m = \int_{S^2_R} \mathbf{B} \cdot d\mathbf{S} = \int_{S^2_R} (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$$

$$= \lim_{C \to 0} \oint_C \mathbf{A} \cdot d\mathbf{r} = 0$$

where $C$ is any closed curve over the sphere $S^2_R$ of radius $R$. 
A way out of this puzzle is to admit that the process of shrinking a closed curve on any surface around the origin will inevitably encounter a singularity.
The locus of singularities is a string connecting the monopole, called the Dirac string.
Local existence of magnetic potentials

Let $S$ be a Dirac string connecting the monopole at the origin with the infinity of $\mathbb{R}^3$. Consider the open set

$$U = \mathbb{R}^3 \setminus S$$

Then there is nothing preventing us from solving the equation

$$B = \nabla \times A$$

over $U$ so that $B$ is represented over $U$ (at least). However, we are not satisfied since $B$ should only be singular at the origin but not on the entire string $S$ which could well be an artifact.
From local to global representation

In order to find a way to globally represent the magnetic field $\mathbf{B}$ of the monopole over the entire space

$$\mathcal{M} = \mathbb{R}^3 \setminus \{0\}$$

we pick two strings $S^+$ and $S^-$, which are the positive and negative $x^3$-axes, respectively, form

$$U^+ = \mathbb{R}^3 \setminus S^+, \quad U^- = \mathbb{R}^3 \setminus S^-$$

and solve $\mathbf{B} = \nabla \times \mathbf{A}$ over $U^+$ and $U^-$, separately, to get $\mathbf{A}^+$ and $\mathbf{A}^-$. Since $\mathcal{M} = U^+ \cup U^-$, we are able to represent $\mathbf{B}$ globally from its local representations over $U^+$ and $U^-$. Puzzle solved?
Puzzle not yet solved

Consistency with physics
Motion of a test particle of mass $\mu$ and electric charge $e$ in $B$ is quantum mechanically governed by the gauged Schrödinger equation

$$i \frac{\partial \psi}{\partial t} = - \frac{1}{2\mu} (\nabla - eA)^2 \psi$$

which enjoys the gauge invariance

$$\psi \mapsto e^{i\omega} \psi, \quad A \mapsto A + \frac{1}{e} \nabla \omega$$

In other words, physical consistency requires that we must obey the gauge invariance in the Schrödinger equation when considering transition between the two magnetic potentials $A^+$ and $A^-$ over the interaction $U^+ \cap U^-$. 
Fiber bundle terminology

“Fiber bundle” (principal $U(1)$-bundle) $P \to \mathcal{M}$:

\[
\Omega = e^{i\omega} \in U(1), \quad \psi \mapsto \Omega \psi, \quad \mathbf{A} \mapsto \mathbf{A} + \frac{i}{e} \Omega \, d\Omega^{-1}
\]

The relation $\mathbf{B} = \nabla \times \mathbf{A}$ may be translated as

\[
\mathbf{B} = d\mathbf{A} \quad \text{(exactness)}
\]

which is locally valid over the coordinate patches $U^+$ and $U^-$, respectively, but can never be valid globally over the entire $\mathcal{M}$. The quantity that measures the “distance” from such a global “exactness” is an integer $N$ called the first Chern class which lies in the second cohomology of $\mathcal{M}$:

\[
N = c_1(P) \in H^2(\mathcal{M}, \mathbb{Z}) = H^2(S^2, \mathbb{Z}) = \mathbb{Z} = \text{Topology}
\]
The Dirac quantization formula

\[ \frac{2e}{\hbar} m = N, \quad N = \pm 1, \pm 2, \ldots \]

which explains why all electric charges are integer multiples of a basic unit charge.
Schwinger’s dyons

The motion of a dyon of electric and magnetic charges, \( e_1, m_1 \), in the electromagnetic field of another dyon of electric and magnetic charges, \( e_2, m_2 \), results in the Schwinger quantization formula

\[
2(m_1 e_2 - m_2 e_1) = N, \quad N = \pm 1, \pm 2, \cdots
\]
Modern physics

Monopoles and dyons are inevitable predictions rather than conceptual constructs

T. T. Wu, C. N. Yang, ’t Hooft, Polyakov, Julia, Zee, Bogomol’nyi, Prasad, Sommerfield, Jackiw, Witten
Typical variational structure (once again)

\[ I(A_0, A) = \int \left\{ -|\nabla A_0|^2 + |\nabla \times A|^2 + \text{other terms} \right\} \]

Monopoles – positive definite case ($A_0 \equiv 0$)

Dyons – indefinite case (when both $A_0$ and $A$ are present)

Constraints: finite energy; nontrivial topology
Abelian Higgs model (simplest gauge theory)

\( \phi \) – complex scalar field
\[ D_A \phi = \nabla \phi + i A \phi \] – gauge-covariant derivative

Action functional is
\[
I(\phi, A_0, A) = \int_{\mathbb{R}^3} \left\{ -|\nabla A_0|^2 + |\nabla \times A|^2 - A_0^2 |\phi|^2 + |D_A \phi|^2 + \frac{\lambda}{2} (|\phi|^2 - 1)^2 \right\}
\]

Energy is
\[
E(\phi, A_0, A) = \int_{\mathbb{R}^3} \left\{ |\nabla A_0|^2 + |\nabla \times A|^2 + A_0^2 |\phi|^2 + |D_A \phi|^2 + \frac{\lambda}{2} (|\phi|^2 - 1)^2 \right\}
\]
Equations of motion are

\[ D_A^2 \phi = \lambda (|\phi|^2 - 1) \phi - A_0^2 \phi, \]
\[ \nabla \times \nabla \times A = i \frac{1}{2} (\phi D_A \phi - \bar{\phi} D_A \phi), \]
\[ \Delta A_0 = |\phi|^2 A_0. \]

"Theorem", Trivial!

"Proof". \( \pi_2(S^1) = 0. \)

Open Problem \# 1. Obtain an analytic proof that any finite energy solution must be gauge-equivalent to the trivial solution

\[ \phi = 1, \quad A_0 = 0, \quad A = 0. \]
Modification (the advent of non-Abelian theory)

We need to enlarge the range of $\phi$ so that the set of images of $\phi$ at infinity become $S^2$. Thus, from

$$\pi_2(S^2) = 0$$

we may expect to get a big zoo of monopoles and dyons.

Hence, we modify $\phi$ so that it takes values in $\mathbb{R}^3$, with the natural symmetry group $SO(3)$.

For sake of topological simplicity, we further replace $SO(3)$ by $SU(2)$ since the latter is a universal double covering of $SO(3)$, which is simply connected, and the Lie algebras of $SO(3)$ and $SU(2)$ are identical so that we do not lose any gauge fields. In doing so, $\phi$ lies in the adjoint representation of $SU(2)$ instead.
Non-Abelian Gauge Field Theory

For convenience, we use iso-vectors for elements in $su(2)$.

Thus we have

\[ A_\mu \in \mathbb{R}^3 \ (\mu = 0, 1, 2, 3) \] – gauge field

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + eA_\mu \times A_\nu \] – curvature tensor

\[ D_\mu \phi = \partial_\mu + eA_\mu \times \phi \] – gauge-covariant derivative

Non-Abelian Yang–Mills–Higgs action density is

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} \cdot F^{\mu\nu} + \frac{1}{2} D_\mu \phi \cdot D^\mu \phi - \frac{\lambda}{4} (|\phi|^2 - 1)^2 \]
Static case $\partial_0 = 0$

Lagrangian density

$$\mathcal{L} = -\frac{1}{2} |\partial_i A_0 + e A_i \times A_0|^2 - \frac{1}{2} e^2 |A_0 \times \phi|^2 + \frac{1}{4} |F_{ij}|^2 + \frac{1}{2} |D_i \phi|^2 + \frac{\lambda}{4} (|\phi|^2 - 1)^2$$

Energy density

$$\mathcal{H} = \frac{1}{2} |\partial_i A_0 + e A_i \times A_0|^2 + \frac{1}{2} e^2 |A_0 \times \phi|^2 + \frac{1}{4} |F_{ij}|^2 + \frac{1}{2} |D_i \phi|^2 + \frac{\lambda}{4} (|\phi|^2 - 1)^2$$

Topology

$$\text{deg}(\phi) \equiv \text{deg} \left( \phi \big|_{S^2_\infty} \right) = N \quad \text{or} \quad [\phi \big|_{S^2_\infty}] \in \pi_2(S^2)$$
Equations of motion

We have

\[ D_i D_i \phi = -e^2 A_0 \times (A_0 \times \phi) + \lambda \phi (|\phi|^2 - 1) \]
\[ D_j F_{ij} = e (A_0 \times F_{i0} + \phi \times D_i \phi) \]
\[ \Delta A_0 = -e \partial_i (A_i \times A_0) \]
\[ + e (\partial_i A_0 + e A_i \times A_0) \times A_i + e^2 \phi \times (A_0 \times \phi) \]

The \( A_0 \) sector looks hard.

Q: Is \( A_0 \) component important for electricity?
Electromagnetism

The 't Hooft electromagnetic tensor is proposed to be

$$F_{\mu \nu} = \frac{\phi}{|\phi|} \cdot F_{\mu \nu} - \frac{1}{e|\phi|^3} \phi \cdot (D_\mu \phi \times D_\nu \phi)$$

The induced electric field $E = (E^i)$ is given by

$$E^i = F^i_0 = \frac{\phi}{|\phi|} \cdot (-\partial_i A_0 + eA_0 \times A_i) - \frac{1}{|\phi|^3} \phi \cdot ([A_0 \times \phi] \times D_i \phi)$$

which dictates the condition

$$A_0 \neq 0$$

in order to have a nontrivial electric sector.
Magnetic charge

The magnetic charge $Q_m$ can be obtained through computing the total flux of the magnetic field induced to yield the quantization formula

$$eQ_m = N \quad (= \text{deg}(\phi))$$

(Arafune, Freund, and Goebel)

Facts

The magnetic charge $Q_m$ has a topological origin

Although the above quantization formula generalizes that of Dirac, the electric charge $e$ is put in by hand externally which may not be the electric charge induced from the ’t Hooft tensor
Electric charge

The electric charge $Q_e$ induced from the 't Hooft tensor can be computed as the total flux generated from the electric field which gives us

$$Q_e = \frac{1}{4\pi} \int_{\mathbb{R}^3} \partial_i F_{0i}$$

which is not topological and requires

$$A_0 \neq 0$$

as observed already.

In other words, we need to obtain solutions with $A_0 \neq 0$ in order that both magnetic and electric charges are present (dyons).
Existence of Dyons

For any \( N \in \mathbb{Z} \), obtain a finite-energy \( N \)-dyon solution with nonvanishing magnetic and electric charges \( Q_m \) and \( Q_e \).

Remark: Too ambitious a problem!
Special cases

\[ A_0 = 0 \text{ ("temporal gauge" = monopole case)} \]

The equations become

\[
\begin{align*}
D_i D_i \phi &= \lambda \phi (|\phi|^2 - 1) \\
D_j F_{ij} &= e \phi \times D_i \phi
\end{align*}
\]

\( N = 1 \), radially symmetric solution due to Belavin, Polyakov, Schwartz, and Tyupkin

\( N = 1, \lambda = 0 \) (BPS), reduction to the self-dual or BPS equations

\[
B_i = \pm D_i \phi, \quad B_i = \frac{1}{2} \epsilon_{ijk} F_{jk}
\]

Prasad–Sommerfield, Weinberg, Taubes, Ward, Atiyah, Manton
Theorem 1 (forthcoming 2010)

For $N = 1$, there is a one-parameter family of dyon solutions with

$$Q_m = \frac{1}{e} \text{ (topological)}$$

$$Q_e = \text{any value in an open interval (non-topological)}$$

Remark: In particular, $Q_e$ is not quantized
Special case $\lambda = 0$

Julia and Zee found an explicit family of dyon solutions for which the electric and magnetic charge relation is given by

$$Q_e = \sinh \alpha Q_m, \quad -\infty < \alpha < \infty$$

where the parameter $\alpha$ determines the asymptotic value of the modulus of the Higgs field,

$$\lim_{|x| \to \infty} |\phi(x)| = \cosh \alpha$$
Open problems (#2 — #4)

Any dyon solution for \( N \geq 2 \) and \( \lambda > 0 \)?

What mechanism fixes \( Q_e \)?

How to compute dyons?
Progress in other dyon problems

Dyons in the electroweak theory of Weinberg–Salam

Dyons in gauged Skyrme model
(joint work with F. H. Lin, 2010)

$Q_e$ is still not fixed
Dimension reduction

Can we lower the dimension so that electric charge is fixed by topology?

Motivation: Vortex-lines vs monopoles

Answer: Yes (?) and No (?)

About the “No” answer – this is the Julia–Zee theorem which is a fundamental property of classical gauge field theory

About the “Yes” answer – this is the existence theory for Chern–Simons vortices
For the Abelian Higgs model over $\mathbb{R}^2$, Julia and Zee proved, assuming radial symmetric and sufficient decay rate at infinity, that the field configuration must stay in the temporal gauge

$$A_0 = 0$$

In other words, the Abelian Higgs model over $\mathbb{R}^2$ must be electrically neutral and must be the same as the classical Ginzburg–Landau model for superconductivity.

A generally accepted principle in classical gauge field theory

In classical gauge field theory, Abelian or non-Abelian, finite-energy static solutions must stay in the temporal gauge

\[ A_0 = 0 \]

Thus, the “Julia–Zee theorem” is rigorously established.
1. From Maxwell equations to monopoles and dyons
2. Monopoles and dyons in gauge field theory
3. The Julia–Zee theorem
4. The Chern–Simons vortices
5. Self-dual Chern–Simons equations
6. Summary

**PDE problems**

**Abelian case**

\[
D_i D_i \phi = 2V'(|\phi|^2)\phi - A^2_0 \phi
\]

\[
\partial_j F_{ij} = \frac{i}{2}(\overline{\phi} D_i \phi - \phi \overline{D_i \phi})
\]

\[
\Delta A_0 = |\phi|^2 A_0
\]

Energy density over \( \mathbb{R}^2 \) is

\[
\mathcal{H} = \frac{1}{2} |\partial_i A_0|^2 + \frac{1}{2} A^2_0 |\phi|^2 + \frac{1}{4} F^2_{ij} + \frac{1}{2} |D_i \phi|^2 + V(|\phi|^2)
\]

Remark: It seems quite trivial that \( A_0 = 0 \) should follow readily from the finite-energy condition. However, the real situation is trickier because the method is expected to work over \( \mathbb{R}^2 \) only.
Non-Abelian case

The $A_0$-equation is

$$\Delta A_0 + \partial_i(A_i \times A_0) + A_i \times \partial_i A_0 + A_i \times (A_i \times A_0) = \phi \times (A_0 \times \phi)$$

The part of the energy density over $\mathbb{R}^2$ that involves $A_0$ is

$$\frac{1}{2} |\partial_i A_0 + (A_i \times A_0)|^2 + \frac{1}{2} |A_0 \times \phi|^2$$

Remark

It may be interesting to recall similar results regarding entire solutions of PDEs such as the Liouville theorem and the Bernstein theorem
The Chern–Simons vortices over $\mathbb{R}^2$

Extra topological term added – the Chern–Simons term
In 3-dimensional spacetime and in terms of differential forms, the term reads

$$A \wedge dA$$

Existence of electrically and magnetically charged vortices
(Dyons in $\mathbb{R}^2$)

Two areas of research:
Self-dual case (extensive work) (Italian school, in particular)
Non-self-dual case (hard situation) – progress report
Abelian Case (the simplest Chern–Simons–Higgs model)

The Lagrangian density is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\kappa}{4} \epsilon^{\mu\nu\alpha} A_\mu F_{\nu\alpha} + \frac{1}{2} D_\mu \phi \overline{D^\mu \phi} - \frac{\lambda}{8} (|\phi|^2 - 1)^2$$

$\phi$ – complex scalar

$D_\mu \phi = \partial_\mu \phi + i A_\mu \phi$

$\kappa > 0$ (the Chern–Simons coupling constant)
Static equations

\[ D_i D_i \phi = \frac{\lambda}{2} (|\phi|^2 - 1) \phi - A_0^2 \phi \]

\[ \partial_j F_{ij} = \kappa \epsilon_{ij} \partial_j A_0 + \frac{i}{2} (\bar{\phi} D_i \phi - \phi \bar{D}_i \phi) \]

\[ \Delta A_0 = \kappa F_{12} + |\phi|^2 A_0 \]

\[ F_{12} \quad \text{– magnetic field} \]

We now formally compute the magnetic and electric charges.
Magnetic charge

We have

\[ Q_m = \frac{1}{2\pi} \int_{\mathbb{R}^2} F_{12} \, dx \]

\[ = c_1 = \text{the first Chern class} \]

\[ = N = \text{the winding number of } \phi \text{ near infinity} \]

\[ = \text{topological invariant} \]
Electric charge

In order to compute the electric charge, recall that the current density is given by

\[ J^\mu = \frac{i}{2}(\bar{\phi} D^\mu \phi - \phi D^\mu \bar{\phi}) \]

Hence the electric charge density is given by

\[ \rho = J^0 = \frac{i}{2}(\bar{\phi} D^0 \phi - \phi D^0 \bar{\phi}) = -A_0 |\phi|^2 \]

So

\[ Q_e = \frac{1}{2\pi} \int_{\mathbb{R}^2} \rho \, dx \]

\[ = -\frac{1}{2\pi} \int_{\mathbb{R}^2} A_0 |\phi|^2 \, dx = \text{topological?} \]
Theorem 3


For any $N \in \mathbb{Z}$, the static Chern–Simons–Higgs equations on $\mathbb{R}^2$ have a finite-energy smooth solution of topological class $N$. The solution decays fast enough at infinity so that

$$\int_{\mathbb{R}^2} \Delta A_0 \, d\mathbf{x} = 0$$

As a consequence, the magnetic and electric charges are all quantized topologically and obey the formulas

$$Q_m = N, \quad Q_e = \kappa N$$

Remark: This concludes our "Yes" answer discussion
Proof of Theorem 3

The action functional is of the form

\[ I = I^+ - I^- \]

\[ = \int_{\mathbb{R}^2} \left( \frac{1}{2} F_{12}^2 + \frac{1}{2} |D_i \phi|^2 + \frac{\lambda}{8} (|\phi|^2 - 1)^2 \right) \, dx \]

\[ - \int_{\mathbb{R}^2} \left( \frac{1}{2} |\nabla A_0|^2 + \kappa A_0 F_{12} + \frac{1}{2} |\phi|^2 A_0^2 \right) \, dx \]

The energy is

\[ E = I^+ + \int_{\mathbb{R}^2} \left( \frac{1}{2} |\nabla A_0|^2 + \frac{1}{2} |\phi|^2 A_0^2 \right) \, dx \]

The admissible class is

\[ \mathcal{C} = \left\{ (\phi, A_0, A_i) \mid E(\phi, A_0, A_i) < \infty, \quad \delta I^- = 0 \right\} \]
Proof continued

From $\delta I^-$, we get

$$\int_{\mathbb{R}^2} \left( \nabla A_0 \cdot \nabla \eta + \kappa F_{12} \eta + |\phi|^2 A_0 \eta \right) \, dx = 0, \quad \forall \eta$$

Taking $\eta = A_0$ as a test function, we have

$$\int_{\mathbb{R}^2} ( |\nabla A_0|^2 + |\phi|^2 A_0^2 ) \, dx = -\kappa \int_{\mathbb{R}^2} F_{12} A_0 \, dx$$

Inserting this into the action functional $I$, we find

$$I = E = \text{positive definite}$$

Q.E.D.
Self-dual Chern–Simons equations

Two prototype equations
The Liouville equation (integral, “non-relativistic”, “doable”)

\[ \Delta u = \pm e^u \]

Extension: Toda systems

\[ \Delta u_a = -K_{ab}e^{u_b}, \quad a, b = 1, \ldots r, \quad r = \text{rank of gauge group} \]

The “Jaffe–Taubes” equation (non-integrable, “relativistic”)

\[ \Delta u = e^u - 1 \]

Full of challenges and many still intractable (recent focus)
(contributors from Italian school include: G. Tarantello, M. Nolasco, T. Ricciardi, D. Bartolucci)
Example 1

(solved with C. S. Lin and A. Ponce, J. Funct. Anal., 2007)

The governing equations over $\mathbb{R}^2$ are

$$\Delta u = e^v(e^u - 1)$$
$$\Delta v = e^u(e^v - 1)$$

How to approach this problem?

(Lack of \textit{apriori} estimates or compactness; no maximum principle)

Variational structure?
Example 1 continued

With

\[ u + v = f, \quad u - v = g \]

we transform the equations into

\[
\begin{align*}
\Delta f &= 2f - \frac{1}{2}(f+g) - \frac{1}{2}(f-g) \\
\Delta g &= \frac{1}{2}(f+g) - \frac{1}{2}(f-g)
\end{align*}
\]

which are the Euler–Lagrange equations of the action functional

\[
l(f, g) = \int_{\mathbb{R}^2} \left( \frac{1}{2}|\nabla f|^2 - \frac{1}{2}|\nabla g|^2 + 2f - 2\frac{1}{2}(f+g) - 2\frac{1}{2}(f-g) \right) \, dx
\]

which is indefinite
Example 2 (unsolved; non-relativistic; Open Problem # 5)

The governing equations over \( \mathbb{R}^2 \) are

\[
\Delta u = e^v - 1 \\
\Delta v = e^u - 1
\]

With the same change of variables, \( f = u + v \) and \( g = u - v \), we have the new equations

\[
\Delta f = e^{\frac{1}{2}(f+g)} + e^{\frac{1}{2}(f-g)} - 2 \\
\Delta g = -e^{\frac{1}{2}(f+g)} + e^{\frac{1}{2}(f-g)}
\]

which are the Euler–Lagrange equations of the action functional

\[
l(f, g) = \int \left( \frac{1}{2} |\nabla f|^2 - \frac{1}{2} |\nabla g|^2 + 2e^{\frac{1}{2}(f+g)} - 2e^{\frac{1}{2}(f-g)} - 2f \right) \, dx
\]

which is again indefinite ("most off-diagonal").

The governing equations over $\mathbb{R}^2$ are

$$
\Delta u_a = \lambda \left( K_{ab} K_{bc} e^{u_b + u_c} - K_{ab} e^{u_b} \right), \quad a, b, c = 1, \ldots, r
$$

$(K_{ab})$ is the Cartan matrix and $r$ is the rank of the gauge group $G$.

Suppose (a very general condition for standard vacua to exist)

$$
\sum_{b=1}^{r} (K^{-1})_{ab} > 0, \quad a = 1, \ldots, r
$$

Existence of topological solutions satisfying

$$
\lim_{|x| \to \infty} u_a(x) = \ln \left( \sum_{b=1}^{r} (K^{-1})_{ab} \right), \quad a = 1, \ldots, r
$$

**Method:** Calculus of variations; positive-definite functional; suitable transformations
Example 3 continued

For $G = SU(N)$, $N \geq 3$, $r = N - 1$, and the matrix $K$ is

$$K = \begin{pmatrix}
2 & -1 & \cdots & \cdots & 0 \\
-1 & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & -1 \\
0 & \cdots & \cdots & -1 & 2
\end{pmatrix}$$

When $K$ is not symmetric, it is a curious inquiry whether there is a solution theory when $K$ is replaced by $K^\tau$.

Indeed, in the original physics literature, $K$ is wrongly given as $K^\tau$ which defied solution for a while.

This leads to our **Open Problem # 6**
Example 3 continued

It will be interesting to develop an existence theory when the domain $\mathbb{R}^2$ is replaced by a closed 2-surface such as $T^2$.

There are some results due to M. Nolasco and G. Tarantello when

$$G = SU(3)$$

A general existence study is yet to be fully carried out

Open Problem # 7
Summary

There are locally concentrated static solutions carrying both electric and magnetic charges called dyons.

Dyons are topologically originated.

Dyons are realized as critical points of indefinite action functionals.

The magnetic charge of a dyon is quantized topologically.

The electric charge of a dyon takes continuous values and is non-topological.

Dyons cannot exist in two spatial dimensions in classical theory.

Dyons can be generated in two spatial dimensions if the Chern–Simons terms are added in classical models so that both electric and magnetic charges are topologically quantized.
Quotation from P. Dirac (again)

“I played with a few equations.”

— Words of Paul Dirac when answering questions from an interviewer

Encouraging?

— End of Talk —