Perelman's Dilaton

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Topics

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- Ricci flow as a constrained gradient flow.

Notation

- Let M be a smooth finite dimensional manifold and g a smooth Riemannian metric on it. We will call the pair (M, g) a *Riemannian manifold*.
- We will denote with

$$\Gamma_{ij}^{k} = \frac{1}{2} g^{kr} [\partial_i g_{rj} + \partial_j g_{ir} - \partial_r g_{ij}]$$

the Christoffel symbols of the Levi-Civita connection associated to the Riemannian metric g.

Curvature Tensors

- We define the Riemann tensor by mean of the commutation rule for the covariant derivative associated to the Levi-Civita connection

$$\nabla_i \nabla_j \omega_k - \nabla_j \nabla_i \omega_k = \mathrm{R}_{ijk}{}^{l} \omega_l = \left[\partial_j \Gamma^l_{ik} - \partial_i \Gamma^l_{jk} + \Gamma^r_{ik} \Gamma^l_{jr} - \Gamma^r_{jk} \Gamma^l_{ir} \right] \omega_l \,,$$

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- We define the Ricci tensor and the scalar curvature respectively as

$$\mathrm{R}_{ij} = g^{kl}\mathrm{R}_{kilj} \quad \text{and} \quad \mathrm{R} = g^{ij}\mathrm{R}_{ij}\,.$$

The Ricci Flow

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- Given an admissible metric g_0 on M, it is always possible take it as a good initial datum for a Ricci flow (the equation is weakly parabolic).
- Using Riemannian normal coordinates, one can check that

$$\partial_t \Gamma^k_{ij} = -g^{kr}[\nabla_i \mathrm{R}_{rj} + \nabla_j \mathrm{R}_{ir} - \nabla_r \mathrm{R}_{ij}]\,.$$

Hamilton's Result

Theorem

The Ricci flow is not the gradient flow of any smooth functional of the Riemannian metric and its spatial derivatives.

Proof.

It is possible to argue by contraddiction computing the second variation and checking that its symmetry properties are not satisfied if the first variation is the Ricci tensor. □

Perelman's *gradient-like* Formulation I

- Consider the Hilbert functional associated to a Riemannian metric $\mathcal{H} = \int_M \mathrm{R} dV$ and its first variation $\partial_s \mathcal{H} = \int_M \langle h_{ij}, \frac{1}{2}\mathrm{R} g_{ij} - \mathrm{R}_{ij} \rangle dV$.

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- There are problems due to the variation of the volume. If we want to get rid of them, we are forced to introduce some "*auxiliary quantity*".

Perelman's *gradient-like* Formulation II

Theorem (Perelman)

Let $\partial_s g_{ij} = h_{ij}$ and $\partial_s f = l$ be variations of the Riemannian metric and of a smooth function $f : M \to \mathbb{R}$. Then, the variation of the functional

$$\mathcal{F}(g, \partial g, \partial^2 g, f) = \int_M (R + |\nabla f|^2) e^{-f} dV$$

is given by

$$\partial_s \mathcal{F} = \int_M [-h^{ij}(\mathrm{R}_{ij} + \nabla^2_{ij} f) + (\frac{tr_g h}{2} - l)(2\Delta f - |\nabla f|^2 + R)] e^{-f} dV\,.$$

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- We check that the Hilbert's functional of the extended manifold can reproduce the values of the Perelman's functional for the original manifold.
- We show that the projection on the original manifold of the (suitably constrained) gradient flow of the Hilbert functional of the extended manifold is the (diffeo-modified) Ricci flow.

The Setting

Let (M^m, g) and (N^n, h) be two closed Riemannian manifolds of dimension m and n respectively and $f : M \to \mathbb{R}$ be a smooth function on M. Consider $\widetilde{M} = M \times N$ with the metric

$$\widetilde{g} = e^{-Af} g \oplus e^{-Bf} h,$$

where A and B are real constants.

Computations

$$\widetilde{\mathrm{R}}_{jl} = \mathrm{R}_{jl} + \nabla^2_{jl} f\left(\frac{Am+Bn}{2} - A\right) + \frac{A}{2} g_{jl}\left[\Delta f - |\nabla f|^2\left(\frac{Am+Bn}{2} - A\right)\right] + \frac{1}{4} df_j df_l (2ABn + (m-2)A^2 - B^2 n).$$

$$\widetilde{\mathrm{R}}_{\alpha\beta} = \mathrm{R}_{\alpha\beta} + \frac{B}{2} e^{(A-B)f} h_{\alpha\beta}\left[\Delta f - |\nabla f|^2\left(\frac{Am+Bn}{2} - A\right)\right]$$

$$\widetilde{\mathrm{R}} = e^{Af}\mathrm{R}^M + e^{Bf}\mathrm{R}^N + e^{Af}\Delta f(Am+Bn-A) + \frac{e^{Af}}{4}|\nabla f|^2 (4ABn - 2ABmn + 3mA^2 - 2A^2 - m^2A^2 - B^2 n - B^2 n^2)\,.$$

Ansatz

$$2ABn + (m-2)A^2 - B^2 n = 0 \tag{C1}$$

$$\frac{Am+Bn}{2} - A = 1 \qquad \Longleftrightarrow \qquad A(m-2) + Bn = 2\,. \tag{C2}$$

$$\theta := \frac{A}{B} \qquad \text{then} \qquad (m-2)\theta^2 + 2n\theta - n = 0$$

Lemma

If $m+n>2$, we can always find two non zero constants A and B satisfying (C1) *and* (C2)

The *static* Problem

Theorem

Let (M^m, g) and (N^n, h) be two closed Riemannian manifolds of dimension m and n respectively, with $m + n > 2$ and $f : M \to \mathbb{R}$ be a smooth function on M. Consider on $\widetilde{M} = M \times N$ the metric $\widetilde{g}$ given by

$$\widetilde{g} = e^{-Af} g \oplus e^{-Bf} h\,,$$

where A and B are constants satisfying conditions $(\mathrm{C1})$ and $(\mathrm{C2})$. Then:

$$\int_{\widetilde{M}} \widetilde{\mathrm{R}}\, d\widetilde{\mu} = \mathrm{Vol}(N, h)\mathcal{F}(g, f) + \left(\int_M e^{(B-A-1)f}\, d\mu\right) \int_N \mathrm{R}(N, h)\, d\sigma$$

The Flow

- Consider $\widetilde{M} = M \times N$ with a time dependent metric $\widetilde{g}(t)$ for $t \in [0, T]$ such that $\widetilde{g}(0) := \widetilde{g} = \hat{g} \oplus \varphi h$. Where $\varphi : M \to \mathbb{R}$ is a smooth function and (N, h) is Ricci flat and has unit volume.

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- Let the metric evolve by the gradient of the Hilbert functional with the constraint $\partial_t(\varphi^{-\theta}\tilde{\mu}) = 0$.

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- Let the metric evolve by the gradient of the Hilbert functional with the constraint $\partial_t(\varphi^{-\theta}\tilde{\mu}) = 0$.
- The evolution for h, shows that it evolves just by multiplication by a positive factor. It follows that h is constant.
- Defining $f := -\frac{1}{B}\log\varphi$, one gets $\widetilde{g} = e^{-Af} g \oplus e^{-Bf} h$ and $\delta\widetilde{g} = e^{-Af}\left(\delta g - Ag\delta f\right) \oplus e^{-Bf}\left(-Bh\delta f\right)$.

The constrained Gradient Flow

Theorem

Suppose there exists a unique solution of the gradient flow which preserves the warped product. Then:

$$\delta \int_{\widetilde{M}} 2\widetilde{\mathrm{R}}\, d\widetilde{\mu} = -2 \int_M \langle \mathrm{Ric}(M,g) + \nabla^2 f \,|\, \delta g \rangle e^{-f}\, d\mu\,.$$

Hence, the associated flow of the metric $\widetilde{g} = e^{-Af} g \oplus e^{-Bf} h$ *is described by*

$$\begin{cases} \partial_t g = -2(\mathrm{Ric}(M,g) + \nabla^2 f) \\ \partial_t h = 0 \\ \partial_t f = -\Delta f - \mathrm{R}(M,g)\,. \end{cases}$$

Applications

- We can study the properties of the constrained gradient flow on the extended manifold to obtain informations on the original one.

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- We can study the properties of the constrained gradient flow on the extended manifold to obtain informations on the original one.
- We can choose A and B in order to reproduce other functionals on the original manifold.