March 25, 2019

SISSA
Entrance
Examination

Astroparticle Physics Curriculum

SOLVE two out of the four problems. If you hand in more than two, then state clearly which two problems you wish to be evaluated (otherwise, the two with the *lowest* score will be considered).
Answer two out of the six questions concisely (no more than one page per question).

Problem 1

Consider the following Friedmann and continuity equations:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_{\mathrm{T}} - \frac{k}{a^2} \tag{1}$$

and

$$\dot{\rho_{\mathrm{T}}} = -3H(\rho_{\mathrm{T}} + p_{\mathrm{T}})\,, \tag{2}$$

where a is the scale factor, H the Hubble parameter, derivatives are made with respect to time, k is a constant of values (-1,0,1) according to the geometry of the Universe, and ρ_{T} and p_{T} are the energy density and pressure of the cosmological fluid (G the gravitational constant and $c = 1$ in the eqs. above).

1. Consider now the two cases $\rho_{\mathrm{T}} = \rho_{\mathrm{m}}$ and $\rho_{\mathrm{T}} = \rho_{\mathrm{s}}$). The first case is a form of matter with $p_{\mathrm{m}} = 0$ and the second is another fluid component whose equation of state is $p_{\mathrm{s}} = \gamma\rho_{\mathrm{s}}$ with $-1 \leq \gamma < -1/3$. Derive the evolution of the two energy densities as a function of the scale factor and of their values at $a = 1$ (present time).

2. Using the Friedmann and continuity equations above derive an expression for the second derivative of the scale factor $\ddot{a}$ (second Friedmann equation).

3. Having defined $\Omega_{\mathrm{i}}(a) = \rho_{\mathrm{i}}(a)/\rho_{\mathrm{cr}}(a)$, $\rho_{\mathrm{cr}}(a) = 3H^2(a)/(8\pi G)$ for $i = (m, s)$, and $1 + z = 1/a$, find an expression for the parameter $\Omega_{\mathrm{m},0} = \Omega_{\mathrm{m}}(a = 1)$ as a function of the two following quantities *only*: 1) $z_\star$, the redshift at which the condition $\dot{a} = \ddot{a} = 0$ is satisfied (the relation between redshift and scale factor is $1 + z = 1/a$); 2) $n = 3 + 3\gamma$.

4. Calculate $\Omega_{\mathrm{m},0}$ in the limit in which $n \to 2$ and study the $\Omega_{\mathrm{m},0}(z_\star)$ function obtained in this limit. Discuss in particular what happens at $z_\star \to 0$.

5. Calculate the value of n for which the present age of a flat $k = 0$ Universe with $\rho_{\mathrm{T}} = \rho_{\mathrm{s}}$ is three times the age of a flat $k = 0$ Universe with $\rho_{\mathrm{T}} = \rho_{\mathrm{m}}$.

Problem 2

Consider the Schwarzschild geometry,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{1 - \frac{2M}{r}} dr^2 + r^2 d\Omega^2, \tag{1}$$

where we are using geometric units $G = c = 1$. The differential equations for affinely parametrized radial null geodesics in the equatorial plane ($\theta = \pi/2$) can be reduced to a single equation,

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V_{\text{eff}} = \frac{1}{2}E^2. \tag{2}$$

where

$$V_{\text{eff}} = \frac{1}{2}\left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2}$$

where E and L are respectively the conserved energy and the angular momentum L on these geodesics.

1. Show that there are null geodesics that follow circular trajectories, and calculate the corresponding radius R. This radius determines what is known as the *photon sphere*. Are these trajectories stable or unstable?

2. Show that there is a maximum angular momentum $L_\star$ with which both ingoing and outgoing radial null geodesics can cross the photon sphere ($dr/d\lambda \neq 0$), and calculate the value of $L_\star$. What happens to an outgoing null geodesic from inside the photon sphere endowed with an angular momentum larger than $L_\star$?

3. Let us imagine that we place a spherical perfect mirror at a given radius $r = 2M + \delta$ with $0 < \delta < M$, and that we send a photon radially from the photon sphere to the center of the gravitational potential. Calculate the time that it takes for the photon to be reflected in the mirror and reach us again. The time must be calculated in our clocks, assuming that we are static observers at $r = R = 3M$. What happens if we remove the mirror? You may need the following result:

$$\int \frac{x}{x - x_0} dx = x + x_0 \ln(x - x_0) + C. \tag{3}$$

Problem 3

Consider the production of a neutral scalar mesons π^0, in the interaction of a photon γ with a charged nucleus N initially at rest:

$$\gamma + N \to \pi^0 + N\,. \tag{1}$$

Let M be the nucleus rest mass, and m the meson rest mass. [You can work in natural units where $c = 1$.]

1. Compute the minimum photon energy k such that the meson can be produced.

2. For the process initiated by a photon of minimum energy k, compute the velocity V of the produced meson and the corresponding relativistic factor $\gamma(V)$, in terms of the masses m, M.

3. Once the neutral meson is produced, it rapidly decays into two photons. For the process initiated by a photon of minimum energy k, compute the energies of the photons coming from the meson decay in terms of the masses m and M, in the lab frame where the target nucleus is at rest. Discuss the limiting case where $m \ll M$.

4. Given the decay channel of π^0 into 2γ, estimate the suppression factor of the related decay mode of π^0 into an electron-positron pair.

Problem 4

Consider a massive vector field $B^\mu(x)$ in four dimensions with Lagrangian density:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 B_\mu B^\mu - B^\mu J_\mu \,, \tag{1}$$

where $F_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu$ and the current $J^\mu(x)$ is some external source for the field $B^\mu(x)$. [The adopted signature is $+---$.]

1. Derive the equations of motion.

2. Show that if $m \neq 0$, the equations of motion imply that $\partial_\mu B^\mu = 0$ in case the current satisfy $\partial_\mu J^\mu = 0$.

3. Show that B_0 is non-dynamical, being the canonical momentum conjugate to B_0 vanishing, and that B_0 can be completely eliminated in terms of the other fields and the external source.

4. Suppose now that $m = 0$; even though it does not follow from the equations of motion, can one still assume $\partial_\mu B^\mu = 0$?

Questions

1. Give a few examples of astrophysical tracers of the matter density field that can be used for cosmological studies. Briefly outline the main scientific questions that they can address.

2. List a few properties that particle dark matter candidates must obey.

3. You are standing in an elevator without access to the exterior world. The elevator does not appear to move and you experience a constant downward force proportional to your mass. How could you discriminate between being in an elevator standing still at some floor of a building on Earth or alternatively being uniformly accelerated in flat spacetime by some rocket? Why doing so would not be a violation of Einstein equivalence principle?

4. Idealising the Earth to be a spherical body without angular momentum, could you determine its radial density profile by measuring its gravitational field with orbiting satellites?

5. Is the baryon number a conserved quantity in the Standard Model of particle physics?

6. Why would the observation of neutrinoless double beta decay discriminate between the Majorana and Dirac nature of neutrinos?